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A STRAIN-ENERGY EXPRESSION FOR THIN CYLINDRICAL SHELLS

By

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LIST OF SYMBOLS

Coordinates: r, z, θ. see Fig. 1

a   mean - radius

h   shell thickness

u   longitudinal displacement (positive in direction of z-axis)

v   tangential displacement (positive in direction of positive θ)

w   radial displacement (positive inward)

u_z, u_θ ......w_{zz}, ... derivatives of u, v, w with respect to subscripts

E, ν, Young's modulus, Poisson's ratio.

A STRAIN-ENERGY EXPRESSION FOR THIN CYLINDRICAL SHELLS

Differential equations for the determination of the displacements of thin shells, and an expression for the strain energy in terms of the displacements were first derived by Love. (1)   However, Love already realized and stated that his expression for the strain energy was not sufficiently accurate, and only applicable if the bending part of the strain energy was small compared with the membrane-stress part.   That there is a discrepancy between the differential equations and the strain-energy expression becomes evident if an extremum principle is used to derive a set of differential equations for the displacements which are then found to differ from the differential equations derived directly.

Subsequently, further objections were raised even against the differential equations, the principal one being that these equations lack certain symmetries.   These objections are discussed in detail in recent papers by Osgood and Joseph, (2) Langhaar (3) and Vlasov (4).

Approaching the problem on a higher level, Vlasov's paper starts with the general equations of elasticity and derives a new set of differential equations for thin shells of any shape.   He gives also a simplified set for cylindrical shells, showing the necessary symmetries.   It is interesting to note that Vlasov's equations agree with the differential equations for cylindrical shells derived in a direct and fairly elementary manner by Flügge. (5) It appears, therefore, that Flügge's approach is sufficient to treat the case of cylindrical shells.

For many engineering applications it would be advantageous to know the strain-energy expression, which was not derived by Flügge or Vlasov.   A more accurate strain-energy expression than the one given by Love was recently derived by Langhaar. (3)   When this expression is used to obtain the differential equations for the cylindrical shell it is found that they disagree with those derived by Flügge and Vlasov.

Another expression, based on a suggestion by Osgood, is used by Salerno and Levine (6), but it also does not lead to the usual set of differential equations.

The differential equations for the cylindrical shell given by Flügge and confirmed by Vlasov are in general use and may be presumed correct.   As none of the strain-energy expressions are consistent with these equations, the writers have used Flügge's approach to obtain a strain-energy expression which is consistent with his differential equations.

Flügge's book contains expressions for the strains [Eqs. (68), p. 115] from which the strain energy can be computed by integration over the volume of the shell in an obvious manner.   The strain energy for plane stress is

$$U = \frac{E}{2(1 - \nu^2)} \iiint \left[\epsilon_z^2 + \epsilon_\theta^2 + \frac{1 - \nu}{2} \gamma_{z\theta}^2 + 2\nu \epsilon_z \epsilon_\theta \right] dr dz r d\theta \tag{1}$$

where the term $rd\theta$ (not $ad\theta$) is used, allowing for the different length of circumferential fibers through the thickness of the shell.

The strains are:

$$\epsilon_z = u_z + (r - a)w_{zz}$$

$$\epsilon_\theta = \frac{v_\theta}{a} + \frac{r-a}{a}\frac{w_{\theta\theta}}{r} - \frac{w}{r}$$

$$\gamma_{z\theta} = \frac{u_\theta}{r} + \frac{r}{a}v_z + w_{z\theta}\left(\frac{r-a}{a} + \frac{r-a}{r}\right) \tag{2}$$

where the signs differ from Flügge's because of the change in direction of positive deflections w. After integration with respect to r from $a - h/2$ to $a + h/2$ the strain energy becomes finally

$$U = \frac{E}{2(1-\nu^2)}\frac{h}{a}\iint\left[a^2 u_z^2 + (v_\theta - w)^2 + 2a\nu u_z(v_\theta - w) + \frac{1-\nu}{2}(u_\theta + av_z)^2\right]dzd\theta$$

$$+ \frac{E}{24(1-\nu^2)}\frac{h^3}{a^3}\iint\left[a^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 + \frac{1-\nu}{2}(aw_{z\theta} - u_\theta)^2\right.$$

$$\left. + \frac{3(1-\nu)}{2}a^2(v_z + w_{z\theta})^2 + 2\nu a^2 w_{zz}(w_{\theta\theta} + v_\theta) + 2a^3 u_z w_{zz}\right]dzd\theta \tag{3}$$

Logarithmic terms occurring in the integration leading to (3) were approximated by expanding in series of ascending powers of h/a and neglecting terms above the cubic one.

