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SOME METHODS OF COMPARING SOCIOMETRIC MATRICES

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with discussion by

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SOME METHODS OF COMPARING SOCIOMETRIC MATRICES[1/]

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Overview

An approach often used in sociometric investigations is to have each of the members of a group of n individuals rank all the others according to some characteristic such as wisdom, leadership ability, likeability, etc., with the object of studying, and perhaps improving, the structure of the group with respect to this relationship. One of the great desiderata in this connection is a fruitful method of comparing results obtained from the same group (a) at different times but with respect to the same characteristic or (b) at the same time but with respect to different characteristics. Again, when the members of two groups of the same size are in one-to-one correspondence, it may be desirable to compare results obtained from the two groups with respect to the same characteristic. This paper presents two measures which may be useful in making comparisons such as these.

For the investigation of what is called here the <u>hierarchical structure</u> of a group, this paper introduces first a function "h" of the ranks. This h, called the <u>hierarchy index</u>, ranges from the value 0 when the members of

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the group are indicated by the data to be "equal" with respect to the characteristic in question, to the value 1 when the most extreme type of hierarchical relationship appears. Differences between the h-values obtained from two sets of observations, whether from the same group or from different groups, may be tested for significance.

The paper next introduces a coefficient of agreement "θ" as another method of comparing the group structures associated with two sets of data from groups of the same size. The coefficient θ ranges from -1 when the data display opposite hierarchical characteristics, to +1 when they display the same hierarchical characteristics.

The measures θ and h appear to be of considerable sensitivity. However, their usefulness in sociometric research will depend ultimately on the significance of the concepts on which they are based and on the appropriateness of the manner in which the measures arithmetize those concepts. These issues can be decided only by the test of actual use of the measures.

1. Assumptions and Definitions. Let us assume that in a group of n individuals, each member ranks the other n-1 members, from highest to lowest, according to some agreed-on characteristic. More specifically, we assume that each member assigns the other members the ranks

$$n-1, n-2, \ldots, 3, 2, 1$$

in some order, so that in fact no two individuals are assigned the same rank by any one member. It is convenient to assign the members of the group the names "1", "2", ..., "n". Then the ranking data may be recorded systematically in the form of a table:

Ratees

		1	2	. . .	n
	1				
	2				
Raters					
	n				

Here the entries 1, 2, ..., n in the outer column are the names of the members assigning the ranks and the entries in the top row are the names of the members being ranked.

It is desirable for the purposes of what follows to enter a zero in the first row and first column, the second row and second column, and so on throughout the table. This is as though each individual were required to assign himself the rank zero. If we denote the entry in the $j^{\underline{th}}$ row and $k^{\underline{th}}$ column of the body of the table by s_{jk}, then

$$(1.1) \qquad s_{jk} = \begin{cases} p \text{ if "j" assigns "k" the rank } p \\ 0 \text{ if } j = k. \end{cases}$$

We then have the square array of ranks:

$$\begin{bmatrix} 0 & s_{12} & s_{13} & \cdot & \cdot & \cdot & s_{1n} \\ s_{21} & 0 & s_{23} & \cdot & \cdot & \cdot & s_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ s_{n1} & s_{n2} & s_{n3} & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

We shall call this the <u>data matrix</u> S of the group with respect to the characteristic in question. In experimental work, it is important to remember that this data matrix is not necessarily a stable property of the group, but may change drastically with time. Indeed, the proper control of such change might well be a primary objective in some circumstances.

In S we now form the column totals to get the score structure "s" of S:

$$s = (s_1, s_2, \dots, s_n)$$

where

$$(1.2) \qquad s_k = s_{1k} + s_{2k} + \dots + s_{nk} = \sum_{j=1}^{k} s_{jk}.$$

The total score, s_k, received by the k^{th} individual will be called simply his score. Two score structures differing at most in the order of the integers appearing in them will be called similar.

