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ON THE USE OF SINGULAR YIELD CONDITIONS AND ASSOCIATED FLOW RULES

by

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On the use of singular yield conditions and associated flow rules[1]

By William Prager[2]

Abstract. It is well known that the use of Tresca's yield condition frequently leads to a simpler system of equations for the stresses in a plastic solid than the use of the yield condition of Mises. In most cases where Tresca's yield condition has been used for this reason, the flow rule associated with the Mises condition has been retained, however. Following Koiter* (1), it is shown that further simplification results from the use of the flow rule associated with the Tresca condition. The reason for this is discussed in connection with two examples concerning the finite enlargement of a circular hole in an infinite sheet of perfectly plastic or work-hardening material. The second example is probably the first non-trivial case in which a problem of finite plastic deformation of a work-hardening material has been treated in closed form by the use of incremental stress-strain relations.

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INTRODUCTION

The mechanical behavior of a perfectly plastic solid is defined by the yield condition and the flow rule. The yield condition specifies the states of stress under which plastic flow will occur. For each of these states of stress the flow rule specifies the components of the plastic strain rate to within an arbitrary common factor. This positive factor is required by the assumption that the solid is inviscid: the state of stress necessary to cause a certain type of plastic flow does not depend on the speed of deformation. In the following, a set of strain rate components that are defined only to within a common positive factor will be said to specify a "flow mechanism".

For an isotropic perfectly plastic solid, the orientation of the principal axes of stress does not enter into the yield condition. This condition therefore assumes the form

$$f(\sigma_1,\sigma_2,\sigma_3) = 0, \qquad [1]$$

where the function f must be symmetric in the principal stresses σ_1, σ_2, σ_3. If the solid retains its isotropy during plastic flow, the principal axes of the strain rate must coincide with those of the stress. The flow rule therefore reduces to a relation between the principal stresses σ_1, σ_2, σ_3 and the principal strain rates e_1, e_2, e_3.

It is convenient to represent the yield condition [1] geometrically by considering σ_1, σ_2, and σ_3 as the rectangular coordinates of a point on the "yield surface". The flow rule may then be taken to specify a direction at each point of the yield surface, the three direction cosines being proportional to e_1, e_2, e_3.

Mises (2) proposed the flow rule that associates with each point of the yield surface the direction of the exterior normal of this surface at this point:

$$e_1 : e_2 : e_3 = \frac{\partial f}{\partial \sigma_1} : \frac{\partial f}{\partial \sigma_2} : \frac{\partial f}{\partial \sigma_3} . \qquad [2]$$

In Eq. [2] it has been assumed that the sign of the yield function f has been so chosen that the exterior normal of the yield surface indicates the direction of increasing values of f.

The idea of using the yield function f as the "plastic potential" proved useful for the formulation of flow rules for crystals and other anisotropic solids. In recent years, the concept of the plastic potential has gained added importance because the theory of limit analysis is based on this relation between the yield condition and the flow rule (see, for instance, (3), (4)).

The flow rule [2] presupposes that there is a uniquely determined exterior normal at each point of the yield surface. Not all yield conditions used in the mathematical theory of plasticity satisfy this regularity requirement. For example, the yield surface corresponding to Tresca's condition of constant maximum shearing stress is a regular hexagonal prism. At a point on an edge of this prism, the exterior normal is not defined, and there arises the question how the flow rule [2] should be modified at such a singular point.

Consider first a point P on the edge formed by two adjacent flat or curved faces of a yield surface. Each of these faces has a unique exterior normal at P, and each of these normals may be considered as representing a flow mechanism that is possible under the state of stress represented by P. It is then natural to assume that other possible flow mechanisms can be obtained by combining these two fundamental mechanisms in a linear fashion and with positive coefficients.

