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DEPARTMENT OF CIVIL ENGINEERING

FORCED MOTIONS OF ELASTIC RODS

by

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FORCED MOTIONS OF ELASTIC RODS

Statement of Problem

An approximate, one-dimensional, theory of vibrations of rods (1)[1], in which the effect of radial inertia and radial shear deformation are taken into account, is contained in the equations of motion

$$
\begin{aligned}
a^2 x_1^2 \mu u'' - 8 x_1^2(\lambda+\mu)u - 4a x_1^2 \lambda w' + 4aR &= \rho a^2 \ddot{u} \\
2a\lambda u' + a^2(\lambda+2\mu)w'' + 2aZ &= \rho a^2 \ddot{w}
\end{aligned}
\qquad (1)
$$

the stress-displacement relations

$$
\begin{aligned}
2P_r = 2P_\theta &= x_1^2\left[2a(\lambda+\mu)u + a^2\lambda w'\right] \\
2P_z &= 2a\lambda u + a^2(\lambda+2\mu)w' \\
4Q &= x^2 a^2 \mu u'
\end{aligned}
\qquad (2)
$$

the boundary conditions

1. At each point along the length of the bar, one member of each of the products $\dot{u}R$ and $\dot{w}Z$ must be specified.
2. At each end of the bar one member of each of the products $\dot{u}Q$ and $\dot{w}P_z$ must be specified. (3)

and the initial conditions

The initial displacements and velocities must be specified. (4)

---

1. Numbers in parentheses refer to the Bibliography at the end of the paper.

In [1] to [4] the symbols have the following meanings:

λ, μ	Lamé's constants of elasticity
ρ	density
u, w	radial and axial displacements, respectively
a	radius of the rod
R, Z	radial and tangential components of traction, respectively, on the cylindrical surface of the rod
χ, χ_1	correction factors
P_r, P_θ, P_z, Q	rod-stress components, defined in (1)

Primes indicate differentiation with respect to z, the coordinate along the axis of the rod, and dots indicate differentiation with respect to time.

The object of this paper is to describe the development of a formal solution of the above equations for a rod of finite length, ℓ, and any admissible combination of time-dependent boundary conditions and arbitrary initial conditions.

The solution will be developed by making use of Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} = Q_n \qquad [5]$$

where T is the kinetic energy of the mechanical system considered, V is the potential energy, q_n is the n-th generalized coordinate, $\dot{q}_n$ the n-th generalized velocity, and Q_n the n-th generalized force. As a convenience in specifying the boundary conditions, the following notation will be employed:

$$B_1 = u \quad \text{or} \quad Q \quad \text{at} \quad z = 0$$

$$B_2 = w \quad \text{or} \quad P_z \quad \text{at} \quad z = 0$$

$$B_3 = u \quad \text{or} \quad Q \quad \text{at} \quad z = \ell$$

$$B_4 = w \quad \text{or} \quad P_z \quad \text{at} \quad z = \ell$$

Accordingly, the boundary conditions [3] become

$$B_i = f_i(t) \qquad i = 1, 2, 3, 4 \qquad [3']$$

in which the four functions $f_i(t)$ are prescribed.

The initial conditions [4] are specified by the four arbitrary functions

$$\begin{aligned} u(z, 0) &= u_0(z) \\ w(z, 0) &= w_0(z) \\ \dot{u}(z, 0) &= \dot{u}_0(z) \\ \dot{w}(z, 0) &= \dot{w}_0(z) \end{aligned} \qquad [4']$$

Kinetic Energy

The form of the kinetic energy of the rod may be obtained by performing the appropriate integrations in the expression of the kinetic energy in terms of the general three-dimensional displacements. Thus, in general, the kinetic energy is

$$T = \frac{\rho}{2} \int_V (\dot{u}_r^2 + \dot{u}_\theta^2 + \dot{u}_z^2) \, d\tau \qquad [6]$$

where u_r, u_θ, u_z are the components of displacement in cylindrical coordinates and the integration is throughout the volume of the rod. With the assumed displacements of the one-dimensional theory (1), which were

$$\begin{aligned} u_r &= \frac{r}{a} u(z,t) \\ u_\theta &= 0 \\ u_z &= w(z,t) \end{aligned} \qquad [7]$$

the kinetic energy of the rod becomes

$$T = \frac{\rho}{2} \int_0^{2\pi} \int_0^a \int_0^l \left(\frac{r^2}{a^2} \dot{u}^2 + \dot{w}^2 \right) r \, dr \, d\theta \, dz \qquad [8]$$

