ON SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY THE

HAHN-BANACH THEOREM

BY

P. R. GARABEDIAN AND MAX SHIFFMAN

TECHNICAL REPORT NO. 9

JANUARY 21, 1953

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

ON SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY THE
HAHN-BANACH THEOREM

by

P. R. Garabedian and Max Shiffman

1. Introduction.

In this paper we show how to construct the Green's and Neumann's
functions of a linear elliptic partial differential equation by a method
based on linear functionals. The construction depends essentially on the
Hahn-Banach extension theorem [1], which has been used recently in a similar
connection by Lax [5]. Lax's approach differs from ours in that, while his
proof centers about a bounded linear functional based on inhomogeneous
boundary conditions, our proof centers rather about a functional based on
the solution of an inhomogeneous differential equation. This latter point
of view has the advantage that in constructing the Green's and Neumann's
functions we need only a fundamental solution of the differential equation
in the small, and, indeed, we can succeed with merely a parametrix, which is
readily obtained for a linear elliptic partial differential equation, whereas
all earlier existence proofs of the type studied here [3, 4, 5] are based on
the previous knowledge of a fundamental solution in the large. Our considera-
tions are also advantageous for domains with general boundaries and for
Riemannian manifolds.

For earlier work on application of the Hahn-Banach theorem to boundary
value problems, we refer to the papers of Caccioppoli [2] and of Miranda [6].
It should be mentioned at the outset that the proof of the Hahn-Banach theorem
in the form in which we shall need it in this paper does not require trans-
finite induction, since the Banach space of continuous functions which we use
is separable.

2. The Neumann's function.

Let $P = P(x) \geqslant \varepsilon > 0$ be a positive, continuously differentiable function of the point x in a plane domain D bounded by a finite number of simple closed analytic curves C. Denote by B the Banach space of all functions f which are continuous in D+C, with the norm

$$\|f\| = \max |f| \quad ,$$

and denote by B′ the subspace of B of functions

(1) $$f = \triangle u - Pu \quad ,$$

where u is any function, continuous with its first and second derivatives in D+C, whose inner normal derivative vanishes on C,

(2) $$\frac{\partial u}{\partial \nu} = 0 \quad .$$

For any point w in D, we define a linear functional $L_w\{f\}$ on B′ by the formula

(3) $$u(w) = L_w\left\{\triangle u - Pu\right\} \quad .$$

That this linear functional is uniquely defined and bounded can be shown by the following argument.

First, if

(4) $$\triangle U - PU \leqslant 0$$

in D and

(5) $$\frac{\partial U}{\partial \nu} = 0$$

on C, then $U \geqslant 0$ in D. For if $U < 0$ in some subregion of D, we could apply Green's theorem to that subregion to obtain

(6) $$\iint [(\nabla U)^2 + PU^2] d\tau = -\iint U[\triangle U - PU] d\tau - \int U \frac{\partial U}{\partial \nu} ds \quad ,$$

where $d\tau$ is the area element and where s denotes the arc length along the

boundary curves of the region $U < 0$. By (4), (5) and the condition $U < 0$, it is clear that the right-hand side of (6) is non-positive, whereas it is equally clear that the left-hand side of (6) is positive. This contradiction yields the original statement $U \geqslant 0$. Uniqueness of the functional L_w for given $\Delta u - Pu$, as defined in (3), is an immediate consequence of this.

Second, let $v_1 = 1/\varepsilon$, so that

(7) $\qquad\qquad \Delta v_1 - Pv_1 \leqslant -1$,

and let $M = \|f\|$. Then in D

$$\Delta(Mv_1 - u) - P(Mv_1 - u) \leqslant -M - f = -\|f\| - f \leqslant 0 \qquad,$$

and on C

$$\frac{\partial}{\partial \nu} (Mv_1 - u) = 0 \qquad,$$

whence $Mv_1 - u \geqslant 0$. It follows from the definition (3) that for f in B',

(8) $\qquad\qquad L_w\{f\} \leqslant \frac{1}{\varepsilon} \|f\|$,

which proves that L_w is a bounded functional.

We conclude from the Hahn-Banach theorem that the bounded linear functional L_w can be extended from the vector space B' to operate on all elements of the Banach space B and to satisfy (8) there. By applying the extended functional in a suitable way, we shall set up an expression which we shall prove to be the Neumann's function of the linear elliptic partial differential equation

(9) $\qquad\qquad \Delta w - Pw = 0$

in D.

