UNCLASSIFIED

ANALYSIS OF PLATES CONTINUOUS OVER

FLEXIBLE BEAMS

By

J. G. Sutherland, L. E. Goodman, and N. M. Newmark

Technical Report

to

Office of Naval Research

Contract N6ori-71, Task Order 6

Project NR-064-183

January 1953

University of Illinois

Urbana, Illinois

CONTENTS

CONTENTS (Cont'd)

LIST OF FIGURES

LIST OF TABLES

SYNOPSIS

The Ritz energy method is used to obtain solutions for an interior panel of a plate or slab which is continuous over a rectangular grid of flexible beams supported by columns at their intersections. It is assumed that the plate is uniformly loaded over its whole area, and that parallel beams are of equal stiffness and are spaced regularly. The plate deflection is represented by an infinite series of polynomial functions derived by W. J. Duncan for beams clamped at both ends; these functions lead to relatively accurate results with a reasonable amount of numerical work. Deflections, moments and average moments for cases covering a wide range of beam-plate rigidities and width-length ratios are presented in tabular and graphical form.

ANALYSIS OF PLATES CONTINUOUS OVER FLEXIBLE BEAMS

I. INTRODUCTION

1. Object and Scope of Investigation

Some of the most important special problems of Elastic Theory are those involving a medium-thick rectangular plate supported along its boundaries and bent by loads applied to one of its faces. In many cases, particularly those in which two opposite edges of the plate are simply supported, theoretically complete solutions of the problems have been obtained. However, for many other cases exact solutions have not been found; the most notable example is the case of a uniformly loaded plate fixed against displacement and rotation at its edges. In studying such problems recourse has been had to experimental methods[1] or to approximate numerical methods of solution. In the present work, an approximate numerical solution is obtained for the technically important case of a plate supported by elastic beams.

The problem considered is that of an interior panel of a plate continuous over a rectangular grid of flexible beams which are supported at their intersections by columns. It is assumed that parallel beams are of equal stiffness and are uniformly spaced and that the system has a large number of panels in both directions. The plate is considered to be acted upon by a lateral load which is uniformly distributed over its whole area. Under these conditions, the bending in all interior panels

---

[1] For instance for the clamped plate problem see references (1) and (2) of the Bibliography.

may be assumed to be identical, consequently only one interior panel need be considered. Apart from their intrinsic importance, results of the analysis for this case of uniform loading of the entire plate can be used to obtain moments, shears and deflections for several other loadings by superposition with known solutions.

The problem described has been solved by the use of a method which embodies the customary assumptions for the flexure of medium-thick plates. In brief these are:

1. The plate is of uniform thickness and is homogeneous and isotropic, at least so far as strains and deformations parallel to the bounding surfaces are concerned.
2. Hooke's law applies to the horizontal strains, which are assumed to vary linearly through the thickness. This implies that the deformation due to shear is negligible compared with that due to the direct bending stresses.
3. Compressive stresses between planes parallel to the bounding surfaces are small compared to the flexural stresses and may be neglected.
4. The resultant of the direct stresses acting on any cross-section of the plate is assumed to be a pure couple. It can be shown (3)* that this is equivalent to requiring that the deflections of the plate (measured from a developable surface) be small in comparison with the plate thickness. These assumptions follow very closely those which are customary in the ordinary theory of flexure of beams.

---

* Numbers in parentheses, unless otherwise identified, refer to the Bibliography.

In addition to these limitations the problem has been simplified by assuming that the widths of the beams and the cross-sectional dimensions of columns are small compared with the panel dimensions. This simplification will not have much effect on the deflections and moments near the centers of the panels but, in some cases, will lead to excessively high moments near the columns.

The most common structure similar to the plate-beam system described is the two-way concrete slab. This type of structure differs from the idealized problem in that reinforced concrete is neither homogeneous nor isotropic nor perfectly elastic. Furthermore the supporting beams, when of concrete, are usually built monolithically with the slab. The effect of casting the slab monolithically with the supporting beams can be approximately accounted for, in the usual manner, by assuming that part of the slab is effective as a flange on the beam. In general, the ordinary isotropic theory of flexure of plates, together with experimental data, has provided a satisfactory basis for design of slab elements of present day reinforced concrete structures.

