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Summary

An investigation is made into the stability of four types of two-dimensional free surface flows of an ideal fluid when subjected to small perturbations.

For the case of a bounded hollow vortex flow only neutrally stable perturbations are found, and the propagation of these perturbations is compared to the propagation of gravity waves in water. The impinging of a jet upon a plate of finite width is also found to be a stable configuration. A series of orifice flows is investigated, all of whose perturbations are found to be stable with the exception of an isolated unstable perturbation in the case of one member of the series, namely the flow through a Borda mouthpiece. Finally the existence of unstable perturbations is indicated in the case of equal and opposite impinging jets.

Acknowledgements

The authors wish to express their appreciation for the assistance of L. C. Maximon for many valuable comments and suggestions. Grateful acknowledgement is made to Mrs. Marion Porritt for the typing and to Miss Eleanor Addison for the figures.

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ON THE STABILITY OF SOME FLOWS OF AN IDEAL FLUID WITH FREE SURFACES[1]

By

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I. Introduction

Steady state plane flows of an incompressible inviscid fluid with free surfaces were originally studied by Helmholtz [2]* and Kirchoff [3], and have since been thoroughly reported in the literature. Their work was an attempt to improve the classical solutions of flow around sharp corners which are physically unacceptable because they give rise to infinite velocities at the corners. Helmholtz and Kirchoff reasoned that as the velocity becomes large, the pressure in the fluid decreases to the value at which the fluid goes over into the vapor state. This gives rise to a so called cavitated region bounded by a "free surface" over which the pressure is assumed to be maintained constant and uniform.

One also deals with steady free surface flows in the case of jets flowing in an ambient constant pressure atmosphere.

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1. The results presented in this report were obtained in the course of research conducted under contract N7onr-35807 sponsored by the Mechanics Branch, Office of Naval Research.

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* Numbers in brackets refer to the bibliography at the end of the paper.

Several examples of such problems are treated by Milne-Thompson [4].

Very little work has been done in the past concerning time dependent flows with free surfaces. Lord Kelvin [5] discussed the vibrations of a hollow columnar vortex flow. Some unsteady free surface flows under the influence of external forces, such as gravity waves in water are discussed in Lamb [6] and the investigation there is extended to include the effects of surface tension and viscosity.

Recently Ablow and Hayes [1] developed a theory of the small perturbations of the two-dimensional flow of a perfect fluid in the presence of a free surface without external forces. They then used their theory to study two specific problems, namely the flow around a hollow vortex and the flow through a Borda mouthpiece.

The present investigation will concern itself with an extension of the work of Ablow and Hayes [1] to some free surface flows of jets as well as to a number of generalizations of problems treated in [1]. Our primary concern will be to obtain information concerning the stability of these flows.

II. Résumé of Basic Theory

The basic theory underlying the methods used in this report has been discussed in detail in the work of Ablow and Hayes [1]. For the sake of convenience however, a brief outline of the important results will be given.

A. Assumptions

We shall be dealing with perturbations of steady state flows which do not fill the entire plane. They can be conveniently divided into three categories: (1) flows which are cavitated due to the fact that there is a minimum pressure the fluid can sustain; (2) jet type flows; (3) flows which are a combination of (1) and (2). In all cases there exist in the steady flow free surfaces along which the pressure remains constant and uniform.

The fluid is assumed homogeneous, incompressible and inviscid. Both the steady and perturbed states are assumed to be irrotational and two-dimensional.

All quantities are written in non-dimensional form through the use of a characteristic length, pressure and velocity in such a manner as to make the steady state velocity along the free surface of unit magnitude.

Under the assumptions made the flows must satisfy Bernoulli's equation in the form

$$p + \frac{1}{2}\rho q^2 + \rho\dot{\varphi} = C(t) \tag{2.1}$$

where p is the pressure, ρ the density, q the velocity, φ the velocity potential and $C(t)$ is a function of time alone. The dot indicates partial differentiation with respect to time.

Furthermore, we can introduce a complex potential f and a complex velocity w such that

$$w = \frac{df}{dz} \tag{2.2}$$

where $w = u - iv$ and u and v are the cartesian velocity components in the $z = x + iy$ plane.

B. Basic Flow

The basic steady flow satisfies the steady form of (2.1)

$$p_o + \frac{1}{2}\rho q_o^2 = \text{constant} \tag{2.3}$$

where the use of a zero subscript denotes the basic steady flow. Since

$$w_o = \frac{df_o}{dz_o} \quad \text{and} \quad q_o = |w_o|$$

we can write (2.3) as

$$p_o + \frac{1}{2}\rho w_o \bar{w}_o = \text{constant}$$

where the bar indicates the operation of taking the complex conjugate.

C. Perturbation Relations

We shall now give the steady state basic flow a small perturbation in terms of a small real parameter ϵ in the form

$$z = z_o + \epsilon z_1(z_o,t)$$

$$f(z,t) = f_o(z_o) + \epsilon f_1(z_o,t)$$

$$w(z,t) = w_o(z_o) + \epsilon w_1(z_o,t)$$

$$p(z,t) = p_o(z_o) + \epsilon p_1(z_o,t)$$

where f and w are analytic functions of z_o. All subsequent relations will be linearized by neglecting terms of order ϵ^2 and higher. Hence all perturbations are small perturbations in that they are correct only to first order in ϵ. It is convenient to perturb the independent variable z_o, although the perturbations in f, w and p are given in terms of the fixed point z_o.

The perturbations given above are not independent since we can derive the following relations from (2.1) and (2.2)

$$w_o^2 \, z_1' + w_1 = w_o \, f_1' \qquad (2.4)$$

$$p_1 + \rho R\ell \, [w_1 \, \bar{w}_o + \dot{f}_1 - w_o \, \dot{z}_1] = 0 \qquad (2.5)$$

where the prime indicates partial differentiation with respect to f_o. Thus, we see that only two of the four perturbations are independent.

We note that, when properly chosen, two different sets of perturbations, e.g., (z_3, f_3) and (z_4, f_4) may represent the same physical perturbation. Their difference, namely $z_I = z_3 - z_4$, $f_I = f_3 - f_4$, will leave the flow unchanged and the perturbation (z_I, f_I) will be called an invariant perturbation.

We define a stationary perturbation (z_2, f_2) to be one in which any given physical perturbation is evaluated at a fixed point z_o of the basic flow, i.e., one for which the space variable is not perturbed ($z_2 \equiv 0$). We can now find, corresponding to a given perturbation (z_1, f_1), a unique stationary form by superposing on (z_1, f_1) the invariant perturbation $z_I = -z_1$.

Using (2.4) and (2.5) we can derive the following relations between the stationary perturbations f_2, w_2 and p_2

$$w_2 = w_o f_2' \tag{2.6}$$

$$p_2 = -\rho R\ell \, [w_o \bar{w}_o f_2' + \dot{f}_2] \tag{2.7}$$

In this formulation there is only one independent perturbation quantity, say f_2, restricted only by the condition that it be admissible under the boundary conditions of the problem.

In subsequent work, for the sake of compactness, we shall not change the name of a function after a change of independent variable, e.g., we shall write

$$f(z_o) = f\,[w_o(z_o)] = f(w_o).$$

D. Free Surface Condition

There are two conditions that must hold on the free surface. First, the free surface pressure remains constant, and second, a particle originally on the free surface remains on the free surface in the perturbed state. From (2.5) we see that we can satisfy the first condition by demanding that

$$R\ell \, [f_2' + \dot{f}_2 + w_o' \dot{z}_1] = 0 \tag{2.8}$$

where we have used the relation between f_1 and its stationary form f_2

$$f_2 = f_1 - w_o z_1.$$

The second condition can be shown to imply that

$$\operatorname{Im} [f_2' + (w_o z_1)' + (w_o z_1)^{\bullet}] = 0. \qquad (2.9)$$

We can satisfy (2.8) identically if we set the expression in the bracket equal to $i\chi(z_o,t)$, where the function χ is real on the free surface and otherwise arbitrary. Solving for z_1 we have

$$z_1 = \frac{1}{w_o'} \left\{ i\chi - D[f_2] \right\} \qquad (2.10)$$

where the operator $D[\ ] \equiv \frac{\partial[\ ]}{\partial \sigma_o} + \frac{\partial[\ ]}{\partial t}$. Now substituting for z_1 from (2.10) in (2.9) we find, after some reduction, that the free surface boundary condition is

$$\operatorname{Im} \left\{ D\ [f_2 - \omega D\ [f_2]\ ] - \dot{f}_2 \right\} = 0 \qquad (2.11)$$

where $\omega \equiv w_o/w_o'$.

Adopting the notation

$$H \equiv L\ [f_2] \equiv D\ [f_2 - \omega D\ [f_2]\ ] - \dot{f}_2 \qquad (2.12)$$

(2.11) becomes with w_o as the independent variable

$$H(w_o) = \overline{H(w_o)} \qquad \text{on} \qquad w_o \bar{w}_o = 1. \qquad (2.13)$$

E. Other Boundary Conditions

The additional boundary conditions depend mainly upon the particular flow considered. We shall here discuss some boundary conditions that occur in most of the problems to be investigated.

In general, from a physical point of view, we shall demand that all perturbation quantities be regular at any regular, finite interior point of the basic flow. The term regular as applied to a function implies that the function is developable in a Taylor series, while by a regular point of the basic flow we mean a point at which the basic flow potential is regular.

Most of the flows we shall consider originate at infinity, i.e., have a source point at infinity. Since we do not wish the perturbations to alter the fundamental nature of the basic flow, we require that the pressure and velocity perturbations vanish at the source point.

(1) At upstream infinity (source point).

(a) the perturbation of velocity vanishes

$$\lim_{z_0 \to \infty} w_2 = 0$$

(b) the perturbation of pressure vanishes. From (2.7) this implies

$$\lim_{z_0 \to \infty} \mathcal{R}\ell\, [w_0 \bar{w}_0 f_2' + \dot{f}_2] = 0$$

using condition 1(a), 1(b) reduces to

$$\lim_{z_0 \to \infty} \mathcal{R}\ell\, [\dot{f}_2] = 0.$$

(2) Along any fixed wall in the flow the perturbed flow

can have no component normal to the wall. This condition is satisfied if

$$\frac{w_2}{w_0} \text{ is real on a fixed wall.}$$

(3) If in the basic flow the free surface originates at a sharp edge, it must continue to originate there in the perturbed flow if no infinite velocities are to occur. We choose z_1 such that a point z_0 on the basic flow free surface goes into a point z on the perturbed free surface. Our boundary condition demands that the basic flow free surface and perturbed free surface coincide at a sharp edge where the free surface first originated. We can satisfy our boundary condition if $z_1 = 0$ there which becomes from (2.10)

$$0 = z_1 = \frac{1}{w_0'} \left\{ iX - D\,[f_2] \right\}$$

(4) In subsequent work we shall consider a flow involving a hollow vortex. This problem requires some special considerations.

(a) $\Delta z_1 = 0$

where $\Delta(\;)$ indicates the change in a quantity after going around a closed contour encircling the vortex. This condition then ensures that the perturbed free streamline remains closed.

(b) $\Delta f_2 = i\,C(t)$

where $C(t)$ is a real valued function of time.

This ensures that the circulation about any closed contour moving with the fluid remains constant, a consequence of Kelvin's theorem (see [6, p. 36]).

F. <u>Symmetry Considerations</u>

In cases where the basic flow has a velocity distribution which is a symmetric function of z_o (symmetric basic flow) certain simplifications can be made in the problem. In functional notation a symmetric function F(v) satisfies

$$F^s(v) = \overline{F^s(\overline{v})}$$

while an anti-symmetric function satisfies

$$F^a(v) = - \overline{F^a(\overline{v})} \ .$$

We can combine both relations in a convenient notation

$$F^{\substack{s\\a}}(v) = \pm \overline{F^{\substack{s\\a}}(\overline{v})}$$

where the + and - signs go with the symmetric and anti-symmetric parts respectively.

Certain operations performed on a symmetric or anti-symmetric function preserve these properties. It can be shown for example that the operations of differentiation and integration are symmetry preserving, i.e., the symmetry or anti-symmetry of the function remain unchanged. Also if

$$F^{\substack{s\\a}}(v) = \pm \overline{F^{\substack{s\\a}}(\overline{v})}$$

and we transform to a new variable [illegible] such that

$$u(v) = \overline{u(\overline{v})}$$

then

$$\overset{s}{F^{n}}(u) = \pm \overline{\overset{s}{F^{n}}(\overline{u})}$$

(i.e., the function retains its symmetry or anti-symmetry properties in the u plane).

The important consequence of these considerations is embodied in a theorem which will be stated here without proof, (for proof see [1, pp 26 et seq.]).

Symmetry Theorem: If the basic flow is symmetric, any perturbation can be represented as the sum of a symmetric and an anti-symmetric perturbation each of which satisfies all boundary conditions and so is an admissible perturbation in its own right.

G. Separation of Time Dependence

At this point in the development of the theory, the only restriction placed on the perturbation in the potential is that it shall satisfy all applicable boundary conditions. We shall attack the problem by assuming solutions of the form

$$f_2 = G_1(w_o)e^{\lambda t} + G_2(w_o)e^{\overline{\lambda} t} . \qquad (2.14)$$

We anticipate that this choice of time dependence will lead to an eigenvalue problem for the determination of the functions G_1 and G_2. We expect to find that the boundary conditions can be satisfied only for certain specific values of λ and that a general solution will be a sum of all such elementary forms of f_2. The primary concern will be to determine the magnitude of the real part of all admissible λ since this will indicate the stability of the flow. We shall have

unstable perturbations for	$\mathcal{R}\ell \{\lambda\} > 0$
neutrally stable perturbations for	$\mathcal{R}\ell \{\lambda\} = 0$
stable perturbations for	$\mathcal{R}\ell \{\lambda\} < 0.$

It might seem sufficient to assume f_2 in the form

$$f_2 = G(w_o)e^{\lambda t} \tag{2.14a}$$

since if λ and $\bar{\lambda}$ were both eigenvalues, both would be found among the admissible values of λ. The form (2.14) has been chosen because it is found that the elementary form (2.14a) is not capable of satisfying all the boundary conditions, whereas the form (2.14) can represent an admissible perturbation.

Substitution of the form for f_2 from (2.14) in our previous expression (2.12) for the operator H gives

$$H(w_o) = L_\lambda [G_1]e^{\lambda t} + L_{\bar{\lambda}} [G_2]e^{\bar{\lambda} t} \tag{2.15}$$

where

$$L_\lambda[G] = \frac{w_o^2}{\omega} G_{w_o w_o} + 2\lambda w_o G_{w_o} + \lambda G[\omega' + \lambda\omega] . \tag{2.16}$$

Our free surface boundary condition (2.13) becomes

$$L_\lambda [G_1(w_o)] = \overline{L_{\bar{\lambda}} [G_2(w_o)]} . \tag{2.17}$$

In the event the basic flow is symmetric we can decompose f_2 into symmetric and anti-symmetric components and write

$$f_2 = f_2^s + f_2^a$$

where

$$f_2^{\substack{s\\a}} = G_1^{\substack{s\\a}} e^{\lambda t} + G_2^{\substack{s\\a}} e^{\bar{\lambda} t} \tag{2.18}$$

and we can show that

$$G_2^{\substack{s\\a}} (w_o) = \pm \overline{G_1^{\substack{s\\a}} (\bar{w}_o)} \ . \tag{2.19}$$

With (2.19) we can eliminate G_2 from our free surface boundary condition (2.17) which then becomes

$$L_\lambda \ [G_1^{\substack{s\\a}} (w_o)] = \pm L_\lambda \ [G_1^{\substack{s\\a}} (\frac{1}{w_o})] \ . \tag{2.20}$$

This condition is to be applied on the free surface $w_o \bar{w}_o = 1$ or $\bar{w}_o = (\frac{1}{w_o})$. We can use analytic continuation, however, and demand that it hold over the entire w_o plane.

The remainder of this report will be devoted to solving the perturbation equation (either (2.17) or (2.20)) for several different types of problems.

In the remainder of the work the subscript zero used in denoting the basic flow velocity w_o will be dropped.

III. Hollow Vortex Bounded by Cylindrical Walls

The first problem to be investigated is a generalization of one treated in [1].

A. Basic Flow

The basic flow is a cyclic irrotational motion with circular streamlines bounded on the outside by a solid circular wall and on the inside by a concentric circular hollow vortex forming a constant pressure surface. We shall denote the radius of the undisturbed free surface by a and of the fixed walls by b, or in dimensionless form by 1 and $b/a = 1/\alpha$ (see Figure 1).

We can readily find the form of the basic flow potential in the hodograph plane (see Figure 2) as (see [4 pp 316 et seq.])

$$f_o = i \log w \tag{3.1}$$

and we find

$$\omega = i$$
$$\omega' = 0 \tag{3.2}$$

where we recall that the prime denotes $\partial/\partial f_o$.

B. Form of the Perturbation Potential

In the hodograph plane the entire physical flow is contained in the annulus bounded by $|w| = 1$ and $|w| = \alpha = a/b$. Since we expect the perturbed flow to have the same fundamental nature as the basic flow, we can allow singularities of the perturbed flow only at singular points of the basic flow. In this case the only such point is $w = 0$. With this in mind, we shall assume as the most general form for f_2

$$f_2 = G_1(w)e^{\lambda t} + G_2(w)e^{\bar{\lambda} t} \tag{3.3}$$

where

$$G_1(w) = B_1 \log w + \sum_{-\infty}^{\infty} a_r w^r$$

and

$$G_2(w) = B_2 \log w + \sum_{-\infty}^{\infty} b_k w^k$$

with B_1, B_2, a_r and b_k arbitrary constants.

C. Application of the Boundary Conditions

We shall now apply the relevant boundary conditions of Section II to the general form for f_2, (3.3).

(1) $\Delta f_2 = i\, C(t)$ where Δf_2 represents the change in f_2 after proceeding about a closed contour encircling the origin and $C(t)$ is a real function of time. This boundary condition ensures constancy of circulation for any contour encircling the singularity.

The application of this boundary condition gives

$$B_2 = \bar{B}_1$$

(2) Wall streamline boundary condition

We shall insist that the surface $|w| = a$ which is the map of the fixed wall remain a streamline, i.e., that

$$\frac{w_2}{w} \text{ be real on } |w| = a.$$

The expression for w_2/w is found from (2.6) to be

$$\frac{w_2}{w} = w' f_{2w} = \frac{w}{\omega} f_{2w} \, .$$

Making the appropriate substitutions we find

$$-iw\left\{\left[\frac{B_1}{w} + \sum_{r=1}^{\infty} r a_r w^{r-1}\right] e^{\lambda t} + \left[\frac{\bar{B}_1}{w} + \sum_{\kappa=1}^{\infty} \kappa b_\kappa w^{\kappa-1}\right] e^{\bar{\lambda} t}\right\}$$

is real on $|w| = a$. This will be true for all time t if and only if

$$\left\{-i\left[B_1 + \sum_{r=1}^{\infty} r a_r w^{r}\right]\right\} = \overline{\left\{-i\left[\bar{B}_1 + \sum_{\kappa=1}^{\infty} \kappa b_\kappa w^{\kappa}\right]\right\}} \, .$$

Performing the indicated conjugate operation we have

$$-i[B_1 + \sum_{r=-\infty}^{\infty} r a_r w^r] = i[B_1 + \sum_{k=-\infty}^{\infty} k \bar{b}_k \bar{w}^k] .$$

Substituting $\bar{w} = a^2/w$ and equating coefficients of like powers of w we find

$$B_1 = - B_1$$

and hence

$$B_1 = 0$$

and

$$\bar{b}_{-n} = a^{2n} a_n \tag{3.4}$$

or

$$b_n = a^{-2n} \bar{a}_{-n} .$$

(3) The free surface boundary condition is given by (2.17) as

$$L_\lambda[G_1(w)] = \overline{L_{\bar{\lambda}}[G_2(w)]} \quad \text{on} \quad w\bar{w} = 1,$$

which in expanded form is

$$\frac{w^2}{\omega} G_{1ww} + 2\lambda w G_{1w} + \lambda^2 \omega G_1 = \overline{[\frac{w^2}{\omega} G_{2ww} + 2\bar{\lambda} w G_{2w} + \bar{\lambda}^2 \omega G_2]}$$

on $w\bar{w} = 1$. Upon substituting in the above for G_1, G_2 and ω and using the results of (3.4) we obtain after some algebraic manipulation, the following relation

$$\sum_{r=-\infty}^{\infty} a_r[i(a\lambda - ir)^2 + ir]w^r = \sum_{k=-\infty}^{\infty} a_{-k} a^{-2k}[-i(a\lambda + ik)^2 - ik]w^{-k}, \tag{3.5}$$

If we now equate coefficients of w^n we find λ as the solution of a quadratic equation in n which gives rise to an infinite set of λ's, say λ_n, where

$$\lambda_n = i(n \pm \sqrt{N(n)}) \tag{3.6}$$

and

$$N(n) = n\left[\frac{1-\alpha^{2n}}{1+\alpha^{2n}}\right]$$

for all integer n.

We can now find the final forms of $G_1(w)$ and $G_2(w)$ in the expression (3.3) for an elementary solution of the perturbation potential f_2. We find two such elementary solutions. Corresponding to

$$\lambda_{n_+} = i(n + \sqrt{N(n)})$$

we have

$$\begin{aligned} f_{2n_+} = {} & a_{n_+} w^n e^{i(n+\sqrt{N(n)})t} + \\ & + \alpha^{2n}\bar{a}_{n_+} w^{-n} e^{-i(n+\sqrt{N(n)})t} \end{aligned} \tag{3.7}$$

and corresponding to

$$\lambda_{n_-} = i(n - \sqrt{N(n)})$$

we have

$$\begin{aligned} f_{2n_-} = {} & a_{n_-} w^n e^{i(n-\sqrt{N(n)})t} + \\ & + \alpha^{2n}\bar{a}_{n_-} w^{-n} e^{-i(n-\sqrt{N(n)})t} \end{aligned} \tag{3.8}$$

where a_{n_+} and a_{n_-} are arbitrary constants. We note that each elementary solution involves a term in w^n and a term in w^{-n}. The solution for a general admissible perturbation will be found as the sum over all n of such elementary solutions.

D. Final Forms of the Perturbation Quantities

If we write a solution for a general admissible perturbation as a sum of elementary solutions of the form (3.7) and (3.8), after some reduction, we arrive at the relation

$$\begin{aligned} f_2 = A_o &+ \sum_{n=1}^{\infty} \alpha^{-n} C_n e^{i(n+\sqrt{N(n)})t} w^n \\ &+ \sum_{n=1}^{\infty} \alpha^{n} \bar{C}_n e^{-i(n+\sqrt{N(n)})t} \frac{1}{w^n} \\ &+ \sum_{n=1}^{\infty} \alpha^{-n} D_n e^{i(n-\sqrt{N(n)})t} w^n \\ &+ \sum_{n=1}^{\infty} \alpha^{n} \bar{D}_n e^{-i(n-\sqrt{N(n)})t} \frac{1}{w^n} \end{aligned} \tag{3.9}$$

where A_o, C_n and D_n are constants to be determined by the initial conditions of the specific perturbation to be investigated.

We can get a much clearer physical picture from the form of the perturbations of z_1 at the free surface. From ([illegible]) we can write z_1 as

$$z_1 = \frac{1}{w'} \left\{ iX - f_2' - \dot{f}_2 \right\} .$$

We note that on the basic flow free surface the actual steady state velocity $\bar{w} = u + iv$ can be represented by $\bar{w} = ie^{i\theta}$. This implies $w = -ie^{-i\theta}$. With this in mind and substituting in the above for f_2 from (3.9) we have

$$
\begin{aligned}
z_1 = -\bar{w}\chi &+ \sum_{n=1}^{\infty} a^{-n}\sqrt{(N)}(-1)^n C_n \bar{w}\, e^{[i(n+\sqrt{N})t-in\theta]} \\
&- \sum_{n=1}^{\infty} a^{n}\sqrt{N}(1)^n \bar{C}_n \bar{w}\, e^{[-i(n+\sqrt{N})t+in\theta]} \\
&- \sum_{n=1}^{\infty} a^{-n}\sqrt{N}(-1)^n D_n \bar{w}\, e^{[i(n-\sqrt{N})t-in\theta]} \\
&+ \sum_{n=1}^{\infty} a^{n}\sqrt{N}(1)^n \bar{D}_n \bar{w}\, e^{[-i(n-\sqrt{N})t+in\theta]} .
\end{aligned}
\tag{3.10}
$$

E. Discussion of Results

(1) Stability of basic flow.

We have shown in (3.6) that the only allowable eigenvalues λ are

$$\lambda_n = i(n \pm \sqrt{N(n)}).$$

Thus λ_n is a pure imaginary and we may conclude immediately that the basic flow is neutrally stable when subjected to small perturbations.

(2) Wave character of the perturbations.

The angular velocity of the fluid particles on the basic flow free surface is unity. In equation (3.10) we have written an expression for the perturbations of the free surface which specifies such perturbations up to an arbitrary function χ. Aside from this arbitrariness, we note that the perturbation z_1 is made up of a wave pattern whose components travel at angular velocities equal to $(1 \pm \sqrt{N}/n)$, i.e., they either lead or lag the fluid particles with an angular velocity of $\sqrt{N}/n$.

(3) Analogy with water waves.

The square of the linear velocity of propagation of the perturbations along the free surface is seen to be

$$c^2 = \frac{N(n)}{n^2} = \frac{1}{n}\left[\frac{1 - \alpha^{2n}}{1 + \alpha^{2n}}\right]. \qquad (3.11)$$

The wavelength γ of these perturbations is

$$\left(\frac{\gamma}{a}\right) = \frac{2\pi}{n}.$$

Defining the depth of fluid from the free surface to the walls as h we have

$$h = b - a$$

or dividing through by a we have

$$\frac{h}{a} = \frac{1}{\alpha} - 1. \qquad (3.12)$$

We shall consider the situation where the ratio h/γ is maintained constant, while at the same time we make the dimensionless depth (h/a) very small. We have from (3.12)

$$\alpha = \frac{1}{1 + \frac{h}{a}}$$

which we can expand in terms of h/a as

$$\alpha = 1 - \frac{h}{a} + \text{terms of higher order}$$

hence

$$\alpha^{2n} = \left[\left(1 - \frac{h}{a}\right)^{\frac{1}{\frac{h}{a}}}\right]^{2n\frac{h}{a}}$$

which in the limit $h/a \to 0$ becomes

$$\alpha^{2n} = e^{-2nh/a} .$$

Substituting in (3.11) we find

$$c^2 = \frac{(\frac{1}{a})\Upsilon}{2\pi}\left[\frac{1 - e^{-\frac{2nh}{a}}}{1 + e^{-\frac{2nh}{a}}}\right]$$

$$c^2 = \frac{(\frac{1}{a})\Upsilon}{2\pi} \tanh \frac{2\pi h}{\Upsilon} . \qquad (3.12)$$

But the above is the same as the square of the velocity of propagation of a water wave in a constant gravity field if one replaces g by 1/a (see [6 p 367]). Thus we see that the analogy is complete in the limiting case $h/a \longrightarrow 0$. This is as we might expect since for h/a very small the centrifugal forces throughout the fluid are approximately constant and equal to 1/a, and since they produce the wavelike disturbances by a mechanism analogous to that operating in gravity waves.

(4) Limiting case with $b \to \infty$.