For the construction, we need a so-called parametrix $S(w,z)$ for (9). Such a parametrix is a function of w which has a logarithmic singularity at $w = z$ and is such that $\Delta S - PS$ is continuous, even at the point z. More precisely, we shall require certain regularity properties of $\Delta S - PS$ as a function of z,

and for our present purposes we select $S(w,z)$ to be a symmetric function of w and z. Let

(10)
$$g(w,z) = \log \frac{1}{|w - z|} + \ldots$$

be a fundamental solution of Laplace's equation in a region E. When E is a semi-circle, it is an elementary problem to construct g explicitly in such a way that g vanishes on the circular arc bounding E and has zero normal derivatives on the diameter bounding E. We set

$$I_1(w,z) = \frac{1}{2\pi} \iint_E P(t)g(w,t)g(z,t)d\tau \qquad ,$$

$$I_2(w,z) = \frac{1}{2\pi} \iint_E P(t)g(w,t)I_1(z,t)d\tau \qquad ,$$

and we set

(11)
$$S_0(w,z) = g(w,z) - I_1(w,z) + I_2(w,z) \qquad ,$$

whence by Poisson's equation

$$\Delta S_0 - PS_0 = -PI_2 \qquad ,$$

which shows, for our purposes, that S_0 is a parametrix for (9). If E is the above semi-circle and g is explicitly constructed in the manner described for a semi-circle, then S_0 will have a normal derivative which vanishes on the diameter bounding E. For z in the neighborhood of a point z_0 in which we are interested, we obtain for all w a suitable parametrix S whose normal derivative vanishes on C by multiplying S_0 by a function δ, with continuous derivatives of the first few orders, which is identically 1 near z_0, but which vanishes identically outside a small circle about z_0.

Having obtained a parametrix $S(w,z)$ in this manner, we define the Neumann's function $N(w,z)$ for (9) in D by the formula

(12)
$$N(w,z) = S(w,z) - L_w \left\{ [\Delta - P(t)]S(t,z) \right\} \qquad ,$$

and we proceed to show that N has, as a function of the point z, the character-
istic properties of the Neumann's function. It is easily verified from the
definition (3) that N is independent of the choice of the parametrix S, since
the normal derivative of S vanishes on C and the difference of two such
parametrices is regular.

First, since L_w is a bounded functional, the quantity $L_w[-\{\Delta - P(t)\}S(t,z)]$
depends continuously on z, and, indeed, we can differentiate with respect to z
under the functional L_w. We obtain

(13) $[\Delta - P(z)]N(w,z) = [\Delta - P(z)]S(w,z) - L_w\{[\Delta - P(t)]K[\Delta - P(z)]S(t,z)\} = 0$.

The expression on the right vanishes, according to (3), because, as a function
of w, the expression $[\Delta - P(z)]S(w,z)$ has continuous second derivatives in D+C
and has a normal derivative which vanishes on C. Thus N satisfies the partial
differential equation (9) as a function of z.

Second, $N(w,z)$ has a logarithmic singularity at the point z = w, since
the second term on the right in (12) is bounded.

Third, we can calculate the normal derivative of N at a point z_o of C
conveniently by making a special choice of E and g in (11). The curves C have
an analytic representation z = z(s) in terms of a parameter s, and we choose E
to lie in D and to correspond in the s-plane to a small semi-circle whose
bounding diameter lies on the real axis and whose center maps into z_o. We
choose g to be the fundamental solution of the Laplace equation in the semi-
circle which vanishes on the circular boundary and has normal derivative zero
on the rectilinear boundary. Since z = z(s) is a conformal mapping, g satisfies
Laplace's equation in E also, and substitution into (11) yields a parametrix S_o
with vanishing normal derivatives with respect to both z and w on an arc of C
including the point z_o. Multiplication by a function δ which is identically 1

in a small neighborhood of s_o, identically zero outside a slightly larger neighborhood of s_o, and has a vanishing normal derivative on C, yields our choice of S. (A selection of δ as a function of $|s|$ will satisfy these conditions.)

With this choice of S in (12), it is permissible to compute the normal derivative of N with respect to s at s_o by differentiating under the bounded functional L_w. We find

$$(14) \qquad \frac{\partial N}{\partial \nu} - \frac{\partial S}{\partial \nu} - L_w\left\{(\Delta - P)\frac{\partial S}{\partial \nu}\right\} = 0 \qquad ,$$

since $\frac{\partial S}{\partial \nu} = \delta \cdot \frac{\partial S_o}{\partial \nu} \equiv 0$ throughout D.

We have thus shown that N has the three characteristic properties of the Neumann's function of (9) in D, as a function of s, and we conclude therefore that the Neumann's function exists and is equal to N.

We remark that it is possible to generalize the above existence proof to partial differential equations other than (9). Also, the restriction independent variables is unnecessary, and indeed, for more independent variables the only additional complication comes in deriving the boundary condition (13), which has been based here on conformal mapping. However, for general elliptic equations in several variables we can treat the easy case of a plane boundary without any loss of generality, since we can always find a transformation of coordinates which takes a small portion of an analytic surface into a portion of a plane and this transformation will merely carry one elliptic equation into another which can be discussed equally well. In another direction, we can generalize the existence proof by replacing the boundary condition (2) by

$$(15) \qquad \frac{\partial u}{\partial \nu} - \lambda u = 0 \qquad ,$$

where λ is a given positive function on C. In this case, we obtain the existence of a Green's function of the third kind. Also, we note that the domain D may be a multisheeted Riemann surface over the plane, instead of a schlicht region.