Two-way concrete slabs are usually designed without accounting for the flexibility of the supporting beams, that is, the beam deflections are usually assumed to be small in comparison with the deflections of the central portions of the loaded panels. The present work gives the theoretical distribution of moments in an interior panel for supporting beams of any stiffness. The solutions include, as limiting cases of the general problem, the known results for plates fixed against rotation and deflection at their supports, and for plates supported by a rectangular array of columns without connecting beams (flat slabs).

2. Acknowledgment

This investigation has been in part supported by the Mechanics Branch, Office of Naval Research, Department of the Navy, in connection with a research program entitled "Numerical and Approximate Methods of Stress Analysis," in progress in the Structural Research Laboratory, of the Department of Civil Engineering, and in the Engineering Experiment Station of the University of Illinois, under Contract N6ori - 71, Task order 6. The material in this report has been drawn from a dissertation of the same title submitted to the Graduate Faculty of the University of Illinois by Mr. J. G. Sutherland, January 1953.

II. METHOD OF SOLUTION

3. Notation

The following nomenclature is used. Any consistent set of units can be employed; however, for reference, the units are given in the engineering inch-pound-second system.

a = length of panel, in the x-direction (in)

b = width of panel, in the y-direction (in)

$c = b/a$ = ratio of panel width to panel length (nondimensional)

m,n = integers ranging from one to infinity

q = intensity of uniform load on the plate (lb/in^2)

r,s = integers; particular values of m and n respectively

w = lateral deflection of the plate (in)

x,y = Cartesian coordinates, directed along the beams (in)

x_1,y_1 = Cartesian coordinates with their origin at the center of the panel (in)

D = flexural rigidity of the plate (in lb)

E_a,E_b = moduli of elasticity of the beams of span length a and b, respectively (lb/in^2)

I_a,I_b = second moments of area of the cross-sections of the beams of span length a and b, respectively (in^4)

M_x,M_y = bending moments per unit length of sections of a plate perpendicular to x- and y-axes, respectively; the superscript "o" is used to indicate that Poisson's ratio is zero (lb)

$M_1,M_2..M_8$ = average bending moments per unit length on various sections of the plate; the superscript "o" is used to indicate that Poisson's ratio is zero (lb)

M_a,M_b = moments in the beams of span length a and b, respectively (in lb)

$N_1, N_2 \ldots N_8$ = moment coefficients for computing average moments for any value of Poisson's ratio (lb)

S_m = a set of symmetric functions (nondimensional)

U = strain energy due to bending (in lb)

V = total potential energy of one panel (in lb)

W = potential energy of the load, q, for one panel of the system (in lb)

$\alpha_m = m\pi c/2$ = term appearing in the exact solution for a plate supported by columns only (nondimensional)

$\alpha_{mn}, \beta_m, \gamma_n$ = coefficients defining the deflected shape of the plate (nondimensional)

$\lambda_a = E_a I_a / aD$ = ratio of beam rigidity to plate rigidity for each beam parallel to the x-axis (nondimensional)

$\lambda_b = E_b I_b / bD$ = ratio of beam rigidity to plate rigidity for each beam parallel to the y-axis (nondimensional)

μ = Poisson's ratio

$\xi = x/a$ = dimensionless coordinate in the x-direction

$\eta = y/b$ = dimensionless coordinate in the y-direction

4. Résumé of the Ritz Energy Method as Applied to Plate Equilibrium Problems

By suitably restricting the deflected shape of a continuous elastic system, it can be reduced to a similar but simpler system having a finite number of degrees of freedom. This idea was first used by Lord Rayleigh(4) who applied it in approximate calculations of the fundamental frequencies of vibration of complicated structures. In these applications the restrictions are equivalent to constraints acting on the structure. W. Ritz(5) later developed the same idea in a more general form; he discussed in detail the solution of the problem of the rectangular plate clamped at its boundaries, the solution of boundary-value problems governed by Laplace's equation, the solution of second-order differential equations with variable coefficients, and the solution of the problem of the vibrating string. The general method is known as the Rayleigh-Ritz energy method; in applications to plate equilibrium problems it is usually referred to as the Ritz method. The Rayleigh-Ritz method has been one of the most useful approximate numerical methods in the field of structural analysis; it has yielded satisfactory solutions for numerous problems in equilibrium, buckling and vibration.