The data matrix S associated with a given group is not uniquely defined since a different assignment of "names" to the members of the group results in a symmetrical permutation of the rows and columns of S. It is a matter of definition that this does not change the structure of the group. For example, if n = 3, we might obtain for one method of naming the members of the group, the data matrix

$$S = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

If now we interchange the names of members "1" and "3", we in effect interchange the first and third rows as well as the first and third columns of S. We thus obtain the data matrix

$$S_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

It will be noted that S and S_1, like any two data matrices differing only because of different arrangement of the members of the group, have score structures which are similar according to our previous definition. The converse is not true, however. That is, the data matrices of groups having similar score structures cannot always be transformed one into the others simply by rearranging the individuals of the group.

From the definition of the ranking operation, it is clear that one, but not more than one, of the s_k's, can assume the value $(n-1)^2 = (n-1)\cdot(n-1)$ and that no higher score is attainable. Similarly, one of the s_k's, but not more than one, can assume the value $(n-1)\cdot 1$ which is the lowest score attainable. The scores s_k have the total

$$(1.3) \qquad \sum_{k=1}^{n} s_k = n\left[(n-1) + (n-2) + \dots + 2 + 1\right] = \frac{n^2(n-1)}{2}$$

since the expression in brackets is the sum of the entries in any one of the n rows. It follows that each score structure is a partitioning of $n^2(n-1)/2$ into n parts s_k such that

$$n-1 \leq s_k \leq (n-1)^2, \qquad k = 1, 2, \dots, n$$

and such that not more than one s_k is $(n-1)^2$ or $(n-1)$.

The number of distinct data matrices S which are possible for a given value of n is rather large even when n is small. In fact, there are $(n-1)!$ ways of filling the off-diagonal positions in each row, so that there are $\left[(n-1)!\right]^n$ such matrices possible. For $n=3$, this is $(2!)^3 = 8$, but for $n=4$ it is $(3!)^4 = 1296$.

By way of illustrating some of the ideas so far presented, we list all possible data matrices for the case $n=3$. The matrices

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

each has the score structure (3,3,3). Also, interchanging the names of individuals 2 and 3 will reduce one matrix to the other. The matrices

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

all have a score structure similar to (4,3,2) and again, any one may be obtained from any other by symmetric permutation of the rows and columns.

This last mentioned property does not persist beyond $n = 3$. Indeed, the matrices

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 2 & 1 \\ 2 & 3 & 0 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$$

both have score structure (8,7,6,3) but one cannot be reduced to the other by symmetrical permutation of the rows and columns. In the first the four 3's are distributed among three individuals, whereas in the second they are distributed between two individuals.

The large number of data matrices possible for given n emphasizes the importance of the score structure as a method of condensing the information contained in the matrices. When $n = 4$, we have 1296 matrices, as noted above, but there are only 16 score structures, as will appear later.

2. The Hierarchy Index. Among the score structures two extreme cases are of particular interest. The equality is the structure with

$$(2.1) \qquad s_1 = s_2 = \dots = s_n = \frac{n(n-1)}{2}$$

that is, the case when every member of the group has the same score. An example of this is given by the matrix:

$$\begin{bmatrix} 0 & n-1 & \dots & 1 \\ 1 & 0 & \dots & 2 \\ 2 & 1 & \dots & 3 \\ 3 & 2 & \dots & 4 \\ \dots & \dots & \dots & \dots \\ n-2 & n-3 & \dots & n-1 \\ n-1 & n-2 & \dots & 0 \end{bmatrix}$$

At the other extreme is the score structure which we call the extreme hierarchy. In the case of this score-structure, we have one individual with the highest possible score, thereafter one with the resulting next-highest possible score, and so on. For example we may have

$$S = \begin{bmatrix} 0 & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 0 & n-2 & \cdots & 2 & 1 \\ n-1 & n-2 & 0 & \cdots & 2 & 1 \\ n-1 & n-2 & n-3 & \cdots & 2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot \\ n-1 & n-2 & n-3 & \cdots & 2 & 1 \\ n-1 & n-2 & n-3 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{bmatrix}$$

with

$$(2.2) \quad \begin{cases} s_1 = (n-1)\,(n-1) \\ s_2 = (n-2)\,(n-2) + 1\cdot(n-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_k = (n-k)\,(n-k) + (k-1)\,(n-k+1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{n-1} = 1\cdot 1 + (n-2)\cdot 2 \\ s_n = (n-1)\cdot 1 \end{cases}$$

Any score structure similar to (2.2) is also an example of the extreme hierarchy. A little thought will make it clear that any data matrix whose score structure is an extreme hierarchy can be reduced to the form given above by suitable numbering of the members of the group.