3. The Green's function.

In this section we study the Dirichlet problem for a linear elliptic partial differential equation in several independent variables by a technique analogous to that of Section 2. Our objective is to give a more penetrating analysis of the boundary condition, with hypotheses on the boundary far less restrictive than analyticity. This can be achieved by appropriate use of the principle of majorization and by a stronger use of the Hahn-Banach theorem. We present the material for three independent variables in space for the sake of simplicity, and we merely remark that for any finite number of dimensions the steps to be carried out are similar.

Let D be a bounded domain in space with boundary C having the following property: Each point of C can be touched by a solid sphere R which is otherwise disjoint from D+C. Let $P(z) \geqslant 0$ be a non-negative continuously differentiable[1]/ function of the point z in a neighborhood of D+C, and let B' denote the linear space of all functions

$$f = \triangle u - Pu$$

in D+C with u twice continuously differentiable in D+C and zero on C. We let B denote the Banach space of all continuous functions in D+C with the norm

$$\|f\| = \max |f| \qquad .$$

---

[1]/The considerations apply if, in place of continuous derivatives, $P(z)$ satisfies a Lipschitz condition $|P(z) - P(z')| \leqslant K|z-z'|$.

For any point w in D we define a linear functional L_w over B' by the formula

(16) $\qquad L_w[\Delta u - Pu] = u(w)$.

That this functional is uniquely defined follows from the theorem that a solution of $\Delta u - Pu = 0$ in D which vanishes on C must be identically 0, since it cannot have a positive maximum or negative minimum in the interior of D.

We prove that, for f in B',

(17) $\qquad L_w\{f\} \leq b(f)$,

where

(18) $\qquad b(f) = \inf v(w)$

among all twice continuously differentiable functions v with

(19) $\qquad \Delta v - Pv \leq f$

in D, and $v \geq 0$ on C. For the proof, we note that since

$$\Delta(v-u) - P(v-u) \leq f - f = 0 \quad ,$$

$v-u$ cannot have a negative minimum inside D. Therefore, since $v-u \geq 0$ on C, we must have $v-u \geq 0$ in D also, which yields the desired estimate (17). In fact, since u is itself an admissible function in the determination of $b[f]$, we have equality in (17) for f in B'. In particular, $b[0] = 0$.

The functional $b[f]$ defined in (18), (19) has a meaning for any f in the entire space B. For, let v_2 be the function

$$v_2 = K - \frac{1}{6}(x^2 + y^2 + z^2) \quad ,$$

where K is a large constant such that $v_2 \geq 0$ in $D+C$. We have

$$\Delta v_2 - Pv_2 \leq -1 \quad ,$$

and $\|f\| v_2$ is an admissible function for (18). Thus

$$(20) \qquad b[f] \leqslant v_2(w) \cdot ||f|| \leqslant K \cdot ||f|| \qquad .$$

Also, we notice that

$$(21) \qquad b[f] \leqslant 0$$

if $f \geqslant 0$ throughout D, since the function $v \equiv 0$ is then admissible in (18).

The functional $b[f]$ is positively homogeneous and subadditive; i.e.,

$$b[tf] = tb[f] \quad \text{if} \quad t > 0, \quad b[f_1 + f_2] \leqslant b[f_1] + b[f_2] \qquad .$$

The latter inequality follows from the fact that if v' and v'' are admissible functions for the determination of $b[f_1]$ and $b[f_2]$, respectively, then $v' + v''$ is admissible for $b[f_1 + f_2]$. We remark that since we already know that $b[0] = 0$, the subadditivity shows that $b[f] \geqslant -b[-f] \geqslant -K||f||$, whence $b[f]$ is finite and likewise bounded from below.

We now use the Hahn-Banach theorem to extend $L_w[f]$ as a linear functional over the entire space B still satisfying the inequality (17). Applying the extended functional to a parametrix for the equation

$$\Delta u - Pu = 0$$

in a suitable way, we shall obtain the Green's function in D for this equation.