The analytical equilibrium problem for plates is that of finding a deflection function $w(x,y)$ which satisfies the equation $\nabla^4 w = q/D$ within the boundary of the plate and which also satisfies certain conditions on the boundary. Ritz observed that the mathematical problem is the same as that of rendering stationary the integral expression for the total potential energy of the system. With the ordinary theory of flexure for plates and beams, the total potential energy, V, can be expressed in terms of the deflection function, w, and its second derivatives. The problem is then to find the function $w(x,y)$ which minimizes V and satisfies the prescribed boundary conditions for the system under consideration. Direct application of the calculus of variations to minimize V leads to the analytical problem; that is, to the governing differential equation together with the so-called "natural" boundary conditions. Instead of following this procedure, Ritz assumed the deflection, $w(x,y)$, as an infinite series of "admissible" functions and adjusted the coefficients in this series so as to minimize the total potential energy, V. To be admissible, a function must satisfy the prescribed boundary conditions of the problem. Although it is not required, it is desirable to choose functions which also satisfy the natural boundary conditions. In order to be assured that the assumed series of functions can accurately approximate the actual deflection, it is also necessary that the functions form a complete set. If now, the assumed deflection function is substituted into the integral equation for the total potential energy, this energy is obtained as a quadratic function of the unknown coefficients. The variational problem is thus reduced to an ordinary minimum problem. On equating to zero the partial derivative of V taken with respect to each coefficient, there is obtained an infinite system of linear simultaneous equations which define the values of these coefficients. An approximation to the deflection function is obtained by solving a limited number of these equations.

5. Potential Energy of the Loaded Plate-Beam System

Consider a plate carrying a uniformly distributed load and supported on a rectangular grid of beams with columns at their intersections as was previously described. The flexural strain energy in one panel of the plate is [3]

$$U_{pl.} = \iint_A \left\{ \frac{D}{2}\left[w_{xx} + w_{yy}\right]^2 - D(1-\mu)\left[w_{xx}\,w_{yy} - w_{xy}^2\right] \right\} dA \qquad (1)$$

where A is the region within the panel boundaries. The potential energy of the uniform load, q, on the plate is

$$W = -\iint_A q\,w\,dA \quad , \qquad (2)$$

and the flexural strain energy of a beam of length L is

$$U_{bm.} = \frac{EI}{2}\int_L (w_{xx})^2\, dx \quad . \tag{3}$$

Consequently, for coordinate axes directed along the beams, the total potential energy of one panel of the loaded plate is

$$\begin{aligned} V = &\int_0^b\int_0^a \left\{ \frac{D}{2}\left[w_{xx}+w_{yy}\right]^2 - D(1-\mu)\left[w_{xx}w_{yy} - w_{xy}^2\right]\right\} dx\,dy \\ &+ \int_0^b\int_0^a wq\; dx\,dy + \frac{E_aI_a}{2}\int_0^a \left[w_{xx}(x,0)\right]^2 dx + \frac{E_bI_b}{2}\int_0^b \left[w_{yy}(0,y)\right]^2 dy \quad . \end{aligned} \tag{4}$$

The deflection is zero at the corner of each panel and, from symmetry, the slope of the deflection surface at an interior panel boundary in the direction normal to the boundary is also zero. Therefore, the problem reduces to that of finding a symmetrical function $w(x,y)$ which minimizes V, given by equation (4), and satisfies the imposed boundary conditions,

$$\begin{aligned} w(0,0) &= w(0,b) = 0 \\ w_x(0,y) &= w_y(x,0) = 0 \end{aligned} \tag{5}$$

If the methods of the calculus of variations are applied to equation (4), one finds for the first variation of the potential energy

$$\begin{aligned} \delta V = &\int_0^a \left[E_aI_a\, w_{xxxx}(x,0) - 2D\, w_{yyy}(x,0)\right] \delta w(x,0)\; dx \\ &+ \int_0^b \left[E_bI_b\, w_{yyyy}(0,y) - 2D\, w_{xxx}(0,y)\right] \delta w(0,y)\; dy \\ &+ \int_0^b\int_0^a \left[D\nabla^4 w - q\right] \delta w(x,y)\; dx\,dy \qquad = 0 \quad . \end{aligned} \tag{6}$$