In the work above, if we let the radius of the cylindrical wall b tend to infinity, we find that we have reproduced in detail the results of [1] in which the case of a hollow vortex in an unbounded fluid was treated. Thus, as one might expect, the limiting case of a hollow vortex bounded by infinitely large walls is the same as the unbounded hollow vortex.

In the limiting process $\alpha \longrightarrow 0$ giving

$$\lambda_n \to i(n \pm \sqrt{n}).$$

In this case the velocity of propagation of waves on the free surface is analogous to the velocity of propagation of deep water waves. This result can also be obtained by letting $h/\gamma \longrightarrow \infty$ in (3.12).

(5) Comparison with previous work.

The results of this section have previously been obtained by another method by Lord Kelvin [5]. However, Kelvin treats a three-dimensional disturbance of the basic flow. If one considers the special case in which Kelvin's three-dimensional disturbance becomes two-dimensional, the results of [5] and this section are identical in all detail.

IV. Generalized Orifice Flows

Each of the flows to be considered in this section represents the draining of an infinite reservoir through an orifice. The sides of the orifice are made up of two semi-infinite planes inclined to each other at an angle of $2\pi/n$ radians; where $n = 2^p$, $p = 0, 1, 2, \ldots$ (see Figure 3). When $p = 0$ the configuration becomes the Borda mouthpiece which has been treated in [1], and hence this section is essentially a generalization of that problem.

A. Basic Flow Equations

The flow in the physical plane is mapped onto a sector of the unit circle in the hodograph plane bounded by radii inclined at an angle of $\pm\pi/n$ radians to the positive real axis (see Figure 4).

To find the potential for which the circular arc and radii are streamlines we image the sector to cover the entire unit circle. The potentials of the basic flows for all n are similar. Each may be thought of as being due to the presence of a source at the origin and sinks on the unit circle at the nth roots of unity. The sum of the strengths of the n sinks is twice the strength of the source. The potential for all values of n becomes

$$f_o(w) = \log \frac{w^{n/2}}{(w^n - 1)} \, . \qquad (4.1)$$

In terms of a new variable $\zeta = w^n$

$$f_o(\zeta) = \frac{1}{2} \log \frac{\zeta}{(\zeta - 1)^2} \qquad (4.2)$$

provides a single representation for the potential of all the flows considered here. We may now evaluate

$$\omega(\zeta) = \frac{-n}{2} \frac{\zeta + 1}{\zeta - 1}$$

and

$$\omega'(\zeta) = \frac{-2n\zeta}{(\zeta - 1)(\zeta + 1)} \, . \qquad (4.3)$$

B. Derivation of the Perturbation Equation.

We shall now derive the equations governing the perturbation potential in order that it may satisfy the wall streamline boundary condition and the free surface condition. All other boundary conditions will then be applied to the solutions of these perturbation equations in order to obtain the admissible perturbations.

(1) Wall streamline boundary condition

The wall streamline boundary condition demands that the perturbation velocity have no component normal to the walls. This condition will be satisfied if

$$\frac{w_2}{w} \text{ is real on the wall.}$$

We have seen that in the hodograph plane the entire physical flow is contained in a sector of the unit circle of $2\pi/n$ radians. Let us transform to a new variable $\eta = iw^{n/2}$. In the η plane the flow is contained in the upper half of the unit circle $|\eta| < 1$ and the walls have gone into the real axis (see Figure 5). The boundary condition now becomes

$$\frac{w_2}{w} \text{ is real on } \eta \text{ real.}$$

From (2.6)
$$\frac{w_2}{w} = w' f_{2w} = \frac{w}{\omega} f_{2w} = \frac{w}{\omega} f_{2\eta} \frac{d\eta}{dw} ;$$

hence
$$\frac{w_2}{w} = - \eta \frac{(\eta^2 + 1)}{(\eta^2 - 1)} f_{2\eta} \text{ is real on } \eta \text{ real.}$$

Since the coefficient of $f_{2\eta}$ is real for real η, we must have f_2 itself real on the real axis. This implies that f_2 is a symmetric function of η.

(2) Symmetry and analyticity considerations

As explained previously we demand that the perturbations be non-singular at all regular points of the flow. The basic flow is regular in $|\eta| < 1$ except for [illegible] isolated singularity at $\eta =$ [illegible]. Any admissible perturbed flow then can only have an isolated singularity at the origin. Hence $f_{2\eta}$ can be represented by a Laurent series in $|\eta| < 1$. Integrating termwise, an operation which preserves symmetry, gives

$$f_2 = 2c \log \eta + F(\eta) \qquad -\pi \leq \arg \eta \leq \pi \qquad (4.4)$$

where $F(\eta)$ is an analytic function of η, i.e., has a Laurent series. With the choice of argument above f_2 is a symmetric function of η. We note that c must be a real function of time in order that f_2 may satisfy the condition 4(b) of section II, namely constancy of circulation.

The basic flows for all n have symmetric velocity distributions. Hence the perturbations in the hodograph plane or, after the symmetry preserving transformation $\zeta = w^n$, in the ζ plane, can be decomposed into symmetric and anti-symmetric components. Expressed in the η plane, ($\eta = i\zeta^{1/2}$), the ζ plane symmetry relation

$$f_2^{\genfrac{}{}{0pt}{}{s}{a}}(\zeta) = \pm \overline{f_2^{\genfrac{}{}{0pt}{}{s}{a}}(\bar{\zeta})}$$

becomes

$$f_2^{\genfrac{}{}{0pt}{}{s}{a}}(\eta) = \pm \overline{f_2^{\genfrac{}{}{0pt}{}{s}{a}}(-\bar{\eta})} \qquad (4.5)$$

as may be seen in Figure [illegible]. Note that the s and a superscript notation still refers to the symmetric and anti-symmetric components of f_2 in the ζ plane. We have already determined that f_2 is a symmetric function of η, hence

$$\overline{f_2(-\bar{\eta})} = f_2(-\eta).$$

Substituting this relation into (4.5) shows that

$$f_2^{\overset{s}{a}}(\eta) = \pm f_2^{\overset{s}{a}}(-\eta). \qquad (4.6)$$

Thus f_2 represents a ζ plane symmetric or anti-symmetric perturbation depending upon whether it is an even or an odd function of η respectively.

The term

$$2c \log \eta = c \log \zeta + ic\pi$$

is symmetric except for the anti-symmetric constant $ic\pi$ which can be absorbed in f_2^a.

ζ behaves like η^2. Hence an even function of η with a Laurent series development (a series containing only even powers of η) becomes a Laurent series in ζ containing all powers of ζ. On the other hand an odd function of η having a Laurent series (a series containing only odd powers of η) looks like $\zeta^{1/2}$ multiplied by a Laurent series containing all powers of ζ when expressed in the ζ plane. Thus we can write

$$f_2 = f_2^s + f_2^a$$

where

$$f_2^s = c \log \zeta + F^s \qquad (4.7)$$

and

$$f_2^a = \zeta^{1/2} F^a$$

where F^s and F^a are analytic functions of ζ, i.e., have Laurent series.

As in section III we shall assume the functional form of the time dependence of f_2 to be

$$f_2 = G_1 e^{\lambda t} + G_2 e^{\overline{\lambda} t} . \tag{4.8}$$

Symmetry of the basic flow relates G_2 to G_1. The symmetry relation in the hodograph plane (see (2.19)), after the symmetry preserving transformation $\zeta = w^n$, becomes

$$G_2^{\substack{s\\a}} (\zeta) = \pm G_1^{\substack{s\\a}} (\overline{\zeta}). \tag{4.9}$$

Consistent with equations (4.7 - 4.9) we can represent the symmetric and anti-symmetric components of G_1 and G_2 in the form

$$\begin{aligned} G_1^s &= \alpha \log \zeta + g^s(\zeta) \\ G_2^s &= \overline{\alpha} \log \zeta + \overline{g^s(\overline{\zeta})} \\ G_1^a &= g^a(\zeta) \\ G_2^a &= - \overline{g^a(\overline{\zeta})} \end{aligned} \tag{4.10}$$

where g^s and $\zeta^{-1/2} g^a$ are analytic in $|\zeta| < 1$ and α is a complex constant. It can easily be verified that the forms assumed in (4.10) satisfy equations (4.7 - 4.9).

(3) Free surface condition

The free surface condition in the w plane is (2.16)

$$L_\lambda [G_1(w)] = \overline{L_{\overline{\lambda}} [G_2(w)]} \quad \text{on } w \overline{w} = 1.$$

After transforming to the ζ plane this condition becomes

$$L_\lambda [G_1(\zeta)] = \overline{L_{\overline{\lambda}} [G_2(\zeta)]} \quad \text{on } \zeta \overline{\zeta} = 1$$

where we now understand L_λ to be the transformed differential operator. Now using the symmetry relation (4.9) to eliminate G_2 we obtain

$$L_\lambda\,[G_1^{\overset{s}{a}}(\zeta)] = L_\lambda\,[G_1^{\overset{s}{a}}(\tfrac{1}{\zeta})].$$

The differential operator L_λ in the ζ plane is found to be

$$L_\lambda\,[G(\zeta)] = -2n\zeta^2\,\frac{(\zeta-1)}{(\zeta+1)}\left[G_{\zeta\zeta} + G_\zeta\left(\frac{\frac{n-1}{n}}{\zeta} - \frac{\lambda(\zeta+1)}{\zeta(\zeta-1)}\right)\right.$$
$$\left. + G\left(\frac{\lambda}{\zeta(\zeta-1)^2} + \frac{\lambda^2(\zeta+1)^2}{4\zeta^2(\zeta-1)^2}\right)\right]. \qquad (4.11)$$

Since the symmetric and anti-symmetric components satisfy the boundary conditions independently, we can substitute the forms for G_1^s and G_1^a from (4.10) into the above and find the following functional relations

$$2a\,\frac{\zeta-1}{\zeta+1} + h^s(\zeta) = 2a\,\frac{\frac{1}{\zeta}-1}{\frac{1}{\zeta}+1} + h^s\,(\tfrac{1}{\zeta}) \qquad (4.12a)$$

$$h^a(\zeta) = -\,h^a\,(\tfrac{1}{\zeta}) \qquad (4.12b)$$

where

$$h^{\overset{s}{a}}(\zeta) = L_\lambda\,[g^{\overset{s}{a}}(\zeta)]. \qquad (4.13)$$

We know that in $|\zeta| < 1$ g^s and $\zeta^{-1/2}g^a$ are analytic functions of ζ. With this in mind we can, by means of equation (4.13), determine that in $|\zeta| < 1$ $h^a(\zeta)$ behaves like $\zeta^{1/2}$ multiplied by an analytic function of ζ, while $h^s(\zeta)$ behaves like an analytic function of ζ. For convenience we may represent these functions in the following manner

$$h^{a}(\zeta) = \frac{\zeta - 1}{\zeta^{1/2}} \sum_{k=-\infty}^{\infty} a_k \left[\frac{\zeta}{(\zeta - 1)^2}\right]^k \tag{4.14a}$$

$$h^{s}(\zeta) = -2a\,\frac{\zeta - 1}{\zeta + 1} + \sum_{k=-\infty}^{\infty} b_k \left[\frac{\zeta}{(\zeta - 1)^2}\right]^k \tag{4.14b}$$

where the constants a_k and b_k are unknown.

These forms have the behavior demanded above and in addition satisfy (4.12a) and (4.12b) termwise. Our aim is to find functions $g^{\overset{s}{a}}(\zeta)$ satisfying the relation (4.13) with $h^{\overset{s}{a}}(\zeta)$ having the forms given in (4.14a) and (4.14b). To do this, we regard (4.13) as an inhomogeneous differential equation for $g^{\overset{s}{a}}(\zeta)$. We note that the inhomogeneous term $h^{\overset{s}{a}}(\zeta)$ is known in form only and hence the solutions $g^{\overset{s}{a}}(\zeta)$ will retain some arbitrariness which, for a given problem, ought to be determined by the initial conditions.

Knowing $g^{\overset{s}{a}}(\zeta)$, G_1 and G_2 can then be found from (4.10), thus determining f_2 and all other perturbation quantities.

C. <u>Other Boundary Conditions</u>

The restrictions which must be placed on the solutions of the perturbation equation in order that they may satisfy the remaining boundary conditions are derived in Appendix A, part 1. They are as follows

(1) The edge of the orifice ($\zeta = -1$).

The free surface shall continue to originate from the edge of the orifice, i.e., $z_1 = 0$ there

$$g^s_\zeta(-1) = \alpha, \qquad g^a_\zeta(-1) = 0.$$

(2) Upstream infinity ($\zeta = 0$).

(a) The perturbation velocity w_2 goes to zero.

$n = 1$ $\quad g^s$ and $\zeta^{1/2} g^a$ are regular at $\zeta = 0$

$n = 2, 4, 8, \ldots$ $\quad g^s$ and $\zeta^{-1/2} g^a$ are regular at $\zeta = 0$

(b) The pressure perturbation p_2 shall be zero.

$$\zeta^{-1} g^s \quad \text{and} \quad \zeta^{-1/2} g^a \quad \text{are regular at } \zeta = 0.$$

(3) Downstream infinity ($\zeta = 1$).

No disturbances originating at downstream infinity shall be propagated upstream in the jet

$$(\zeta - 1)^{-\lambda}\, g^s(1) \quad \text{and} \quad (\zeta - 1)^{-\lambda}\, g^a(1) \quad \text{exist.}$$

D. Solutions of the Perturbation Equation

The development of the anti-symmetric and symmetric solutions of the perturbation equation (4.13) is carried out in Appendix A, part II. We shall here use the results of that work.

(1) Anti-symmetric solutions.

The general form of the anti-symmetric solution of the perturbation equation valid about $\zeta = 0$ can be written as a linear combination of the two complementary solutions plus a particular integral.

$$g^a_H = A^a_H\, K^{(1)} + B^a_H\, K^{(2)} + \sum_{r=1}^{\infty} E^a_r\, K^a_{H+r} \, . \qquad (4.15)$$

The integer R used to describe a solution g_R^a is a number chosen so that E_0^a is the first non-zero E^a appearing in the solution. Admissible values of R will be determined by the application of the remaining boundary conditions. $K^{(1)}$ and $K^{(2)}$ are two linearly independent solutions of the homogeneous equation developed about the point $\zeta = 0$. The particular integral is represented by the terms

$$\sum_{r=0}^{\infty} E_r^a K_{R+r}^a ,$$

where the general term

$$E_r^a K_{R+r}^a$$

is a particular integral of the differential equation when the inhomogeneous side of the equation $h^a(\zeta)$, consists of the single term

$$\frac{\zeta - 1}{\zeta^{1/2}} a_{R+r} \left[\frac{\zeta}{(\zeta - 1)^2}\right]^{R+r},$$

(see (4.14a)). A_R^a, B_R^a and E_r^a are arbitrary constants.

The solution (4.15) is valid up to the nearest singular point, i.e., within the unit circle. Since boundary conditions will have to be applied not only within but also at points on the unit circle, we must find solutions valid at these points. To do this we study the indicial equation and hence the form of the complementary solution appropriate to the point in question and then match this form of the solution with the solution about the origin by the process of analytic continuation. A new form for the particular integral may then be found. This method must be continued until we obtain solutions near all points at which boundary conditions must be applied.

(2) Symmetric solutions.

The general form of the symmetric solution can be written in a manner similar to that used in writing the anti-symmetric solution. Thus

$$g_R^s = A_R^s K^{(1)} + B_R^s K^{(2)} + K_\alpha^s + \sum_{r=0}^{\infty} E_r^s K_{R+r}^s \qquad (4.16)$$

where the additional term K_α^s is the solution for a particular integral of the differential equation upon substitution for the inhomogeneous part of the differential equation only the term $-2\alpha \frac{\zeta - 1}{\zeta + 1}$ appearing in expression (4.14b) for $h^s(\zeta)$.

E. Application of the Boundary Conditions to the Solutions

The details of applying the boundary conditions of section C to the solutions of the perturbation equation are carried out in Appendix A, part III. The results of this work are collected below.

(1) Anti-symmetric perturbations.

The values of the index R which are found to be admissible depend upon whether we demand that $\zeta^{1/2}g^a$ or $\zeta^{-1/2}g^a$ be regular at $\zeta = 0$ as stipulated in boundary condition 2. With $\zeta^{1/2}g^a$ regular at $\zeta = 0$ we may have $R = 0, 1, 2, 3, \ldots$, while with $\zeta^{-1/2}g^a$ regular R may be 1, 2, 3, ... Since we demand that both the velocity and pressure perturbations vanish at upstream infinity (boundary conditions 2(a) and 2(b)), we see that the value $R = 0$ is not admissible even for $n = 1$.

With these values of R we can meet all boundary conditions by satisfying no more than three linear homogeneous equations in the unknown coefficients A_R^a, B_R^a and E_r^a, and by restricting the values of λ so that

$$R\ell \left\{ \lambda \right\} \leq 2 - 2(R + r).$$

(2) Symmetric perturbations.

Analogously in the symmetric case, demanding that g^s be regular at $\zeta = 0$ gives the allowable values of R as $R = 0, 1, 2, 3, \ldots$, while demanding that $\zeta^{-1} g^s$ be regular gives $R = 1, 2, 3, \ldots$. As before the pressure condition at upstream infinity eliminates $R = 0$. This boundary condition also determines that $\alpha \equiv 0$.

Now, to meet all boundary conditions we must satisfy no more than three linear homogeneous equations in the unknowns A_R^s, B_R^s and E_r^s. In this case the restriction on λ becomes

$$R\ell \left\{ \lambda \right\} \leq 1 - 2(R + r).$$

F. Stability of the Basic Flow

(1) Stability to anti-symmetric perturbations.

We have found that the application of our boundary conditions demands that we satisfy three linear homogeneous equations in the unknowns A_R^a, B_R^a and E_r^a. In general we are assured of a non-trivial solution if we have more unknowns than equations. Thus we can take as a solution involving four unknown constants

$$g_R^a = A_R^a\, K^{(1)} + B_R^a\, K^{(2)} + \sum_{r=0}^{\infty} E_r^a\, K_{R+r}^a \, .$$

With this solution our restriction on λ

$$R\ell \left\{\lambda\right\} \leq 2 - 2(R + r)$$

with $R = 1, 2, 3, \ldots$, implies that

$$R\ell \left\{\lambda\right\} \leq -2$$

which would indicate stable flow.

There exist any number of other possible solutions g_R^a containing one or more terms $E_r^a\, K_{R+r}^a$ in addition to $E_o^a\, K_R^a$. Any of these will make $R\ell \left\{\lambda\right\}$ even smaller than -2.

There is one special case for which a greater $R\ell \left\{\lambda\right\}$ may perhaps occur. If it is possible to satisfy non-trivially the three homogeneous equations with three unknown constants, then the solution

$$g_R^a = A_R^a\, K^{(1)} + B_R^a\, K^{(2)} + E_o^a\, K_R^a$$

is admissible and gives

$$R\ell \left\{\lambda\right\} \leq 0.$$

This perturbation is still a neutrally stable or stable perturbation. This situation will be discussed further in section VI.

There is another possible type of solution g^a which satisfies all boundary conditions. This solution consists of the complementary solutions, i.e., $E_r^a = 0$ for all r, and therefore $h_\zeta^a = L_\lambda\,[g^a(\zeta)] \equiv 0$. In this case the values of λ are restricted to

$$\lambda = -(2N+1) \pm \sqrt{\frac{2}{n}(2N+1)} \qquad \begin{array}{l} N = 0, 1, 2, \ldots \\ n = 1, 2, 4, 8, \ldots \end{array}$$

which gives $R\ell\{\lambda\} < 0$ for all n and N except N = 0, n = 1 for which

$$\lambda = -1 \pm \sqrt{2}\ .$$

(2) Stability to symmetric perturbations.

In the case of symmetric perturbations we have to satisfy three linear homogeneous relations in the unknowns A_R^s, B_R^s and E_r^s. Thus, if we take as a solution

$$g_R^s = A_R^s\,K^{(1)} + B_R^s\,K^{(2)} + \sum_{r=0}^{1} E_r^s\,K_{R+r}^s$$

our restriction on λ

$$R\ell\{\lambda\} \leq 1 - 2(R + r)$$

implies that

$$R\ell\{\lambda\} \leq -3$$

which would indicate stable flow.

In this case if we consider the possibility of the existence of a non-trivial solution

$$g_R^s = A_R^s\,K^{(1)} + B_R^s\,K^{(2)} + E_0^s\,K_R^s$$

we find

$$R\ell\left\{\lambda\right\} \leq -1$$

which still indicates a stable flow.

As in the anti-symmetric case there exists the possibility of satisfying all boundary conditions with the complementary solutions alone provided

$$\lambda = -2N \pm 2\sqrt{\frac{N}{n}} \qquad N = 1, 2, 3, \ldots$$
$$n = 1, 2, 4, 8, \ldots .$$

This gives $R\ell\left\{\lambda\right\} < 0$ for all possible choices of n and N.

G. Conclusions

Usually in the solution of an eigenvalue problem in an infinite domain one expects to find a one-dimensional continuum of eigenvalues λ and the set of eigenfunctions corresponding to these eigenvalues. The results of the present analysis have only given upper bounds on the $R\ell\left\{\lambda\right\}$ and hence λ has only been restricted to lie in a portion of the complex plane.

With one exception, namely that of the Borda mouthpiece (n = 1), the application of the boundary conditions to the solutions of the perturbation equation has shown that there exist no non-trivial admissible perturbations with $R\ell\left\{\lambda\right\} > 0$. We conclude from this that the generalized orifice flows for n = 2, 4, 8, ... are stable when subjected to small perturbations.

In the case of the Borda mouthpiece there does however appear to be an admissible perturbation, composed of only the complementary solution of the perturbation equation, with $R\ell\{\lambda\} > 0$. We note that this unstable perturbation occurs for only one isolated value of permissible λ namely λ real and $\lambda = -1 + \sqrt{2}$. Since we have not been able to determine a one-dimensional continuum of eigenvalues and corresponding eignefunctions, i.e., elementary solutions, it is not quite evident whether this unstable perturbation would always appear as a component of any general perturbation thus rendering the Borda mouthpiece flow generally unstable.

V. Equal and Opposite Jets

In this section we shall consider the flow made up of two equal and opposite two-dimensional jets impinging upon each other (see Figure 7).

A. Basic Flow

The complex potential of the basic flow may be found from an investigation of the nature of the flow in the hodograph plane (see Figure 8). The flow may be thought of as arising from the presence of two sources at $w = \pm 1$ and two sinks at $w = \pm i$, all of the same strength. With this in mind we can write the potential as

$$f_o(w) = \log \frac{w^2 - 1}{w^2 + 1} . \tag{5.1}$$

For convenience we shall make a transformation

$$\zeta = w^2 .$$

In terms of the new variable ζ

$$f_o(\zeta) = \log \frac{\zeta - 1}{\zeta + 1} \, . \tag{5.2}$$

Then

$$\begin{aligned} \omega(\zeta) &= \frac{4\zeta}{(\zeta - 1)(\zeta + 1)} \\ \omega'(\zeta) &= \frac{-2(\zeta^2 + 1)}{(\zeta - 1)(\zeta + 1)} \, . \end{aligned} \tag{5.3}$$

B. Derivation of Perturbation Equation

The differential equation governing the perturbation potential will be derived by applying symmetry considerations and the boundary condition on the free surface. All other boundary conditions will then be applied to the solutions of the perturbation differential equation.

(1) Symmetry and analyticity considerations.

In the basic flow, as represented in the hodograph plane, the imaginary axis is an axis of symmetry. We shall first decompose the velocity perturbations into components which are symmetric and anti-symmetric with respect to the imaginary axis. Expressed mathematically as a boundary condition, we have either

Case (a) $\frac{w_2}{w}$ real on imaginary w axis for the symmetric case, or

Case (b) $\frac{w_2}{w}$ pure imaginary on imaginary w axis for the anti-symmetric case.

Let us examine the implication of the above with respect to the form of the perturbation potential. For convenience we shall define a new variable v by the transformation

$$v = iw.$$

Since $\zeta = w^2$ we have also

$$v = i\zeta^{1/2}.$$

From equation (2.6) we have

$$\frac{w_2}{W} = w' f_{2w}$$

which becomes in terms of v

$$\frac{w_2}{W} = \frac{W}{\omega} f_{2v} \frac{dv}{dw} .$$

Upon substitution we find

$$\frac{w_2}{W} = \frac{(v^2 + 1)(v^2 - 1)}{4v} f_{2v} \tag{5.4}$$

with transformed boundary conditions

case (a) $\frac{w_2}{W}$ real on v real

case (b) $\frac{w_2}{W}$ pure imaginary on v real.

Since, in (5.4), the coefficient of f_{2v} is real on v real, these conditions imply

case (a) f_{2v} is a symmetric function of v

case (b) f_{2v} is an anti-symmetric function of v.

The basic flow potential is a regular function in

$|v| < 1$. We shall insist that the perturbed flow also be regular at all regular points of the base flow. Hence after integrating f_{2v}, which operation preserves symmetry, we still have

case (a) $f_2(v)$ is a symmetric and regular function of v

case (b) $f_2(v)$ is an anti-symmetric and regular function of v.

The basic flow has a symmetric velocity distribution about the real axis and the results of section II allow us to decompose our perturbations into symmetric and anti-symmetric perturbations with respect to the real axis in either the hodograph plane or the transformed ζ plane. In all further express.ons the symmetric and anti-symmetric notation will refer to properties in the ζ plane.

Case (a) $f_2(v)$ a symmetric function of v implies

$$f_2(v) = \overline{f_2(\bar{v})}. \qquad (5.5)$$

The symmetry relation in the ζ plane states that

$$f_2^{\substack{s\\a}}(\zeta) = \pm \overline{f_2^{\substack{s\\a}}(\bar{\zeta})}.$$

This becomes in the $v = i\zeta^{1/2}$ plane

$$f_2^{\substack{s\\a}}(v) = \pm \overline{f_2^{\substack{s\\a}}(-\bar{v})}.$$

Applying (5.5) to this expression gives

$$f_2^{\substack{s\\a}}(v) = \pm f_2^{\substack{s\\a}}(-v).$$

Hence a ζ plane symmetric or anti-symmetric perturbation is represented by an even or odd function of v respectively. Since we know $f_2(v)$ is a regular function in $|v| < 1$ we can write

$$f_2^s\ (v) = \sum_{n=0}^{\infty} a_n v^{2n} \tag{5.6}$$
$$f_2^a\ (v) = \sum_{n=0}^{\infty} b_n v^{2n+1}$$

Since v and ζ are related by $\zeta = -v^2$ we have

$$f_2^s\ (\zeta) = \sum_{n=0}^{\infty} a_n' \zeta^n \tag{5.7}$$

$$f_2^a\ (\zeta) = \zeta^{1/2} \sum_{n=0}^{\infty} b_n' \zeta^n$$

implying that $f_2^s\ (\zeta)$ and $\zeta^{-1/2}\ f_2^a\ (\zeta)$ are regular functions of ζ in $|\zeta| < 1$.

We assume the usual form for time dependence

$$f_2^{\overset{s}{a}} = G_1^{\overset{s}{a}}\ (\zeta) e^{\lambda t} + G_2^{\overset{s}{a}}\ (\zeta) e^{\bar{\lambda} t} .$$

G_1 and G_2 may be related by using symmetry of the basic flow giving

$$G_2^{\overset{s}{a}}\ (\zeta) = \overline{G_1^{\overset{s}{a}}\ (\bar{\zeta})}.$$

Consistent with the above equations, we may define G_1 and G_2 in the following manner

$$G_1^s = g^s\ (\zeta)$$
$$G_2^s = \overline{g^s\ (\bar{\zeta})} \tag{5.8}$$
$$G_1^a = g^a\ (\zeta)$$
$$G_2^a = -\overline{g^a\ (\bar{\zeta})}$$

where $g^s(\zeta)$ and $\zeta^{-1/2} g^a(\zeta)$ are regular functions of ζ in $|\zeta| < 1$.