Let E be a (small) region containing z, and let $g(w,z) = \frac{1}{|w-z|} + \ldots$ be a fundamental solution for Laplace's equation which is non-negative in E and symmetric in w and z ($|w-z|$ means the distance between w, z). For example, $g(w,z)$ could be the Green's function for Laplace's equation for a sphere enclosing E. Set

$$(22) \qquad I_j(w,z) = \frac{1}{4\pi} \iiint_E P(t) g(w,t) I_{j-1}(z,t) d\tau \qquad ,$$

$j = 1, 2, \ldots$, where $I_0(w,z) = g(w,z)$ and $d\tau$ is the volume element. We have $I_j(w,z) \geqslant 0$ in E. Then

(23)
$$S_0(w,z) = \sum_{j=0}^{3} (-1)^j I_j(w,z)$$

is a parametrix for the equation $\Delta u - Pu = 0$, and

(24)
$$\Delta S_0 - PS_0 = PI_3 \geqslant 0 \quad .$$

For z in the neighborhood of a point z_0 in which we are interested, we obtain for all w in D a suitable parametrix $S_1(w,z)$ by multiplying S_0 by a function δ, with continuous derivatives of the first few orders, which is identically 1 near z_0, but which vanishes identically outside a small sphere about z_0.

We now define the Green's function $G(w,z)$ in D by the formula

(25)
$$G(w,z) = S(w,z) - L_w[\{\Delta - P(t)\} S(t,z)] \quad ,$$

where $S(w,z)$ is a parametrix in D which vanishes when w is on C. We show directly that G has as a function of z in D the characteristic properties of the Green's function. Note that the formula (25) is independent of the particular $S(w,z)$ used, by virtue of (16

First, the second term on the right in (25) is bounded for z near w by virtue of (17) and (20), so that G must have a pole at $z = w$. Second, since the functional L_w is a bounded linear functional, the second term on the right in (25) is continuous in z for z in D. Indeed, we can differentiate with respect to the coordinates of z under the operator L_w, and we find

$$\{\Delta - P(z)\} G = \{\Delta - P(z)\} S - L_w[\{\Delta - P(t)\}\{\Delta - P(z)\} S] = 0 \quad .$$

The last equality follows from (16), since $\{\Delta - P(z)\} S(w,z)$ is a twice continuously differentiable function of w which vanishes for w on C.

Third, we prove that $G \geqslant 0$. Let σ be a large positive number and let $F(S)$ be a function of S which is identically S for $S \leqslant \sigma$, which is continuous

with its first two derivatives for all S, which has a finite limit as $S \longrightarrow +\infty$, and which has a non-positive second derivative $F''(S)$ for $S > \sigma$. With the parametrix S_0 constructed in (23), we have[1]

(26) $\quad \{\Delta - P\} F(S_0) - \{\Delta - P\} S_0 = F''(\nabla S_0)^2 + (F' - 1)(\Delta S_0 - P S_0)$

$$+ (F' S_0 - F) P \leqslant 0 \quad,$$

by (24) and the inequalities $F'S - F \leqslant 0$, and $F' - 1 \leqslant 0$, which follow from $F'' \leqslant 0$.

Using the parametrix S_1 constructed in the paragraph immediately following (24), we therefore have for sufficiently large σ

$$(\Delta - P)F(S_1) - (\Delta - P)S_1 \leqslant 0 \quad .$$

It now follows that

$$F(S_1) = L_w[(\Delta - P)F(S_1)] \geqslant L_w[(\Delta - P)S_1] \quad,$$

since the linear functional $L_w[f] \leqslant 0$ when f is everywhere non-negative, by virtue of (21) and (17). Therefore

$$G = S_1 - L_w[(\Delta - P)S_1] \geqslant S_1 - F(S_1) \geqslant 0 \quad,$$

which is the desired result.

Finally, we prove that G has boundary values 0. Let z_0 be a point of C, let R be a small sphere in the exterior of D and touching C at z_0, and let R' be a small sphere containing z_0 and orthogonal to R. Denote the region outside R and inside R' by E, and let $g(w,z)$ be the Green's function for Laplace's equation in E. For this choice of E and g, define $I_j(w,z)$ as in (22) and set

---

[1] For the general differential equation, F can always be constructed so that the first term corresponding to $F''(\nabla S_0)^2$ is the dominant term, and we still have the inequality.

$$S_0^*(w,z) = \sum_{j=0}^{4} (-1)^j I_j(w,z) \quad .$$

Then,

(27) $\quad \Delta S_0^* - P S_0^* = -P I_4(w,z) \leqslant 0 \quad .$

Also, for w on the boundary of E, we have $g(w,z) = 0$ and $I_j(w,z) = 0$ from (22).
Therefore, $S_0^*(w,z) = 0$ on the boundary of E. Since S_0^* cannot have a negative
minimum in the interior of E by (27), it follows that

$$S_0^* \geqslant 0$$

throughout E.

Multiply S_0^* by a non-negative function δ, continuous with its first
few derivatives, which is identically 1 in a small neighborhood of z_0 and
which is identically 0 in a slightly larger neighborhood of z_0. This yields
a parametrix S_1^* throughout D which is non-negative on C and approaches 0
as $z \rightarrow z_0$.