Since $\delta w(x,y)$ is an arbitrary function which need satisfy only the imposed boundary conditions, each bracketed quantity must vanish identically. Rendering V stationary thus implies satisfaction of the governing differential equation and of the natural boundary conditions. On applying the variational methods to the second term in the integral for the strain energy of the plate, it is found that the boundary conditions alone are sufficient to insure that the variation of this integral vanish. It thus follows that this term is invariant and can be dropped from equation (4). H. Langhaar[6] has proven that this term is invariant when, and only when, the boundary is a geodesic of the deformed surface. L. E. Goodman has observed [6] that for a uniformly loaded infinite plate supported on either (a) a rectangular gridwork of flexible beams supported at their intersections by columns, the beams in either direction being of uniform stiffness, or, (b) a gridwork of flexible beams in the shape of a regular polygon also supported at the corners by columns, the beams form a boundary along which the deflection of the deformed surface is a geodesic.

III. W. J. DUNCAN'S S-FUNCTIONS

6. Definitions and Boundary Properties of the S-Functions

A set of simple polynomial functions with properties very suitable to the present problem has been constructed by W. J. Duncan[7]. Since only the symmetric functions have been used in the present work, they will be the only ones discussed. The general symmetric function can be defined as

$$S_m(\xi) = \frac{\sqrt{4m+1}}{2m!} \cdot \frac{d^{2m-2}}{d\xi^{2m-2}}\left[\xi^{2m}(1-\xi)^{2m}\right] , \qquad (7)$$

or,

$$S_m(\xi) = \frac{(4m-1)P_{2m+2}(2\xi-1) - 2(4m+1)P_{2m}(2\xi-1) + (4m+3)P_{2m-2}(2\xi-1)}{4(4m-1)(4m+3)\sqrt{4m+1}} , \qquad (8)$$

in which P_m is the Legendre polynomial of the m-th degree. The corresponding sequence of anti-symmetric functions can be obtained from the sequence for $S_m(\xi)$ by replacing m with (m+1/2). The polynomial forms of the first three symmetric functions are

$$\begin{aligned} S_1(\xi) &= \frac{\sqrt{5}}{2}(\xi^2 - 2\xi^3 + \xi^4) \\ S_2(\xi) &= \frac{1}{2}(3\xi^2 - 20\xi^3 + 45\xi^4 - 42\xi^5 + 14\xi^6) \qquad (9) \\ S_3(\xi) &= \frac{\sqrt{13}}{2}(\xi^2 - 14\xi^3 + 70\xi^4 - 168\xi^5 + 210\xi^6 - 132\xi^7 + 33\xi^8). \end{aligned}$$

The first and second differential coefficients of the functions are

$$S'_m(\xi) = \frac{P_{2m+1}(2\xi-1) - P_{2m-1}(2\xi-1)}{2\sqrt{4m+1}} , \qquad (10)$$

and

$$S_m''(\xi) = \sqrt{4m+1}\ P_{2m}(2\xi-1) \quad . \tag{11}$$

These functions were derived for application to doubly built-in beams, consequently they satisfy the following boundary conditions,

$$\begin{aligned} S_m(0) &= S_m(1) = 0 \\ S_m'(0) &= S_m'(1) = 0 \ . \end{aligned} \tag{12}$$

Numerical values of the first four functions and their second derivatives are presented in Table 1. This table was prepared with the aid of the British Association Tables of Legendre Polynomials (8).

7. Integral Properties of the S-Functions

In applying sequences of functions to engineering problems, it is necessary to evaluate various integrals which contain the functions and their derivatives. The following integrals comprise those which occurred in solving the problem under discussion. These integrals have been evaluated using equations (8) and (11) and well known integral properties of the Legendre polynomial functions (9).