Case (b) Applying similar arguments we find, omitting the detailed steps,

$$
\begin{aligned}
G_1^s &= g^s(\zeta) \\
G_2^s &= \overline{g^s(\bar{\zeta})} \\
G_1^a &= g^a(\zeta) \\
G_2^a &= -\overline{g^a(\bar{\zeta})}
\end{aligned}
\tag{5.9}
$$

where now $\zeta^{-1/2} g^s(\zeta)$ and $g^a(\zeta)$ are regular functions of ζ in $|\zeta| < 1$.

(2) Free surface condition

In a manner similar to that of section IV, the free surface boundary condition demands that the symmetric and anti-symmetric perturbations of equations (5.8) and (5.9) satisfy an inhomogeneous differential equation

$$L_\lambda [g^{s}_{a}(\zeta)] = h^{s}_{a}(\zeta)$$

where

$$
\begin{aligned}
h^s(\zeta) &= h^s\left(\frac{1}{\bar{\zeta}}\right) \\
h^a(\zeta) &= -h^a\left(\frac{1}{\bar{\zeta}}\right) .
\end{aligned}
\tag{5.10}
$$

The differential equations for $g^s(\zeta)$ and $g^a(\zeta)$ of case (b) are identical in form with (5.10) above.

The differential operator L_λ in the ζ plane becomes

$$L_\lambda\,[G(\zeta)] = \frac{4\zeta^2}{\omega}\left\{G_{\zeta\zeta} + G_\zeta\left(\frac{\frac{1}{2}}{\zeta} + \frac{2\lambda}{\zeta - 1} - \frac{2\lambda}{\zeta + 1}\right)\right.$$

$$\left. + G\left(\frac{1}{\zeta(\zeta - 1)(\zeta + 1)}\left[\frac{2\lambda^2 - 2\lambda}{(\zeta - 1)} + \frac{2\lambda^2 + 2\lambda}{(\zeta + 1)} - 2\lambda\right]\right)\right\} \qquad (5.11)$$

(3) Form of $h(\zeta)$

Case (a) We know $\zeta^{-1/2}g^a$ and g^s are regular in $|\zeta| < 1$. Substituting into

$$L_\lambda\,[g^{\substack{s\\a}}(\zeta)] = h^{\substack{s\\a}}(\zeta)$$

we find that $h^s(\zeta)$ is a regular function of ζ and $h^a(\zeta)$ equals $\zeta^{1/2}$ multiplied by a function of ζ with a simple pole at $\zeta = 0$. For convenience we shall choose

$$h^a(\zeta) = \frac{\zeta - 1}{\zeta^{1/2}} \sum_{r=-\infty}^{\infty} a_r\left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^r$$

$$h^s(\zeta) = \sum_{r=-\infty}^{\infty} b_r\left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^r \qquad (5.12)$$

where a_r and b_r are unknown. These forms have the behavior demanded above and in addition satisfy termwise the functional relations

$$h^{\substack{s\\a}}(\zeta) = \pm\, h^{\substack{s\\a}}\left(\frac{1}{\zeta}\right).$$

Case (b) Similar arguments in this case lead to the following choices for $h^{\substack{s\\a}}(\zeta)$

$$h^a(\zeta) = \frac{\zeta - 1}{\zeta + 1} \sum_{r=-\infty}^{\infty} a_r\left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^r$$

$$h^s(\zeta) = \frac{\zeta + 1}{\zeta^{1/2}} \sum_{r=-\infty}^{\infty} b_r\left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^r \qquad (5.13)$$

C. Other Boundary Conditions

The remaining boundary conditions which the solutions of the perturbation equation must satisfy are derived in Appendix B, part I. We shall merely collect the results here. They are

(1) At $\zeta = 0$. We insure the previously determined behavior of g^s and g^a by demanding that

case (a) g^s and $\zeta^{-1/2}g^a$ are regular functions of ζ as $\zeta \longrightarrow 0$.

case (b) $\zeta^{-1/2}g^s$ and g^a are regular functions of ζ as $\zeta \longrightarrow 0$.

(2) Upstream infinity ($\zeta = 1$).

(i) The perturbation velocity w_2 vanishes

case (a) and case (b) $\lim\limits_{\zeta \longrightarrow 1} (\zeta - 1) g^{\substack{s \\ a}}_{\zeta} = 0.$

(ii) The perturbation pressure p_2 vanishes.

case (a) and case (b) $\lim\limits_{\zeta \longrightarrow 1} g^{\substack{s \\ a}} = 0.$

(3) Downstream infinity ($\zeta = -1$).

No disturbances originating at downstream infinity shall be propagated upstream in the jets.

case (a) and case (b) $\lim\limits_{\zeta \longrightarrow 1} (\zeta + 1)^{-\lambda} g^{\substack{s \\ a}}$ exists.

D. Complete Solutions of the Perturbation Equation

The detailed derivation of the solutions of the perturbation equations (5.10) is carried out in Appendix B, part

II. The results of this work are used here.

Case (a) (1) anti-symmetric solutions.

As in section IV near $\zeta = 0$ we may write the complete solution of the differential equation governing the anti-symmetric perturbations as

$$g_R^a = A_R^a K^{(1)} + B_R^a K^{(2)} + \sum_{r=0}^{\infty} E_r^a K_{R+r}^a \qquad (5.14)$$

where $K^{(1)}$ and $K^{(2)}$ are the linearly independent solutions of the homogeneous equation developed about $\zeta = 0$, and the terms in the summation represent the particular integral. The individual terms of the summation $E_r^a K_{R+r}^a$ represent the solution for a particular integral with the inhomogeneous term $h^a(\zeta)$ in (5.10) replaced by the single term

$$\frac{\zeta - 1}{\zeta^{1/2}} a_{R+r} \left[\frac{(\zeta - 1)^2}{(\zeta + 1)^2}\right]^{R+r}$$

(see (5.12)). The significance of the index R, which is an integer used to denote the solution g_R^a has been explained previously in section IV.

(2) Symmetric solutions.

The complete symmetric solution can be written in the form

$$g_R^s = A_R^s K^{(1)} + B_R^s K^{(2)} + \sum_{r=0}^{\infty} E_r^s K_{R+r}^s \ . \qquad (5.15)$$

$K^{(1)}$ and $K^{(2)}$ are as defined in (5.14) and the term $E_r^s K_{R+r}^s$ appears as the solution of a particular integral when $h^s(\zeta)$ in

(5.10) is replaced by the single term

$$b_{R+r}\left[\frac{(\zeta-1)^2}{(\zeta+1)^2}\right]^{R+r},$$

(see (5.12)).

Case (b) The solutions for the symmetric and anti-symmetric solutions of case (b) differ only in minor detail from those given above and will not be written explicitly. They may be found in Appendix B, part II.

E. <u>Application of the Boundary Conditions</u>

The details of applying the boundary conditions to the symmetric and anti-symmetric solutions of the perturbation equations for case (a) and case (b) are carried out in Appendix B, part III. We shall examine here the significant results of this work.

We note first that symmetric or anti-symmetric solutions, made up of only the complementary solutions of the perturbation equation, can easily be shown to be non-trivial only if

$$\mathcal{R}\ell\left\{\lambda\right\} < 0.$$

This result holds for both case (a) and case (b).

We shall now discuss the application of the boundary conditions to the solutions of the perturbation equation containing terms from the particular integral.

<u>Case (a)</u> (1) Anti-symmetric solutions.

Applying the boundary condition at $\zeta = 0$ yields two linear homogeneous equations that must be satisfied by the unknown coefficients A_R^a, B_R^a and E_r^a.

As a result of applying the boundary condition at $\zeta = -1$ we find that λ and the index R must satisfy the inequality

$$2R + R\ell\left\{\lambda\right\} \leq -1.$$

The boundary condition at $\zeta = 1$ specifies that $g^a(-1) = 0$. A consideration of the behavior of the various terms of the solution shows that the boundary condition may be satisfied in several different manners depending upon the relative magnitude of R and $R\ell\left\{\lambda\right\}$. If we examine all possible situations, noting that at the same time the inequality between R and $R\ell\left\{\lambda\right\}$, found from the boundary condition at $\zeta = -1$ must always hold, we find that in any non-trivial solution capable of satisfying all boundary conditions

$$R\ell\left\{\lambda\right\} < 0.$$

(2) Symmetric solutions.

In the case of the symmetric solution, the boundary condition at $\zeta = 0$ will be satisfied provided A_R^s and E_r^s satisfy a single linear homogeneous equation. The coefficient B_R^s is as yet arbitrary.

The inequality between R and $R\ell\left\{\lambda\right\}$ resulting from applying the boundary condition at $\zeta = -1$ is

$$2R + R\ell\left\{\lambda\right\} \leq 1 - 2r_{max}$$

where r_{max} is the greatest r for which E_r^s is not zero. As yet the boundary conditions have yielded no information concerning the least value of r_{max} for which a solution will be non-trivial.

Proceeding to the boundary conditions at $\zeta = 1$ we find we must again investigate several possibilities depending upon the relative magnitudes of R and $R\ell\left\{\lambda\right\}$. As opposed to the anti-symmetric case, we now find one possibility, with $r_{max} = 0$ and the coefficients A_R^s, B_R^s and E_0^s satisfying a linear homogeneous equation, for which there exists a solution with

$$0 \leq R\ell\left\{\lambda\right\} < 1$$

corresponding to the index $R = 0$. We seem assured of a non-trivial solution for the coefficients A_R^s, B_R^s and E_0^s since we must satisfy only two linear homogeneous equations in these three unknowns (one arising from the condition at $\zeta = 0$, the other from the condition at $\zeta = 1$). All other possibilities of satisfying the boundary condition at $\zeta = 1$ yield no further perturbations with $R\ell\left\{\lambda\right\} > 0$.

Case (b) Proceeding as in case (a) we find no admissible symmetric or anti-symmetric perturbations with $R\ell\left\{\lambda\right\} > 0$.

F. Conclusions as to the Stability of the Basic Flow

As in the previous problem our analysis has not established a one-dimensional continuum of eigenvalues λ with corresponding eigenfunctions, but rather has only succeeded in restricting the range of the eigenvalue λ to a portion of the complex plane.

The investigation has shown that non-trivial symmetric perturbations with $1 > \mathcal{R}\ell\{\lambda\} > 0$ exist in case (a). Since this range of λ, with positive real part greater than zero, covers a strip of finite width in the complex λ plane, one might expect that a general disturbance would be unstable.

We note that case (a) corresponds to a flow for which both the steady and perturbed velocity has a zero component normal to the imaginary axis in either the physical plane or the hodograph plane. In effect case (a) may be thought of as the impinging of a finite jet on an infinite plate or wall. It will be of interest to compare the results of case (a) with those of the next section which will discuss the stability of a finite jet impinging on a finite plate. Of special interest will be the limit case when we permit the ratio of plate width to jet width to tend to infinity.

Except for the unstable perturbation discussed above, our analysis has shown that all other admissible perturbations have $\mathcal{R}\ell\{\lambda\} < 0$ and hence are stable perturbations.

VI. Jet Impinging on a Finite Plate

In this section we shall consider the stability of a flow over an obstacle. The particular flow chosen is that of a jet originating at infinity impinging normally on a plate of finite width (see Figure 9). Asymptotically, the jets leaving the upper and lower edges of the plate become straight jets inclined at angles $\pm\,\theta$ to the horizontal.

Initially the authors intended to consider the stability of the cavitated flow configuration produced by a finite plate placed normal to the free stream direction of an originally uniform stream (Helmholtz plate problem). The effect of the finite plate is to leave the originally uniform flow at upstream and downstream infinity unchanged. In the hodograph plane this results in mapping both upstream and downstream infinity into a single point and one is faced with the problem of applying, at a single point in the hodograph plane, boundary conditions pertaining to completely different points of the physical plane. From a mathematical point of view the effect is to give rise to an irregular singular point in the perturbation differential equation.

In trying to find solutions to this problem an attempt was made to differentiate between upstream and downstream infinity by artificially separating slightly the source and sink representing these points in the hodograph plane. It was then realized that this could be done in a straight forward manner by considering the limiting case of a much more general problem, namely that of a jet impinging on a finite plate.

A. Basic Flow Solutions

The form of the basic flow potential is most readily determined in the hodograph plane, (see Figure 10), where the flow may be thought of as having arisen from the presence of unit sources at $w = \pm 1$ and sinks of strength $\frac{1}{2}$ on the unit circle at $w = \pm a$ and $\pm \bar{a}$ (where argument of $a = \theta$, the jet inclination at downstream infinity). From the above, we can find

$$f_o(w) = \log \frac{(w^2 - 1)^2}{(w^2 - a^2)(w^2 - \bar{a}^2)} . \qquad (6.1)$$

After a transformation $\zeta = w^2$ the potential in the ζ plane becomes

$$f_o(\zeta) = \log \frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})} \qquad (6.2)$$

where $b = a^2$. Then

$$\omega(\zeta) = 2\zeta \frac{\{2(\zeta - b)(\zeta - \bar{b}) - (\zeta - 1)[(\zeta - b) + (\zeta - \bar{b})]\}}{(\zeta - 1)(\zeta - b)(\zeta - \bar{b})} \qquad (6.3)$$

and

$$\omega'(\zeta) = 2 - \frac{2\zeta}{(\zeta-1)(\zeta-b)(\zeta-\bar{b})} \left\{ \frac{2(\zeta-b)^2(\zeta-\bar{b})^2 - (\zeta-1)^2[(\zeta-b)^2 + (\zeta-\bar{b})^2]}{2(\zeta-b)(\zeta-\bar{b}) - (\zeta-1)[(\zeta-b)+(\zeta-\bar{b})]} \right\} \qquad (6.4)$$

The asymptotic inclination of the downstream jets, as characterized by a parameter $\beta = \sin\theta$, can be related to the dimensionless ratio (d/D) of plate width to original jet width. This can be accomplished by considering the integral

$$\int_0^{\frac{1}{2}(d/D)} dz = \frac{1}{2}\left(\frac{d}{D}\right).$$

In terms of hodograph variables we can write

$$dz = \frac{1}{w}\left(\frac{df_o}{dw}\right) dw.$$

Thus we have

$$\int_0^{\frac{1}{2}(d/D)} dz = \int_0^{-i} \frac{1}{w} \left(\frac{df_0}{dw}\right) dw = \frac{1}{2} \left(\frac{d}{D}\right).$$

Now using expression (6.1) for f_0, which involves the unknown sink position a, we finally arrive at

$$\frac{d}{D} = \frac{\pi}{2} \left[1 - \sqrt{1 - \beta^2}\right] + \frac{\beta}{2} \log \left[\frac{1 + \beta}{1 - \beta}\right]. \qquad (6.5)$$

The limit cases $\beta = 1$ and $\beta = 0$ correspond to the impinging of a jet of finite width on an infinite plate, and the impinging of an infinite stream upon a finite plate (Helmholz plate problem) respectively. These, in turn, correspond to making the sink position a approach i or + 1 respectively. The steady state form of the potential, (6.2), in these limiting cases goes over into the known form of the potential with d/D = ∞ or 0 respectively. Care must be taken in arriving at the proper limit for the Helmholtz plate problem since no simple source sink configuration results, but rather one made up of a quadrupole and a doublet at w = + 1.

B. Derivation of the Perturbation Equation

(1) Wall streamline condition.

The boundary condition on the imaginary axis (map of the plate) in the w plane is that the perturbed flow have no component normal to the axis.

We shall transform from the hodograph plane w to a new plane η by means of the relation

$$\eta = i w.$$

This transformation merely rotates the image of the plate onto the real axis in $|\eta| \leq 1$. The boundary condition will be satisfied if

$$\frac{w_2}{w} \text{ is real on } \eta \text{ real.}$$

Evaluating w_2/w, we have that

$$\frac{\eta(\eta^2+1)(\eta^2+b)(\eta^2+\bar{b})}{2\eta^2\left\{2(\eta^2+b)(\eta^2+\bar{b}) - (\eta^2+1)[(\eta^2+b)+(\eta^2+\bar{b})]\right\}} f_{2\eta}$$

must be real on η real. Since the coefficient of $f_{2\eta}$ in the expression above is itself real on η real we infer that $f_{2\eta}$ must be real on η real and hence a symmetric function of η. The basic flow is everywhere regular in $|\eta| < 1$ from which we can deduce that $f_{2\eta}$ is both a regular and symmetric function of η in $|\eta| < 1$. After integrating $f_{2\eta}$ we find $f_2(\eta)$ is a regular and symmetric function of η in $|\eta| < 1$, i.e.,

$$f_2(\eta) = \overline{f_2(\bar{\eta})} \,. \tag{6.6}$$

(2) Symmetry and analyticity considerations

The basic flow is symmetric in the ζ plane. Hence, we may decompose the perturbations into symmetric and anti-symmetric components. Symmetry in the ζ plane, which is expressed by

$$f_2^{\substack{s\\a}}(\zeta) = \pm\, \overline{f_2^{\substack{s\\a}}(\bar{\zeta})}$$

becomes in terms of η

$$f_2^{\substack{s\\a}}(\eta) = \pm\, \overline{f_2^{\substack{s\\a}}(-\bar{\eta})}.$$

Using the relation (6.6) we have

$$f_2^{\substack{s\\a}}(\eta) = \pm f_2^{\substack{s\\a}}(-\eta). \tag{6.7}$$

Thus a ζ plane symmetric or anti-symmetric perturbation is represented by an even or odd function of η respectively, and we can write

$$f^s = \sum_0^\infty a_n \eta^{2n}$$

$$f^a = \sum_0^\infty b_n \eta^{2n+1}.$$

Since $\zeta = -\eta^2$ these relations transform in the ζ plane to

$$f_2^s = F^s(\zeta)$$

$$f_2^a = \zeta^{1/2} F^a(\zeta)$$

where F^s and F^a are regular functions of ζ in $|\zeta| < 1$.

We shall assume the form of the f_2 time dependence as

$$f_2 = G_1(\zeta)e^{\lambda t} + G_2\zeta e^{\bar{\lambda} t}.$$

In addition we know that, due to symmetry of the basic flow, we have the relation

$$G_2^{\substack{s\\a}}(\zeta) = \pm \overline{G_1^{\substack{s\\a}}(\bar{\zeta})}. \tag{6.8}$$

Consistent with the above we may define

$$\begin{aligned} G_1^s &= g^s(\zeta) \\ G_2^s &= \overline{g^s(\bar{\zeta})} \\ G_1^a &= g^a(\zeta) \\ G_2^a &= -\overline{g^a(\bar{\zeta})} \end{aligned} \tag{6.9}$$

where g^s and $\zeta^{-1/2}g^a$ are regular functions of ζ in $|\zeta| < 1$.

(2) Free surface boundary condition

For a symmetric basic flow the free surface condition is

$$L_\lambda\,[G_1^{\overset{s}{a}}(\zeta)] = L_\lambda\,[G_1^{\overset{s}{a}}(\tfrac{1}{\zeta})]. \qquad (6.10)$$

The form of the differential operator L_λ in the ζ plane is

$$L_\lambda[G(\zeta)] = \frac{2\zeta(\zeta-1)(\zeta-b)(\zeta-\bar b)}{2(\zeta-b)(\zeta-\bar b)-(\zeta-1)[(\zeta-b)+(\zeta-\bar b)]}\left\{ G_{\zeta\zeta}\right.$$

$$+\,G_\zeta\left[\frac{(\zeta-1)(\zeta-b)(\zeta-\bar b)+4\lambda\zeta\,\{2\zeta(\zeta-b)(\zeta-\bar b)-(\zeta-1)[(\zeta-b)+(\zeta-\bar b)]\}}{2\zeta(\zeta-1)(\zeta-b)(\zeta-\bar b)}\right]$$

$$+\ \lambda G\left[\frac{2(\zeta-b)(\zeta-\bar b)-(\zeta-1)[(\zeta-b)+(\zeta-\bar b)]}{\zeta(\zeta-1)(\zeta-b)(\zeta-\bar b)}\right.$$

$$-\frac{2(\zeta-b)^2(\zeta-\bar b)^2-(\zeta-1)^2[(\zeta-b)^2+(\zeta-\bar b)^2]}{(\zeta-1)^2(\zeta-b)^2(\zeta-\bar b)^2}$$

$$\left.\left.+\frac{\lambda\{2(\zeta-b)(\zeta-\bar b)-(\zeta-1)[(\zeta-b)+(\zeta-\bar b)]\}^2}{(\zeta-1)^2(\zeta-b)^2(\zeta-\bar b)^2}\right]\right\} \qquad (6.11)$$

If we substitute in (6.10) the forms for G_1^s and G_1^a, (6.9), we arrive at the following relations

$$\begin{aligned} h^s(\zeta) &= \ \ h^s(\tfrac{1}{\zeta}) \\ h^a(\zeta) &= -\,h^a(\tfrac{1}{\zeta}) \end{aligned} \qquad (6.12)$$

where

$$h^{\overset{s}{a}}(\zeta) = L_\lambda\,[g^{\overset{s}{a}}(\zeta)]\,. \qquad (6.13)$$

The form of the differential operator L_λ and our knowledge of the functional behavior of g^s and g^a in $|\zeta| < 1$ together with (6.13) shows that in $|\zeta| < 1$ $h^s(\zeta)$ is a regular function of ζ and that $h^a(\zeta)$ behaves like $\zeta^{1/2}$ multiplied by a function with a simple pole at $\zeta = 0$. For convenience we shall represent h^s and h^a in the following manner

$$h^a(\zeta) = \frac{\zeta - 1}{\zeta^{1/2}} \sum_{k=-\infty}^{\infty} a_k \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^k \tag{6.14a}$$

$$h^s(\zeta) = \sum_{k=-\infty}^{\infty} b_k \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^k \tag{6.14b}$$

with a_k and b_k unknown constants. These forms for h^s and h^a possess the required behavior in $|\zeta| < 1$ and in addition satisfy the functional relations (6.12) termwise.

C. <u>Other Boundary Conditions</u>

The remaining boundary conditions are derived in detail in Appendix C, part I. They are

(1) At $\zeta = 0$. We insure the previously determined behavior for $g^{\substack{s\\a}}$ by demanding that

g^s and $\zeta^{-1/2}g^a$ are regular functions of ζ as $\zeta \longrightarrow 0$.

(2) At the edges of the plate ($\zeta = -1$).

The perturbed free surface shall continue to originate at the edges of the plate, i.e., $z_1 = 0$ there

$$g_\zeta^{\substack{s\\a}}(-1) = 0.$$

(3) Upstream infinity ($\zeta = 1$).

(a) The perturbation velocity $w_2 = 0$

$$\lim_{\zeta \to 1} (\zeta - 1)\, g_{\zeta}^{\overset{s}{a}} = 0.$$

(b) The perturbation pressure $p_2 = 0$

$$\lim_{\zeta \to 1} g^{\overset{s}{a}} = 0.$$

(4) Downstream infinity ($\zeta = b$ or $\bar{b}$).

The jets leaving the edges of the plate behave asymptotically like straight jets. No disturbances originating at downstream infinity are propagated upstream in the jet

$$\lim_{\zeta \to b} (\zeta - b)^{-\lambda} g^{s} \quad \text{and} \quad \lim_{\zeta \to \bar{b}} (\zeta - \bar{b})^{-\lambda} g^{\overset{s}{a}} \quad \text{exist.}$$

D. <u>Complete Solutions of the Perturbation Equation</u>

The details of writing the complete symmetric and anti-symmetric solutions of the perturbation equation are carried out in Appendix C, part II. We can now write the forms of the complete solutions.

(1) Anti-symmetric solutions

The complete anti-symmetric solution near $\zeta = 0$ is

$$g_{R}^{a} = A_{R}^{a}\, K^{(1)} + B_{R}^{a}\, K^{(2)} + \sum_{r=0}^{\infty} E_{r}^{a}\, K_{R+r}^{a} \qquad (6.15)$$

where A_{R}^{a}, B_{R}^{a} and E_{r}^{a} are unknown coefficients and $K^{(1)}$ and $K^{(2)}$ are the complementary solutions developed about $\zeta = 0$. The term $E_{r}^{a}\, K_{R+r}^{a}$ is a particular integral of (6.13) with $h^{a}(\zeta)$ replaced by

$$h^a_{R+r} = \frac{\zeta - 1}{\zeta^{1/2}} a_{R+r} \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^{R+r} .$$

(2) Symmetric solutions

The complete symmetric solution is

$$g^s_R = A^s_R K^{(1)} + B^s_R K^{(2)} + \sum_{r=0}^{\infty} E^s_r K^s_{R+r} \qquad (6.16)$$

where A^s_R, B^s_R and E^s_r are unknown coefficients and $E^s_r K^s_{R+r}$ is a particular integral of (6.13) with $h^s(\zeta)$ replaced by

$$h^s_{R+r} = b_{R+r} \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^{R+r} .$$

E. Application of the Boundary Conditions

The application of the boundary conditions to the solutions written above (see Appendix C, part III for details) yields the following results.

It can readily be shown that there are no non-trivial solutions made up of only the terms from the complementary solutions (i.e., E^a_r or E^s_r zero for all r) capable of satisfying all boundary conditions.

Seeking admissible perturbations among the solutions containing terms of the particular integral yields the following

(1) Anti-symmetric solutions

The boundary conditions at $\zeta = -1$ and $\zeta = 0$ demand that a total of three linear homogeneous equations be satisfied in the unknowns A^a_R, B^a_R and E^a_r.

The boundary condition at $\zeta = b$ or $\zeta = \bar{b}$ (both give identical information) will be satisfied if R and λ satisfy the inequality

$$R + R\ell\left\{\lambda\right\} \leq 1 - r_{max}\ .$$

One of the linear homogeneous equations noted above involves only the unknowns E_r^a and moreover the coefficient of E_o^a is non-zero. If $r_{max} = 0$ we would have $E_o^a = 0$. Since we have assumed $E_o^a \neq 0$ we can set 1 as a lower bound for r_{max} making the inequality

$$R + R\ell\left\{\lambda\right\} \leq 0.$$

The boundary condition at $\zeta = 1$ may be satisfied in several different manners depending upon the relative magnitude of R and $R\ell\left\{\lambda\right\}$. Examining all possibilities we find that consistent with the requirements of the other boundary conditions all non-trivial solutions have

$$R\ell\left\{\lambda\right\} \leq 0.$$

(2) Symmetric solutions

In this case the boundary conditions at $\zeta = 0$ and $\zeta = -1$ give only two linear homogeneous equations to be satisfied by A_R^s, B_R^s and E_r^s.

The inequality found by applying the boundary condition at $\zeta = b$ is

$$R + R\ell\left\{\lambda\right\} \leq 1 - r_{max}\ .$$

The form of the linear homogeneous equations above are now not incompatible with the choice $r_{max} = 0$.