For any parametrix S which vanishes on C, the difference $S_1^* - S$ is twice
continuously differentiable in D and is non-negative on C. Hence, by (17),
(18), and (19), with $f = (\Delta - P)(S_1^* - S)$ and $v = S_1^* - S$, we obtain

$$L_w[(\Delta - P)(S_1^* - S)] \leqslant S_1^* - S \quad ,$$

or

(28) $\quad G \leqslant S_1^* - L_w[(\Delta - P)S_1^*] \quad .$

Letting $z \rightarrow z_0$, we have $S_1^*(w,z) \rightarrow 0$ and $(\Delta - P(t))S_1^*(t,z) \rightarrow 0$ uniformly in t;
this last limiting relation is clear outside any neighborhood of z_0, while
inside a small neighborhood of z_0 it follows from (27). Consequently

(29) $\quad \overline{\lim_{z \rightarrow z_0}} \ G(w,z) \leqslant 0 \quad ,$

and since $G \geqslant 0$, we can make the final desired conclusion

(30) $\lim_{z \to 0} G(w,z) = 0$.

These calculations show that G is the Green's function of D for the linear elliptic partial differential equation $\Delta u = Pu$. This construction of G gives a remarkably simple proof of the existence of the Green's function, without previous knowledge of a fundamental solution in the large. In the special case $P \equiv 0$ we obtain the Green's function of Laplace's equation.

Incidentally, the symmetry of the Green's function is also contained in the above. Since $G(w,z)$ is the Green's function for $\Delta u = Pu$ as a function of the second variable, with singularity at the first variable, the function $G(z,w)$ is an admissible singularity function $S(w,z)$ in (25). We immediately obtain, therefore,

(31) $G(w,z) = G(z,w)$.

Of course, solvability of the Dirichlet problem is obtained once G is known.

4. <u>General domains</u>.

The argument of the preceding section can easily be generalized to the case of a domain D, bounded or unbounded, with an arbitrary boundary C. Let D lie in n-dimensional Euclidean space, where $n \geqslant 3$, and let $P(z) \geqslant 0$ be an arbitrary non-negative function over D, continuous with continuous first derivatives in D. In 2 dimensions, we suppose either that D is bounded, or if D is unbounded, that $P(z) > 0$. Consider the function

$$v_3 = g(|z|) \qquad ,$$

where $|z| = r$ is the distance from z to the origin, and

$$g(r) = \int_r^\infty \frac{r\,dr}{1+r^n} \qquad ,$$

so that

$$\Delta g(r) = - \frac{n}{(1+r^n)^2} < 0 \quad .$$

The function v_3 is positive, is continuous and has continuous first and second derivatives, and

$$-w_3 = (\Delta - P)v_3 < 0 \quad ,$$

so that $w_3 > 0$. In 2 dimensions, with D unbounded and $P > 0$, select $v_3 \equiv 1$, and $w_3 = P > 0$. Define the norm of a function f which is continuous in D by

$$(32) \qquad \|f\| = \sup \left| \frac{f}{v_3} \right| \quad ,$$

and let B denote the Banach space of the functions f with finite norm. Let B' denote the subspace of functions f in B given by

$$f = \Delta u - Pu \quad ,$$

where u is continuous together with first and second derivatives in D, and $u \to 0$ as the boundary or the point at infinity (if D is unbounded) is approached. Finally, define the linear functional L_w over B as in (16).

Proceeding as in Section 3, define the bound $b[f]$ for any f in B as in (18), (19), with the condition $v \geqslant 0$ on C replaced by $\underline{\lim} \, v \geqslant 0$ as the boundary C or the point at infinity is approached. Inequality (17) applies. An admissible function in (19) is $\|f\| v_3$. For,

$$(\Delta - P) \|f\| v_3 = - \|f\| w_3 \leqslant - \frac{|f|}{v_3} v_3 \leqslant f \quad .$$

Thus

$$b[f] \leqslant v_3(w) \|f\|$$

as in (20), and we also have (21).

Now extend the functional $L_w[f]$ throughout B so that the inequality (17) is retained, and define the function $G(w,z)$ as in Section 3 by (25), where $S(w,z)$ is a parametrix which vanishes on C and at infinity. Section 3 shows

that $G(w,z)$, as a function of z, satisifies the equation

$$\Delta u - Pu = 0 \quad,$$

and has a fundamental singularity at $z = w$. As a function of the second variable z, G is therefore a fundamental solution over D of $\Delta u = Pu$.

We shall prove that, among all fundamental solutions, G is characterized by the property of being the smallest positive fundamental solution. This replaces the condition $G = 0$ on the boundary C because of the arbitrariness of C.