In the extreme hierarchy (2.2) we have $s_k - s_{k+1} = n-2$ so that the scores of s are equally spaced n-2 units apart, beginning with $(n-1)^2$ and ending with n-1. It is thus a simple matter to write down the score structure for an extreme hierarchy when n is given.

In order to obtain a measure of where a given score structure falls between the equality and the extreme hierarchy, we note first that the mean score of a score structure is

$$(2.3) \qquad \bar{s} = \frac{1}{n}\sum_{k=1}^{n} s_k = \frac{n\ (n-1)}{2}$$

and that the variance is therefore

$$(2.4) \qquad \text{Var.} = \frac{1}{n}\sum_{k=1}^{n} \left(s_k - \frac{n(n-1)}{2}\right)^2 .$$

The maximum variance of the s's will evidently be obtained when they have the values listed in (2.2). The variance in this case is given by

$$(2.5) \quad \text{Var}_{\max} = \frac{1}{n}\sum_{k=1}^{n}\left[(n-k)^2 + (k-1)(n-k+1) - \frac{n\ (n-1)}{2}\right]^2$$

$$= (n^2 - 1)(n-2)^2/12.$$

We may therefore define a <u>hierarchy index</u>, h, as the ratio of the actually observed variance to the maximum possible variance. From (2.4) and (2.5), we see then that

$$(2.6\ ) \qquad h = \frac{12}{n(n^2-1)(n-2)^2} \sum_{k=1}^{n}\left(s_k - \frac{n(n-1)}{2}\right)^2 , \qquad (n \geq 3).$$

This reduces without difficulty to the form:

$$(2.7) \qquad h = \frac{12}{n(n^2-1)(n-2)^2} \left(\Sigma {s_k}^2 - \frac{n^3(n-1)^2}{4}\right) .$$

From the definition it follows at once that the minimum value of h is zero. This value is obtained when the score structure is the equality. Similarly, the maximum value of h is 1, which is obtained when the score structure is an extreme hierarchy.

It should be noted that we can have a hierarchy in the sense that

$$s_1 > s_2 > \cdots > s_n,$$

for example, without having the extreme case defined by (1.4). For such a hierarchy, $h < 1$. An example is given by the data matrix

$$\begin{bmatrix} 0 & 4 & 3 & 1 & 2 \\ 4 & 0 & 3 & 2 & 1 \\ 4 & 3 & 0 & 2 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 3 & 4 & 2 & 1 & 0 \end{bmatrix}$$

with $s = (15, 14, 10, 6, 5)$, $h = 41/45 = .91^+$. However, if we have an extreme hierarchy where $n = 5$, the matrix has a score structure similar to $s = (16, 13, 10, 7, 4)$, with $h=1$, of course.

Another interesting special structure occurs only when n is even, because it takes an odd number of persons to form an equality. With an even number of persons, we can have

$$s_1 = (n-1)^2, \quad s_2 = s_3 = \cdots = s_n = \frac{(n-1)^2 + 1}{2}$$

so that if $n > 2$, we have one leader, the other members of the group constituting among themselves an equality. Such a score-structure may be called an extreme leadership. Denoting h by h_1 in this case, we find

$$h_1 = \frac{3}{n+1}$$

which approaches 0 for large values of n. Even though the value of h_1 is small, an extreme leadership might have great social importance.

This demonstrates that h is not to be regarded as a measure of the possible social importance of a given score structure. It is only a measure of resemblance to the extreme hierarchy.

To give a final illustration of the behavior of h, we list in Table 1 the 16 score structures for n=4 and the corresponding values of h.