Performing the integrations with respect to θ and r, the volume integral in [8] reduces to a simple integral, extended over the length l of the rod,

$$T = \frac{\pi \rho a^2}{2} \int_0^l \left(\frac{\dot{u}^2}{2} + \dot{w}^2 \right) dz \qquad [9]$$

Potential Energy

The potential energy of the rod is obtained, similarly, by performing appropriate integrations in the general expression for the potential energy. The latter is

$$V = \int_V W \, d\tau \qquad [10]$$

where the integration is over the volume of the rod and

$$W = \frac{1}{2}\left[\sigma_{rr}\frac{\partial u_r}{\partial r} + \cdots \sigma_{\theta z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) + \cdots\right] \qquad [11]$$

The assumed displacements [7] are now inserted into [11] and the integrations are performed with respect to r and θ, thus obtaining an expression in terms of u, w, P_r, P_θ, P_z and Q. Use of the stress-displacement relations [2] reduces the result to

$$V = \pi \int_0^\ell \left[2(\lambda+\mu)u^2 + 2\lambda a u w' + \frac{a}{2}(\lambda+2\mu)w'^2 + \frac{\mu \chi^2 a^2}{4} u'^2\right] dz \qquad [12]$$

For the sake of simplicity, the correction factor χ, has been set equal to unity.

Free Vibrations

As a preliminary to the study of forced vibrations, we shall be concerned with the problem of free vibrations, specified by the homogeneous differential equations of motion

$$\begin{aligned} a^2\chi^2\mu u'' - 8(\lambda+\mu)u - 4a\lambda w' &= \rho a^2 \ddot{u} \\ 2a\lambda u' + a^2(\lambda+2\mu)w'' &= \rho a^2 \ddot{w} \end{aligned} \qquad [13]$$

and homogeneous boundary conditions

$$B_i = 0 \qquad i = 1, 2, 3, 4 \qquad [14]$$

Considering solutions in the form

$$u(z,t) = U(z)\sin\omega t$$
$$w(z,t) = W(z)\sin\omega t \qquad [15]$$

we may show that the equations of motion [13], together with the boundary conditions [14], are satisfied for an infinite set of discrete frequencies ω_n, each of which corresponds to a mode shape given by functions $U_n(z)$ and $W_n(z)$, which are determined except for a multiplying factor, common to both functions.

The kinetic energy of the rod, vibrating in the n-th principal mode, is, according to [9],

$$T_n(t) = \frac{\omega_n^2 \pi \rho a^2}{2}\int_0^l \left(\tfrac{1}{2}U_n^2 + W_n^2\right) dz \cos^2\omega_n t \qquad [16]$$

and its maximum value is

$$T_{n\,max} = \frac{\omega_n^2 \pi \rho a^2}{2}\int_0^l \left(\tfrac{1}{2}U_n^2 + W_n^2\right) dz \qquad [17]$$

Similarly, the maximum potential energy of the rod is, from [12],

$$V_{n\,max} = \pi\int_0^l \Big[2(\lambda+\mu)U_n^2 + 2\lambda a U_n W_n' + \frac{a^2}{2}(\lambda+2\mu)W_n'^2 + \frac{\mu x^2 a^2}{4} U_n'^2\Big] dz \qquad [18]$$

Employing the principle of conservation of energy, we restate, for future reference, the well known relationship

$$T_{n\,max} = V_{n\,max} \qquad [19]$$

Time-Dependent Boundary Displacements

Whenever a number, m, of time-dependent displacements is specified at the ends of the rod, in a forced vibration problem, it is convenient to solve m static problems in advance; each problem being governed by the equilibrium equations

$$\begin{aligned} a^2 x^2 \mu u'' - 8(\lambda+\mu) u - 4a\lambda w' &= 0 \\ 2a\lambda u' + a^2(\lambda+2\mu) w'' &= 0 \end{aligned} \qquad [20]$$

The boundary conditions for each static problem are obtained by setting one of the m end displacements equal to unity and letting the remaining three conditions be homogeneous. For example, if the boundary conditions of the forced vibration problem are

$$\begin{aligned} u(0,t) &= f_1(t) \\ P_2(0,t) &= f_2(t) \\ Q(\ell,t) &= f_3(t) \\ w(\ell,t) &= f_4(t) \end{aligned}$$

then $m = 2$ and two static problems have to be solved: one with boundary conditions

$$\begin{aligned} u(0,t) &= 1 \\ P_2(0,t) &= 0 \\ Q(\ell,t) &= 0 \\ w(\ell,t) &= 0 \end{aligned} \qquad [21]$$

and the other with boundary conditions

$$
\begin{aligned}
u(0,t) &= 0\\
P_z(0,t) &= 0\\
Q(\ell,t) &= 0\\
w(\ell,t) &= 1
\end{aligned}
\qquad [22]
$$

The m static solutions $u(z)$, $w(z)$, which are readily obtained, will be designated $g_{iu}(z)$, $g_{iw}(z)$, where i runs from 1 to m.