First, $G(w,z) \geqq 0$ throughout D, just as in Section 3. Second, we shall show that the important symmetry property

$$(33) \qquad G(w,z) = G(z,w)$$

is easily contained in our formulation. Let $S'(w,z)$ be a parametrix with non-negative boundary values and non-negative values at infinity (i.e., let $\lim S'(w,z) \geqq 0$ as $w \to C$ or the point at infinity, if D is unbounded). Then, by (17) and (18),

$$S'(w,z) - S(w,z) \geqq L_w[(\Delta - P)(S'(t,z) - S(t,z))] \quad,$$

or

$$(34) \qquad S'(w,z) - L_w[(\Delta - P)S'(t,z)] \geqq G(w,z) \quad.$$

Now, we have already proved that G, as a function of its second point, is a fundamental solution with singularity at the first point. Thus, $G(z,w)$ (note the location of z and w) is a function of w which is continuous and has continuous first and second derivatives throughout D, except at the point z. Since $G(z,w) \geqq 0$, the function $G(z,w)$ is therefore admissible as a choice for the parametrix $S'(w,z)$ in (34), and we obtain

$$G(z,w) \geqq G(w,z) \quad.$$

This inequality holds for any locations of the points z, w, so that by interchanging z,w the reverse inequality follows and the symmetry property

(33) is established.

The symmetry property (33) shows that $G(w,z)$ is a well-behaved function of its first point w as well, and is a fundamental solution as a function of this first point with singularity at the second point.

Third, $G(w,z)$ is the smallest positive fundamental solution in D with singularity at z. For, if $F(w,z)$ is such a positive fundamental solution, it can be used as the parametrix $S'(w,z)$ in (34), with the desired result

$$F(w,z) \geqslant G(w,z) \qquad .$$

We mention the following additional property of $G(w,z)$: If we exclude a neighborhood Ω of the singular point z, the function $G(w,z)$ is bounded for w outside this neighborhood. If this were not so, use the function $F(S_o)$ constructed in connection with (26), for sufficiently large σ. Define the function $\emptyset(w,z)$ as equal to $G(w,z)$ for w in Ω, and equal to $F(G(w,z))$ for w outside Ω. The function $\emptyset(w,z)$ is an admissible parametrix in (34) with the result

$$\emptyset(w,z) - L_w[(\triangle - P)\emptyset(t,z)] \geqslant G(w,z) \qquad .$$

But $(\triangle - P)\emptyset(t,z) \leqslant 0$ by (26), and (21) now gives

$$\emptyset(w,z) \geqslant G(w,z) \qquad .$$

Selecting a point w outside Ω where $\emptyset(w,z) \neq G(w,z)$, we contradict the inequality $F(S_o) < S_o$, or $\emptyset(w,z) < G(w,z)$, at such a point. Therefore, $G(w,z)$ is bounded outside Ω. (Indeed, we have proved that $G(w,z)$, for w outside Ω, attains its maximum on the boundary of Ω.)

We shall conclude with several remarks concerning the generality of the method. First, the considerations apply equally well to an arbitrary elliptic partial differential equation satisfying a maximum principle. All that is required is the construction of a parametrix, and this is done in a standard

manner using a number of terms of a Neumann expansion analogous to (23). Also, the construction of a particular function analogous to v_3 can be performed under reasonable assumptions concerning the behavior of the coefficients of the differential equation on the boundary C. The fact that the boundary values of G are zero, as in Section 3, can be obtained by using the infinite Neumann series for S_o^*. Of course, if the equation is not self-adjoint, the adjoint equation will enter into the considerations.

Second, the considerations apply to Riemannian manifolds in place of domains in Euclidian space. Integrals of the first, second and third kinds, with or without periods, can be constructed.

REFERENCES

[1] Banach, S., *Théorie des opérations linéaires*, Warsaw, 1932.

[2] Caccioppoli, R., "Sui teoremi d'esistenza di Riemann," *Annali della R. Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche*, Ser. II, Vol. 7 (1938), p. 177-187.

[3] Garabedian, P. R., "The classes L_p and conformal mapping," *Trans. Amer. Math. Soc.*, Vol. 69 (1950), pp. 392-415.

[4] Garabedian, P. R. and Schiffer, M., "On existence theorems of potential theory and conformal mapping," *Ann. Math.*, Vol. 52 (1950), pp. 164-187.

[5] Lax, P. D., "On the existence of Green's function," *Proc. Amer. Math. Soc.*, Vol. 3 (1952), pp. 526-531.

[6] Miranda, C., "Sul principio di Dirichlet per le funzioni armoniche," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, Vol. 3 (1947), pp. 55-59.