TABLE 1

SCORE STRUCTURES AND VALUES OF h FOR n=4

SCORE STRUCTURE*	h	SCORE STRUCTURE	h
9753	1	8754	5/10
9744		8664	4/10
9663	9/10	8655	3/10
8853		7764	
8844	8/10	7755	2/10
9654	7/10	7665	1/10
8763		6666	0
9555	6/10		
7773			

It is easy to verify that there actually are data matrices giving rise to each of the listed score structures.

3. Comparison of Score Structures. It is desirable for certain purposes to compare matrices of rankings such as described above, which are obtained at different times, or with respect to different characteristics, from the same group. Suppose, for example, that the experience and education of a group were designed so as to produce eventually a group with the extreme hierarchy as its structure. Progress toward this end could be measured by observing the progress of h toward the value + 1.

Suppose, on the other hand, that we wish to compare two groups of individuals in which the members of the groups can be placed in a logical

*In this and following tables, commas are omitted from the score structures.

one-to-one correspondence. Here corresponding members of the two groups would be assigned the same number "name", of course. As a pair of such groups we could use two baseball teams, for example. Then the score structures of the two groups may be compared simply by computing the product-moment correlations of these score structures. For fixed n, the correlation coefficient in this case takes on one of a finite set of values.

Let $s = (s_1, s_2, \ldots s_n)$ and $s' = (s'_1, s_2', \ldots, s_n')$ be two score structures obtained from the same group or from two groups, under the hypotheses stated above. We then define the <u>coefficient of agreement</u>, θ, of the two score structures to be

$$\theta = \frac{\text{Cov}(s, s')}{\sqrt{\text{Var}(s)\cdot\text{Var}(s')}}$$

Since

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} s'_k = \frac{n(n-1)}{2}$$

this may be written as

$$\theta = \frac{\sum_{k=1}^{n}\left(s_k - \frac{n(n-1)}{2}\right)\left(s'_k - \frac{n(n-1)}{2}\right)}{\sqrt{\sum_{k=1}^{n}\left(s_k - \frac{n(n-1)}{2}\right)^2 \cdot \sum_{k=1}^{n}\left(s'_k - \frac{n(n-1)}{2}\right)^2}} \tag{3.1}$$

which reduces readily to

$$\theta = \frac{\Sigma\, s_k s'_k - \frac{n^3(n-1)^2}{4}}{\sqrt{\left(\Sigma s^2_k - \frac{n^3(n-1)^2}{4}\right)\left(\Sigma s'^2_k - \frac{n^3(n-1)^2}{4}\right)}} \tag{3.2}$$

Since either n or n-1 is even, $\frac{n^3(n-1)^2}{4}$ will always be an integer.

If we put

$$d_k = s_k - s'_k$$

we have

$$\Sigma d^2_k = \Sigma s^2_k + \Sigma s'^2_k - 2\Sigma s_k s'_k$$

so that the formula (3.2), on elimination of $\Sigma s_k s'_k$, may be written in the form:

$$\theta = \frac{\Sigma s^2_k + \Sigma s'^2_k - \frac{1}{2} n^3(n-1)^2 - \Sigma d^2_k}{\sqrt{\left[2\Sigma s^2_k - \frac{1}{2} n^3(n-1)^2\right]\left[2\Sigma s'^2_k - \frac{1}{2} n^3 (n-1)^2\right]}} \tag{3.3}$$

Since the s's are ordinarily not very large, formula (3.2) is not too inconvenient, especially if a table of products is available. However, (3.3) enables us to work with a table of squares, and even for small n it is apt to be more convenient than (3.2) since s_k may be as large as $(n-1)^2$.