In the case of Tresca's yield condition, for instance, the faces of the prismatic yield surface lie in the planes

$$\sigma_1 - \sigma_2 = \pm \sigma, \quad \sigma_1 - \sigma_3 = \pm \sigma, \quad \sigma_2 - \sigma_3 = \pm \sigma, \qquad [3]$$

where σ is the yield stress in simple tension. For points on one of the adjacent faces $\sigma_1 - \sigma_2 = \sigma$ and $\sigma_1 - \sigma_3 = \sigma$, the flow rule [2] furnishes

$$e_1:e_2:e_3 = 1:-1:0 \qquad [4]$$

and

$$e_1:e_2:e_3 = 1:0:-1, \qquad [5]$$

respectively. The first flow mechanism represents pure shear in the σ_1, σ_2 plane; the second mechanism, pure shear in the σ_1, σ_3 plane. The point representing the state of simple tension $\sigma_1 = \sigma$ lies on the common edge of these two faces. According to the flow rule proposed above, the flow mechanisms possible under this state of simple tension are characterized by

$$e_1:e_2:e_3 = 1:-\lambda:-(1-\lambda), \qquad [6]$$

where $0 \leq \lambda \leq 1$. The three terms on the right-hand side of Eq. [6] are obtained by multiplying the corresponding terms of Eqs.

[4] and [5] by λ and $1 - \lambda$, respectively, and adding.

While the type of singular point just discussed is the only one occurring in connection with Tresca's yield condition, other types, such as a vertex of a polyhedron or a cone, can be treated in a similar manner.

At first glance it might seem that this modification of the flow rule [2] at singular points of the yield surface would greatly complicate the mathematical treatment of problems of plastic flow. As was pointed out by Koiter (1), however, the contrary is true: a considerable simplification results from the joint use of Tresca's yield condition and the associated flow rule. The reason for this will be explained in connection with the following examples which will also demonstrate that the simplification is by no means restricted to the case of small plastic deformations considered by Koiter.

TRESCA'S YIELD CONDITION IN PROBLEMS OF PLANE STRESS

In problems of plane stress one of the principal stresses, say σ_3, vanishes, and the yield condition can be represented by a curve or polygon in the σ_1, σ_2 plane. If Tresca's yield condition is used, the yield polygon is formed by the lines with the equations

$$\sigma_1 - \sigma_2 = \pm \sigma, \quad \sigma_1 = \pm \sigma, \quad \sigma_2 = \pm \sigma \qquad [7]$$

obtained from [3] by setting $\sigma_3 = 0$. Figure 1 shows this hexagon which is an oblique section of the afore-mentioned hexagonal prism. The axis of this prism passes through the origin O and forms equal angles with the positive axes of σ_1, σ_2, and σ_3

(the latter being normal to the plane of Fig. 1).

Consider a generic point P on one side of the yield hexagon, for instance on AB. The exterior normal of the three dimensional yield surface at P is not contained in the plane of the figure. The projection of this normal on the plane of the figure is normal to AB, however. Thus, the ratio $e_1:e_2$ for the state of stress represented by P may be obtained as the ratio between the direction cosines of the normal to the yield hexagon at P. Once the ratio $e_1:e_2$ is known, the condition of incompressibility

$$e_1 + e_2 + e_3 = 0 \qquad [8]$$

yields the ratios $e_1:e_2:e_3$.

All states of stress occurring in the following examples will turn out to be represented by points on the side AB of the yield hexagon. With $0 \leq \lambda \leq 1$, the modified flow rule then furnishes the following information.

(a) State of stress represented by point A:

$$\sigma_1 = -\sigma, \quad \sigma_2 = \sigma_3 = 0 \quad \text{and} \quad e_1:e_2:e_3 = -1:\lambda:-(1-\lambda). \qquad [9]$$

(b) State of stress represented by interior point of segment AB:

$$\sigma_1 - \sigma_2 = -\sigma, \quad \sigma_3 = 0 \quad \text{and} \quad e_1:e_2:e_3 = -1:1:0. \qquad [10]$$

(c) State of stress represented by point B:

$$\sigma_2 = \sigma, \quad \sigma_1 = \sigma_3 = 0 \quad \text{and} \quad e_1:e_2:e_3 = -\lambda:1:-(1-\lambda). \qquad [11]$$