Contract Nonr 225(11)
(NR-041-086)

Chief of Naval Research	Office of Naval Research
Office of Naval Research	Department of the Navy
Department of the Navy	Washington 25, D. C.
Washington 25, D. C.	Attn: Code 438 — 2
Attn: Code 432 — 5	

Scientific Section
Office of Naval Research
Department of the Navy
1000 Geary Street
San Francisco 9, California — 2
  Attn: Dr. J. D. Wilkes — 1

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
  Attn: Re3d — 1
      Re6a — 1

Director, Naval Research Lab.
Washington 25, D. C.
  Attn: Technical Information
      Officer — 9

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C. — 1

Chief, Bureau of Ships
Asst. Chief for Research and
  Development
Department of the Navy
Washington 25, D. C. — 1

Dr. Edward Paulson
Head, Statistics Branch
Office of Naval Research
Department of the Navy
Washington 25, D. C.
  Attn: Code 433 — 1

Director, ONR
Branch Office
495 Summer Street
Boston 10, Mass. — 1

Planning Research Division
Deputy Chief of Staff
Comptroller, U. S. A. F.
The Pentagon
Washington 25, D. C. — 1

Director, ONR
Branch Office
844 N. Rush Street
Chicago 11, Illinois — 1

Headquarters, U. S. A. F.
Director of Research and Development
Washington 25, D. C. — 1

Director, ONR
Branch Office
346 Broadway
New York 13, N. Y. — 1

Chairman
Research and Development Board
The Pentagon
Washington 25, D. C. — 1

Director, ONR
Branch Office
1030 E. Green St.
Pasadena 1, California — 1

Asst. Chief of Staff, G-4
  for Research and Development
U. S. Army
Washington 25, D. C. — 1

Office of the Asst. Naval
  Attaché for Research
Naval Attaché
American Embassy
Navy No. 100
Fleet Post Office
New York, N. Y. — 10

Chief of Naval Operations
Operations Evaluation Group-
  OP342E
The Pentagon
Washington 25, D. C. — 1

Underwater Ordnance Dept.
Naval Ordnance Test Station
Pasadena Annex
3202 E. Foothill Boulevard
Pasadena 8, California   1

Office of Ordnance Research
Duke University
2127 Myrtle Drive
Durham, North Carolina   1

Commanding Officer
Ballistic Research Laboratory
Aberdeen Proving Grounds
Aberdeen, Maryland
    Attn: Mr. R. H. Kent   1

Director, David Taylor Model Basin
Carderock, Maryland
    Attn: L. Landweber   1
         P. Eisenberg   1
         Hydromechanics Lab.   1
         Technical Library   1

Director, National Bureau of Standards
Washington, D. C.
    Attn: Natl. Hydraulics Lab.   1

Commanding General
U. S. Proving Grounds
Aberdeen, Maryland   1

Commander
U. S. Naval Ordnance Test Station
Inyokern, China Lake, California   1

N. A. C. A.
1724 F St., N.W.
Washington 25, D. C.
    Attn: Chief, Office of Aero-
         nautical Intelligence   1

Hydrodynamics Laboratory
National Research Laboratory
Ottawa, Canada   1

Director
Penn. State School of Engineering
Ordnance Research Laboratory
State College, Pa.   1

Commanding Officer
Naval Ordnance Laboratory
White Oak, Silver Spring 19, Md.
    Attn: Technical Library   1

Ames Aeronautical Laboratory
Moffett Field
Mountain View, California
    Attn: Technical Librarian   1

Library
California Institute of Technology
Pasadena 4, California   1

Engineering Societies Library
29 W. 39th Street
New York, N. Y.   1

John Crerar Library
Chicago 1, Illinois   1

National Bureau of Standards Library
3rd Floor North West Building
Washington 25, D. C.   1

Library
Massachusetts Inst. of Technology
Cambridge 39, Mass.   1

Louisiana State University Library
University Station
Baton Rouge 3, La.   1

Librarian
G-16, Littauer Center
Harvard University
Cambridge 38, Mass.   1

Prof. L. V. Ahlfors
Dept. of Mathematics
Harvard University
Cambridge 38, Mass.   1

Prof. Peter G. Bergmann
Dept. of Physics
Syracuse University
Syracuse 10, N.Y.   1

Prof. G. Birkhoff
Dept. of Mathematics
Harvard University
Cambridge 38, Mass.   1

Prof. H. Busemann
Dept. of Mathematics
University of So. California
Los Angeles 7, California — 1

Dr. F. H. Clauser
Dept. of Aeronautical Engineering
Johns Hopkins University
Baltimore 18, Maryland — 1

Dr. Milton Clauser
Dept. of Aeronautical Engineering
Purdue University
Lafayette, Indiana — 1

Dr. E. P. Cooper
U. S. Naval Shipyard
U. S. Navy Radiological Defense
  Laboratory
San Francisco, California — 1

Prof. R. Courant
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. — 2

Dr. A. Craya
Dept. of Aeronautical Engineering
Columbia University
New York 27, N. Y. — 1

Dr. K. S. M. Davidson
Experimental Towing Tank
Stevens Inst. of Technology
711 Hudson Street
Hoboken, N. J. — 1