In the event that both s and s' are similar to the score structure of the extreme hierarchy, we have

$$\Sigma s^2_k = \Sigma s'^2_k = \sum_{k=1}^{n} \left[(n-k)^2 + (k-1)(n-k+1)\right]^2$$

$$= \frac{1}{6}n^2 (2n^4 - 5n^3 + 3n^2 + 2n - 2) .$$

Then

$$2\Sigma s^2_k - \frac{1}{2} n^3 (n-1)^2 = \frac{1}{6} n (n^2-1)(n-2)^2$$

and θ assumes the much simpler, special form

$$\theta_1 = 1 - \frac{6\Sigma d^2}{n(n^2-1)(n-2)^2} . \tag{3.4}$$

It is interesting to note that when s is the extreme hierarchy:

$$s = \left((n-1)^2,\ (n-2)^2 + (n-1),\ \ldots\ ,\ (n-k)^2 + (k-1)(n-k+1)\ \ldots \right)$$

and s' is the reverse extreme hierarchy, namely

$$s' = \left(0^2 + (n-1)\cdot 1,\ 1^2 + (n-2)\cdot 2,\ \ldots\ ,\ (k-1)^2 + (n-k)\cdot k,\ \ldots \right)$$

then Σd^2 takes on its maximum value which is in fact

$$(\Sigma d^2)_{max} = \frac{n(n^2-1)(n-2)^2}{3}$$

so that

$$\theta_1 = 1-2\cdot \frac{\Sigma d^2}{(\Sigma d^2)_{max}} .$$

We complete the definition of θ as follows. In the event that one or both of s, s' should be the equality, the expression (3.2) for θ is undefined. In order to preserve the symmetry of the distribution of values of θ, we define θ to be zero in such a case.

In order now to illustrate the computation of θ, let us consider a society of five individuals for whom are obtained the three following score-structures, say with respect to three different characteristics: $s = (16, 13, 10, 7, 4)$, $s' = (15, 14, 10, 6, 5)$ and $s'' = (9, 9, 9, 9, 14)$. For s and s' we have $\Sigma d^2 = 4$, $\Sigma s^2_k = 590$ and $\Sigma {s'}_k^2 = 582$.

Then

$$\theta(s,s') = \frac{590 + 582 - 1000 - 4}{\sqrt{(2\cdot 590 - 1000)(2\cdot 582 - 1000)}} = \frac{168}{\sqrt{180\cdot 164}} = 0.9891.$$

On the other hand, for s and s" we have $\Sigma d^2 = 170$, $\Sigma s_k^{\,2} = 590$ and $\Sigma s''^{\,2}_k = 520$. Hence, in this case,

$$\theta(s,s'') = \frac{590 + 520 - 1000 - 170}{\sqrt{(2\cdot 590 - 1000)(2\cdot 520 - 1000)}} = \frac{-60}{\sqrt{180\cdot 40}} = -0.3545$$

These results seem intuitively to be quite satisfactory.

The measures θ and h are of course related, although the formula is not simple. From (2.7) we have, after a little manipulation,

$$\frac{n(n^2-1)(n-2)^2 h}{6} = 2\Sigma s^2_k - \frac{n^3(n-1)^2}{2}$$

so that, writing a similar formula for h' in terms of $\Sigma s'^{\,2}_k$, we obtain after substitution into (3.3):

$$(3.4) \qquad \theta = \frac{1}{\sqrt{hh'}}\left(\frac{h + h'}{2} - \frac{6\Sigma d^2_k}{n(n^2-1)(n-2)^2}\right),$$

in the event that $hh' \neq 0$.

4. Statistical Considerations. In order to evaluate the significance of changes in the score structure of a group, or the significance of a difference in the score structures of two comparable groups, we need certain statistical tables. For instance, when n = 3, we consider Tables 2 and 3. In these, the probabilities for h are computed, on the assumption that the rankings are random phenomena.

Table 2

PROBABILITY DISTRIBUTION OF h FOR n = 3

Score structure	Number of matrices with a similar score structure	h	Pr(h)	Pr(h ≥ listed value)
432	6	1	0.75	0.75
333	2	0	0.25	1

TABLE 3

PROBABILITY TABLE FOR $h_2 - h_1$, $n = 3$

Values of $h_2 - h_1$	No. of pairs of matrices giving listed value of $h_2 - h_1$	$Pr(h_2 - h_1)$	$Pr(h_2 - h_1 \geq$ listed value)
-1	12	$\frac{3}{16}$	1
0	40	$\frac{10}{16}$	$\frac{13}{16}$
+1	12	$\frac{3}{16}$	$\frac{3}{16}$

Suppose, for example, that as a result of training, h for a group of 3 men rises from 0 to +1. The probability that this is the result of pure chance rankings is $\frac{3}{16}$. Suppose next the method of training applied to five such groups results in the same change. The probability that this happens by pure chance is $\left(\frac{3}{16}\right)^5 = 0.00023$ so that we may conclude that the method of training is effective in bringing about this change if we set the level of significance at 0.001, for example.