Again the boundary condition at $\zeta = 1$ may be satisfied in several ways depending upon the relative magnitude of R and $R\ell\left\{\lambda\right\}$. An investigation of one of these possibilities shows that if a third linear homogeneous equation is satisfied by A_R^s, B_R^s and E_r^s with $r_{max} = 0$, then there exist admissible perturbations with $R\ell\left\{\lambda\right\} > 0$. However, we would then require three linear homogeneous equations to be satisfied non-trivially by the three unknowns A_R^s, B_R^s and E_0^s. This would require that the determinant of the coefficients be equal to zero. These coefficients appear to be unrelated and it seems improbable that their determinant is zero. It does not seem feasible, however, to evaluate the coefficients in closed form in order to demonstrate this fact mathematically. Assuming the determinant to be non-zero implies that r_{max} must be greater than zero. This strengthens the inequaltiy found at $\zeta = b$ and we can then show that no admissible perturbations have $R\ell\left\{\lambda\right\} > 0$.

F. <u>Conclusions as to the Stability of the Basic Flow</u>

Although we have not found a one-dimensional continuum of eigenvalues λ, our present analysis has been able to restrict the possible values of λ to be completely in the left half of the complex λ plane. This shows that any admissible perturbation has $R\ell\left\{\lambda\right\} \leq 0$ and we conclude that a jet impinging normally upon a finite plate gives a neutrally stable or stable flow configuration.

The results of this analysis are valid for any finite but non-zero ratio of plate width to jet width (d/D). By making this ratio as small as we please we can consider the flow of a jet of arbitrarily great width past a finite plate. But this is a proper physical interpretation of the Helmholtz plate problem since one never has a truly infinite stream in reality. From this point of view the stability conclusions reached in this section apply to the Helmholtz plate problem.

On the other hand, we find a different situation if we make the ratio (d/D) very large but still finite. In this case the point $\zeta = -1$ continues to be an ordinary point of the perturbation differential equation and the boundary condition $z_1 = 0$ must always be satisfied there. However if one actually sets $d/D = \infty$ the basic flow of this section goes over exactly into the basic flow of case (a) of section V. Now the point $\zeta = -1$ is a regular singular point corresponding to upstream infinity and the boundary condition $z_1 = 0$ no longer applies. A reexamination of the work of the present section will show that it was the presence of this additional boundary condition which precluded the existence of any non-trivial unstable perturbations. Thus, as far as stability is concerned, the limiting case $(d/D) \rightarrow \infty$ corresponding to the impinging of a finite jet on a plate of extremely large width does not approach the case of a jet impinging on a truly infinite wall which in turn represents a special case (case (a) section V) of two equal and opposite jets.

VII. Concluding Remarks

This report has presented the results of an investigation into the stability of four types of two-dimensional free surface flows of an ideal fluid when subjected to small perturbations.

The perturbations of a hollow vortex flow bounded by cylindrical walls were shown to be neutrally stable and the propagation of these disturbances was compared with the propagation of gravity waves in water. The impinging of a jet on a finite plate was found to be a stable flow configuration. In the case of a series of orifice flows all perturbations were found to be stable with the exception of an isolated unstable perturbation of the flow through a Borda mouthpiece. The existence of unstable perturbations was indicated in the case of impinging equal and opposite jets.

Ablow and Hayes have previously shown in [1] that certain simplifications are possible when the basic flow has a symmetric velocity distribution. The only anti-symmetric problem treated in this report was the bounded hollow vortex. In the course of the investigation the authors attempted to treat the perturbations of a hollow vortex bounded by parallel plane walls as an additional example of an anti-symmetric basic flow. In this case, however, the basic flow potential was found as an expression involving elliptic functions and the complexity of the resulting expressions made a solution intractable.

In the remaining problems the method of attack used in this report did not result in the determination of a one-dimensional continuum of permissible eigenvalues and corresponding eigenfunctions, but rather, only gave upper bounds on $R\ell\{\lambda\}$. As a consequence it has not been possible to form any sort of set of elementary perturbations capable of generating a general perturbation through a process similar to Fourier integration.

z_0 plane

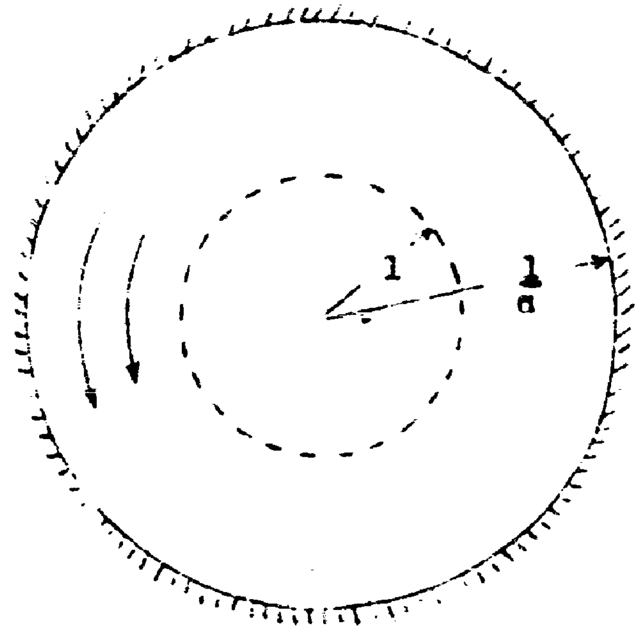

Fig. 1

w_0 plane

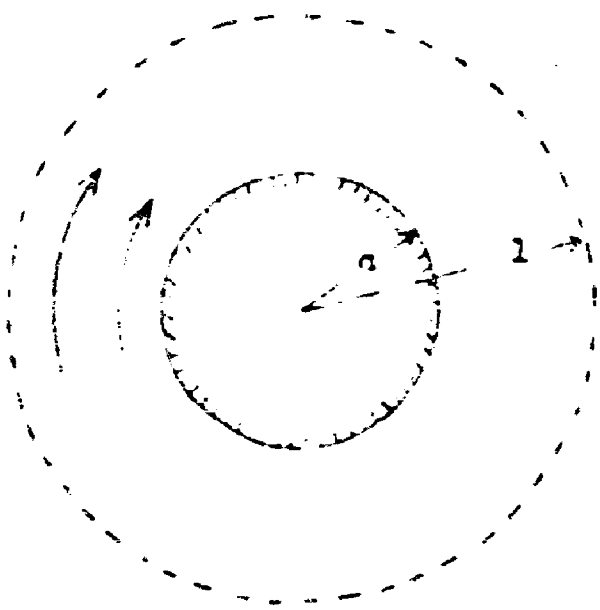

Fig. 2

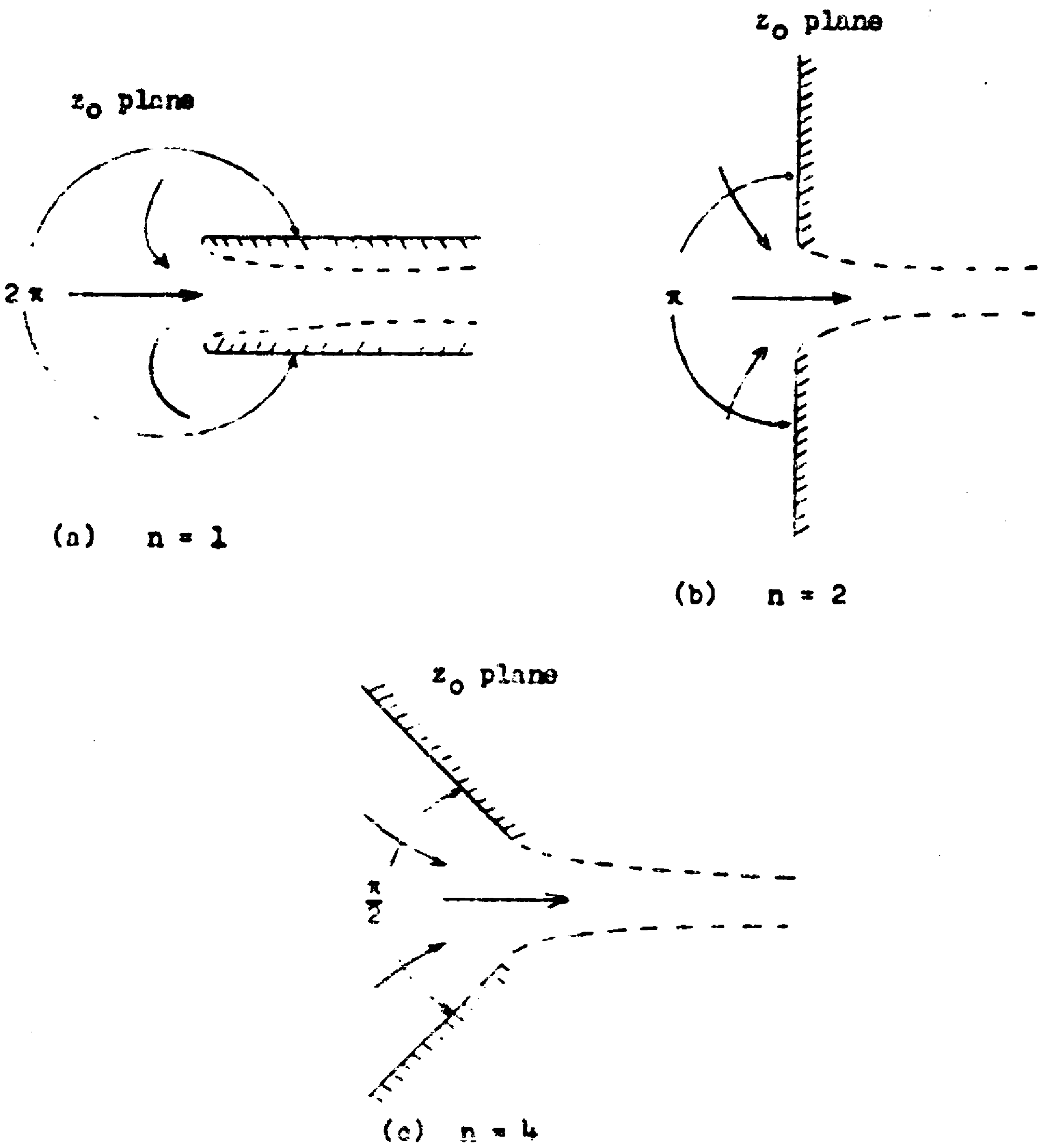

Fig. 3

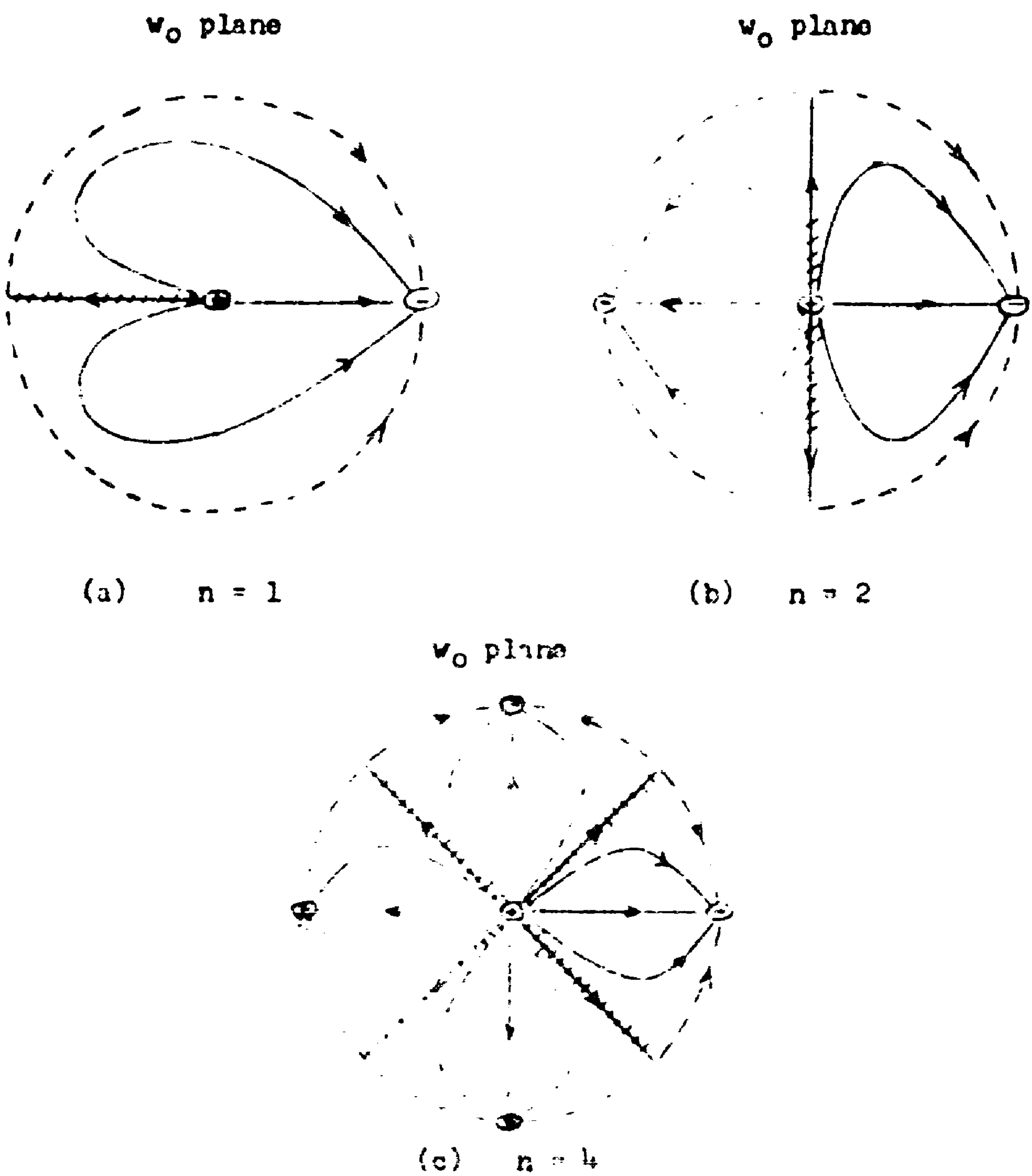

Fig. 4

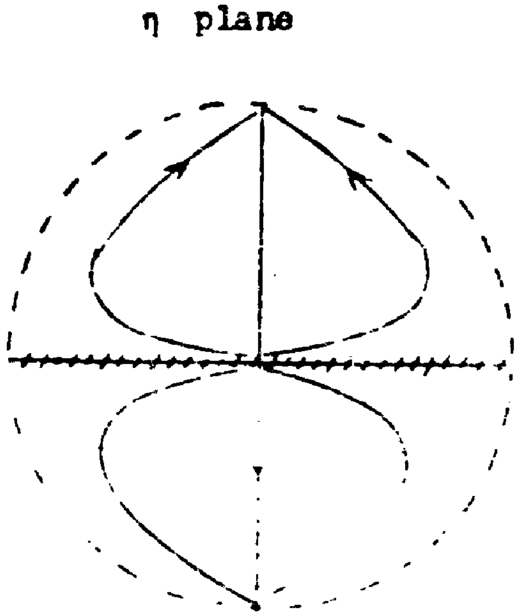
η plane

Fig. 5

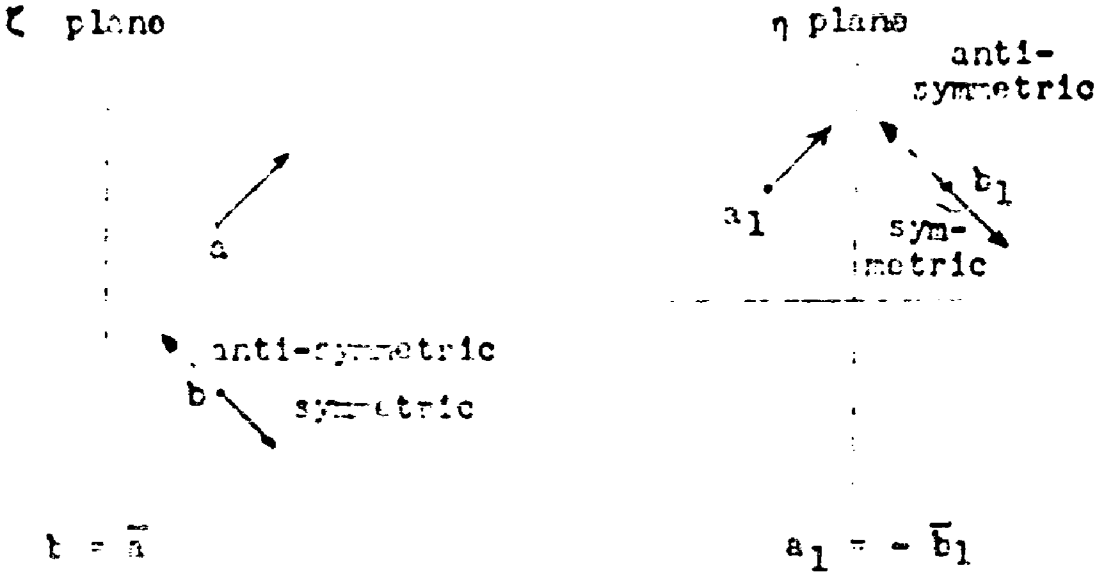
ζ plane
a
anti-symmetric
b
symmetric
$b = \bar{a}$
η plane
anti-symmetric
a_1
b_1
sym-metric
$a_1 = -\bar{b}_1$

Fig. 6

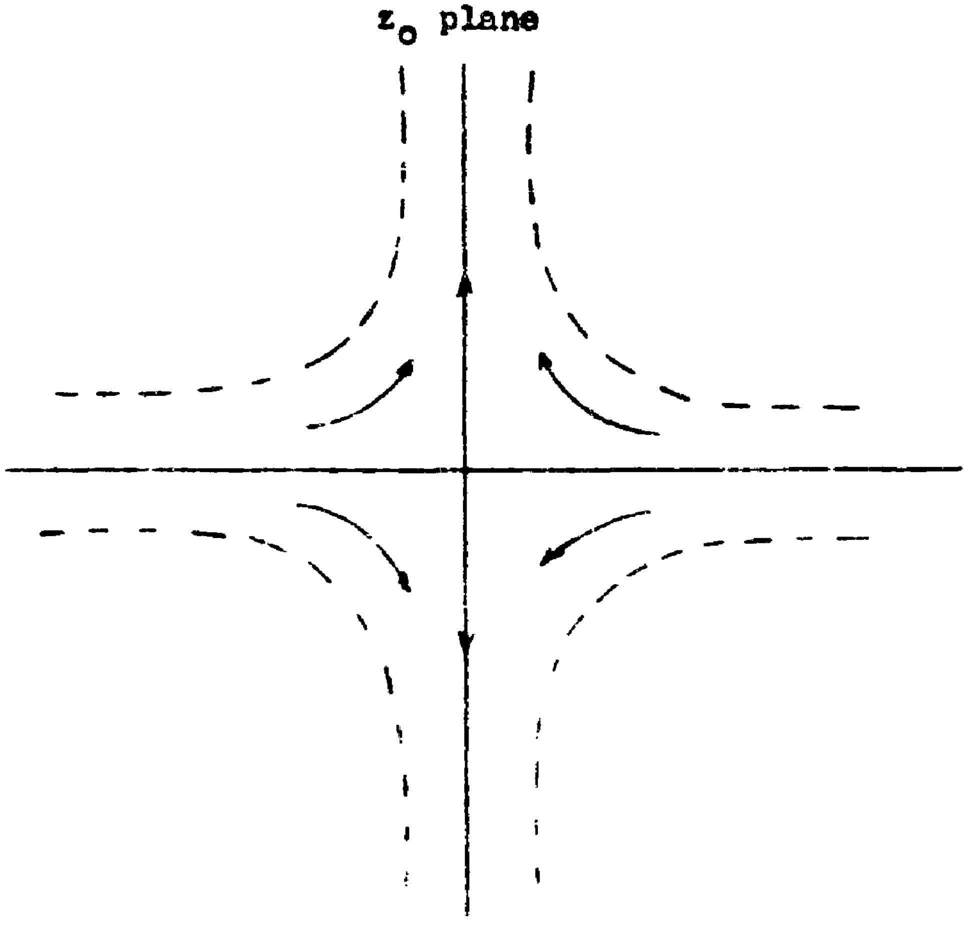

Fig. 7

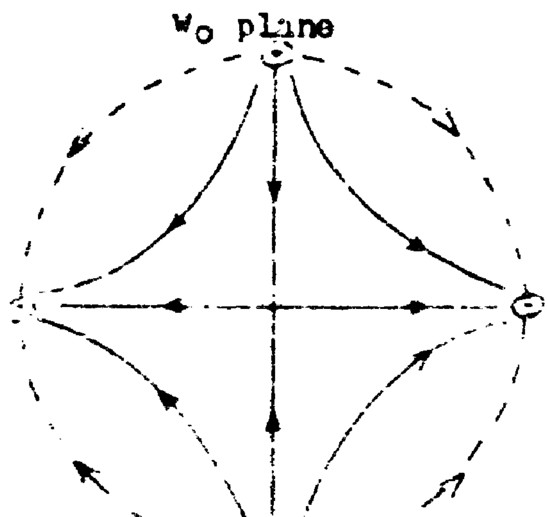

Fig. 8

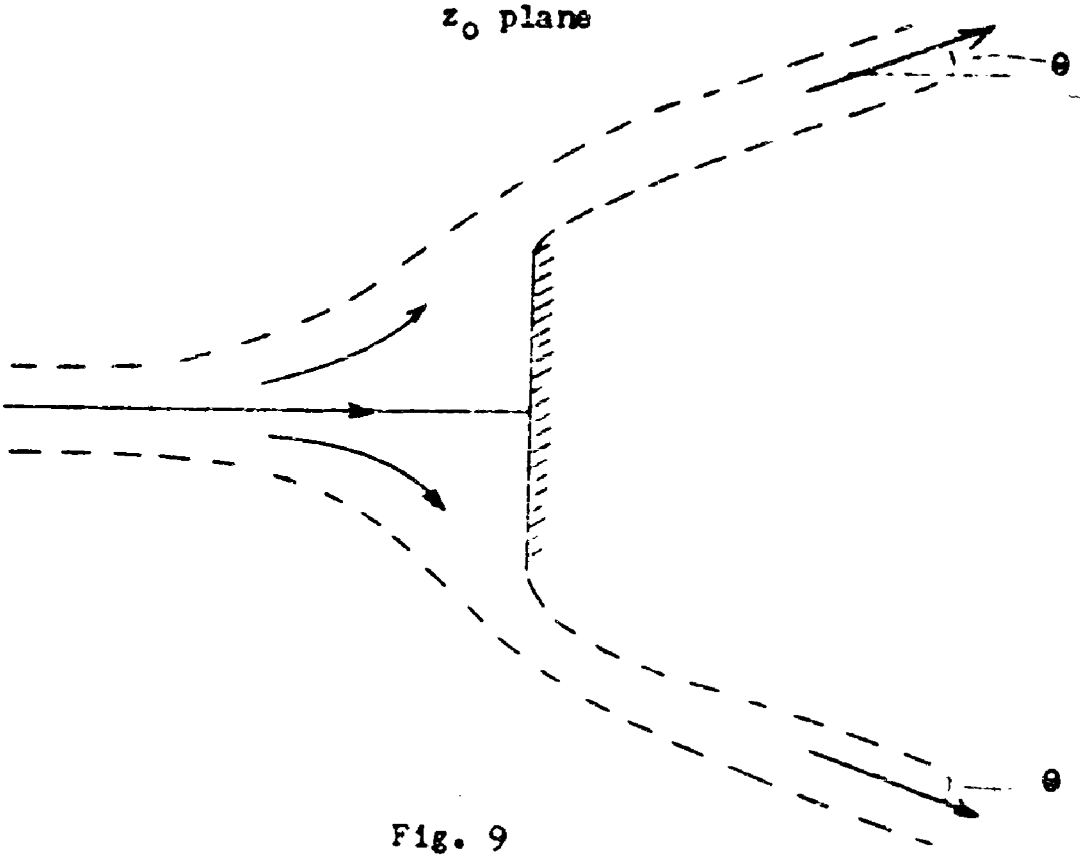

Fig. 9

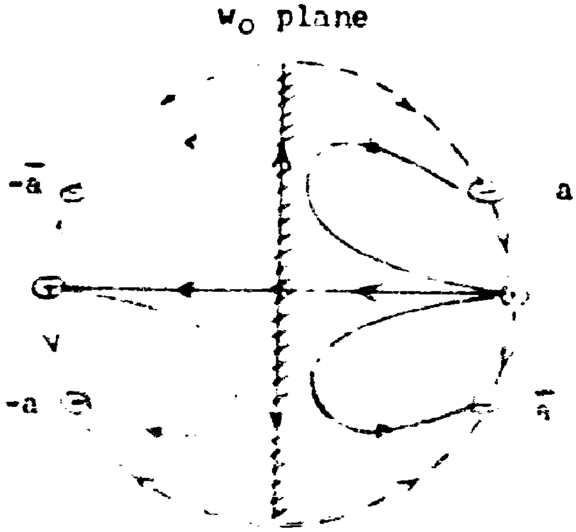

Fig. 10

Appendix A

Generalized Orifice Flow

Part I. Derivation of the Boundary Conditions

(1) The edge of the orifice ($\zeta = -1$).

In the basic flow the free surface originates at the point $\zeta = -1$ which is the map of the edge of the orifice. Since we are dealing with idealized sharp edged orifices, it is reasonable physically to demand that the free surface continue to originate here in any perturbed state. As discussed in section II, this condition will be satisfied if

$$z_1 = \frac{1}{w'}\left\{ i\chi - \dot{f}_2 - f_2' \right\} = 0.$$

In terms of ζ we have

$$-\frac{n}{2}\,\frac{\zeta+1}{\zeta-1}\,\frac{1}{\zeta^{1/n}}\left\{ i\chi - \lambda G_1 e^{\lambda t} - \lambda G_2 e^{\bar{\lambda} t} \right\}$$

$$-\, n\zeta^{\frac{n-1}{n}}\left\{ G_{1\zeta} e^{\lambda t} + G_{2\zeta} e^{\bar{\lambda} t} \right\} = 0 \quad \text{at } \zeta = -1.$$

The point $\zeta = -1$ is a regular point of the basic flow and hence we can expect χ, G_1 and G_2 to have a finite value here. Hence, to satisfy the boundary condition for all time t, we need demand only that

$$G_{1\zeta} = G_{2\zeta} = 0 \quad \text{at} \quad \zeta = -1.$$

Substituting for G_1 and G_2 from (4.10) gives

$$-a + g_\zeta^s(-1) + g_\zeta^a(-1) = 0$$

$$-\bar{a} + \overline{g_\zeta^s(-1)} - \overline{g_\zeta^a(-1)} = 0.$$

Taking the conjugate of the second equation, we have

$$- a + g_\zeta^s (-1) + g_\zeta^a (-1) = 0$$

$$- a + g_\zeta^s (-1) - g^a(\zeta)(-1) = 0$$

giving as solutions

$$g_\zeta^s (-1) = a$$

$$g_\zeta^a (-1) = 0.$$

(2) Upstream infinity ($\zeta = 0$).

In order to preserve the nature of the basic flow we demand that any perturbations in velocity or pressure vanish at upstream infinity.