General Forced Vibration Problem

The solution of the general forced vibration problem, specified by [1'], [3'] and [4'] will be sought in the form

$$
\begin{aligned}
u(z,t) &= \sum_{i=1}^{m} g_{iu}(z)\, f_i(t) + \sum_{n=1}^{\infty} U_n(z)\, q_n(t)\\
w(z,t) &= \sum_{i=1}^{m} g_{iw}(z)\, f_i(t) + \sum_{n=1}^{\infty} W_n(z)\, q_n(t)
\end{aligned}
\qquad [23]
$$

All the functions in [23] have been introduced in preceding sections, and are assumed to have been determined, with the exception of $q_n(t)$, which will be found from Lagrange's equations [5]. It should be noted that [23] satisfies the boundary conditions [3'].

Inserting the forms [23] into the expression [9] for the kinetic energy T, we obtain

$$
\begin{aligned}
T = \frac{\pi \rho a^2}{2}\int_0^{\ell} \Big\{ \tfrac{1}{2}\Big[\sum_{i=1}^{m} g_{iu}\dot f_i + \sum_{n=1}^{\infty} U_n \dot q_n\Big]^2 \\
+\Big[\sum_{i=1}^{m} g_{iw}\dot f_i + \sum_{n=1}^{\infty} W_n \dot q_n\Big]^2 \Big\}\, dz
\end{aligned}
\qquad [24]
$$

Thus

$$\frac{\partial T}{\partial q_n} = 0 \qquad [25]$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} = \frac{\pi \rho a^2}{2}\int_0^l \Big[\sum_{i=1}^{m} g_{iu} U_n \ddot{f}_i + 2\sum_{i=1}^{m} g_{iw} W_n \ddot{f}_i\Big]\, dz$$
$$+ T_{n\,max}\frac{2}{\omega_n^2}\ddot{q}_n \qquad [26]$$

where use has been made of [17] and of

$$\int_0^l \Big(\frac{1}{2} U_r U_s + W_r W_s\Big)\, dz = 0\,, \qquad \omega_r \neq \omega_s \qquad [27]$$

The relationship [27], which expresses the property of orthogonality of the principal modes of free vibration, can be obtained by substituting the assumed displacements [7] and [15] into Clebsch's theorem (3) and performing the integrations with respect to r and θ. Alternatively, [27] may be obtained from [13] and [14], using Rayleigh's method (4).

Insertion of the displacements [23] into the expression [12] for the potential strain energy V yields

$$V = \pi\int_0^l \Big\{ 2(\lambda+\mu)\Big[\sum_{i=1}^{m} g_{iu} f_i + \sum_{n=1}^{\infty} U_n q_n\Big]^2$$
$$+ 2\lambda a\Big[\sum_{i=1}^{m} g_{iu} f_i + \sum_{n=1}^{\infty} U_n q_n\Big]\Big[\sum_{i=1}^{m} g'_{iw} f_i + \sum_{n=1}^{\infty} W'_n q_n\Big]$$
$$+ \frac{a^2}{2}(\lambda+2\mu)\Big[\sum_{i=1}^{m} g'_{iw} f_i + \sum_{n=1}^{\infty} W'_n q_n\Big]^2$$
$$+ \frac{\mu z^2 a^2}{4}\Big[\sum_{i=1}^{m} g'_{iu} f_i + \sum_{n=1}^{\infty} U'_n q_n\Big]^2 \Big\}\, dz \qquad [28]$$