If the yield stress σ remains constant during plastic flow, the solid is called perfectly plastic; if the yield stress increases during plastic flow, the solid is called work-hardening. For the most general type of work-hardening solid the form of the yield condition could also change during plastic flow, for instance the hexagon of Fig. 1 could gradually change into an ellipse. This would correspond to a gradual transition from the yield condition of Tresca to that of Mises. This case will not be considered in this paper, however. It will be assumed that during plastic flow the yield hexagon remains centered at the origin and merely increases in size. In the following the yield stress of the virgin material will be denoted by σ_0 and it will be assumed that the rate of hardening $\dot{\sigma}$ is proportional to the rate D at which mechanical energy is dissipated during plastic flow:

$$\dot{\sigma} = \alpha D , \qquad [12]$$

where α is a constant.

For the three cases considered above $D = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3$ has the following values:

$$\text{(a)} \quad D = - \sigma e_1, \qquad [13]$$

$$\text{(b)} \quad D = (\sigma_1 - \sigma_2) e_1 = - \sigma e_1, \qquad [14]$$

$$\text{(c)} \quad D = \sigma e_2. \qquad [15]$$

FIRST EXAMPLE

[illegible]he first example consider the finite enlargement of a circular hole in a sheet of a perfectly plastic material. This problem has been treated by Taylor (5) who used Tresca's yield condition but the flow rule associated with Mises' yield condition. The comparison of Taylor's work and the following analysis reveals the great simplification achieved by the use of the appropriate flow rule.

A circular hole of the radius a_o in an infinite plate of the uniform thickness h_o is to be enlarged to the radius $a = 1.4\ a_o$ application of a gradually increasing uniform pressure to the edge of the hole. Since finite plastic deformations will be considered, elastic deformations will be neglected. The material at a sufficiently large distance from the hole must then be treated as rigid because it will not reach the yield limit. Under these circumstances, radial displacement of the material near the hole is made possible only by a thickening of the sheet.

In the elastic part of the sheet the radial stress σ_r and the hoop stress σ_Θ are inversely proportional to the square of the radius r, and $\sigma_\Theta = - \sigma_r > 0$. Thus, the state of stress at the elastic-plastic interface is represented by the center of the segment AB in Fig. 1. Just inside this interface the yield condition and flow rule are therefore given by Eqs. [10] with the subscripts 1, 2, 3 standing for r, Θ, z, respectively. Since $e_z = 0$, there is no thickening of the sheet and radial displacement is therefore impossible. The material just inside the elastic-plastic interface accordingly remains rigid even though the stresses σ_r and σ_Θ satisfy the yield condition

$$\sigma_r - \sigma_\Theta = - \sigma_o. \qquad [16]$$

The condition of radial equilibrium in this rigid plastic zone is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\Theta}{r} = 0. \qquad [17]$$

From [16], [17] and the boundary condition $\sigma_r = -\sigma_\theta = -\sigma_0/2$ at the elastic plastic interface ($r = \rho_0$) it follows that

$$\sigma_r = -\sigma_0(\tfrac{1}{2} + \log \frac{\rho_0}{r}), \qquad [18]$$
$$\sigma_\theta = \sigma_0(\tfrac{1}{2} - \log \frac{\rho_0}{r})$$

in the rigid plastic zone. The hoop stress is therefore decreasing as one progresses from the elastic-plastic interface into the rigid plastic zone. It follows from the second Eq. [18] that the hoop stress vanishes for

$$r = \sigma_0 / \sqrt{e} = 0.606\rho_0. \qquad [19]$$

This radius which will occur frequently in the following work will be denoted by ρ. The state of stress at $r = \rho$ is represented by the point A in Fig. 1.

Just inside the circle $r = \rho$, the circumferential strain rate e_θ must vanish on account of the rigidity of the surrounding material. The radial strain rate e_r need not vanish, however. The flow mechanism is therefore represented by the normal to the side AF of the yield hexagon.

There are now two possibilities regarding the variation of stress and flow mechanism inside the circle $r = \rho$. If the point representing the state of stress moves from A towards F, the flow rule requires that $e_\theta = 0$. This means that the radial velocity v must remain zero inside the circle $r = \rho$, because $e_\theta = v/r$. The sheet would therefore remain rigid even inside the circle $r = \rho$. If, on the other hand, the point representing the state of stress remains at A, the greater degree of freedom in the choice of the flow mechanism at this singular point allows a plastic deformation of the sheet inside the circle $r = \rho$.