The first term of equation (3), proportional to h/a, is identical with the strain energy of the membrane stresses given by Love and all other authors. The second term, proportional to h^3/a^3, differs from the respective expressions of Love,

$$\frac{E}{24(1-\nu^2)}\frac{h^3}{a^3}\iint\left[a^4 w_{zz}^2 + (w_{\theta\theta} + v_\theta)^2 + 2\nu a^2 w_{zz}(w_{\theta\theta} + v_\theta) + 2(1-\nu)a^2(w_{z\theta} + v_z)^2\right]dzd\theta \tag{4}$$

of Langhaar,

$$\frac{E}{24(1-\nu^2)}\frac{h^3}{a^3}\iint\left[a^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 + 2\nu a^2 w_{zz}(w_{\theta\theta} + w) + 2(1-\nu)a^2(w_{z\theta} + v_z)(w_{z\theta} - u_\theta)\right]dzd\theta \tag{5}$$

and of Salerno and Levine,

$$\frac{E}{24(1-\nu^2)}\frac{h^3}{a^3}\iint\left[a^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 + 2\nu a^2 w_{zz}(w_{\theta\theta} + w) + 2(1-\nu)a^2(w_{z\theta} + \frac{1}{2}v_z - \frac{1}{2}u_\theta)^2\right]dzd\theta \tag{6}$$

It should be emphasized that the numerical differences between the four expressions are quite small if compared to the total strain energy. On the other hand, if the terms proportional to h^3/a^3 only are compared the differences may be substantial. However, these terms in themselves are quite small compared to the membrane strain energy, unless the state of deformations considered is inextensional, or nearly so. In the inextensional case, the decisive h^3/a^3 term in Eq. (3) and the corresponding terms listed in Eqs. (4), (5) and (6) become entirely identical, and in a close to inextensional case the differences are not large. In this respect the reader is referred to the discussion of Langhaar's paper by Koiter. (7)

It appears therefore that the difference in the value of U found from Eq. (3) and the other expressions considered is presumably not very important. The advantage of Eq. (3) lies essentially in the fact that the principle of stationary potential energy when used with the expression (3) actually furnishes the accepted differential equations.

As Langhaar's expression has been derived on a fundamentally strict, geometrical basis, the source of the difference between Eqs. (3) and (5) deserves examination. One cause for the difference is the fact that Langhaar neglects certain terms which correspond to the non-linearity in the stress distribution, stating that their effect is quite small. That this is a contributory cause can be verified by comparing the expressions for the strains, which are found to agree only up to the linear terms in z. A second cause is the fact that in Eq. (1) Flügge uses the expression $rd\theta$ for the length of a circumferential fiber, while Langhaar uses the approximate value $ad\theta$ in the expression for the element of volume.

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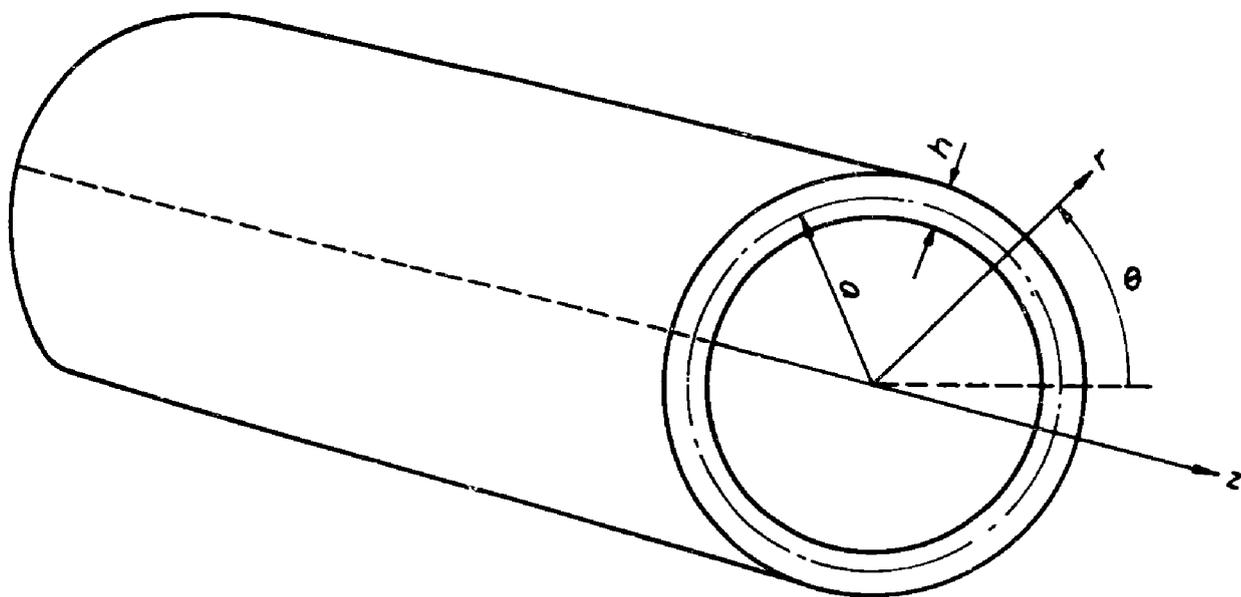
FIG. 1