(a) $\lim\limits_{\zeta \to 0} w_2 = 0$

from (2.6)

$$w_2 = wf_2' .$$

In terms of ζ

$$w_2 = - 2 \zeta^{\frac{n+1}{n}} \left(\frac{\zeta - 1}{\zeta + 1}\right) f_{2\zeta} .$$

The boundary condition implies

$$\lim_{\zeta \to 0} \zeta^{\frac{n+1}{n}} [G_{1\zeta} e^{\lambda t} + G_{2\zeta} e^{\bar{\lambda} t}] = 0.$$

As in boundary condition (1) substitution for G_1 and G_2 from (4.10) shows that

$$\lim_{\zeta \to 0} \zeta^{\frac{n+1}{n}} g_\zeta^s = \lim_{\zeta \to 0} \zeta^{\frac{n+1}{n}} g_\zeta^a = 0.$$

We know that in $|\zeta| < 1$ g^s and $\zeta^{-1/2} g^a$ are analytic functions of ζ, hence we may write

$$g^s = \sum_{-\infty}^{\infty} a_r \zeta^r$$

$$g^a = \sum_{-\infty}^{\infty} b_r \zeta^{r+1/2}.$$

Applying to g^s the boundary condition,

$$\lim_{\zeta \to 0} \zeta^{\frac{n+1}{n}} g^s_\zeta = 0,$$

we see immediately that

$$a_r = 0 \quad \text{for} \quad r \leq -1$$

for all n. By means of a similar treatment for the expression for g^a we can find that

$$b_r = 0 \quad r \leq -2 \quad \text{for } n = 1$$

$$b_r = 0 \quad r \leq -1 \quad \text{for } n = 2, 4, 8.$$

The above then imply the following

$g^s(\zeta)$ is a regular function of ζ at $\zeta = 0$

for all n.

$\zeta^{1/2} g^a(\zeta)$ is regular at $\zeta = 0$

for n = 1.

$\zeta^{-1/2} g^a(\zeta)$ is regular at $\zeta = 0$

for all other n.

(b) $\lim\limits_{\zeta \to 0} p_2 = 0$

From boundary condition (1(b)) of section II this requires

$$\lim_{\zeta \to 0} R\ell \left\{ \dot{f}_2 \right\} = 0$$

or

$$\lim_{\zeta \to 0} R\ell \left\{ \lambda G_1 e^{\lambda t} + \bar{\lambda} G_2 e^{\bar{\lambda} t} \right\} = 0.$$

Hence

$$G_1(0) = G_2(0) = 0.$$

From (4.10) we find that this requires that

$$g^s(0) = g^a(0) = a = 0$$

and as in boundary condition 2(a) we can finally establish that

$$\zeta^{-1} g^s \quad \text{and} \quad \zeta^{-1/2} g^a$$

are regular functions of ζ at $\zeta = 0$ for all n.

(3) Downstream infinity ($\zeta = 1$).

In the basic flow, as the jet proceeds downstream from the orifice, it asymptotically approaches a uniform straight jet. The behavior of a uniform straight jet when subjected to small disturbances is readily investigated by using the methods of [6, chapter IX].

It is found that such a flow configuration is neutrally stable. Any small disturbance of the jet is propagated downstream unchanged with the uniform velocity of the jet.

We shall demand that in the limit as we approach downstream infinity our flow behave like a straight jet. We can express this by demanding that the change of the velocity perturbation as seen by an observer moving with the jet be zero. Mathematically this can be expressed by demanding that the material derivative of w_2 vanish, i.e.,

$$\frac{\partial w_2}{\partial t} + U \cdot \frac{\partial w_2}{\partial s} = 0 \quad \text{as } \zeta \longrightarrow 1$$

where

s = distance along the jet

and

U = velocity of basic flow at $\longrightarrow 1$.

We recall that the asymptotic velocity of the jet $U = 1$ and we note that

$$f_o = U \cdot s + \text{const.} = s + \text{const.}$$

and

$$s = U \cdot t + \text{const.} = t + \text{const.}$$

hold asymptotically. Combining these we have

$$f_o = s + \text{const.}$$
$$t = f_o + \text{const.}$$

With this in mind our downstream infinity boundary condition becomes simply

$$\frac{dw_2}{df_o} = 0 \quad \text{as} \quad \zeta \longrightarrow 1$$

where t has been replaced by f_o + const. in $w_2(f_o,t)$. We have

$$w_2 = -2\zeta^{\frac{n+1}{n}} \frac{\zeta - 1}{\zeta + 1} [c_1\zeta^{\lambda t} + c_2\zeta^{\bar{\lambda} t}].$$

Substituting

$$t = f_o + x = \frac{1}{2} \log \frac{\zeta}{(\zeta - 1)} + \alpha \quad (\alpha = \text{const.})$$

we find

$$w_2 = -2\zeta^{\frac{n+1}{n}} \frac{\zeta-1}{\zeta+1} [G_{1\zeta}(\zeta-1)^{-\lambda}\zeta^{\lambda/2} c_1 + G_{2\zeta}(\zeta-1)^{-\bar{\lambda}}\zeta^{\bar{\lambda}/2} c_2]$$

where c_1 and c_2 are non-zero constants. Our boundary condition reduces to

$$\frac{d}{df_o}[(\zeta - 1)^{1-\lambda} G_{1\zeta}] = \frac{d}{df_o}[(\zeta - 1)^{1-\bar{\lambda}} G_{2\zeta}] = 0.$$

In terms of ζ this becomes

$$(\zeta - 1)\frac{d}{d\zeta}[(\zeta - 1)^{1-\lambda} G_{1\zeta}] = (\zeta - 1)\frac{d}{d\zeta}[(\zeta - 1)^{1-\bar{\lambda}} G_{2\zeta}] = 0$$

$$\text{as} \quad \zeta \longrightarrow 1.$$

The point $\zeta = 1$ is seen from an examination of the equations governing G_1 and G_2 (4.11) to be a regular singular point of the differential equation. From this we can infer that G_1 and G_2 will behave either like some power of $(\zeta - 1)$, or like log $(\zeta - 1)$ multiplied by some power of $(\zeta - 1)$. With this information it can be shown that we can satisfy the boundary condition above by demanding that

$$(\zeta - 1)^{-\lambda} G_1 \quad \text{and} \quad (\zeta - 1)^{-\bar{\lambda}} G_2 \quad \text{exist at } \zeta = 1.$$

Substituting for G_1 and G_2 from (4.10) our boundary condition finally becomes

$$(\zeta - 1)^{-\lambda} g^s \quad \text{and} \quad (\zeta - 1)^{-\lambda} g^s \quad \text{exist at } \zeta = 1.$$

Part II. Derivation of the solutions of the perturbation equation

The perturbation equation (4.13)

$$L_\lambda\,[g^{\overset{s}{a}}(\zeta)] = h^{\overset{s}{a}}(\zeta)$$

in expanded form is

$$g^{\overset{s}{a}}_{\zeta\zeta} + g^{\overset{s}{a}}_{\zeta}\left(\frac{\frac{n-1}{n}}{\zeta} - \frac{\lambda(\zeta+1)}{\zeta(\zeta-1)}\right)$$

$$+\, g^{\overset{s}{a}}\left(\frac{\lambda}{\zeta(\zeta-1)^2} + \frac{\lambda^2(\zeta+1)^2}{4\zeta^2(\zeta-1)^2}\right) = \frac{-(\zeta+1)}{2n\zeta^2(\zeta-1)}\,h^{\overset{s}{a}}\,. \qquad (A.1)$$

The differential equation (A.1) is a linear second order equation with regular singular points at $\zeta = 0, 1, \infty$, and hence falls into the class of a Fuchsian differential equation of the second order.

Following the method of [7, pp. 155 et seq.], we can put equation (A.1) into standard form and determine the roots of the indicial equation at the finite singular points. It is not necessary to write this form explicitly, since a knowledge of the location of the singular points and the roots of the indicial equation, or the exponents of the singularities as they are generally denoted, completely determines the form of the solutions of the equation.

We can readily find the exponents of the singularities to be

$$\mu = \frac{(1 - n\lambda) - \sqrt{1 - 2n\lambda}}{2n}\,, \qquad \nu = \frac{(1 - n\lambda) + \sqrt{1 - 2n\lambda}}{2n}$$

at $\zeta = 0$

and

$$\lambda, \quad 1 + \lambda$$
$$\text{at } \zeta = 1.$$

If we make the following change of dependent variable

$$g = \zeta^{\mu}(\zeta - 1)^{\lambda} H \tag{A.2}$$

where g and H stand for either g^s and H^s or g^a and H^a, the superscripts having been dropped for convenience, we again obtain a differential equation with regular singularities at $\zeta = 0, 1, \infty$ and with exponents

$$0, \; \frac{\sqrt{1 - 2n\lambda}}{n} \quad \text{at } \zeta = 0$$

$$0, \quad 1 \quad \text{at } \zeta = 1.$$

The differential equation for H is the standard form of the Hypergeometric equation.

Solutions of the Homogeneous Equation

We are now in a position to write the solutions of the homogeneous equation, i.e., (A.1) with the right hand side set equal to zero. In general the two linearly independent solutions about a regular singular point $\zeta = a$ with exponents β_i are of the form

$$H^{(i)} = (\zeta - a)^{\beta_i} F^{(i)}$$

where $F^{(i)}$ is regular and non-zero at $\zeta = a$. Using the transformation (A.2) we can then find two linearly independent solutions for g which we shall denote by $K^{(i)}$.

We now write the forms of the solutions for g about the points $\zeta = 0$, 1 and -1.

$\underline{\zeta = 0}$

$$K^{(1)} = \zeta^{\nu}(\zeta - 1)^{\lambda}F^{(1)}$$
$$K^{(2)} = \zeta^{\mu}(\zeta - 1)\,F^{(2)} \qquad (A.3)$$

where $F^{(1)}$ and $F^{(2)}$ are regular and non-zero at $\zeta = 0$. The above is valid provided $\frac{\sqrt{1 - 2n\lambda}}{2n}$ is not an integer. If such is the case the exponents of the singularities differ by an integer and further considerations show that $K^{(2)}$ must be replaced by

$$K^{(2)*} = c_2\zeta^{\nu} \log \zeta(\zeta - 1)^{\lambda}F^{(1)} + \zeta^{\mu}(\zeta - 1)^{\lambda}F^{(2)*} \qquad (A.4)$$

where c_2 = constant and $F^{(2)*}$ is regular and non-zero at $\zeta = 0$.

$\underline{\zeta = 1}$

$$K^{(3)} = \zeta^{\mu}(\zeta - 1)^{1+\lambda}F^{(3)}$$
$$K^{(4)} = c_4\zeta^{\mu}(\zeta - 1)^{1+\lambda}\log(\zeta - 1)F^{(3)} + \zeta^{\mu}(\zeta - 1)^{\lambda}F^{(4)} \qquad (A.5)$$

with c_4 = constant, and $F^{(3)}$ and $F^{(4)}$ regular and non-zero at $\zeta = 1$. In any region where both the solutions about $\zeta = 0$ and the solutions about $\zeta = 1$ are valid, we know that a linear homogeneous relation connecting any three of them must exist. Thus we can write

$$K^{(1)} = a_1K^{(3)} + b_1K^{(4)}$$
$$K^{(2)} = a_2K^{(3)} + b_2K^{(4)} \text{ or } K^{(2)*} = a_2^*K^{(3)} + b_2^*K^{(4)} \qquad (A.6)$$

where the a's and b's are constants and

$$a_1b_2 - a_2b_1 = \delta_1 \neq 0$$

$$a_1b_2^* - a_2^*b_1 = \delta_1^* \neq 0.$$

$\underline{\zeta = -1}$

An examination of our differential equation (A.1) shows that the point $\zeta = -1$ is an ordinary point of the differential equation, and at such a point we may prescribe both the value of the solution and its first derivative. Further, we can easily show that it is possible to choose two linearly independent solutions, $K^{(5)}$ and $K^{(6)}$, in such a manner that

$$\begin{aligned} K^{(5)}(-1) &= K_\zeta^{(6)}(-1) = 1 \\ K_\zeta^{(5)}(-1) &= K^{(6)}(-1) = 0. \end{aligned} \tag{A.7}$$

As before, we may also write

$$\begin{aligned} K^{(1)} &= a_3K^{(5)} + b_3K^{(6)} \\ K^{(2)} &= a_4K^{(5)} + b_4K^{(6)} \text{ or } K^{(2)*} = a_4^*K^{(5)} + b_4^*K^{(6)} \end{aligned} \tag{A.8}$$

with

$$a_3b_4 - b_3a_4 = \delta_2 \neq 0$$

$$a_3b_4^* - b_3a_4^* = \delta_2^* \neq 0.$$

Particular Integrals

Having found the complementary solutions, the appropriate forms of the particular integral in the neighborhood of the points at which we wish to apply boundary conditions, are readily found. Thus, knowing two linearly independent solutions $K^{(1)}$ and $K^{(3)}$ we can write the particular integral (see [8, p-123]) as

$$
\begin{aligned}
P.I. = K^{(i)} \int_{\zeta_o}^{\zeta} \frac{(\zeta + 1)\, h\,(\zeta)}{\zeta^2(\zeta - 1)\, W\,(K^{(i)}, K^{(j)})}\, K^{(j)} d\zeta \\
- K^{(j)} \int_{\zeta_o}^{\zeta} \frac{(\zeta + 1)\, h\,(\zeta)}{\zeta^2(\zeta - 1)\, W\,(K^{(i)}, K^{(j)})}\, K^{(i)} d\zeta
\end{aligned}
\tag{A.9}
$$

where ζ_o is an arbitrary ordinary point which we shall take as $\zeta = -1$.

$W(K^{(i)}, K^{(j)})$ is the Wronskian of the two solutions and can be found immediately from the differential equation (A.1) as (see [8, p-119]),

$$W(K^{(1)}, K^{(2)}) = A \exp \left\{ - \int [\text{coefficient of } g_\zeta] d\zeta \right\}$$

where A = constant. Substituting from (A.1)

$$
\begin{aligned}
W(K^{(1)}, K^{(2)}) &= A \exp - \int \left[\frac{\frac{n-1}{n}}{\zeta} - \frac{\lambda(\zeta + 1)}{\zeta(\zeta - 1)} \right] d\zeta \\
&= A \zeta^{\frac{(1 - n\lambda)}{n} - 1} (\zeta - 1)^{2\lambda}.
\end{aligned}
\tag{A.10}
$$

Since the solutions about $\zeta = 1$ and $\zeta = -1$ are related to the solutions about $\zeta = 0$ by linear relations we can find

$$
\begin{aligned}
W(K^{(3)}, K^{(4)}) &= \frac{W(K^{(1)}, K^{(2)})}{\delta_1} \\
W(K^{(5)}, K^{(6)}) &= \frac{W(K^{(1)}, K^{(2)})}{\delta_2} .
\end{aligned}
\tag{A.11}
$$

Complete Solutions of the Perturbation Equation

(1) **Anti-symmetric solutions**

The complete anti-symmetric solution is

$$g_R^a = A_R^a \, K^{(1)} + B_R^a \, K^{(2)} + \sum_{r=0}^{\infty} E_r^a \, K_{R+r}^a \tag{4.15}$$

K_{R+r}^a is a term of the particular integral found if $h(\zeta)$ is replaced by

$$h_{R+r}^a = \frac{\zeta - 1}{\zeta^{1/2}} \, a_{R+r} \left[\frac{\zeta}{(\zeta - 1)^2} \right]^{R+r}$$

in the expression (A.9) for the particular integral.

Making the appropriate substitutions we have

about $\zeta = 0$

$$g_R^a = A_R^a \left(\zeta^{\nu}(\zeta - 1)^{\lambda} F^{(1)}\right) + B_R^a \left(\zeta^{\mu}(\zeta - 1)^{\lambda} F^{(2)}\right)$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^a(\zeta - 1)^{\lambda}}{\Lambda} \left\{ \zeta^{\nu} F^{(1)} \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-3/2-\nu} (\zeta-1)^{-2(R+r)-\lambda} F^{(2)} d\zeta \right.$$

$$\left. - \zeta^{\mu} F^{(2)} \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-3/2-\mu} (\zeta-1)^{-2(R+r)-\lambda} F^{(1)} d\zeta \right\} \tag{A.12}$$

In case

$$\frac{\sqrt{1 - 2n\lambda}}{n} = N$$

i.e.,

$$\lambda = N + \frac{(1 - nN)^2}{2n}$$

the two complementary solutions are $K^{(1)}$ and $K^{(2)*}$ and we have

$$g_R^a = A_R^a(\zeta^\nu(\zeta-1)^\lambda F^{(1)}) + B_R^a(c_2\zeta^\nu \log\zeta\ (\zeta-1)^\lambda F^{(1)}$$

$$+\zeta^\mu(\zeta-1)^\lambda F^{(2)*}) + \sum_{r=0}^{\infty} \frac{E_r^a}{\Lambda^*}(\zeta-1)^\lambda \Big\{\zeta^\nu F^{(1)} \int_{-1}^{\zeta} [c_2(\zeta+1)\zeta^{(R+r)-3/2-\mu}\ \times$$

$$\times\ \log\zeta(\zeta-1)^\lambda F^{(1)} + (\zeta+1)\zeta^{(R+r)-3/2-\nu}(\zeta-1)^{-2(R+r)-\lambda}F^{(2)*}]d\zeta$$

$$-[c_2\zeta^\nu \log\zeta\, F^{(1)} + \zeta^\mu F^{(2)*}] \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-3/2-\mu}(\zeta-1)^{-2(R+r)-\lambda}F^{(1)}d\zeta\Big\}$$

(A.13)

<u>about $\zeta = 1$</u>

$$g_R^a = (A_R^a a_1 + B_R^a a_2)[\zeta^\mu(\zeta-1)^{1+\lambda}F^{(3)}] + (A_R^a b_1 + B_R^a b_2)[c_4\zeta^\mu(\zeta-1)^{1+\lambda}\ \times$$

$$\times\ \log(\zeta-1)F^{(3)} + \zeta^\mu(\zeta-1)^\lambda F^{(4)}] + \sum_{r=0}^{\infty} \frac{E_r^a b_2}{\Lambda}(\zeta-1)^\lambda \Big\{\zeta^\mu(\zeta-1)F^{(3)} \int_{-1}^{\zeta} \times$$

$$\times\ [c_4(\zeta+1)\zeta^{(R+r)-3/2-\nu}(\zeta-1)^{-2(R+r)+1-\lambda}\log(\zeta-1)F^{(3)} +$$

$$+\ (\zeta+1)\zeta^{(R+r)-3/2-\nu}(\zeta-1)^{-2(R+r)-\lambda}F^{(4)}]d\zeta - [c_4\zeta^\mu(\zeta-1)\log(\zeta-1)F^{(3)}$$

$$+\ \zeta^\mu F^{(4)}] \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-3/2-\nu}(\zeta-1)^{-2(R+r)+1-\lambda}F^{(3)}d\zeta\Big\}$$

(A.14)

<u>about $\zeta = -1$</u>

$$g_R^a = (A_R^a a_3 + B_R^a a_4)K^{(5)} + (A_R^a b_3 + B_R^a b_4)K^{(6)} + \sum_{r=0}^{\infty} \frac{E_r^a b_2}{\Lambda}\Big\{K^{(5)} \int_{-1}^{\zeta}\ \times$$

$$\times\ (\zeta+1)\zeta^{(R+r)-3/2-\frac{(1-n\lambda)}{n}}(\zeta-1)^{-2(R+r)-2\lambda}K^{(6)}d\zeta - K^{(6)} \int_{-1}^{\zeta}\ \times$$

$$\times(\zeta+1)\zeta^{(R+r)-3/2-\frac{(1-n\lambda)}{n}}(\zeta-1)^{-2(R+r)-2\lambda}K^{(5)}d\zeta\Big\}$$

(A.15)

(2) Symmetric solutions

The complete symmetric solution is

$$g_R^s = A_R^s K^{(1)} + B_R^s K^{(2)} + K_a^s + \sum_{r=0}^{\infty} E_r^s K_{R+r}^s . \qquad (4.16)$$

K_a^s is a term of the particular integral found if $h(\zeta)$ is replaced by

$$h_a^s = -2a \frac{\zeta - 1}{\zeta + 1}$$

in the expression (A.9) and $E_r^s K_{R+r}^s$ is a term of the particular integral found if $h(\zeta)$ is replaced by

$$h_{R+r}^s = b_{R+r} \left[\frac{\zeta}{(\zeta - 1)^2} \right]^{R+r}$$

in the expression (A.9).

Making the appropriate substitutions, we have

<u>about $\zeta = 0$</u>

$$g_R^s = A_R^s(\zeta^{\nu}(\zeta-1)^{\lambda}F^{(1)}) + B_R^s(\zeta^{\mu}(\zeta-1)^{\lambda}F^{(2)}) + \frac{a}{\Lambda n}(\zeta-1)^{\lambda} \left\{ \zeta^{\nu}F^{(1)} \int_{-1}^{\zeta} \times \right.$$

$$\times \zeta^{-1-\nu}(\zeta-1)^{-\lambda}F^{(2)}d\zeta - \zeta^{\mu}F^{(2)} \int_{-1}^{\zeta} \zeta^{-1-\mu}(\zeta-1)^{-\lambda}F^{(1)}d\zeta \left. \right\}$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^s(\zeta-1)^{\lambda}}{\Lambda} \left\{ \zeta^{\nu}F^{(1)} \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-1-\nu}(\zeta-1)^{-2(R+r)-1-\lambda}F^{(2)}d\zeta \right.$$

$$\left. - \zeta^{\mu}F^{(2)} \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-1-\mu}(\zeta-1)^{-2(R+r)-1-\lambda}F^{(1)}d\zeta \right\} \qquad (A.16)$$

The above holds except when $\lambda = N + \frac{(1 - nN)^2}{2n}$ $N = 0, 1, 2, \ldots$. In the latter case the appr riate solution is

$$g_R^s = A_R^s(\zeta^\nu(\zeta-1)^\lambda F^{(1)}) + B_R^s(c_2\zeta^\nu \log\zeta(\zeta-1)^\lambda F^{(1)} + \zeta^\mu(\zeta-1)^\lambda F^{(2)*})$$

$$+ \frac{a}{\Lambda^* n}(\zeta-1)^\lambda \left\{ \zeta^\nu F^{(1)} \int_{-1}^{\zeta} [c_2\zeta^{-1-\mu} \log\zeta(\zeta-1)^{-\lambda} F^{(1)} + \zeta^{-1-\nu}(\zeta-1)^{-\lambda} \times \right.$$

$$\times F^{(2)*}]d\zeta - [c_2\zeta^\nu \log\zeta F^{(1)} + \zeta^\mu F^{(2)*}] \int_{-1}^{\zeta} \zeta^{-1-\mu}(\zeta-1)^{-2(R+r)-1-\lambda} \times$$

$$\left. \times F^{(1)} d\zeta \right\} + \sum_{r=0}^{m} \frac{E_r^s(\zeta-1)^\lambda}{\Lambda^*} \left\{ \zeta^\nu F^{(1)} \int_{-1}^{\zeta} [c_2(\zeta+1)\zeta^{(R+r)-1-\mu} \times \right.$$

$$\times \log\zeta(\zeta-1)^{-2(R+r)-1-\lambda} F^{(1)} + (\zeta+1)\zeta^{(R+r)-1-\nu}(\zeta-1)^{-2(R+r)-1-\lambda} \times$$

$$\times F^{(2)*}]d\zeta - [c_2\zeta^\nu \log\zeta F^{(1)} + \zeta^\mu F^{(2)*}] \int_{-1}^{\zeta} (\zeta+1)\zeta^{(R+r)-1-\mu} \times$$

$$\left. \times (\zeta-1)^{-2(R+r)-1-\lambda} F^{(1)} d\zeta \right\} \tag{A.17}$$

<u>about $\zeta = 1$</u>

$$g_R^s = (A_R^s a_1 + B_R^s a_2)[\zeta^\mu(\zeta-1)^{1+\lambda} F^{(3)}] + (A_R^s b_1 + B_R^s b_2)[c_4\zeta^\mu(\zeta-1)^{1+\lambda} \times$$

$$\times \log(\zeta-1)F^{(3)} + \zeta^\mu(\zeta-1)^\lambda F^{(4)}] + \frac{a b_1}{\Delta n}\zeta^\mu(\zeta-1)^\lambda \left\{ (\zeta-1)F^{(3)} \int_{-1}^{\zeta} \times \right.$$

$$\times [c_4\zeta^{-1-\nu}(\zeta-1)^{1-\lambda} \log(\zeta-1)F^{(3)} + \zeta^{-1-\nu}(\zeta-1)^{-\lambda} F^{(4)}]d\zeta$$

$$-[c_4(\zeta-1)\log(\zeta-1)F^{(3)}+F^{(4)}]\int_{-1}^{\zeta}\zeta^{-1-\nu}(\zeta-1)^{1-\lambda}F^{(3)}d\zeta\Big\} + \sum_{r=0}^{\infty}\frac{E_r^s\delta_1}{\Lambda}\ x$$

$$x\ \zeta^{\mu}(\zeta-1)^{\lambda}\Big\{(\zeta-1)F^{(3)}\int_{-1}^{\zeta}[c_4(\zeta+1)\zeta^{(R+r)-1-\nu}(\zeta-1)^{-2(R+r)-\lambda}\ x$$

$$x\ \log(\zeta-1)F^{(3)}+(\zeta+1)^{(R+r)-1-\nu}(\zeta-1)^{-2(R+r)-1-\lambda}F^{(4)}]d\zeta\ -[c_4(\zeta-1)\ x$$

$$x\ \log(\zeta-1)F^{(3)}+F^{(4)}]\int_{-1}^{\zeta}(\zeta-1)\zeta^{(R+r)-1-\nu}(\zeta-1)^{-2(R+r)-\lambda}F^{(3)}d\zeta\Big\}\tag{A.18}$$

<u>about $\zeta = -1$</u>

$$g_R^s = (A_R^s a_3+B_R^s b_3)K^{(5)}+(A_R^s b_3+B_R^s b_4)K^{(6)}+\frac{\alpha\delta_2}{\Lambda n}\Big\{K^{(5)}\int_{-1}^{\zeta}\zeta^{-1-\frac{(1-n\lambda)}{n}}\ x$$

$$x\ (\zeta-1)^{-2\lambda}K^{(6)}d\zeta-K^{(6)}\int_{-1}^{\zeta}\zeta^{-1-\frac{(1-n\lambda)}{n}}(\zeta-1)^{-2\lambda}K^{(5)}d\zeta\Big\}+\sum_{r=0}^{\infty}\frac{E_r^s\delta_2}{\Lambda}\ x$$

$$x\ \Big\{K^{(5)}\int_{-1}^{\zeta}(\zeta+1)\zeta^{(R+r)-1-\frac{(1-n\lambda)}{n}}(\zeta-1)^{-2(R+r)-1-2\lambda}K^{(6)}d\zeta\ -K^{(6)}\ x$$

$$x\int_{-1}^{\zeta}(\zeta+1)\zeta^{(R+r)-1-\frac{(1-n\lambda)}{n}}(\zeta-1)^{-2(R+r)-1-2\lambda}K^{(5)}d\zeta\Big\}\tag{A.19}$$

Part III. <u>Application of Boundary Conditions</u>

We shall now apply the boundary conditions as derived in part I of this appendix to the complete solutions of the perturbation equation found in the preceding section

(1) Anti-symmetric solutions

i) About $\zeta = -1$; $g^a_\zeta(-1) = 0$.

An inspection of the solution, equation (A.15), shows that

$$\frac{d}{d\zeta}\left(K^a_{R+r}\right) \text{ is zero at } \zeta = -1.$$

Now

$$\frac{d}{d\zeta}(g^a_R) = 0 = (A^a_R a_3 + B^a_R a_4)K^{(5)}_\zeta(-1) + (A^a_R b_3 + B^a_R b_4)K^{(6)}_\zeta(-1).$$

Substituting from (A.7) $K^{(5)}_\zeta(-1) = 0$ and $K^{(6)}_\zeta(-1) = 1$, we obtain

$$A^a_R b_3 + B^a_R b_4 = 0. \tag{A.20}$$

ii) About $\zeta = 0$; $\zeta^{1/2} g^a$ or $\zeta^{-1/2} g^a$ is a regular function of ζ.