$$\begin{aligned} \int_0^1 S_m(\xi)\, d\xi &= 0 \qquad && m \neq 1 \\ &= \frac{\sqrt{5}}{60} && m = 1 \end{aligned} \tag{13}$$

$$\int_0^1 S_m''(\xi)\, d\xi = 0 \qquad \text{all } m \tag{14}$$

$$\begin{aligned} \int_0^1 S_m''(\xi)\, S_n''(\xi)\, d\xi &= 0 \qquad && n \neq m \\ &= 1 && n = m \end{aligned} \tag{15}$$

$$\int_0^1 S_m(\xi)\, S_n''(\xi)\, d\xi = 0 \qquad n \neq m\,,\ n \neq m \pm 1$$

$$= \frac{1}{4(4m+3)\sqrt{(4m+1)(4m+5)}} \qquad n = m+1 \tag{16}$$

$$= \frac{-1}{2(4m-1)(4m+3)} \qquad n = m$$

$$= \frac{1}{4(4m-1)\sqrt{(4m+1)(4m-3)}} \qquad n = m-1$$

$$\int_0^1 S_m(\xi)\, S_n(\xi)\, d\xi = 0 \qquad n \neq m\,,\ n \neq m \pm 1\,,\ n = m \pm 2$$

$$= \frac{3}{8(4m-1)(4m-3)(4m+3)(4m+5)} \qquad n = m$$

$$= \frac{-1}{4(4m-1)(4m+3)(4m-5)\sqrt{(4m+1)(4m-3)}} \qquad n = m-1 \tag{17}$$

$$= \frac{1}{16(4m-1)(4m-3)(4m-5)\sqrt{(4m+1)(4m-7)}} \qquad n = m-2$$

$$\int_0^\xi S_m(\xi)\, d\xi = \frac{\sqrt{5}}{2}\,\xi^3\left(\frac{1}{3} - \frac{\xi}{2} + \frac{\xi^2}{5}\right) \qquad m = 1$$

$$= \frac{1}{8(4m-1)(4m-3)(4m+3)(4m+5)\sqrt{4m+1}}\Big[(4m-1)(4m-3)P_{2m+3}(2\xi-1)$$

$$-3(4m-3)(4m+3)P_{2m+1}(2\xi-1) + 3(4m-1)(4m+5)P_{2m-1}(2\xi-1) \tag{18}$$

$$-(4m+3)(4m+5)P_{2m-3}(2\xi-1)\Big]\,, \qquad m > 1$$

$$\int_0^\xi S_m''(\xi)\, d\xi = \frac{P_{2m+1}(2\xi-1) - P_{2m-1}(2\xi-1)}{2\sqrt{4m+1}}\,, \quad \text{all } m \tag{19}$$

Numerical values for some definite integrals of the forms of equations (16), (17), (18) and (19) are given in Table 2.

IV. SOLUTION OF THE PROBLEM USING THE S-FUNCTIONS

8. The Equations for the Deflection Function Coefficients

It is desired to assume the deflection function w(x,y) as a sequence of symmetrical functions which are capable of approximating the deflected shape of the plate as closely as may be desired. Each function of this sequence must satisfy the prescribed boundary conditions given by equation (5). These conditions are fulfilled by assuming a deflection function in the following form,

$$\frac{w}{qa^4/D} = \sum_1^\infty\sum_1^\infty \alpha_{mn} S_m(\xi) S_n(\eta) + \sum_1^\infty \beta_m S_m(\xi) + \sum_1^\infty \gamma_n S_n(\eta) \ . \qquad (20)$$

The coefficients, α_{mn}, β_m and γ_n , are to be determined such that the total potential energy V will be a minimum. Omitting the second term from the first integral of equation (4) and writing the equation in dimensionless form, the expression for V becomes,

$$V = a^2 c\int_0^1\int_0^1 \left\{\frac{D}{2a^4}\left[w_{\xi\xi} + w_{\eta\eta}/c^2\right]^2 - qw\right\} d\xi\, d\eta$$
$$+ \frac{\lambda_a D}{2a^2}\int_0^1 \left[w_\xi(\xi,0)\right]^2 d\xi + \frac{\lambda_b D}{2a^2c^2}\int_0^1\left[w_{\eta\eta}(0,\eta)\right]^2 d\eta \ . \qquad (21)$$