Since the sheet thickness h will not remain constant inside the circle $r = \rho$, the equation of radial equilibrium has the form

$$\frac{\partial(h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\theta)}{r} = 0. \qquad [20]$$

Since $\sigma_2 = -\sigma_0$, $\sigma_\theta = 0$ when the state of stress is represented by the point A, this equilibrium condition may be written as follows:

$$\frac{\partial}{\partial r}(rh) = 0. \qquad 21$$

While it is convenient to use the terms "velocity", "strain rate", and "rate of dissipation of mechanical energy", it must be remembered that the plastic solids considered here are inviscid. Consequently, the flow process is independent of the time scale, and any variable that increases monotonically during the flow process may be used as a measure of "time". In the following, the radius ρ will be used in this manner.

If the radial velocity is denoted by v, the radial and circumferential strain rates are $e_r = \partial v/\partial r$, and $e_\theta = v/r$. The strain rate in the direction normal to the sheet is $(1/h)Dh/d\rho$, where $Dh/D\rho = \partial h/\partial\rho + v\partial h/\partial r$ denotes the "material" derivative of h. The condition of incompressibility requires that the sum of the three strain rates vanish:

$$\frac{1}{h}\frac{\partial h}{\partial \rho} + \frac{v}{h}\frac{\partial h}{\partial r} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0. \qquad [22]$$

Equa Equations [21], [22], and the boundary conditions

$$h = h_0, \qquad v = 0 \qquad \text{at } r = \rho \qquad [23]$$

define the functions $h = h(r,\rho)$, $v = v(r,\rho)$. From Eq. [21] and the first boundary condition [23] it follows that

$$h = h_o \frac{\rho}{r} . \qquad [24]$$

With this expression for h, Eq. [22] reduces to a differential equation for v. The solution satisfying the second boundary condition [23] is

$$v = 1 - \frac{r}{\rho} . \qquad [25]$$

(Since the radius ρ is used as a measure of "time", the "velocity" must be dimensionless.) From Eq. [25] the strain rates are obtained as follows

$$e_r = \frac{\partial v}{\partial r} = -\frac{1}{\rho}, \quad e_\theta = \frac{v}{r} = \frac{1}{r} - \frac{1}{\rho} . \qquad [26]$$

If the material that at the "time" ρ is at the radius r was initially at the radius r_o, conservation of mass requires that

$$2\pi \int_r^\rho hr\, dr = \pi h_o(\rho^2 - r_o^2). \qquad [27]$$

Substitution of [24] into [27] yields

$$r = \frac{\rho^2 + r_o^2}{2\rho} . \qquad [28]$$

Equation [24] can therefore be written in the form

$$h = 2h_o \frac{\rho^2}{\rho^2 + r_o^2} . \qquad [29]$$

According to [9], the solution just obtained will be valid only if $e_1 : e_2$ lies between -1:0 and -1:1. The first of these bounds is attained at $r = \rho$ where $e_\theta = 0$ by the second Eq. [26]. The other bound is attained at $r = \rho/2$ where Eqs. [26]

furnish $e_r = 1/\rho$, $e_\theta = 1/\rho$. Equation [28] shows that $r = \rho/2$ corresponds to $r_o = 0$. Thus, the solution remains valid even in the case where a pin hole is enlarged to some finite radius.

Formally the results obtained above agree with those furnished by the "simplified analysis" which Taylor (5) attributes to Bethe. Taylor rejects this analysis because the ratios of the principal strain rates vary while the ratios of the principal stresses are constant. As has been shown above, Bethe's results can be obtained from a consistent theory, and only experiment can decide whether the behavior of a given material agrees better with the formulas of Taylor or Bethe.

SECOND EXAMPLE

The second example differs from the first one only in so far as the sheet material is now supposed to be work-hardening in accordance with Eq. [12]. Since no hardening takes place where the material remains rigid, the stresses outside the ci e $r = \rho$ are the same as in the previous example. To keep track of the progressive hardening of the material inside the circle $r = \rho$, it will be necessary to use the Lagrangian coordinate r_o as one of the independent variables, the other independent variable being ρ as before.