Prof. R. J. Duffin
Dept. of Mathematics
Carnegie Inst. of Technology
Pittsburgh 13, Pa. — 1

Dr. Carl Eckart
Scripps Inst. of Oceanography
La Jolla, California — 1

Prof. A. Erdelyi
Dept. of Mathematics
California Institute of Technology
Pasadena 4, California — 1

Prof. K. O. Friedrichs
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. — 1

Prof. D. Gilbarg
Grad. Inst. for Applied Mathematics
Indiana University
Bloomington, Indiana — 1

Prof. A. T. Ippen
Dept. of Civil & Sanitary Engineering
Mass. Inst. of Technology
Cambridge 39, Mass. — 1

Prof. J. R. Kline
Dept. of Mathematics
University of Pennsylvania
Philadelphia 4, Pa. — 1

Dr. R. T. Knapp
Hydrodynamics Laboratory
California Inst. of Technology
Pasadena 4, California — 1

Prof. P. A. Lagerstrom
Aeronautics Dept.
California Inst. of Technology
Pasadena 4, California — 1

Dr. Martin Lessen
Aeronautical Engineering Dept.
Penn. State College
State College, Pa. — 1

Prof. H. G. Lew
Dept. of Aeronautical Engineering
Penn. State College
State College, Pa. — 1

Prof. C. C. Lin
Dept. of Mathematics
Mass. Inst. of Technology
Cambridge 39, Mass. — 1

Prof. W. T. Martin
Dept. of Mathematics
Mass. Institute of Technology
Cambridge 39, Mass. — 1

Prof. R. von Mises
Dept. of Mathematics
Harvard University
Cambridge 38, Mass. 1

Prof. C. B. Morrey
Dept. of Mathematics
University of California
Berkeley 4, California 1

Prof. Z. Nehari
Dept. of Mathematics
Washington University
St. Louis, Mo. 1

Dr. M. S. Plesset
California Inst. of Technology
Pasadena 4, California 1

Prof. W. Prager
Dept. of Mathematics
Brown University
Providence 12, R. I. 1

Dr. H. Rouse
State Inst. of Hydraulic Research
University of Iowa
Iowa City, Iowa 1

Prof. Charles Saltzer
Case Inst. of Technology
Cleveland, Ohio 1

Prof. A. C. Schaeffer
Dept. of Mathematics
University of Wisconsin
Madison 6, Wisconsin 1

Prof L. I. Schiff
Dept. of Physics
Stanford University
Stanford, California 1

Prof. W. Sears
Grad. School of Aeronautical
   Engineering
Cornell University
Ithaca, N. Y. 1

Prof. Herbert Solomon
Teachers College
Columbia University
New York 27, N. Y. 1

Prof. D. C. Spencer
Fine Hall, Box 708
Princeton, N. J. 1

Prof. J. J. Stoker
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. 1

Dr. V. L. Streeter
Fundamental Mechanics Research
Illinois Inst. of Technology
Chicago 16, Illinois 1

Prof. C. Truesdell
Grad. Inst. for Applied Mathematics
Indiana University
Bloomington, Indiana 1

Prof. Joseph Ullman
Dept. of Mathematics
University of Michigan
Ann Arbor, Michigan 1

Prof. J. K. Vennard
Dept. of Civil Engineering
Stanford University
Stanford, California 1

Prof. S. E. Warschawski
Dept. of Mathematics
University of Minnesota
Minneapolis 14, Minn. 1

Prof. A. Weinstein
Inst. for Fluid Dynamics and
   Applied Mathematics
University of Maryland
College Park, Maryland 1

Prof. A. Zygmund
Dept. of Mathematics
The University of Chicago
Chicago 37, Illinois 1

Los Angeles Engineering Field Office
Air Research and Development Command
5504 Hollywood Boulevard
Los Angeles 28, California
   Attn: Capt. N. E. Nelson 1

Hydrodynamics Laboratory
California Inst. of Technology
1201 East California St.
Pasadena 4, California
  Attn: Executive Committee   1

Navy Department
Naval Ordnance Test Station
Underwater Ordnance Department
Pasadena, California
  Attn: Dr. G. V. Schliestett
       Code P8001   1

Mr. R. T. Jones
Ames Aeronautical Laboratory
Moffett Field
Mountain View, California   1

Prof. P. C. Rosenbloom
Dept. of Mathematics
University of Minnesota
Minneapolis 14, Minn.   1

Mr. J. D. Pierson
Glenn L. Martin Company
Middle River
Baltimore, Maryland   1

Mr. E. G. Straut
Consolidated-Vultee Aircraft
  Corporation
Hydrodynamics Research Lab.
San Diego, California   1

Additional copies for project
leader and assistants and re-
serve for future requirements   50