The energy V is obtained as a quadratic function of the coefficients α_{mn}, β_m and γ_n by substituting equation (20) into equation (21), that is,

$$\frac{V}{q^2a^6c/D} = \frac{1}{2}\int_0^1\int_0^1\Big\{\sum_1^\infty\sum_1^\infty \alpha_{mn}\left[S_m''(\xi)S_n(\eta) + \frac{1}{c^2}S_m(\xi)S_n''(\eta)\right]$$
$$+ \sum_1^\infty \beta_m S_m''(\xi) + \frac{1}{c^2}\sum_1^\infty \gamma_n S_n''(\eta)\Big\}^2 d\xi\, d\eta$$
$$- \int_0^1\int_0^1\left\{\sum_1^\infty\sum_1^\infty \alpha_{mn} S_m(\xi)S_n(\eta) + \sum_1^\infty \beta_m S_m(\xi) + \sum_1^\infty \gamma_n S_n(\eta)\right\} d\xi\, d\eta \qquad (22)$$
$$+ \frac{\lambda_a}{2c}\int_0^1\left\{\sum_1^\infty \beta_m S_m'(\xi)\right\}^2 + \frac{\lambda_b}{2c^3}\int_0^1\left\{\sum_1^\infty \gamma_n S_n''(\eta)\right\}^2 d\eta \ .$$

Let r be any value of m and s be any value of n. On the assumption that the order of summations and integrations can be interchanged and using the properties of the S-functions given by equations (13), (14) and (15), then,

$$\frac{V}{q^2 a^6 c/D} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\alpha_{mn}\alpha_{rs}}{c^2}\int_0^1 S_m'' S_r \, d\xi \int_0^1 S_n S_s'' \, d\eta$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{\alpha_{mn}\alpha_{ms}}{2}\int_0^1 S_n S_s \, d\eta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} \frac{\alpha_{mn}\alpha_{rn}}{2c^4}\int_0^1 S_m S_r \, d\xi \tag{23}$$
$$+ \frac{\sqrt{5}}{60}\sum_{m=1}^{\infty}\alpha_{m1}\beta_m + \frac{\sqrt{5}}{60c^4}\sum_{n=1}^{\infty}\alpha_{1n}\gamma_n + \frac{1}{2}\left(1+\frac{\lambda_a}{c}\right)\sum_{m=1}^{\infty}\beta_m^2$$
$$+ \frac{(1+c\lambda_b)}{2c^4}\sum_{n=1}^{\infty}\gamma_n^2 - \frac{1}{720}\alpha_{11} - \frac{\sqrt{5}}{60}(\beta_1+\gamma_1) \quad .$$

The coefficients are to be chosen such that V will be a minimum; thus for the coefficients β_m and γ_n,

$$\frac{\partial V}{\partial \beta_m} = \frac{\partial V}{\partial \gamma_n} = 0 \quad , \tag{24}$$

which lead to the relationships,

$$\beta_1 = \frac{\sqrt{5}}{60}\frac{(1-\alpha_{11})}{(1+\lambda_a/c)}$$
$$\beta_m = -\frac{\sqrt{5}}{60}\frac{\alpha_{m1}}{(1+\lambda_a/c)} \qquad m \neq 1 \tag{25}$$

$$\gamma_1 = \frac{\sqrt{5}}{60}\frac{(c^4-\alpha_{11})}{(1+c\lambda_b)}$$
$$\gamma_n = -\frac{\sqrt{5}}{60}\frac{\alpha_{1n}}{(1+c\lambda_b)} \qquad n \neq 1$$

On substituting these values of β_m and γ_n into equation (23), the potential energy is obtained as a function of the coefficients α_{mn} alone. Making this substitution and dropping a constant term, one obtains,

$$\frac{V}{q^2a^6c/D} = \sum_1^{\infty}\sum_1^{\infty}\sum_1^{\infty}\sum_1^{\infty} \frac{\alpha_{mn}\alpha_{rs}}{c^2}\int_0^1 S_m'' S_r \, d\xi \int_0^1 S_n S_s'' \, d\eta$$

$$+ \sum_1^{\infty}\sum_1^{\infty}\sum_1^{\infty} \frac{\alpha_{mn}\alpha_{ms}}{2}\int_0^1 S_n S_s \, d\eta + \sum_1^{\infty}\sum_1^{\infty}\sum_1^{\infty} \frac{\alpha_{mn}\alpha_{rn}}{2c^4}\int_0^1 S_m S_r \, d\xi \qquad (26)$$