Partial differentiation with respect to q_n results in

$$\frac{\partial V}{\partial q_n} = \pi \int_0^{\ell} \Big\{ 4(\lambda+\mu) \sum_{i=1}^{m} g_{iu} f_i U_n$$

$$+ 2\lambda a \sum_{i=1}^{m} \left[g_{iu} f_i W_n' + g_{iw} f_i U_n \right]$$

$$+ a^2 (\lambda + 2\mu) \sum_{i=1}^{m} g_{iw}' f_i W_n' + \frac{\mu \kappa^2 a^2}{2} \sum_{i=1}^{m} g_{iu}' f_i U_n' \Big\} dz$$

$$+ 2 V_{n\,max}\, q_n \qquad [29]$$

where use has been made of [18] and of

$$\int_0^{\ell} \Big[4(\lambda+\mu) U_r U_s + 2\lambda a (U_r W_s' + U_s W_r')$$

$$+ (\lambda + 2\mu) a^2 W_r' W_s' + \frac{\mu \kappa^2 a^2}{2} U_r' U_s' \Big] dz = 0 \qquad \omega_r \neq \omega_s \qquad [30]$$

which states, in another way, the orthogonality property of the principal modes and can be transformed to the relationship [27] by use of the equations of motion [13].

Referring to the integral in [29], the primed terms may be integrated by parts. Then, noting that U_n, W_n satisfy [15] and [16], while g_{iu}, g_{iw} satisfy [23] and boundary conditions of the type exemplified by [24], we arrive at

$$\int_0^{\ell} \Big\{ 4(\lambda+\mu) g_{iu} U_n + 2\lambda a (g_{iu} W_n' + g_{iw}' U_n)$$

$$+ (\lambda + 2\mu) a^2 g_{iw}' W_n' + \frac{\mu \kappa^2 a^2}{2} g_{iu}' U_n' \Big\} dz = 0 \qquad [31]$$

Thus, the integral in [29] vanishes and we find, from [29], [31] and [19],

$$\frac{\partial V}{\partial q_n} = 2 q_n V_{n\,max} = 2 q_n T_{n\,max} \qquad [32]$$

The vanishing integral [31] can also be used advantageously to simplify the integral in equation [26]. Assuming, temporarily, the largest possible number of specified, time-dependent displacements at the ends of the rod (m = 4) we obtain, by partial integration,

$$\begin{aligned} &\int_0^{\ell} \Big[\sum_{i=1}^{4} g_{iu} U_n + 2 \sum_{i=1}^{4} g_{iw} W_n \Big] dz \\ &\quad = \frac{4}{\rho \omega_n^2 a^2} \Big[Q_n(0) + P_{2n}(0) - Q_n(\ell) - P_{2n}(\ell) \Big] \end{aligned} \qquad [33]$$

where Q_n and P_{2n} are tractions produced by the displacements of U_n and W_n . Entering this result into [26], we obtain

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} &= \frac{2T_n}{\omega_n^2} \Big[\ddot{f}_1 Q_n(0) + \ddot{f}_2 P_{2n}(0) - \ddot{f}_3 Q_n(\ell) - \ddot{f}_4 P_{2n}(\ell) \Big] \\ &\quad + T_{n\,max} \frac{2}{\omega_n^2} \ddot{q}_n \end{aligned} \qquad [34]$$

The generalized force Q_n in [5], defined as the coefficient of the increment δq_n of the generalized coordinate q_n , in the expression for the work δW_n , done by the applied physical forces in the displacement produced by δq_n , has to be determined on the basis of this definition. We note, at this point, that not only the q_n 's, but also the m functions f_i have to be regarded as generalized coordinates. Consequently, Q_n will not be affected by the presence or absence of any of the functions f_i .

An increment δq_n produces, according to [23] the infinitesimal displacements

$$\begin{aligned} \delta u &= U_n \, \delta q_n \\ \delta w &= W_n \, \delta q_n \end{aligned} \qquad [35]$$

The work δW, done by the applied forces on the surfaces of the rod, has been derived in (1) and is

$$\delta W = 2\pi a \int_0^{\ell} (R \delta u + Z \delta w) \, dz + 2\pi \left[Q(\ell) \delta u(\ell) + P_z(\ell) \delta w(\ell) - Q(0) \delta u(0) - P_z(0) \delta w(0) \right] \qquad [36]$$

Considering the largest possible number of specified, time-dependent forces, inserting [35] in the above expression and factoring δq_n, we obtain the generalized force $Q_n(t)$

$$Q_n(t) = 2\pi a \int_0^{\ell} \left[R U_n + Z W_n \right] dz + 2\pi \left[Q(\ell) U_n(\ell) + P_z(\ell) W_n(\ell) - Q(0) U_n(0) - P_z(0) W_n(0) \right] \qquad [37]$$

Comparing this with [36], Q_n is seen to be the work done by the applied surface tractions in the displacements of the n-th principal mode.