For $n = 3$, we may also construct Table 4 for testing in a similar fashion the significance of observed values of θ.

TABLE 4

PROBABILITY DISTRIBUTION OF θ FOR $n = 3$

Possible values of θ	No. of pairs of matrices giving listed value of θ	$Pr(\theta)$	$Pr\left(\theta \geq \text{listed value}\right)$
-1	6	0.09375	1.0000
$-\frac{1}{2}$	12	0.18750	0.90625
0	28	0.43750	0.71875
$\frac{1}{2}$	12	0.18750	0.28125
1	6	0.09375	0.09375

For $n = 4$, we have Tables 5 and 6.

TABLE 5
PROBABILITY DISTRIBUTION OF h FOR n = 4

Score structure	Number of matrices with a similar score structure	h	Number of score structures with given h	Pr(h)	Pr(h ≥ listed value)
9753	24	1	24	0.0185	0.0185
9744 9663 8853	24 24 24	9/10	72	0.0556	0.0741
8844	24	8/10	24	0.0185	0.0926
9654 8763	96 96	7/10	192	0.1482	0.2407
9555 7773	24 24	6/10	48	0.0370	0.2778
8754	144	5/10	144	0.1111	0.3889
8664	96	4/10	96	0.0741	0.4630
8655 7764	144 144	3/10	288	0.2222	0.6852
7755	120	2/10	120	0.0926	0.7779
7665	264	1/10	264	0.2037	0.9815
6666	24	0	24	0.0185	1.0000
	1296		1296	1.0000	

TABLE 6

PROBABILITY TABLE FOR $h_2 - h_1$, $n = 4$

(For negative values of $h_2 - h_1$, the probabilities may be found by symmetry considerations.)

$h_2 - h_1$	$Pr(h_2 - h_1)$	$Pr(h_2 - h_1 \geq$ value listed)
1	0.00034	0.00034
$\frac{9}{10}$	0.0048	0.0051
$\frac{8}{10}$	0.0134	0.0185
$\frac{7}{10}$	0.0158	0.0343
$\frac{6}{10}$	0.0463	0.0806
$\frac{5}{10}$	0.0336	0.1142
$\frac{4}{10}$	0.0686	0.1823
$\frac{3}{10}$	0.0556	0.2383
$\frac{2}{10}$	0.1070	0.3453
$\frac{1}{10}$	0.0823	0.4276
0	0.1447	0.5724

These last two tables are introduced primarily to give the reader a little more feeling for the behavior of h and of differences in h's. A similar probability table for the values of θ when $n = 4$ would not take too long to construct.

Since experimental n's are apt to be a good deal larger than 4, the application of statistical significance theory to h and θ must await the determination of suitable approximations to their distributions. Our θ seems closely related to Kendall's "coefficient of concordance" W, so that one might reasonably expect adaptations of his methods to yield useful results here.

Discussion by

Lee J. Cronbach

Sociometric methods have been given relatively little study as a formal problem in psychometrics, although a few mathematical treatments of the problem are appearing. Since it appeared probable that a fresh mind, acquainted with matrix algebra, could suggest new analyses of sociometric data, we asked Dr. Hohn to study reports of sociometric studies and to explore whatever leads suggested themselves.

His paper gives a detailed analysis of a particular approach which he calls the hierarchy index. In this comment, I desire to relate his development to comparable procedures used in test psychometrics, and to indicate some possible interpretations.

The first point to be noted is that h, the hierarchy index, is a ratio of variance to maximum variance (V/V max). Such a ratio was once proposed by Ferguson (3) as an index of homogeneity among test items, and is linearly related to the somewhat more familiar index C/C max (C being total interitem covariance) proposed by Loevinger (6). In most studies, these are not superior to coefficient alpha $\frac{n}{(n-1)}\frac{C}{V}$, which is a general form of the Kuder-Richardson coefficient. Alpha is an excellent measure of internal consistency(2).