Again $\sigma_\theta = 0$ inside the circle $r = \rho$. Since $\sigma_r = -\sigma$ is no longer constant, however, the condition of equilibrium [21] must now be replaced by

$$\frac{\partial}{\partial r}(rh\sigma) = \frac{1}{(\partial r/\partial r_o)} \frac{\partial}{\partial r_o}(rh\sigma) = 0. \qquad [30]$$

The solution of this equation that satisfies the boundary conditions $h = h_o$, $\sigma = \sigma_o$ for $r = \rho$ is

$$rh\sigma = \rho h_o \sigma_o . \qquad [31]$$

In accordance with Eqs. [12] and [13], the rate of strain hardening is

$$\frac{\partial \sigma}{\partial \rho} = \alpha D = - \alpha \sigma e_r, \qquad [32]$$

where

$$e_r = \frac{\partial v}{\partial r} = \frac{1}{(\partial r/\partial r_o)} \frac{\partial v}{\partial r_o}$$

$$= \frac{1}{(\partial r/\partial r_o)} \frac{\partial^2 r}{\partial r_o \partial \rho} = \frac{\partial}{\partial \rho} \log \left| \frac{\partial r}{\partial r_o} \right| . \qquad [33]$$

With the use of this expression for e_1, Eq. [32] can be written as follows:

$$\frac{\partial}{\partial \rho}\left(\sigma^{1/\alpha} \left| \frac{\partial r}{\partial r_o} \right| \right) = 0. \qquad [34]$$

The circumferential strain rate is

$$e_\theta = \frac{v}{r} = \frac{1}{r} \frac{\partial r}{\partial \rho} = \frac{\partial}{\partial \rho} \log r, \qquad [35]$$

and the strain rate in the direction normal to the plane of the sheet is

$$e_z = \frac{1}{h} \frac{\partial h}{\partial \rho} = \frac{\partial}{\partial \rho} \log h. \qquad [36]$$

The condition of incompressibility can therefore be written in the form

$$\frac{\partial}{\partial \rho}\left(hr \left| \frac{\partial r}{\partial r_o} \right| \right) = 0. \qquad [37]$$

Those solutions of the differential equations [34] and [37] that are compatible with Eq. [31] and satisfy the boundary condition $\sigma = \sigma_o$ for $r = \rho$ are

$$\sigma = \sigma_o(\frac{\rho}{r_o})^{\alpha/(1+\alpha)} \qquad [38]$$

and

$$r = \frac{\rho}{2 + \alpha} [1 + (1 + \alpha)(\frac{r_o}{\rho})^{(2+\alpha)/(1+\alpha)}] \, . \qquad [39]$$

As before, the solution represented by Eqs. [31], [38], and [39], is valid only as long as the ratio $e_r:e_\theta$ remains below -1. From Eqs. [33] and [35] it is found that this is the case as long as $r > \rho/2$. According to Eq. [39] this means that the following inequalities must be fulfilled if this solution is to be valid:

$$\frac{a_o}{\rho} > (\frac{\alpha}{2})^{\frac{1+\alpha}{2+\alpha}} \, , \qquad [40]$$

$$\frac{a}{a_o} < \frac{1}{2 + \alpha} [1 + (1 + \alpha) \frac{\alpha}{2}](\frac{\alpha}{2})^{- \frac{1+\alpha}{2+\alpha}} \, . \qquad [41]$$

To indicate the manner in which Bethe's solution for a perfectly plastic sheet is modified by work-hardening, assume that $\alpha = 0.5$. For simple tension, this corresponds to one half of a percent increase in yield stress for a strain of one per cent. Figure 2 shows the thickness distribution plotted versus r/ρ and r/r_o. The solid curve corresponds to $\alpha = 0.5$ and the dotted curve to $\alpha = 0$. It is seen that even such a small amount of strain hardening has a significant influence on the thickness distribution. According to Eq. [41] the solution represented by the solid curve in Fig. 2 is valid for $a/a_o < 1.48$.