<u>Case (a) $\lambda \neq N + \frac{(1 - nd)^2}{2n}$</u>

If we demand that $\zeta^{1/2} g^a$ be regular we shall find that permissible values of the index R, are $R = 0, 1, 2, 3, \ldots$. On the other hand if $\zeta^{-1/2} g^a$ is to be regular, R may take on only the values $R = 1, 2, 3, \ldots$. This can be seen by an examination of the two integrals in the solution g^a_R, equation (A.15). In the first integral, for example, the portion of the

integrand
$$\left\{(\zeta + 1)(\zeta - 1)^{-2(R+r)-\lambda}\ F^{(2)}\right\}$$

may be developed in a power series since it is a regular function of ζ in the neighborhood of $\zeta = 0$. Our integrand then looks like

$$\zeta^{(R+r)-3/2-v}\ [a_0 + a_1\zeta + a_2\zeta^2 + \dots]$$

which for $(R+r)^{-3/2-v} \neq -1$ may be integrated termwise, giving the behavior of the integral at the upper limit as

$$\zeta^{(R+r)-1/2-v}\ [b_0 + b_1\zeta + b_2\zeta^2 + \dots\ ].$$

Multiplying by ζ^v we arrive at a function which behaves like

$$\zeta^{(R+r)-1/2} \quad \text{at} \quad \zeta = 0.$$

Now applying the boundary condition $\zeta^{1/2}g^a$ regular, we get a behavior like $\zeta^{(R+r)}$; and applying $\zeta^{-1/2}g^a$ regular, a behavior like $\zeta^{(R+r)-1}$. If R is not restricted as stated above we would have to make all E_r^a zero for r such that (R+r) or (R+r) - 1, respectively is less than zero. Since E_0^a has been assumed non-zero we must have (R+r) or (R+r) - 1, as the case may be, greater than or equal to zero giving permissible values of R as originally stated.

With these values of R the only terms in the solution g_R^a which do not satisfy the boundary conditions can be written in the form

$$[A_R^a + \frac{1}{\Lambda}\sum_{r=0}^{\infty} E_r^a\, C_r^{(2)}(0)]\ K^{(1)} \qquad \text{(A.21a)}$$

when

$$\lambda \neq -(2N+1) + \sqrt{\frac{2}{n}(2N+1)}$$

$$[B_R^a - \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \, C_r^{(1)}(0)] \, K^{(2)} \qquad \text{(A.21b)}$$

when

$$\lambda \neq -(2N+1) - \sqrt{\frac{2}{n}(2N+1)}$$

$$N = 0, 1, 2, 3, \ldots .$$

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \, D_r^{(2)}(0)] \, K^{(1)} \log \zeta \qquad \text{(A.21c)}$$

when

$$\lambda = -(2p+1) + \sqrt{\frac{2}{n}(2P+1)}$$

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \, D_r^{(1)}(0)] \, K^{(2)} \log \zeta \qquad \text{(A.21d)}$$

when

$$\lambda = -(2P+1) - \sqrt{\frac{2}{n}(2P+1)}$$

$$P = R, R+1, R+2, \ldots .$$

We note that it can be shown that

$$\mu = N + \tfrac{1}{2} \quad \text{when } \lambda = -(2N+1) - \sqrt{\frac{2}{n}(2N+1)}$$

and

$$\nu = N + \tfrac{1}{2} \quad \text{when } \lambda = -(2N+1) + \sqrt{\frac{2}{n}(2N+1)}.$$

The following notation has been adopted

$C_r^{(1)}(0)$ is "Hadamards" finite part of the integral in K_{n+r}^a with $F^{(1)}$ appearing in the integrand. This method of evaluation gives the value for an integral whose integrand has at one of the limits of integration, a singular behavior like $1/\zeta^{n+p}$ with $n = 0, 1, 2, \ldots$, and $p < 1$ (see [9]).

$D_r^{(1)}(0)$ is the coefficient of ζ^{-1} in the series expansion of the integrand of the integral in K_{R+r}^a involving $F^{(1)}$.

The terms shown in (A.12a - d) are those terms in g_R^a which do not possess the half-order behavior required by the boundary condition. Thus, in order to satisfy our boundary condition, we must set equal to zero whichever two bracketed quantities in (A.21a - d) are relevant in view of the value of λ. We finally arrive at no more than two linear homogeneous equations to be satisfied by the unknowns A_R^a, B_R^a and E_r^a.

Case (b) $\lambda = N + \frac{(1 - nN)^2}{2n}$

In this case the solution g_R^a valid about $\zeta = 0$ is equation (A.13). The arguments used in the preceding discussion concerning the choice of the index R are still applicable. However, one can show that λ can no longer take on the special values making μ or $\nu = N + \frac{1}{2}$. In this case it can be shown that in order to satisfy the boundary condition we must set equal to zero the coefficient of $K^{(1)}$ and $K^{(2)}$ separately. This again gives rise to two linear homogeneous equations in A_R^a, B_R^a and E_r^a.

iii) About $\zeta = 1$; $(\zeta - 1)^{-\lambda} g^a$ exists.

An examination of the solution g_R^a (A.14) valid at $\zeta = 1$ shows that the terms of the complementary solution satisfy the boundary condition for any value of λ. It remains to examine the particular integral. If we consider a development of the terms in the integrand, which are regular in

$(\zeta - 1)$, in a power series and then integrate termwise, the most singular behavior after multiplying by $(\zeta - 1)^{-\lambda}$ will be like

$$(\zeta - 1)^{-2(R+r)+2-\lambda} .$$

Hence our boundary condition will be satisfied if

$$-2(R + r) + 2 - R\ell\left\{\lambda\right\} \geq 0$$

or

$$R\ell\left\{\lambda\right\} \leq 2 - 2(R + r).$$

(2) Symmetric solutions.

The application of the boundary conditions to the symmetric solutions proceeds in a manner very similar to the anti-symmetric case.

i) About $\zeta = -1$; $g_\zeta^s(-1) = \alpha$.

An inspection of the solution (A.19) shows that

$$\frac{d}{d\zeta}(K^s_{R+r}) = \frac{d}{d\zeta}(K^s_\alpha) = 0 \quad \text{at} \quad \zeta = -1$$

and our boundary condition yields

$$(A^s_R b_3 + B^s_R b_4) = \alpha . \qquad \text{(A.22)}$$

ii) About $\zeta = 0$; g^s or $\zeta^{-1}g^s$ regular at $\zeta = 0$.

<u>Case (a) $\lambda \neq N + \frac{(1 - nN)^2}{2n}$</u>

As in the anti-symmetric case requiring that $\zeta^{-1}g^s$ be regular gives $N = 1, 2, 3, \ldots$, as allowable values of the index. In addition we must have $\alpha \equiv 0$ making $K^s_\alpha \equiv 0$. The terms in g_R^s which will not be regular after applying the boundary condition can be written in the form

$$[A_R^s + \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s \, C_r^{(2)}(0)] \, K^{(1)} \tag{A.23a}$$

when

$$\lambda \neq -2N + 2\sqrt{\frac{N}{n}}$$

$$[B_R^s - \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s \, C_r^{(1)}(0)] \, K^{(2)} \tag{A.23b}$$

when

$$\lambda \neq -2N - 2\sqrt{\frac{N}{n}}$$

$$N = 1, 2, 3, \ldots .$$

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s \, D_r^{(2)}(0)] \, K^{(1)} \log \zeta \tag{A.23c}$$

when

$$\lambda = -2P + 2\sqrt{\frac{P}{n}}$$

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s \, D_r^{(1)}(0)] \, K^{(2)} \log \zeta \tag{A.23d}$$

when

$$\lambda = -2P - 2\sqrt{\frac{P}{n}}$$

$$P = R, R + 1, \ldots .$$

We note that it can be shown that

$$\mu = N \quad \text{when} \quad \lambda = -2N - 2\sqrt{\frac{N}{n}}$$

$$\nu = N \quad \text{when} \quad \lambda = -2N + 2\sqrt{\frac{N}{n}}$$

The application of the boundary condition g^s regular gives as allowable values $R = 0, 1, 2, 3, \ldots$. With these values of R the terms in g_R^s which will not be regular can be written in the form

$$[A_R^s + \frac{a}{\Lambda n} C_a^{(2)}(0) + \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s C_r^{(2)}(0)] K^{(1)} \qquad (A.24a)$$

when

$$\lambda \neq - 2N + 2\sqrt{\frac{N}{n}}$$

$$[B_R^s - \frac{a}{\Lambda n} C_a^{(1)}(0) - \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s C_r^{(1)}(0)] K^{(2)} \qquad (A.24b)$$

when

$$\lambda = - 2N - 2\sqrt{\frac{N}{n}}$$

$$N = 1, 2, 3, \ldots$$

$$[\frac{a}{\Lambda n} D_a^{(2)}(0) + \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s D_r^{(2)}(0)] K^{(1)} \log \zeta \qquad (A.24c)$$

$$\lambda = - 2P + 2\sqrt{\frac{P}{n}}$$

$$[\frac{a}{\Lambda n} D_a^{(1)}(0) + \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^s D_r^{(1)}(0)] K^{(2)} \log \zeta \qquad (A.24d)$$

$$\lambda = - 2P - 2\sqrt{\frac{P}{n}}$$

$$P = r, R + 1, \ldots ,$$

where $C_r^{(1)}(0)$ and $D_r^{(1)}(0)$ have the same meaning as in the anti-symmetric case and the notation $C_a^{(1)}(0)$ and $D_a^{(1)}(0)$ has an analogous meaning with reference to the term K_a^s.

In order to satisfy the boundary condition we must set equal to zero whichever two bracketed quantities in (A.23a-d) or (A.24a - d) are relevant in view of the value of λ. This gives two linear homogeneous equations to be satisfied by the unknowns A_R^s, B_R^s and E_r^s.

Case (b) $\lambda = N + \frac{(1 - nN)^2}{2n}$

In this case the solution for g_R^s valid at $\zeta = 0$ is (A.17). With λ as given above neither μ nor ν can take on integral values. In this case it can be shown that in order to satisfy the boundary condition we must set equal to zero the coefficients of $K^{(1)}$ and $K^{(2)}$ separately. This gives rise to two linear homogeneous equations in the unknowns A_R^s, [illegible], α, and E_r^s.

Applying the boundary condition in the manner used in the anti-symmetric case we find the following restriction on λ

$$Rl\left\{\lambda\right\} \leq 1 - 2(R+r).$$

Appendix B. Equal and Opposite Jets

Part I. Derivation of boundary conditions

(1) About the point $\zeta = 0$.

We have already specified the behavior of the symmetric and anti-symmetric perturbations in $|\zeta| < 1$ for case (a) and (b). We shall insure this behavior by demanding that

Case (a) $g^s(\zeta)$ and $\zeta^{-1/2}g^a(\zeta)$ be regular functions of ζ as $\zeta \to 0$.

Case (b) $\zeta^{-1/2}g^s(\zeta)$ and $g^a(\zeta)$ be regular functions of ζ as $\zeta \to 0$.

(2) Upstream infinity $\zeta = 1$.

i) the perturbation velocity w_z must vanish there, i.e.,

$$\lim_{\zeta \to 1} w_z = 0.$$

But

$$w_2 = wf_2' = w\,\frac{df_2}{d\zeta}\,\frac{d\zeta}{df_0}\,,$$

whence

$$\lim_{\zeta \to 1} \frac{\zeta^{1/2}(\zeta - 1)(\zeta + 1)}{2}\,[G_{1\zeta}e^{\lambda t} + G_{2\zeta}e^{\bar{\lambda}t}] = 0.$$

This implies

$$\lim_{\zeta \to 1}\ (\zeta - 1)G_{1\zeta} = 0$$

$$\lim_{\zeta \to 1}\ (\zeta - 1)G_{2\zeta} = 0.$$

Substituting for G_1 and G_2 from (5.8) or (5.9) respectively, we find case (a) and case (b) $\lim_{\zeta \to 1}\ (\zeta - 1)g^{\frac{g}{a}} = 0.$

ii) the perturbation pressure p_2 vanishes at upstream infinity, i.e.,

$$\lim_{\zeta \to 1}\ p_2 = 0.$$

This implies

$$\lim_{\zeta \to 1}\ R\ell\left\{\dot{f}_2\right\} = 0,$$

hence

$$\lim_{\zeta \to 1}\ R\ell\left\{\lambda G_1 e^{\lambda t} + \bar{\lambda} G_2 e^{\bar{\lambda}t}\right\} = 0.$$

This will be satisfied if

$$\lim_{\zeta \to 1}\ G_1 = 0$$

and

$$\lim_{\zeta \to 1}\ G_2 = 0.$$

Substituting from (5.8) or (5.9) respectively, we find

case (a) and case (b)

$$\lim_{\zeta \to 1}\ g^{\frac{g}{a}} = 0.$$

(3) Downstream infinity $\zeta = -1$.

In the limit as we anproach downstream infinity, the jet is to act like a straight jet. To an observer moving downstream with the asymptotic jet velocity, the perturbation velocity will appear to assume a constant value. Using the same arguments as in Appendix A to Section IV this boundary condition implies

$$(\zeta + 1) \frac{d}{d\zeta}[(\zeta + 1)^{1-\lambda} G_{1\zeta}] = (\zeta + 1) \frac{d}{d\zeta}[(\zeta + 1)^{1-\lambda} G_{2\zeta}] = 0.$$

Using equations (5.8) or (5.9) respectively and observing that the point $\zeta = -1$ is a regular singular point of the perturbation equation, we can satisfy the above boundary condition if

case (a) and case (b)

$$(\zeta + 1)^{-\lambda} g_a^s$$

exists.

Part II. Derivation of solutions of perturbation equation

The perturbation equation (5.10), after substitution of (5.11) for the form of the differential operator L_λ, becomes

$$g_{\zeta\zeta} + g_\zeta \left(\frac{\frac{1}{2}}{\zeta} + \frac{2\lambda}{\zeta - 1} - \frac{2\lambda}{\zeta + 1} \right) + \frac{g}{\zeta(\zeta-1)(\zeta+1)} \left(\frac{2\lambda^2 - 2\lambda}{\zeta - 1} + \frac{2\lambda^2 + 2\lambda}{\zeta + 1} - 2\lambda \right) = \frac{1}{\zeta(\zeta-1)(\zeta+1)} h \qquad (B.1)$$

where g and h represent either symmetric or anti-symmetric quantities, the subscripts having been omitted for convenience.

Equation (B.1) is a non-homogeneous linear ordinary differential equation with regular singular points at $a_1 = 0$, $a_2 = 1$, $a_3 = -1$ and has ∞ as a regular singular point. The exponents of the singularities α_i, β_i are shown below

$$\begin{array}{llll} \alpha_1 = 0, & \beta_1 = \frac{1}{2}, & \text{at} & \zeta = 0 \\ \alpha_2 = -\lambda, & \beta_2 = 1 - \lambda, & \text{at} & \zeta = 1 \\ \alpha_3 = \lambda, & \beta_3 = 1 + \lambda, & \text{at} & \zeta = -1 \\ \alpha_4 = 0, & \beta_4 = -\frac{1}{2}, & \text{at} & \zeta = \infty. \end{array}$$

If we make a transformation of the dependent variable

$$g = (\zeta - a_1)^{\alpha_1}(\zeta - a_2)^{\alpha_2}(\zeta - a_3)^{\alpha_3} H = (\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda} H \quad (B.2)$$

we arrive at the following homogeneous differential equation for H

$$H_{\zeta\zeta} + H_\zeta \left(\frac{1}{2\zeta}\right) + H \left(\frac{\lambda}{\zeta(\zeta - 1)(\zeta + 1)}\right) = 0 \quad (B.3)$$

with the same regular singular points. The new exponents of the singularities are shown below

$$\begin{array}{lll} 0, \frac{1}{2} & \text{at} & \zeta = 0 \\ 0, 1 & \text{at} & \zeta = 1 \\ 0, 1 & \text{at} & \zeta = -1. \end{array}$$

Equation (B.2) is known as Heun's differential equation (see [10] and [11]).

Solutions of the Homogeneous Equation

With the knowledge of the location of the regular singular points and the exponents of the singularities we may write the forms of the solutions of equation (B.3). The application of the transformation (B.2) to these solutions will yield the complementary solutions of equation (B.1) for g which we will designate as $K^{(i)}$.

About $\zeta = 0$

$$K^{(1)} = \zeta^{1/2}(\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda}\, F^{(1)}$$

$$K^{(2)} = (\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda}\, F^{(2)}$$

where $F^{(1)}$ and $F^{(2)}$ are regular and non-zero at $\zeta = 0$.

About $\zeta = 1$

$$K^{(3)} = (\zeta - 1)^{1-\lambda}(\zeta + 1)^{\lambda}\, F^{(3)}$$

$$K^{(4)} = c_4(\zeta - 1)^{1-\lambda}\log(\zeta - 1)(\zeta + 1)^{\lambda}\, F^{(3)} + (\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda}\, F^{(4)}$$

where $F^{(3)}$ and $F^{(4)}$ are regular and non-zero at $\zeta = 1$ and c_4 = constant. In addition we know that in any region of common validity of the solutions about $\zeta = 0$ and $\zeta = 1$ there exist linear relations of the form

$$K^{(1)} = a_1 K^{(3)} + b_1 K^{(4)} \qquad \text{with} \qquad a_1 b_2 - b_1 a_2 = \delta_1 \neq 0$$

$$K^{(2)} = a_2 K^{(3)} + b_2 K^{(4)}$$

<u>About $\zeta = -1$</u>

$$K^{(5)} = (\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda} F^{(5)}$$

$$K^{(6)} = c_6(\zeta - 1)^{-\lambda}(\zeta + 1)^{1+\lambda}\log(\zeta + 1) F^{(5)} + (\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda} F^{(6)}$$

where $F^{(5)}$ and $F^{(6)}$ are regular and non-zero at $\zeta = -1$, and c_6 = constant. Again we may write

$$K^{(1)} + a_3K^{(5)} + b_3K^{(6)} \quad \text{with} \quad a_3b_4 - b_3a_4 = \delta_2 \neq 0$$

$$K^{(2)} = a_4K^{(5)} + b_4K^{(6)}$$

Particular Integrals

Proceeding as in Appendix A to section IV we can write the particular integral as

$$\text{P.I.} = K^{(i)} \int_{\zeta_0}^{\zeta} \frac{h(\zeta)\, K^{(j)}\, d\zeta}{\zeta(\zeta - 1)(\zeta + 1)\, W(K^{(i)}, K^{(j)})} - K^{(j)} \int_{\zeta_0}^{\zeta} \frac{h(\zeta)\, K^{(i)}\, d\zeta}{\zeta(\zeta - 1)(\zeta + 1)\, W(K^{(i)}, K^{(j)})} \tag{B.4}$$

where $K^{(i)}$ and $K^{(j)}$ are two linearly independent complementary solutions and $W(K^{(i)}, K^{(j)})$ is the Wronskian of these two solutions. We may evaluate the Wronskians from the differential equations. They are

$$W(K^{(1)}, K^{(2)}) = \delta_1 W(K^{(3)}, K^{(4)}) = \delta_2 W(K^{(5)}, K^{(6)}) =$$

$$- A\zeta^{-1}(\zeta - 1)^{2\lambda}(\zeta + 1)^{-2\lambda}$$

with Λ = constant.

Complete Solutions of the Perturbation Equation

(1) Anti-symmetric solutions.

The complete anti-symmetric solution is

$$g_R^a = A_R^a K^{(1)} + B_R^a K^{(2)} + \sum_{r=0}^{\infty} E_r^a K_{R+r} \qquad \text{(B.5)}$$

where

$$E_r^a K_{R+r}^a$$

is the particular integral when $h(\zeta)$ in expression (B.4) is replaced by

case (a)

$$h_{R+r}^a = \frac{\zeta - 1}{\zeta^{1/2}} a_{R+r} \left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^{R+r}$$

or

case (b)

$$h_{R+r}^a = \frac{\zeta - 1}{\zeta + 1} a_{R+r} \left[\left(\frac{\zeta - 1}{\zeta + 1}\right)^2\right]^{R+r}$$

Making the appropriate substitutions we have

about $\zeta = 0$

case (a)

$$\begin{aligned} g_R^a &= A_R^a \left[\zeta^{1/2}(\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda} F^{(1)}\right] \\ &+ B_R^a \left[(\zeta - 1)^{-\lambda}(\zeta + 1)^{\lambda} F^{(2)}\right] + \sum_{r=0}^{\infty} \frac{E_r^a}{\Lambda} (\zeta-1)^{-\lambda} \times \\ &\times (\zeta+1)^{\lambda} \left\{\zeta^{1/2} F^{(1)} \int_{\zeta_0}^{\zeta} \zeta^{-1}(\zeta-1)^{2(R+r)+\lambda} \times \right. \\ &\times (\zeta+1)^{-2(R+r)-1-\lambda} F^{(2)} d\zeta - F^{(2)} \int_{\zeta_0}^{\zeta} \zeta^{-1/2} \times \\ &\left. \times (\zeta-1)^{2(R+r)+\lambda} (\zeta+1)^{-2(R+r)-1-\lambda} F^{(1)} d\zeta \right\} \qquad \text{(B.6)} \end{aligned}$$

case (b)

$$g_R^a = A_R^a[\zeta^{1/2}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(1)}] + B_R^a[(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(2)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^a}{\Lambda}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}\left\{\zeta^{1/2}F^{(1)}\int_{\zeta_0}^{\zeta} \zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda} \times\right.$$

$$\left. \times (\zeta+1)^{-2(R+r)-2-\lambda}F^{(2)}d\zeta - F^{(2)}\int_{\zeta_0}^{\zeta} (\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-2-\lambda}F^{(1)}d\zeta\right\} \tag{B.7}$$

<u>about $\zeta = 1$</u>

case (a)

$$g_R^a = (A_R^a a_1 + B_R^a a_2)[(\zeta-1)^{1-\lambda}(\zeta+1)^{\lambda}F^{(3)}] + (A_R^a b_1 + B_R^a b_2)[c_4(\zeta-1)^{1-\lambda} \times$$

$$\times \log(\zeta-1)(\zeta+1)^{\lambda}F^{(3)} + (\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(4)}] + \sum_{r=0}^{\infty} \frac{E_r^a b_1}{\Lambda}(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)^{\lambda}\left\{(\zeta-1)F^{(3)}\int_{\zeta_0}^{\zeta} [c_4\zeta^{-1}(\zeta-1)^{2(R+r)+1+\lambda}\log(\zeta-1) \times\right.$$

$$\times (\zeta+1)^{-2(R+r)-1-\lambda}F^{(3)} + \zeta^{-1}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(4)}]d\zeta$$

$$-[c_4(\zeta-1)\log(\zeta-1)F^{(3)} + F^{(4)}]\int_{\zeta_0}^{\zeta} \zeta^{-1}(\zeta-1)^{2(R+r)+1+\lambda} \times$$

$$\left. \times (\zeta+1)^{-2(R+r)-1-\lambda}F^{(3)}d\zeta\right\} \tag{B.8}$$

case (b)

$$g_R^a = (A_R^a a_1 + B_R^a a_2)[(\zeta-1)^{1-\lambda}(\zeta+1)^{\lambda}F^{(3)}] + (A_R^a b_1 + B_R^a b_2)[c_4(\zeta-1)^{1-\lambda} \times$$

$$\times \log(\zeta-1)(\zeta+1)^{\lambda}F^{(3)} + (\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(4)}] + \sum_{r=0}^{\infty} \frac{E_r^a \delta_1}{\Lambda}(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)^{\lambda}\left\{(\zeta-1)F^{(3)}\int_{\zeta_0}^{\zeta}[(c_4\zeta^{-1/2}(\zeta-1)^{2(R+r)+1+\lambda}\log(\zeta-1) \times\right.$$

$$\times (\zeta+1)^{-2(R+r)-2-\lambda}F^{(3)} + \zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-2-\lambda}F^{(4)}]d\zeta$$

$$-[c_4(\zeta-1)\log(\zeta-1)F^{(3)} + F^{(4)}]\int_{\zeta_0}^{\zeta}\zeta^{-1/2}(\zeta-1)^{2(R+r)+1+\lambda} \times$$

$$\left.\times (\zeta+1)^{-2(R+r)-2-\lambda}F^{(3)}d\zeta\right\} \qquad (B.9)$$

about $\zeta = -1$

case (a)

$$g_R^a = (A_R^a a_3 + B_R^a a_4)[(\zeta-1)^{-\lambda}(\zeta+1)^{1+\lambda}F^{(5)}] + (A_R^a b_3 + B_R^a b_4)[c_6(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)^{1+\lambda}\log(\zeta+1)F^{(5)} + (\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(6)}] + \sum_{r=0}^{\infty} \frac{E_r^a \delta_2}{\Lambda}(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)\left\{(\zeta+1)F^{(5)}\int_{\zeta_0}^{\zeta}[c_6\zeta^{-1}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-\lambda} \times\right.$$

$$\times \log(\zeta+1)F^{(5)} + \zeta^{-1}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(6)}]d\zeta$$

$$-[c_6(\zeta+1)\log(\zeta+1)F^{(5)} + F^{(6)}]\int_{\zeta_0}^{\zeta}\zeta^{-1}(\zeta-1)^{2(R+r)+\lambda} \times$$

$$\left.\times (\zeta+1)^{-2(R+r)-\lambda}F^{(5)}d\zeta\right\} \qquad (B.10)$$

case (b)

$$
\begin{aligned}
g_R^a = {} & (A_R^a a_3 + B_R^a a_4)[(\zeta-1)^{-\lambda}(\zeta+1)^{1+\lambda}F^{(5)}] + (A_R^a b_3 + B_R^a b_4)[c_6(\zeta-1)^{-\lambda} \times \\
& \times (\zeta+1)^{1+\lambda}\log(\zeta+1)F^{(5)} + (\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(6)}] + \sum_{r=0}^{\infty} \frac{E_r^a \delta_2}{\Lambda}(\zeta-1)^{-\lambda} \times \\
& \times (\zeta+1)^{\lambda}\Big\{(\zeta+1)F^{(5)}\int_{\zeta_0}^{\zeta}[c_6\zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda} \times \\
& \times \log(\zeta+1)F^{(5)} + \zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-2-\lambda}F^{(6)}]d\zeta \\
& - [c_6(\zeta+1)\log(\zeta+1)F^{(5)} + F^{(6)}]\int_{\zeta_0}^{\zeta}\zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda} \times \\
& \times (\zeta+1)^{-2(R+r)-1-\lambda}F^{(5)}d\zeta\Big\}
\end{aligned}
\tag{B.11}
$$

(2) Symmetric solutions.