The special formula for h is appropriate to sociometric data where each person ranks every other. Unlike the item-person matrix of test research, this matrix is square and has fixed row means. Moreover, the diagonal entry is ordinarily missing. This means that rows cannot be perfectly correlated. These properties are considered in Dr. Hohn's development. No study has been made of the degree to which, for computational purposes, h is superior or inferior to α or Kendall's W.

The suggestion that sociometric matrices be evaluated in terms of hierarchy is of general usefulness. While the h formula no longer applies, the same general technique may be used for matrices where the person reports (say) his three highest and three lowest choices. This type of index has several related interpretations.

(1) h is a measure of hierarchy. As h becomes closer to 1, the choice relations among a group approach a status system where each person prefers persons of high status and tends not to prefer persons below him. Groups which are divided into cliques will have a lower degree of hierarchy than groups which have a pyramidal system, but will not necessarily be more hierarchical than the group which has random distribution of choices. It may be important to study the conditions under which hierarchy develops, and the differences in performance of groups of different hierarchy.

(2) h is a measure of the extent to which persons constitute a scale in the quality being measured (in the Guttman sense). Just as Guttman can examine whether items can be arrayed in a continuum which is perceived similarly by all persons, so Hohn's index examines whether persons form such a unidimensional continuum. It is of interest to note that some hierarchies which satisfy Guttman's requirements for a scale are not extreme hierarchies in Hohn's definition.

(3) h is a measure of reliability. For the rectangular matrix where raters need not be ratees, Horst has shown that α provides a measure of reliability or agreement of raters (5). α estimates the correlation expected between sets of scores

obtained from two samples of raters. h performs the same function for the square matrix of sociometric ranks. This appears to meet Pepinsky's demand for a measure of reliability of sociometrics, in the sense of consistency over judges (7). No splitting of the group into chance halves is required.

(4) h is a measure of communality of thinking among judges. If a group has a large h, the raters agree on their criteria and frames of reference. A low value of h indicates diversity in members' perceptions. Thus Gage and Exline (using the split-half technique) find greater agreement on ratings of others' productivity than on rating of the same persons' leisure time attractiveness (4). If a group has low internal consistency in ratings of "degree to which each person contributes to the aims of the group", this would suggest the presence of conflicting frames of reference and we might predict that such a group would be inefficient. Roby has done preliminary research of this kind. Studies of change in h over time might reflect development of a common reference frame, especially if h were determined separately for such dimensions as liking and contribution to the task.

We should note that, like ϕ, h depends on the size of the group. It will tend to be larger in a larger group, other things being equal. Therefore h must be interpreted with the size of the group in mind, or some transformation will be required such as the "phi bar" index (2) derived from ϕ.

A variant of conventional internal consistency analysis also may be profitably applied to sociometric data. The common item-test correlation has its analog in the correlation of any row with the marginal row, i.e., with the score structure. This is a measure of the extent to which the individual shares a frame of reference with other raters. One can similarly correlate the row with a row of criterion scores. Anderhalter, Wilkins, and Rigby (1) apply some of these approached to Marine Officer Candidates, and show some evidence that the person who agrees with the marginal rating or with officers' ratings of the candidates, himself tends to receive a high rating. Homogeneity of these groups as judged by the mean row-marginal correlation increased with time.

Hohn's θ is not novel mathematically, being a direct application of product-moment technique. It does, however, draw attention to a possibly fruitful type of analysis which seems not to have been made except in studies of stability of sociometric scores over time. Consider its possible application to a bomber crew, where each man has a designated station. Then a crew where the navigator is rated high, and the flight engineer average, is in some respects different from a crew where the reverse is true. Applying θ to the score structures of many crews would yield a correlation matrix which could perhaps be separated into several types of structure. It is reasonable to suppose that these structures might be significant either as reflections of values within the crew, or as communication networks which influence effectiveness.

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