In computing $\dfrac{d}{dt} \dfrac{\partial T}{\partial \dot{q}_n}$, equation [34], we have assumed, as an extreme case, four time-dependent displacements specified at the ends of the rod, (m = 4). The generalized force Q_n, equation [37], on the other hand, was derived under the presupposition of four time-dependent tractions acting on the ends of the rod, (m = 0). Now, whatever the value of m , the total number of the terms in the bracket of [34] and in the bracket of [37] will always be four. In view of this, it is convenient to define four quantities H_i , $i = 1, 2, 3, 4$ as follows

$$-H_1 = \frac{2\pi}{\omega_n^2} f_1 Q_n(0) \quad \text{or} \quad 2\pi U_n(0) Q(0), \quad \text{according as}$$
$$B_1 = u \quad \text{or} \quad Q$$

$$-H_2 = \frac{2\pi}{\omega_n^2} f_2 P_{2n}(0) \quad \text{or} \quad 2\pi W_n(0) P_2(0), \quad \text{according as}$$
$$B_2 = w \quad \text{or} \quad P_2$$

$$H_3 = \frac{2\pi}{\omega_n^2} f_3 Q_n(\ell) \quad \text{or} \quad 2\pi U_n(\ell) Q(\ell), \quad \text{according as}$$
$$B_3 = u \quad \text{or} \quad Q$$

$$H_4 = \frac{2\pi}{\omega_n^2} f_4 P_{2n}(\ell) \quad \text{or} \quad 2\pi W_n(\ell) P_2(\ell), \quad \text{according as}$$
$$B_4 = w \quad \text{or} \quad P_2$$

Also we define the quantity

$$Q_n^* = \frac{\omega_n^2}{2T_{n\,max}} \left[2\pi a \int_0^\ell (R U_n + Z W_n)\, dz + \sum_{i=1}^{4} H_i \right] \qquad [39]$$

Then Lagrange's equations [5] become

$$\ddot{q}_n + \omega_n^2 q_n = Q_n^* \qquad [40]$$

The general integral of this ordinary differential equation is

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\omega_n} \int_0^t Q_n^*(\tau) \sin \omega_n (t - \tau)\, d\tau \qquad [41]$$

It remains to determine the constants of integration A_n and B_n, employing the initial conditions [4']. At $t = 0$ we have

$$
\begin{aligned}
u(z,0) &= \sum_{i=1}^{m} g_{iu}(z) f_i(0) + \sum_{n=1}^{\infty} A_n U_n \\
w(z,0) &= \sum_{i=1}^{m} g_{iw}(z) f_i(0) + \sum_{n=1}^{\infty} A_n W_n \\
\dot{u}(z,0) &= \sum_{i=1}^{m} g_{iu}(z) \dot{f}_i(0) + \sum_{n=1}^{\infty} B_n U_n \omega_n \\
\dot{w}(z,0) &= \sum_{i=1}^{m} g_{iw}(z) \dot{f}_i(0) + \sum_{n=1}^{\infty} B_n W_n \omega_n
\end{aligned}
\qquad [42]
$$

In view of the orthogonality relationship [27] we multiply the first of equations [42] by $\frac{1}{2} U_s$, the second by W_s, add the two and integrate both sides over the length of the bar.

Solving for A_n we obtain

$$
A_n = \frac{\int_0^{\ell} \left\{ \frac{1}{2}\left[u_0 - \sum_{i=1}^{m} g_{iu}(z) f_i(0)\right] U_n + \left[w_0 - \sum_{i=1}^{m} g_{iw}(z) f_i(0)\right] W_n \right\} dz}{\int_0^{\ell} \left(\frac{1}{2} U_n^2 + W_n^2\right) dz} \qquad [43a]
$$

B_n is obtained from the third and fourth of equations [41] in an analogous manner

$$
B_n = \frac{\int_0^{\ell} \left\{ \frac{1}{2}\left[\dot{u}_0 - \sum_{i=1}^{m} g_{iu}(z) \dot{f}_i(0)\right] U_n + \left[\dot{w}_0 - \sum_{i=1}^{m} g_{iw}(z) \dot{f}_i(0)\right] W_n \right\} dz}{\omega_n \int_0^{\ell} \left(\frac{1}{2} U_n^2 + W_n^2\right) dz} \qquad [43b]
$$

This completes the formal solution of the problem.

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