The complete symmetric solution is

$$
g_R^s = A_R^s K^{(1)} + B_R^s K^{(2)} + \sum_{r=0}^{\infty} E_r^s K_{R+r}^s \tag{B.12}
$$

where

$$
E_r^s \, K_{R+r}^s
$$

is the particular integral when $h(\zeta)$ in expression (B.4) is replaced by

case (a) $$h_{R+r}^s = h_{R+r}\left\{\left(\frac{\zeta-1}{\zeta+1}\right)\right\}^{[illegible] R+r}$$

case (b) $$h_{R+r}^s = [illegible]\, h_{R+r}\left\{\left(\frac{\zeta-1}{\zeta+1}\right)\right\}^{[illegible]+r} .$$

Making the appropriate substitutions we have

<u>about $\zeta = 0$</u>

$$g_R^s = A_R^s[\zeta^{1/2}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(1)}]+B_R^s[(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(2)}]+\sum_{r=0}^{\infty}\frac{E_r^s}{\Lambda}\times$$

$$\times(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}\left\{\zeta^{1/2}F^{(1)}\int_{\zeta_0}^{\zeta}\zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}\times\right.$$

$$\left.\times F^{(2)}d\zeta-F^{(2)}\int_{\zeta_0}^{\zeta}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(1)}d\zeta\right\} \quad (B.13)$$

case (b)

$$g_R^s = A_R^s[\zeta^{1/2}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(1)}]+B_R^s[(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(2)}]+\sum_{r=0}^{\infty}\frac{E_r^s}{\Lambda}\times$$

$$\times(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}\left\{\zeta^{1/2}F^{(1)}\int_{\zeta_0}^{\zeta}\zeta^{-1}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-\lambda}\times\right.$$

$$\left.F^{(2)}d\zeta-F^{(2)}\int_{\zeta_0}^{\zeta}\zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-\lambda}F^{(1)}d\zeta\right\}$$

<u>about $\zeta = 1$</u>

case (a)

$$g_R^s = (A_R^s a_1+B_R^s a_2)[(\zeta-1)^{1-\lambda}(\zeta+1)^{\lambda}F^{(3)}]+(A_R^s b_1+B_R^s b_2)[c_4(\zeta-1)^{1-\lambda}\times$$

$$\times\log(\zeta-1)(\zeta+1)^{\lambda}F^{(3)}+(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(4)}]+\sum_{r=0}^{\infty}\frac{E_r^s b_1}{\Lambda}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}\times$$

$$\times(\zeta-1)F^{(3)}\int_{\zeta_0}^{\zeta}[c_4\zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda}\log(\zeta-1)(\zeta+1)^{-2(R+r)-1-\lambda}F^{(3)}$$

$$+\zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(4)}]d\zeta \; -[c_4(\zeta-1)\log(\zeta-1) \times$$

$$\times F^{(3)}+F^{(4)}]\int_{\zeta_0}^{\zeta} \zeta^{-1/2}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(3)}d\zeta\Big\}$$

case (b)

$$g_R^s = (A_R^s a_1+B_R^s a_2)[(\zeta-1)^{1-\lambda}(\zeta+1)^{\lambda}F^{(3)}]+(A_R^s b_1+B_R^s b_2)[c_4(\zeta-1)^{1-\lambda} \times$$

$$\times \log(\zeta-1)(\zeta+1)^{\lambda}F^{(3)}+(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(4)}]+ \sum_{r=0}^{\infty} \frac{E_r^s\delta_1}{\Lambda}(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}\times$$

$$\times \Big\{(\zeta-1)F^{(3)}\int_{\zeta_0}^{\zeta} [c_4(\zeta-1)^{2(R+r)+\lambda}\log(\zeta-1)(\zeta+1)^{-2(R+r)-\lambda}F^{(3)}$$

$$+(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-\lambda}F^{(4)}]d\zeta \; -[c_4(\zeta-1)\log(\zeta-1)F^{(3)}$$

$$+F^{(4)}]\int_{\zeta_0}^{\zeta} \zeta^{-1}(\zeta-1)^{2(R+r)+\lambda}(\zeta+1)^{-2(R+r)-\lambda}F^{(3)}d\zeta\Big\} \qquad (B.15)$$

<u>about $\zeta = -1$</u>

case (a)

$$g_R^s = (A_R^s a_3+B_R^s a_4)[(\zeta-1)^{-\lambda}(\zeta+1)^{1+\lambda}F^{(5)}]+(A_R^s b_3+B_R^s b_4)[c_6(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)^{1+\lambda}\log(\zeta+1)F^{(5)}+(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(6)}]+ \sum_{r=0}^{\infty} \frac{E_r^s\delta_2}{\Lambda}(\zeta-1)^{-\lambda} \times$$

$$\times (\zeta+1)^{\lambda}\Big\{(\zeta+1)F^{(5)}\int_{\zeta_0}^{\zeta} [c_6\zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-\lambda} \times$$

$$\times \log(\zeta+1)F^{(5)}+\zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda}F^{(6)}]d\zeta \; -$$

$$-[c_6(\zeta+1)\log(\zeta+1)F^{(5)}+F^{(6)}]\int_{\zeta_0}^{\zeta} \zeta^{-1/2}(\zeta-1)^{2(R+r)-1+\lambda} \times$$

$$(\zeta+1)^{-2(R+r)-\lambda}\ F^{(5)}d\,\zeta \Big\} \tag{B.16}$$

case (b)

$$g_R^s = (A_R^s a_3+B_R^s a_4)[(\zeta-1)^{-\lambda}(\zeta+1)^{1+\lambda}F^{(5)}+(A_R^s b_3+B_R^s b_4)[c_6(\zeta-1)^{-\lambda} \times$$

$$\times\ (\zeta+1)^{1+\lambda}\ \log(\zeta+1)F^{(5)}+(\zeta-1)^{-\lambda}(\zeta+1)^{\lambda}F^{(6)}] + \sum_{r=0}^{\infty} \frac{E_r^s \delta_2}{\Lambda}(\zeta-1)^{-\lambda}\times$$

$$\times\ (\zeta+1)^{\lambda}\Big\{(\zeta+1)F^{(5)}\int_{\zeta_0}^{\zeta} [c_6\zeta^{-1}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-1-\lambda} \times$$

$$\times\ \log(\zeta+1)F^{(5)} + \zeta^{-1}(\zeta-1)^{2(R+r)-1+\lambda}(\zeta+1)^{-2(R+r)-2-\lambda}F^{(6)}]d\zeta$$

$$-[c_6(\zeta+1)\log(\zeta+1)F^{(5)}+F^{(6)}]\int_{\zeta_0}^{\zeta} \zeta^{-1}(\zeta-1)^{2(R+r)-1+\lambda} \times$$

$$\times\ (\zeta+1)^{-2(R+r)-\lambda}F^{(5)}d\zeta\Big\} \tag{B.17}$$

Part III. <u>Application of Boundary Conditions</u>

case (a)(i) Anti-symmetric solutions.

We shall now apply the boundary conditions derived in part I of this Appendix to the solutions of part II.

<u>at $\zeta = 0$</u> $\zeta^{-1/2}g^a(\zeta)$ is a regular function of ζ at $\zeta = 0$ where $g^a(\zeta)$ is given by (B.6).

Proceeding as in Appendix A we find that the terms appearing in g_R^a which do not have the required $\frac{1}{2}$ order behavior demanded by the boundary condition can be written in the form

$$[B_R^a - \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \alpha_r^{(1)}] K^{(2)} \tag{B.18}$$

and

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a D_r^{(2)}(0)] \quad K^{(1)} \log \zeta \tag{B.19}$$

where $D_r^{(2)}(0) \neq 0$ is the coefficient of ζ^{-1} in the series development about $\zeta = 0$ of the integrand of the integral in K_{R+r}^a involving $F^{(2)}$ and $\alpha_r^{(1)}$ is the constant contribution from the lower limit of integration of the integral in K_{R+r}^a involving $F^{(1)}$ in the integrand.

Thus in order to satisfy the boundary condition we must equate the coefficients of $K^{(2)}$ and $K^{(1)} \log \zeta$ in (B.18) and (B.19) to zero, thus obtaining two linear homogeneous equations in the unknowns B_R^a and E_r^a.

<u>at $\zeta = -1$</u>

$$(\zeta + 1)^{-\lambda} g^a(\zeta)$$

exists, i.e., is finite at $\zeta = -1$.

In equation (B.10) for g_R^a we notice that all terms of the complementary solutions satisfy the boundary condition identically. Examining the particular integral we note that if

$$-2(R + r_{max}) - \lambda \neq -1$$

where r_{max} = greatest r for which $E_r^a \neq 0$ then the term in $(\zeta + 1)^{-\lambda} g_R^a$ with the least exponent behaves like

$$(\zeta + 1)^{-2(R+r_{max})+1-\lambda} .$$

Thus in order to satisfy the boundary condition, the following inequalities must hold

$$\lambda \text{ not real:} \quad 2R + R\ell\left\{\lambda\right\} \leq 1 - 2r_{max}$$

$$\lambda \text{ real:} \quad 2R + \lambda < 1 - 2r_{max}.$$

We can readily show that 1 is a lower bound for r_{max}, for, if $r_{max} = 0$, then we would have E_0^a as the only term appearing in the summation. But from (B.19) we would then have $E_0^a \equiv 0$ as the only solution. This is not possible since the index R is chosen such that E_0^a is the first non-zero E^a. Hence our inequalities now become

$$\lambda \text{ not real:} \quad 2R + R\ell\left\{\lambda\right\} \leq -1$$

$$\lambda \text{ real:} \quad 2R + \lambda < -1. \qquad \text{(B.20)}$$

We can discard the possibility

$$-2(R + r_{max}) - \lambda = -1$$

since the boundary condition would then demand $E^a_{r_{max}} = 0$ which contradicts the definition of r_{max}.

<u>at $\zeta = 1$</u>

$$\lim_{\zeta \to 1} E^a(\zeta) = 0.$$

One should note that in deriving the boundary conditions at $\zeta = 1$ we found that both

$$(\zeta - 1)E^{a}_{\zeta} \quad \text{and} \quad E^{a}$$

had to be zero as $\zeta \longrightarrow 1$. However, since the point $\zeta = 1$ is a regular singular point of the differential equation, one can readily demonstrate that these conditions are redundant and only one need be applied.

In order to satisfy this boundary condition we must show that in the neighborhood of $\zeta = 1$ the expression (B.8) for $g_R^a(\zeta)$ consists of terms which go to zero as $\zeta \longrightarrow 1$ or that any terms which do not go to zero have a coefficient which is identically zero.

Examining the solution g_R^a we find that as $\zeta \longrightarrow 1$ the term in the complementary solution with least exponent behaves like $(\zeta - 1)^{-\lambda}$, while the term with least exponent in the particular integral behaves either like $(\zeta - 1)^{-\lambda}$ or like $(\zeta - 1)^{2R+2}$, depending upon which of the following applies

$$
\begin{aligned}
&\text{(i)} \quad 2R + 2 < R\ell \left\{- \lambda\right\} \\
&\text{(ii)} \quad 2R + 2 = R\ell \left\{- \lambda\right\} \\
&\text{(iii)} \quad 2R + 2 > R\ell \left\{- \lambda\right\}
\end{aligned}
\qquad \text{(B.21)}
$$

We note here that the term behaving like $(\zeta - 1)^{2R+2}$ is obtained for $r = 0$.

If condition (i) of (B.21) applies the term with least exponent is $(\zeta - 1)^{2R+2}$. This term can be shown to have a non-zero coefficient and hence we can only satisfy the boundary condition by making

$$2R + 2 > 0$$

from which

$$R = 0, 1, 2, \ldots .$$

Hence (B.21) case (i) gives

$$\mathcal{R}\ell\left\{\lambda\right\} < -2.$$

If condition (ii) of (B.21) applies and λ is not real the terms $(\zeta - 1)^{2R+2}$ and $(\zeta - 1)^{-\lambda}$ have different behaviors as $\zeta \to 1$ and our boundary condition still demands that either the coefficient of $(\zeta - 1)^{2R+2}$ equals zero which we have seen to be impossible, or that

$$2R + 2 > 0$$

from which $R = 0, 1, 2, \ldots$. Hence (B.21) case (ii) gives

$$\mathcal{R}\ell\left\{\lambda\right\} \leq -2.$$

The possibility of λ real and $2R + 2 = -\lambda$ must also be considered. Examining (B.8) we find that the exponent

$$2(R + r) + 1 + \lambda = -1$$

for $r = 0$. This will give rise to a term in the particular integral behaving like $(\zeta - 1)^{-\lambda} \log (\zeta - 1)$ and having a non-zero coefficient. The boundary condition will then demand that

$$\mathcal{R}\ell\left\{\lambda\right\} < 0.$$

Finally, if case (iii) of (B.21) applies

$$(\zeta - 1)^{-\lambda}$$

is the most highly singular term. The boundary condition can be satisfied if

$$\mathcal{R}\ell \left\{\lambda\right\} < 0$$

or if we make the coefficient of this term equal to zero. In the latter case we must satisfy a linear homogeneous relation in A^a_R, B^a_R E^a_O. Then, examining the remaining terms, we find terms behaving like

$$(\zeta - 1)^{1-\lambda}, \ (\zeta - 1)^{1-\lambda} \log (\zeta - 1) \ \text{ and } \ (\zeta - 1)^{2R+2} .$$

But the inequality found from the boundary condition at $\zeta = -1$ shows that

$$2R + 2 < 1 - \mathcal{R}\ell \left\{\lambda\right\} .$$

Thus, again, the boundary condition demands that

$$2R + 2 > 0$$

from which

$$R = 0, 1, 2, \ldots .$$

Now from (B.20) we find

$$\begin{array}{ll} \lambda \text{ not real} & \mathcal{R}\ell \left\{\lambda\right\} \leq -1 \\ \lambda \text{ real} & \lambda < -1. \end{array}$$

(2) Symmetric solutions.

Case (a)

The procedures followed here are similar to those of the preceding pages. The results are as follows.

The boundary condition at $\zeta = 0$ demands that

$$\left[A^a_R + \frac{1}{A} \sum E^a_r a^{(1)}_r\right] = 0 . \qquad (B.22)$$

From the boundary condition at $\zeta = -1$ we find that the following inequalities hold

$$\begin{aligned} 2R + R\ell\left\{\lambda\right\} &\leq 1 - 2r_{max} \quad \text{for } \lambda \text{ not real} \\ 2R + \lambda &< 1 - 2r_{max} \quad \text{for } \lambda \text{ real.} \end{aligned} \qquad (B.23)$$

At $\zeta = 1$ we must examine three possible cases

$$\begin{aligned} &(i) \quad 2R + 1 < R\ell\left\{-\lambda\right\} \\ &(ii) \quad 2R + 1 = R\ell\left\{-\lambda\right\} \\ &(iii) \quad 2R + 1 > R\ell\left\{-\lambda\right\} \end{aligned} \qquad (B.24)$$

case (i) the boundary condition demands

$$2R + 1 > 0$$

which gives

$$R = 0, 1, 2, \ldots ,$$

and we find, from the inequality (i) of (B.24),

$$R\ell\left\{\lambda\right\} < -1.$$

case (ii) with λ not real we again must set

$$2R + 1 > 0$$

giving

$$R = 0, 1, 2, \ldots ,$$

and the equality (ii) of (B.24) show that

$$R\ell\left\{\lambda\right\} \leq -1.$$

With λ real and $2R + 1 = -\lambda$ we find terms having a non-zero coefficient behaving like

$$(\zeta - 1)^{-\lambda} \log (\zeta - 1)$$

and the boundary condition can be satisfied only if

$$R\ell \left\{ \lambda \right\} < 0.$$

case (iii) here we can satisfy the boundary condition if we make

$$R\ell \left\{ - \lambda \right\} > 0$$

which implies

$$R \left\{ \lambda \right\} < 0,$$

or we may not restrict λ as yet, but insist that the coefficient of terms behaving like $(\zeta - 1)^{-\lambda}$ be zero. In the latter case we must examine the remaining terms which behave like

$$(\zeta - 1)^{1-\lambda}, \quad (\zeta - 1)^{1-\lambda} \log (\zeta - 1) \quad \text{and} \quad (\zeta - 1)^{2R+1} .$$

Now if $2R + 1 < 1 - R\ell \left\{ \lambda \right\}$, and since the terms behaving like $(\zeta - 1)^{2R+1}$ have a non-zero coefficient, the boundary condition demands that

$$2R + 1 > 0$$

giving

$$R = 0, 1, 2, \ldots ,$$

and the inequality shows

$$R\ell \left\{ \lambda \right\} < 0.$$

We must also examine the possibility

$$2R + 1 \geq 1 - R\ell \left\{ \lambda \right\}.$$

Here we may satisfy the boundary condition by demanding that

$$1 - R\ell\{\lambda\} > 0.$$

We now must satisfy three inequalities simultaneously. They are

$$1 - R\ell\{\lambda\} > 0$$
$$2R + 1 \geq 1 - R\ell\{\lambda\}$$
$$2R + R\ell\{\lambda\} < 1 - 2r_{max}.$$

If we examine, for $r_{max} = 0$, the possible values of R and $R\ell\{\lambda\}$ which will satisfy all three inequalities we find that the only permissible values of R are $R = 0, 1, 2, 3, \ldots$. For $R = 0$ we can show

$$0 \leq R\ell\{\lambda\} < 1;$$

for $R = 1, 2, 3, \ldots$, we find

$$R\ell\{\lambda\} \leq 0.$$

With $2R + 1 \geq 1 - R\ell\{\lambda\}$ we might still satisfy the boundary condition without restricting λ by setting the coefficients of terms behaving like $(\zeta - 1)^{1-\lambda}$, $(\zeta - 1)^{1-\lambda}\log(\zeta - 1)$ equal to zero. This gives rise to two more linear homogeneous equations in A_R^S, B_R^S, E_T^S. With $r_{max} = 0$ we would now have to satisfy a total of four linear homogeneous equations in only three unknowns A_R^S, B_R^S and E_0^S. From the manner in which the coefficients of the unknowns arose, it seems likely that we are justified in assuming these equations to be independent (although the mathematical proof of this would be difficult). Hence to insure a non-trivial solution we must increase the number of unknowns and hence r_{max}. This however strengthens the inequalities of equation (B.24) and we find no further perturbations with $R\ell\{\lambda\} > 0$.

case (b) the application of the boundary conditions to the symmetric and anti-symmetric solutions of case (b) result in relations of the same nature as found for case (a). We shall not include this work here, but shall merely state the results. One does not find any perturbations that can satisfy the boundary conditions with $R\ell\{\lambda\} > 0$.

Appendix C

Finite Jet Impinging on Finite Plate

Part I. Derivation of boundary conditions

1) about $\zeta = 0$.

We have already seen that the solutions about $\zeta = 0$ must satisfy the following boundary conditions

g^s is a regular function of ζ at $\zeta = 0$

$\zeta^{-1/2} g^a$ is a regular function of ζ at $\zeta = 0$.

2) At the edges of the plate $\zeta = -1$.

We demand that the perturbations of the free surface $z_1 = 0$, i.e., that the free surface continue to originate at the edge of the plate. From (2.10) we must have

$$z_1 = \frac{1}{w'}\left\{i\chi - \dot{f}_2 - f_2'\right\} = 0 \quad \text{at } \zeta = -1.$$

Substituting in the above we have

$$2\zeta^{1/2}\left\{\frac{2(\zeta-b)(\zeta-\bar{b}) - (\zeta-1)\left[(\zeta-b) + (\zeta-\bar{b})\right]}{(\zeta-1)(\zeta-b)(\zeta-\bar{b})}\right\}\left[i\chi - (\lambda G_1 e^{\lambda t} + \bar{\lambda} G_2 e^{\bar{\lambda} t})\right] + 2\zeta^{1/2}\left\{G_{1\zeta} e^{\lambda t} + G_{2\zeta} e^{\bar{\lambda} t}\right\} = 0 \text{ at } \zeta = -1.$$

The coefficient of the term

$$\left[i\chi - (\lambda G_1 e^{\lambda t} + \bar{\lambda} G_2 e^{\bar{\lambda} t})\right]$$

can be shown to equal zero at $\zeta = -1$, hence our boundary condition is satisfied if

$$\left\{G_{1\zeta}e^{\lambda t} + G_{2\zeta}e^{\bar{\lambda} t}\right\} = 0 \quad \text{at} \quad \zeta = -1$$

for all time t. This implies

$$G_{1\zeta}(-1) = G_{2\zeta}(-1) = 0.$$

Substituting for G_1 and G_2 from (6.9) we find

$$g^s_\zeta(-1) = g^a_\zeta(-1) = 0.$$

3) Upstream infinity $\zeta = 1$.

(a) The velocity perturbation w_2 must vanish there, i.e.,

$$\lim_{\zeta \to 1} w_2 = 0.$$

But

$$w_2 = \frac{w^2}{\omega} f_{2\zeta} \frac{d\zeta}{dw}.$$

Substituting we have

$$\lim_{\zeta \to 1} \frac{\zeta^{1/2}[(\zeta-1)(\zeta-b)(\zeta-\bar{b})]}{\left\{2(\zeta-b)(\zeta-\bar{b})-(\zeta-1)[(\zeta-b)(\zeta-\bar{b})]\right\}} f_{2\zeta} = 0$$

which will be satisfied if

$$\lim_{\zeta \to 1} (\zeta - 1)G_{1\zeta} = \lim_{\zeta \to 1} (\zeta - 1)G_{2\zeta} = 0.$$

From (6.9) we find

$$\lim_{\zeta \to 1} (\zeta - 1)g^{s}_{\zeta}{}^{a} = 0.$$

(b) The pressure perturbation p_2 must vanish. We can satisfy this boundary condition by demanding that

$$\lim_{\zeta \to 1} R\ell\left\{\dot{f}_2\right\} = 0.$$

This will be satisfied if

$$\lim_{\zeta \to 1} G_1(\zeta) = \lim_{\zeta \to 1} G_2(\zeta) = 0.$$

Using (6.9) we find

$$\lim_{\zeta \to 1} g^{\substack{s\\a}} = 0.$$

4) Downstream infinity $\zeta = b$ and $\zeta = \bar{b}$.

We demand that the asymptotic behavior of the jets leaving the edges of the plate be that of straight jets. This implies that an observer moving with the asymptotic jet velocity sees no change in w_p. Proceeding as in sections IV and V our boundary condition then demands that

$$(\zeta-b)\frac{d}{d\zeta}\left[(\zeta-b)^{1-\lambda}G_{1\zeta}\right] = (\zeta-b)\frac{d}{d\zeta}\left[(\zeta-b)^{1-\bar{\lambda}}G_{2\zeta}\right] = 0$$

and

$$(\zeta-\bar{b})\frac{d}{d\zeta}\left[(\zeta-\bar{b})^{1-\lambda}G_{1\zeta}\right] = (\zeta-\bar{b})\frac{d}{d\zeta}\left[(\zeta-\bar{b})^{1-\bar{\lambda}}G_{2\zeta}\right] = 0.$$

These requirements will be satisfied if

$$(\zeta - b)^{-\lambda}\, g^{\substack{s\\a}} \quad \text{and} \quad (\zeta - \bar{b})^{-\lambda}\, g^{\substack{s\\a}}$$

exist at $\zeta = b$ or $\bar{b}$ respectively.

Part II. <u>Derivation of the Solutions of the Perturbation Equation</u>

If we substitute in the perturbation equation (6.13)

$$L_\lambda\left[g^{\substack{s\\a}}(\zeta)\right] = h^{\substack{s\\a}}(\zeta)$$

the expression (6.1[illegible]) for the differential operator L_λ we arrive at

$$g_{\zeta\zeta} + g_{\zeta}\left[\frac{\frac{1}{2}}{\zeta} + \frac{4\lambda}{\zeta-1} - \frac{2\lambda}{\zeta-b} - \frac{2\lambda}{\zeta-\overline{b}}\right]$$

$$+ g\left[\frac{2\lambda}{\zeta(\zeta-1)} - \frac{\lambda}{\zeta(\zeta-\overline{b})} - \frac{\lambda}{\zeta(\zeta-b)} + \frac{4\lambda^2-2\lambda}{(\zeta-1)^2} + \frac{\lambda^2+\lambda}{(\zeta-b)^2}\right.$$

$$\left. + \frac{\lambda^2+\lambda}{(\zeta-\overline{b})^2} + \frac{2\lambda^2}{(\zeta-b)(\zeta-\overline{b})} - \frac{4\lambda^2}{(\zeta-1)(\zeta-\overline{b})} - \frac{4\lambda^2}{(\zeta-1)(\zeta-b)}\right]$$

$$= \left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\overline{b})}\right] h(\zeta) \qquad \text{(C.1)}$$

where g and h denote either symmetric or anti-symmetric quantities.