CONCLUSION

It has been shown that the joint use of Tresca's yield condition and the associated flow rule can lead to a considerable simplification of the mathematical work in problems of plane plastic stress. The reason for this is the following:

corresponding to the sides and the vertices of the yield polygon, one has either a unique flow mechanism and a one parameter family of principal stress values, or unique values of the principal stresses and a one parameter family of flow mechanisms. In either case considerable simplifications result, and the complete solution is built up of zones in which one or the other type of simplification pertains.

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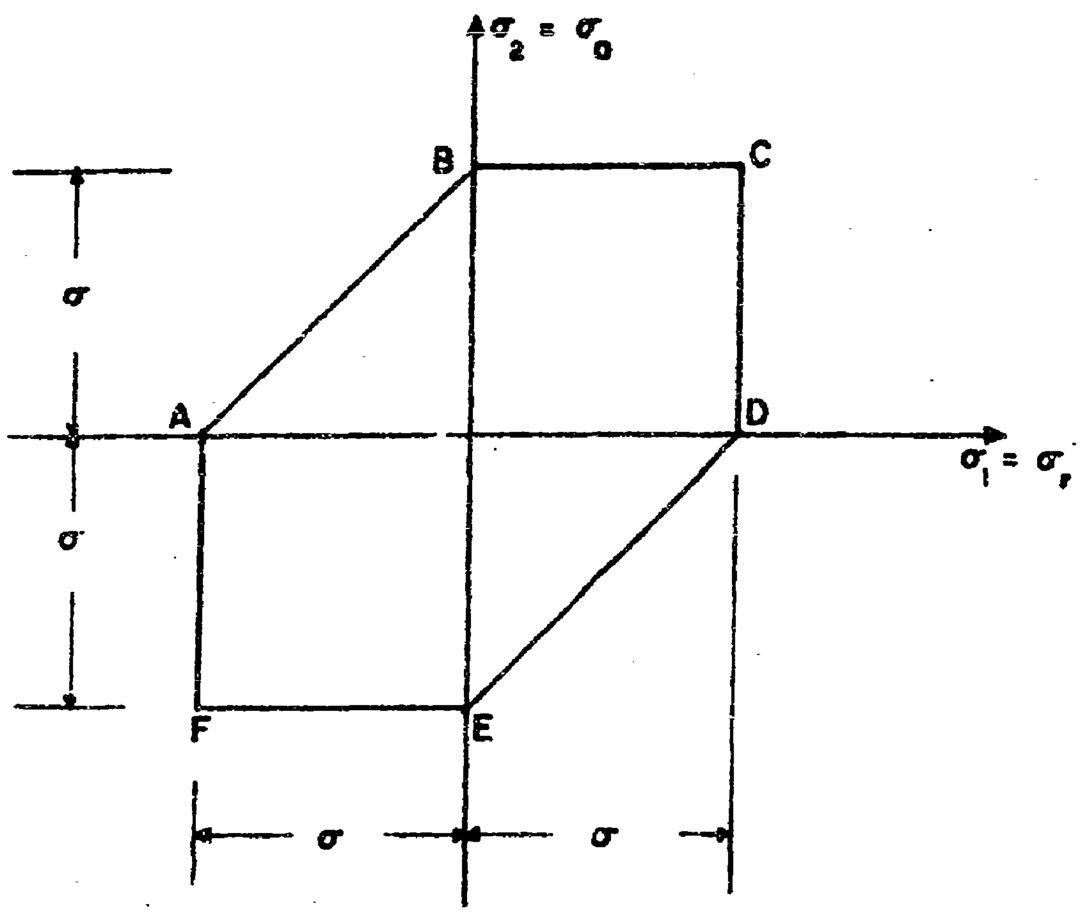

$\sigma_2 = \sigma_\theta$
B
C
A
D
$\sigma_1 = \sigma_r$
F
E
σ
σ
σ
σ

Fig. 1.