Equation (6.1) is a second order ordinary differential equation of Fuchsian type. It has regular singular points at $\zeta = 0$, 1, b, and $\overline{b}$ in the finite plane and has ∞ as a regular singular point. After putting (6.1) into standard form (see [7 pp. 155 et seq.]), we can find the exponents of the singularities in the finite plane

$$\begin{array}{llll} \alpha_1 = 0 & \beta_1 = \frac{1}{2} & \text{at} & \zeta = 0 \\ \alpha_2 = -2\lambda & \beta_2 = 1 - 2\lambda & \text{at} & \zeta = 1 \\ \alpha_3 = \lambda & \beta_3 = 1 + \lambda & \text{at} & \zeta = b \\ \alpha_4 = \lambda & \beta_4 = 1 + \lambda & \text{at} & \zeta = \overline{b}. \end{array}$$

After a transformation of the dependent variable

$$g = \zeta^{\alpha_1}(\zeta-1)^{\alpha_2}(\zeta-b)^{\alpha_3}(\zeta-\overline{b})^{\alpha_4} H = (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\overline{b})^{-\lambda} H \qquad \text{(C.2)}$$

we have the following regular singularities and exponents in the finite plane for the differential equation governing H

$$\begin{array}{llll} 0, & \tfrac{1}{2} & \text{at} & \zeta = 0 \\ 0, & 1 & \text{at} & \zeta = 1 \\ 0, & 1 & \text{at} & \zeta = b \\ 0, & 1 & \text{at} & \zeta = \bar{b}. \end{array}$$

Solutions of the Homogeneous Equation

Knowing the position of the singular points and the exponents of the singularities, we can write the complementary solutions of the differential equation for H. Then applying the transformation above we can find the complementary solutions for g denoted by $K^{(i)}$.

about $\zeta = 0$

$$\begin{aligned} K^{(1)} &= \zeta^{1/2}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\, F^{(1)} \\ K^{(2)} &= (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\, F^{(2)} \end{aligned} \tag{C.3}$$

where $F^{(1)}$ and $F^{(2)}$ are regular and non-zero at $\zeta = 0$.

about $\zeta = 1$

$$\begin{aligned} K^{(3)} &= (\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\, F^{(3)} \\ K^{(4)} &= c_4(\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\log(\zeta-1)\, F^{(3)} \\ &\quad + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\, F^{(4)} \end{aligned} \tag{C.4}$$

where c_4 = constant and $F^{(3)}$ and $F^{(4)}$ are regular and non-zero at $\zeta = 1$. We also know that the following linear homogeneous relations with constant coefficients holds

$$K^{(1)} = a_1 K^{(3)} + b_1 K^{(4)} \quad \text{where} \quad a_1 b_1 - b_1 a_2 = \delta_1 \neq 0$$

$$K^{(2)} = a_2 K^{(3)} + b_2 K^{(4)} \tag{C.5}$$

<u>about $\zeta = b$</u>

$$K^{(5)} = (\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda} F^{(5)}$$

$$K^{(6)} = c_6(\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda} \log(\zeta-b)\, F^{(5)}$$

$$+ (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} F^{(6)} \tag{C.6}$$

where c_6 = constant and $F^{(5)}$ and $F^{(6)}$ are regular and non-zero at $\zeta = b$. In addition we may write

$$K^{(1)} = a_3 K^{(5)} + b_3 K^{(6)} \quad \text{where} \quad a_3 b_4 - b_3 a_4 = \delta_2 \neq 0$$

$$K^{(2)} = a_4 K^{(5)} + b_4 K^{(6)} \tag{C.7}$$

<u>about $\zeta = \bar{b}$</u>

$$K^{(7)} = (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{1+\lambda} F^{(7)}$$

$$K^{(8)} = c_8(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{1+\lambda} \log(\zeta-\bar{b})\, F^{(7)}$$

$$+ (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} F^{(8)} \tag{C.8}$$

where c_8 = constant and $F^{(7)}$ and $F^{(8)}$ are regular and non-zero at $\zeta = \bar{b}$. We have in addition

$$K^{(1)} = a_5 K^{(7)} + b_5 K^{(8)} \quad \text{where} \quad a_5 b_6 - b_5 a_6 = \delta_3 \neq 0$$
$$K^{(2)} = a_6 K^{(7)} + b_6 K^{(8)} \qquad (C.9)$$

<u>about $\zeta = -1$</u>. The point $\zeta = -1$ is an ordinary point of the differential equation (C.1). As a consequence, we may choose as two linear independent complementary solutions $K^{(9)}$ and $K^{(10)}$ such that

$$K^{(9)}(-1) = K_{\zeta}^{(10)}(-1) = 0$$
$$K_{\zeta}^{(9)}(-1) = K^{(10)}(-1) = 1.$$

Moreover we may write

$$K^{(1)} = a_7 K^{(9)} + b_7 K^{(10)} \quad \text{where} \quad a_7 b_8 - b_7 a_8 = \delta_4 \neq 0$$
$$K^{(2)} = a_8 K^{(9)} + b_8 K^{(10)} \qquad (C.10)$$

<u>Complete Solutions of Perturbation Equations</u>

In order to write the complete symmetric or anti-symmetric solutions of the perturbation equation (C.1) we must find the particular integral. The particular integral can be found by using the method of sections IV and V. We shall have need of the Wronskian of the complementary solutions. From (C.1) we can write

$$W(K^{(1)}, K^{(2)}) = A \exp - \int \left[\frac{\frac{1}{2}}{\zeta} + \frac{4\lambda}{\zeta - 1} - \frac{2\lambda}{\zeta - b} - \frac{2\lambda}{(\zeta - \bar{b})} \right] d\zeta$$
$$= A \left[\zeta^{-1/2} (\zeta - 1)^{-4\lambda} (\zeta - b)^{2\lambda} (\zeta - \bar{b})^{2\lambda} \right]. \qquad (C.11)$$

(1) Anti-symmetric solutions

The complete anti-symmetric solution is

$$g_R^a = A_R^a\, K^{(1)} + B_R^a\, K^{(2)} + \sum_{r=0}^{\infty} E_r^a\, K_{R+r}^a \qquad (C.12)$$

where A_R^a, B_R^a and E_r^a are unknown constants and the term $E_r^a\, K_{R+r}^a$ is the solution for a particular integral of (C.1) when $h(\zeta)$ is replaced by

$$h_{R+r}^a = \frac{(\zeta - 1)}{\zeta^{1/2}}\, a_{R+r} \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^{R+r} .$$

Making the appropriate substitutions in (C.12) we have

<u>about $\zeta = 0$</u>

$$g_R^a = A_R^a[\zeta^{1/2}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(1)}] + B_R^a[(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(2)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^a}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\left\{\zeta^{1/2}F^{(1)}\int_{-1}^{\zeta}(\zeta-1)^{2(R+r)+1+2\lambda}\right. \times$$

$$\times\ (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(2)}\left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta$$

$$-F^{(2)}\int_{-1}^{\zeta}\zeta^{1/2}(\zeta-1)^{2(R+r)+1+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda} \times$$

$$\left.\times\ F^{(1)}\left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta\right\} \qquad (C.13)$$

<u>about $\zeta = 1$</u>

$$g_R^a = (A_R^a a_1 + B_R^a a_2)[(\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(3)}] + (A_R^a b_1 + B_R^a b_2) \times$$

$$\times\ [c_4(\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\log(\zeta-1)F^{(3)} + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(4)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^a b_1}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} \Big\{(\zeta-1)F^{(3)} \int_{-1}^{\zeta} [c_4(\zeta-1)^{2(R+r)+2+2\lambda} \times$$

$$\times\ (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}\log(\zeta-1)F^{(3)} + (\zeta-1)^{2(R+r)+1+2\lambda} \times$$

$$\times\ (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(4)}]\Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big]d\zeta$$

$$-[c_4(\zeta-1)\log(\zeta-1)F^{(3)} + F^{(4)}] \int_{-1}^{\zeta} (\zeta-1)^{2(R+r)+2+2\lambda}(\zeta-b)^{-(R+r)-\lambda} \times$$

$$\times\ (\zeta-\bar{b})^{-(R+r)-\lambda}F^{(3)}\Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big]d\zeta\Big\} \qquad (C.14)$$

<u>about $\zeta = b$</u>

$$g_R^a = (A_R^a a_3 + B_R^a a_4)[(\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda}F^{(5)}] + (A_R^a b_3 + B_R^a b_4) \times$$

$$\times\ [c_5(\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda}\log(\zeta-b)F^{(3)} + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(4)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^a b_2}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} \Big\{(\zeta-b)F^{(5)} \int_{-1}^{\zeta} [c_6(\zeta-1)^{2(R+r)+1+\lambda} \times$$

$$\times\ (\zeta-b)^{-(R+r)+1-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}\log(\zeta-b)F^{(5)} + (\zeta-1)^{2(R+r)+1+2\lambda} \times$$

$$\times\ (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(6)}]\Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big]d\zeta -$$

$$-[c_6(\zeta-b)\log(\zeta-b)F^{(5)}+F^{(6)}]\int_{-1}^{\zeta}(\zeta-1)^{2(R+r)+1+2\lambda}(\zeta-b)^{-(R+r)+1-\lambda}\ \times$$

$$\times\ (\zeta-\bar{b})^{-(R+r)-\lambda}F^{(5)}\ \left[\frac{1}{\zeta(\zeta-1)}-\frac{1}{2\zeta(\zeta-b)}-\frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta\ \Big\}\qquad (C.15)$$

<u>about $\zeta = \bar{b}$</u>

$$g_R^a = (A_R^a a_5+B_R^a a_6)[(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(7)}]+(A_R^a b_5+B_R^a b_6)\ \times$$

$$\times[c_8(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{1+\lambda}\log(\zeta-\bar{b})F^{(7)}+(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(8)}]$$

$$+\sum_{r=0}^{\infty}\frac{E_r^a\delta_3}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\Big\{(\zeta-\bar{b})F^{(7)}\int_{-1}^{\zeta}[c_8(\zeta-1)^{2(R+r)+1+\lambda}\ \times$$

$$\times(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)+1-\lambda}\log(\zeta-\bar{b})F^{(7)}+(\zeta-1)^{2(R+r)+1+2\lambda}\ \times$$

$$\times\ (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(8)}]\left[\frac{1}{\zeta(\zeta-1)}-\frac{1}{2\zeta(\zeta-b)}-\frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta$$

$$-[c_8(\zeta-\bar{b})\log(\zeta-\bar{b})F^{(7)}+F^{(8)}]\int_{-1}^{\zeta}(\zeta-1)^{2(R+r)+1+2\lambda}(\zeta-b)^{-(R+r)-\lambda}\ \times$$

$$\times\ (\zeta-\bar{b})^{-(R+r)+1-\lambda}F^{(7)}\ \left[\frac{1}{\zeta(\zeta-1)}-\frac{1}{2\zeta(\zeta-b)}-\frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta\ \Big\}\ (C.16)$$

<u>about $\zeta = -1$</u>

$$g_R^a = (A_R^a a_7+B_R^a a_8)K^{(9)}+(A_R^a b_7+B_R^a b_8)K^{(10)}+\sum_{r=0}^{\infty}\frac{E_r^a\delta_4}{\Lambda}\Big\{K^{(9)}\int_{-1}^{\zeta}\ \times$$

$$\times(\zeta-1)^{2(R+r)+1+4\lambda}(\zeta-b)^{-(R+r)-2\lambda}(\zeta-\bar{b})^{-(R+r)-2\lambda}K^{(10)}\left[\frac{1}{\zeta(\zeta-1)}-\right.$$

$$- \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big] d\zeta - K^{(10)} \int_{-1}^{\zeta} (\zeta-1)^{2(R+r)+1+4\lambda}(\zeta-b)^{-(R+r)-2\lambda} \times$$

$$\times (\zeta-\bar{b})^{-(R+r)-2\lambda} K^{(9)} \Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big] d\zeta \Big\} \qquad (C.17)$$

(2) Symmetric solutions

The complete symmetric solution is

$$g_R^s = A_R^s K^{(1)} + B_R^s K^{(2)} + \sum_{r=0}^{\infty} E_r^s K_{R+r}^s \qquad (C.18)$$

where A_R^s, B_R^s and E_r^s are unknown constants and $E_r^s K_{R+r}^s$ is the solution for a particular integral of (C.1) when $h(\zeta)$ is replaced by

$$h_{R+r}^s = b_{R+r} \left[\frac{(\zeta - 1)^2}{(\zeta - b)(\zeta - \bar{b})}\right]^{R+r}.$$

Substituting in (C.18) we have

<u>about $\zeta = 0$</u>

$$g_R^s = A_R^s[\zeta^{1/2}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(1)}] + B_R^s[(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(2)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^s}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\Big\{ \zeta^{1/2}F^{(1)} \int_{-1}^{\zeta} \zeta^{1/2}(\zeta-1)^{2(R+r)+2\lambda} \times$$

$$\times (\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(2)}\Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big]d\zeta$$

$$- F^{(2)} \int_{-1}^{\zeta} \zeta(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda} F^{(1)} \times$$

$$\times \Big[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\Big]d\zeta \Big\} \qquad (C.19)$$

about $\zeta = 1$

$$g_R^s = (A_R^s a_1 + B_R^s a_2)[(\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(3)}] + (A_R^s b_1 + B_R^s b_2) \times$$

$$\times [c_4(\zeta-1)^{1-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}\log(\zeta-1)F^{(3)} + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(4)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^s \delta_1}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} \Big\{(\zeta-1)F^{(3)} \int_{-1}^{\zeta} [c_4 \zeta^{1/2} \times$$

$$\times (\zeta-1)^{2(R+r)+1+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}\log(\zeta-1)F^{(3)} + \zeta^{1/2} \times$$

$$\times (\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(4)}] \, [\frac{1}{\zeta(\zeta-1)} -$$

$$- \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}] \, d\zeta - [c_4(\zeta-1)\log(\zeta-1)F^{(3)} + F^{(4)}] \int_{-1}^{\zeta} \zeta^{1/2} \times$$

$$\times (\zeta-1)^{2(R+r)+1+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(3)} \times$$

$$\times [\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}] \, d\zeta \Big\} \qquad (C.20)$$

about $\zeta = b$

$$g_R^s = (A_R^s a_3 + B_R^s a_4)[(\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda}F^{(5)}] + (A_R^s b_3 + B_R^s b_4) \times$$

$$\times [c_4(\zeta-1)^{-2\lambda}(\zeta-b)^{1+\lambda}(\zeta-\bar{b})^{\lambda}\log(\zeta-b)F^{(5)} + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(6)}]$$

$$+ \sum_{r=0}^{\infty} \frac{E_r^s \delta_2}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} \Big\{(\zeta-b)F^{(5)} \int_{-1}^{\zeta} [c_6 \zeta^{1/2} \times$$

$$\times (\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)+1-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}\log(\zeta-b)F^{(5)} +$$

$$+ \zeta^{1/2}(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(6)}] \quad x$$

$$x\ [\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}]d\zeta - [c_6(\zeta-b)\log(\zeta-b)F^{(5)} + F^{(6)}] \int_{-1}^{\zeta} x$$

$$x\ \zeta^{1/2}(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)+1-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(5)} \quad x$$

$$x\ [\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}]d\zeta \Big\} \qquad (C.21)$$

<u>about $\zeta = \bar{b}$</u>

$$g_R^s = (A_R^s a_5 + B_R^s a_6)[(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(7)}] + (A_R^s b_5 + B_R^s b_6) \quad x$$

$$x[c_8(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{1+\lambda}\log(\zeta-\bar{b})F^{(7)} + (\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda}F^{(8)}]$$

$$+ \sum_{r=0}^{m} \frac{E_r^s b}{\Lambda}(\zeta-1)^{-2\lambda}(\zeta-b)^{\lambda}(\zeta-\bar{b})^{\lambda} \Big\{ (\zeta-\bar{b})F^{(7)} \int_{-1}^{\zeta} [c_8\zeta^{1/2} \quad x$$

$$x(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)+1-\lambda}\log(\zeta-\bar{b})F^{(7)}$$

$$+\zeta^{1/2}(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)-\lambda}F^{(8)}] \quad x$$

$$x\ [\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-b)}]d\zeta\ - [c_8(\zeta-\bar{b})\log(\zeta-\bar{b})F^{(7)} +$$

$$+F^{(8)}] \int_{-1}^{\zeta} \zeta^{1/2}(\zeta-1)^{2(R+r)+2\lambda}(\zeta-b)^{-(R+r)-\lambda}(\zeta-\bar{b})^{-(R+r)+1-\lambda}F^{(7)} x$$

$$x\ [\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}]\ d\zeta \Big\} \qquad (C.22)$$

about $\zeta = -1$

$$g_R^s = (A_R^s a_7 + B_R^s a_8)K^{(9)} + (A_R^s b_7 + B_R^s b_8)K^{(10)} + \sum_{r=0}^{\infty} \frac{E_r^s b_4}{\Lambda} \left\{ K^{(9)} \int_{-1}^{\zeta} \right. \times$$

$$\times\ \zeta^{1/2}(\zeta-1)^{2(R+r)+4\lambda}(\zeta-b)^{-(R+r)-2\lambda}(\zeta-\bar{b})^{-(R+r)-2\lambda}K^{(10)} \times$$

$$\times\ \left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\right] d\zeta\ - K^{(10)} \int_{-1}^{\zeta} \zeta^{1/2}(\zeta-1)^{2(R+r)+4\lambda} \times$$

$$\left. \times(\zeta-b)^{-(R+r)-2\lambda}(\zeta-\bar{b})^{-(R+r)-2\lambda}K^{(9)}\left[\frac{1}{\zeta(\zeta-1)} - \frac{1}{2\zeta(\zeta-b)} - \frac{1}{2\zeta(\zeta-\bar{b})}\right]d\zeta \right\}$$

(C.23)

Part III. Application of Boundary Conditions

We shall now apply the boundary conditions to the complete symmetric and anti-symmetric solutions of the perturbation equation. Since the method of attack is quite similar to that of previous sections, some algebraic detail will be omitted.

(1) Anti-symmetric solutions

at $\zeta = -1$

$$g_r^a(-1) = 0.$$

The solution g_R^a about $\zeta = -1$ is equation (C.17). We note that the derivative of the particular integral is zero at $\zeta = -1$ and hence we can satisfy the boundary condition if

$$[A_R^a a_7 + B_R^a a_8] = 0. \qquad (C.24)$$

<u>at $\zeta = 0$</u> $\zeta^{-1/2} g^a(\zeta)$ is a regular function of ζ as $\zeta \longrightarrow 0$.

The solution g_R^a valid near $\zeta = 0$ is equation (C.13). We can satisfy the boundary condition by setting equal to zero the coefficients of the terms in g_R^a that do not possess the behavior required by the boundary condition. This gives

$$[B_R^a + \frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \, a_r^{(1)}] = 0 \tag{C.25}$$

$$[\frac{1}{\Lambda} \sum_{r=0}^{\infty} E_r^a \, D_r^{(2)}(0)] = 0 \tag{C.26}$$

where $a_r^{(1)}$ and $D_r^{(2)}(0)$ have been defined previously.

<u>at $\zeta = b$ or $\bar{b}$</u> $\lim_{\zeta \to b} (\zeta - b)^{-\lambda} g^a(\zeta)$ and $\lim_{\zeta \to \bar{b}} (\zeta - \bar{b})^{-\lambda} g^a(\zeta)$

exist, i.e., are finite.

An inspection of the equations (C.15) and (C.16) shows that (C.15) is the same function of b as (C.16) is of $\bar{b}$ and hence we get the same information from both boundary conditions. We shall apply only the boundary condition at $\zeta = b$.

The terms in the complementary solution satisfy the boundary condition identically. If we demand that the exponent of the terms in $(\zeta - b)^{-\lambda} g^a$ arising from the particular integral be such as to give a finite value, we find that

$$R + Re\{\lambda\} \leq 1 - r_{max} \qquad \lambda \text{ not real}$$

$$R + \lambda \leq 1 - r_{max} \qquad \lambda \text{ real.}$$

We can show that as a consequence of equation (C.26) one is a proper lower bound for r_{max}. The inequalities above now give

$$R + R\ell\left\{\lambda\right\} \leq 0 \quad \text{for } \lambda \text{ not real}$$

$$R + \lambda \leq 0 \quad \text{for } \lambda \text{ real.} \tag{C.27}$$

<u>at $\zeta = 1$</u>

$$g^{a}(1) = 0.$$

As in the preceding problem, we note that $\zeta = 1$ is a regular singular point of the differential equation and we need apply only one of the two boundary conditions derived in part I of this Appendix since the other will then be satisfied.

In order to satisfy this boundary condition we must show that in the neighborhood of $\zeta = 1$ the expression (C.14) for $g_R^a(\zeta)$ consists of terms which go to zero as $\zeta \longrightarrow 1$, or that any terms which do not go to zero have a coefficient which is identically zero.

The term with least exponent in the complementary solution behaves like $(\zeta-1)^{-2\lambda}$ while the term from the particular integral with least exponent behaves like $(\zeta-1)^{-2\lambda}$ or $(\zeta-1)^{2R+2}$ depending upon which of the following applies

$$\begin{aligned} &(i) \quad 2R + 2 < R\ell\left\{-2\lambda\right\} \\ &(ii) \quad 2R + 2 = R\ell\left\{-2\lambda\right\} \\ &(iii) \quad 2R + 2 > R\ell\left\{-2\lambda\right\} \end{aligned} \tag{C.28}$$

Case (i): the boundary condition demands $2R + 2 > 0$ giving $R = 0, 1, 2, \ldots$, and the inequality shows $R\ell\left\{\lambda\right\} < -1$.

Case (ii): for λ not real we again must have

$$2R + 2 > 0$$

giving

$$R\ell \{\lambda\} \leq -1.$$

If λ is real we find there will be terms in the particular integral like $(\zeta-1)^{-2\lambda} \log(\zeta-1)$ having a non-zero coefficient. Our boundary condition then demands

$$R\ell \{\lambda\} < 0.$$

Case (iii): in this case we can satisfy the boundary condition by making

$$R\ell \{\lambda\} < 0.$$

Or we may not yet restrict λ, but set the coefficient of terms behaving like $(\zeta-1)^{-2\lambda}$ equal to zero. We must then examine the remaining terms which behave like

$$(\zeta-1)^{1-2\lambda}, \quad (\zeta-1)^{1-2\lambda} \log(\zeta-1), \quad (\zeta-1)^{2R+2}.$$

If now $2R + 2 < 1 - R\ell\{2\lambda\}$, we still must have

$$2R + 2 > 0,$$

giving

$$R = 0, 1, 2, 3, \ldots,$$

and the inequality gives

$$R\ell \{\lambda\} < -\tfrac{1}{2}.$$

If, however, $2R + 2 \geq 1 - R\ell\{2\lambda\}$ and we insist that $1 - R\ell\{2\lambda\} > 0$, we have three inequalities to satisfy simultaneously. They are the two immediately above and (C.27).

From these and the fact that R is an integer we find $R\ell\left\{\lambda\right\} \leq 0$ for all possible values or R and λ.

Again, however, we may satisfy the boundary condition by setting the coefficient of terms like $(\zeta-1)^{1-2\lambda}$ and $(\zeta-1)^{1-2\lambda}\log(\zeta-1)$ equal to zero. These give rise to two linear homogeneous relations in A_R^a, B_R^a and E_r^a. We now find that if we assume $r_{max} = 1$ we are forced to satisfy more equations than the number of unknowns assumed up to now (A_R^a, B_R^a, E_o^a, E_1^a). From the manner in which the equations arose, they would appear to be independent. Hence this would make $r_{max} > 1$ (in order to assure the existence of a non-trivial solution). If r_{max} is at least as great as 2, the inequality (C.27) is strengthened and we find that we can no longer have

$$2R + 2 \geq R\ell\left\{-2\lambda\right\}.$$

Hence we conclude there are no anti-symmetric perturbations with $R\ell\left\{\lambda\right\} > 0$.

(2) Symmetric solutions.

<u>at $\zeta = -1$</u>

$$g_\zeta^s(-1) = 0.$$

The solution for g_R^s is equation (C.23). Applying the boundary condition we find we must satisfy the equation

$$[A_R^s a_7 + B_R^s a_8] = 0. \qquad (C.29)$$

<u>at $\zeta = 0$</u> $g^s(\zeta)$ is a regular function of ζ as $\zeta \to 0$.

Examining the solution near zero (C.19) we find that we can satisfy the boundary condition by making

$$[A_R^s + \frac{1}{\lambda} \sum_{r=0}^{\infty} E_r^s \, a_r^{(1)}] = 0. \qquad (C.30)$$

<u>at $\zeta = b$ or $\bar{b}$</u>: $\lim_{\zeta \to b} (\zeta - b)^{-\lambda} g^s$ and $\lim_{\zeta \to \bar{b}} (\zeta - \bar{b})^{-\lambda} g^s$ exist, i.e., are finite.

Applying this boundary condition at $\zeta = b$ to g_R^s given by (C.21) we find we must satisfy the following inequalities

$$\begin{aligned} R + R\ell\{\lambda\} &\leq 1 - r_{max} \quad \lambda \text{ not real} \\ R + \lambda &< 1 - r_{max} \quad \lambda \text{ real.} \end{aligned} \qquad (C.31)$$

At this point in our development the largest lower bound we can assure for r_{max} is zero.

<u>at $\zeta = 1$</u>: $\lim_{\zeta \to 1} g^s(\zeta) = 0$

g_R^s valid near $\zeta = 1$ is (C.20).

In this case, corresponding to the situation we found in the anti-symmetric case, we must examine the relative size of the exponents

$$2R + 1 \quad \text{and} \quad R\ell\{-2\lambda\}.$$

The possible cases are

$$\begin{aligned} &(i) \quad 2R + 1 < R\ell\{-2\lambda\} \\ &(ii) \quad 2R + 1 = R\ell\{-2\lambda\} \\ &(iii) \quad 2R + 1 > R\ell\{-2\lambda\} \end{aligned} \qquad (C.32)$$

Case (i): since the terms behaving like $(\zeta-1)^{2R+1}$ can be shown to have a non-zero coefficient, we must have

$$2R + 1 > 0$$

which gives

$$R = 0,\ 1,\ 2,\ \dots\ .$$

Then the inequality (C.32) case (i) shows

$$R\ell\left\{\lambda\right\} < -\tfrac{1}{2}\ .$$

Case (ii): for λ not real we must still make $2R + 1 > 0$ which now implies

$$R\ell\left\{\lambda\right\} \leq -\tfrac{1}{2}\ .$$

For λ real. We find terms in the particular integral having non-zero coefficients behaving like $(\zeta-1)^{-2\lambda}\log(\zeta-1)$ and our boundary condition would make

$$R\ell\left\{\lambda\right\} < 0.$$

Case (iii): the boundary condition can be satisfied either (a) by making $R\ell\left\{-2\lambda\right\} > 0$ which implies $R\ell\left\{\lambda\right\} < 0$ or (b), by setting equal to zero the coefficients of terms behaving like $(\zeta-1)^{-2\lambda}$. If (b) is true we must go on and examine terms like

$$(\zeta-1)^{2R+1},\quad (\zeta-1)^{1-2\lambda},\quad (\zeta-1)^{1-2\lambda}\log(\zeta-1).$$

If now $2R + 1 < 1 - R\ell\left\{2\lambda\right\}$ we must have $2R + 1 > 0$ giving $R = 0,\ 1,\ 2,\ \dots$, and the inequality implies

$$R\ell\left\{\lambda\right\} < 0.$$

Another possibility is that $2R + 1 \geq 1 - R\ell\left\{2\lambda\right\}$.

If we satisfy the boundary condition by making $1 - R\ell\{2\lambda\} > 0$ we have to satisfy three inequalities simultaneously, namely

$$2R + 1 \geq 1 - R\ell\{2\lambda\}$$
$$1 - R\ell\{2\lambda\} > 0$$
$$R - R\ell\{\lambda\} \leq 1 - r_{max}.$$

If one assumes $r_{max} = 0$ there appears to be a possible choice of R such that $R\ell\{\lambda\} > 0$. But with $r_{max} = 0$ we find we must satisfy three linear homogeneous equations in the three unknowns A_R^s, B_R^s and E_O^s. This would require that for a non-trivial solution, i.e., A_R^s, B_R^s and E_O^s not all zero, the determinant of the coefficients of the unknowns must be zero. This would not appear to be a likely possibility, although we are not in a position to evaluate the coefficients in closed form in order to show this mathematically. Assuming the determinant not zero we have to make r_{max} at least as great as 1, and in such a case we find there are no possibilities of satisfying our three inequalities with $R\ell\{\lambda\} > 0$.

The remaining possibility is that we satisfy the boundary condition not be demanding $1 - R\ell\{2\lambda\} > 0$, but by setting equal to zero the coefficients of terms behaving like $(\zeta-1)^{1-2\lambda}$ and $(\zeta-1)^{1-2\lambda}\log(\zeta-1)$. Now even without the argument above we can show r_{max} is at least as large as 3 and this would strengthen the inequality of (C.31) so as to preclude the possibility of either case (ii) or case (iii) of (C.32) being applicable. Thus, again we find no possible solutions with $R\ell\{\lambda\} > 0$.

BIBLIOGRAPHY

[1] C. M. Ablow and W. D. Hayes: Perturbation of Free Surface Flows, Technical Report No. 1, Office of Naval Research, Contract N7onr-35807, Graduate Division of Applied Mathematics, Brown University (1951).

[2] Helmholtz: "On the Discontinuous Movements of Fluids", Philosophical Magazine 36 (1868) p. 337.

[3] G. Kirchoff: "Zur Theorie freier Flüssigkeitsstranlen", Journal fur die reine und angewandte Mathematik 70 (1869) p. 289.

[4] L. M. Milne-Thompson: Theoretical Hydrodynamics, Macmillan second edition (1950).

[5] William Thomson: "Vibrations of a Columnar Vortex", Philosophical Magazine 10 (5) (1880) pp. 155-68.

[6] H. Lamb: Hydrodynamics, Dover Publications sixth edition.

[7] A. Forsyth: Theory of Differential Equations, Part III, Vol. IV Cambridge University Press (1902).

[8] E. L. Ince: Ordinary Differential Equations, Dover Publications first American edition.

[9] K. W. Mangler: Improper Integrals in Theoretical Aerodynamics, Royal Aircraft Establishment Farnborough Report Aero. 2424 (1951).

[10] K. Heun: "Zur theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten", Mathematische Annalen 33 (1889) pp. 161-178.

[11] C. Snow: The Hypergeometric and Legendre Functions with Application to Integral Equations of Potential Theory, National Bureau of Standards (1942).