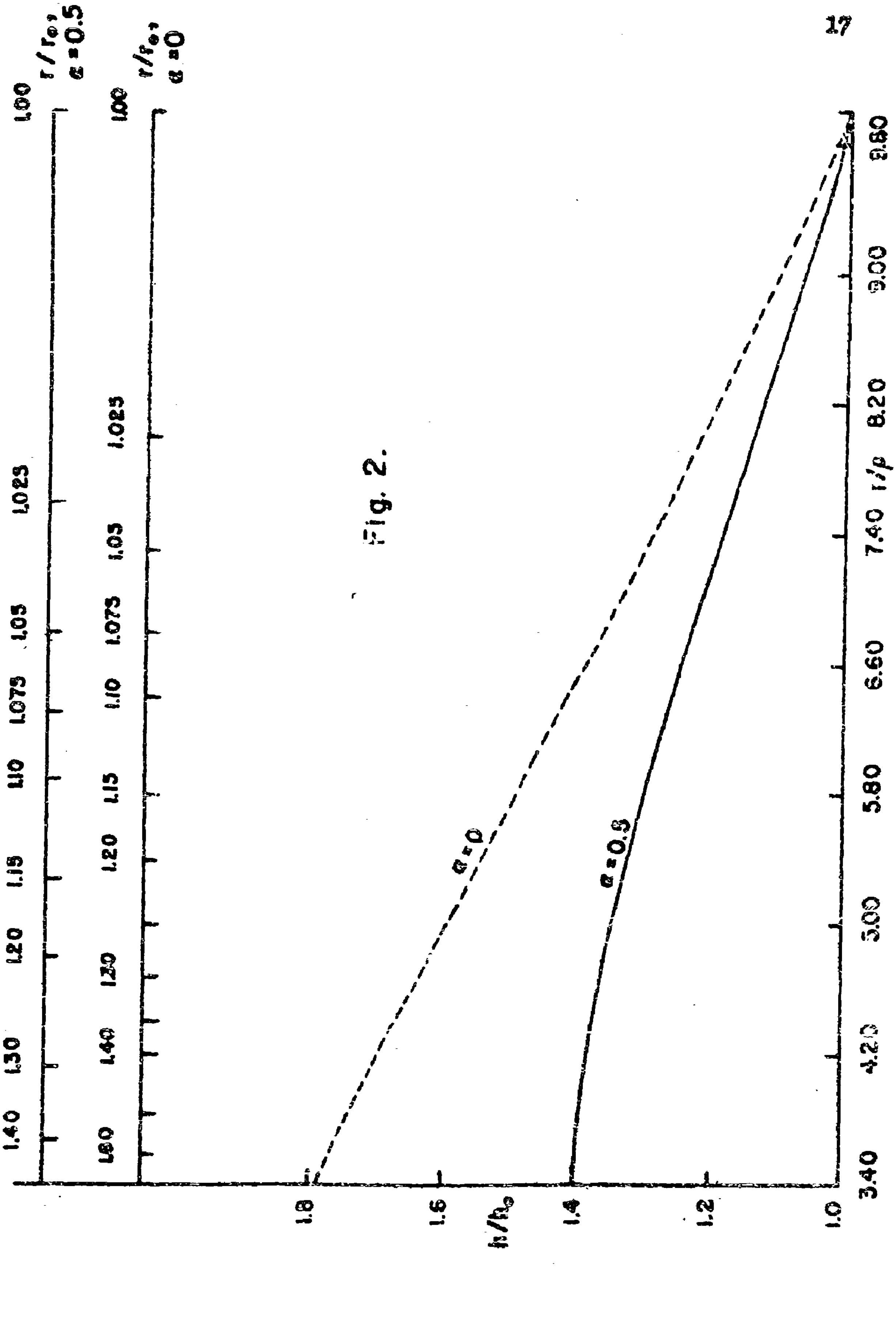
r/r₀, α = 0.5
r/r₀, α = 0
1.00
1.025
1.05
1.075
1.10
1.15
1.20
1.30
1.40
1.60
α = 0
α = 0.5
h/h₀
1.8
1.6
1.4
1.2
1.0
r/ρ
3.40
4.20
5.00
5.80
6.60
7.40
8.20
9.00
9.80

Fig. 2.

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