$$- \frac{1}{1440(1+\lambda_a/c)}\sum_2^{\infty}\alpha_{m1}^2 - \frac{1}{1440c^4(1+c\lambda_b)}\sum_2^{\infty}\alpha_{1n}^2$$

$$- \frac{\alpha_{11}^2}{1440}\left[\frac{1}{(1+\lambda_a/c)} + \frac{1}{c^4(1+c\lambda_b)}\right] + \frac{\alpha_{11}}{720}\frac{(1-\lambda_a\lambda_b)}{(1+\lambda_a/c)(1+c\lambda_b)} .$$

Using this equation and the numerical values of the integrals given in Table 2, the matrix of Table 3 for the total potential energy has been obtained. The total energy is equal to the limiting sum of the entries in the table multiplied by the coefficients at the heads of their respective rows and columns. It should be noted that the matrix is symmetrical about its principal diagonal.

The energy V is minimized by equating the partial derivative with respect to each coefficient α_{mn} to zero. Consequently, the linear simultaneous equations defining these coefficients are obtained by equating each row of Table 3 to zero and cancelling the factor "2" at the top of the column of constants. The properties of the S-functions given by equations (13), (14) and (15) have resulted in the simple relationships of equations (25) between the coefficients β_m and γ_n and the coefficients α_{mn}, and thereby have effected a considerable reduction in the number of simultaneous equations to be solved.

9. **Moments and Average Moments in the Plate and Moments in the Beams.**

Moments in the plate and beams are found from the deflection function w in the usual manner. The plate moments are

$$M_x = -\frac{D}{a^2}\left[\frac{\partial^2 w}{\partial \xi^2} + \frac{\mu}{c^2}\frac{\partial^2 w}{\partial \eta^2}\right] = M_x^o + \mu M_y^o$$
$$M_y = -\frac{D}{a^2 c^2}\left[\frac{\partial^2 w}{\partial \eta^2} + c^2 \mu \frac{\partial^2 w}{\partial \xi^2}\right] = M_y^o + \mu M_x^o \qquad (27)$$

in which the "zero" superscripts are used to indicate that the moment is given for a value of Poisson's ratio equal to zero. Equations (27) are very useful for computing moments at a point for any desirable value of Poisson's ratio. If Poisson's ratio is taken as zero, the moments are

$$M_x^o = -qa^2 \sum_{1}^{\infty}{}_m \sum_{1}^{\infty}{}_n \alpha_{mn} S_m''(\xi) S_n(\eta) - qa^2 \sum_{1}^{\infty}{}_m \beta_m S_m''(\xi)$$
$$M_y^o = -\frac{qa^2}{c^2} \sum_{1}^{\infty}{}_m \sum_{1}^{\infty}{}_n \alpha_{mn} S_m(\xi) S_n''(\eta) - \frac{qa^2}{c^2} \sum_{1}^{\infty}{}_n \gamma_n S_n''(\eta) \qquad (28)$$

In addition to a knowledge of moments at particular points in the plate, it is often desirable to know the maximum average moments in what are commonly called the "middle" and "column" strips of the panel. The term "column strip" is here used to designate the portion of the panel over a beam and its supporting column and having a width equal to half of the panel width; the term "middle strip" is used to designate the middle portion of the panel, one-half the width of the panel. The average moments on the center and end sections of the various column and middle strips are defined as follows:

$$M_1 = M_1^\circ + \mu N_1 = 4\int_0^{1/4} M_x^\circ(\tfrac{1}{2},\eta)\,d\eta + 4\mu\int_0^{1/4} M_y^\circ(\tfrac{1}{2},\eta)\,d\eta$$

$$M_2 = M_2^\circ + \mu N_2 = 4\int_0^{1/4} M_x^\circ(1,\eta)\,d\eta + 4\mu\int_0^{1/4} M_y^\circ(1,\eta)\,d\eta$$

$$M_3 = M_3^\circ + \mu N_3 = 2\int_{1/4}^{3/4} M_x^\circ(\tfrac{1}{2},\eta)\,d\eta + 2\mu\int_{1/4}^{3/4} M_y^\circ(\tfrac{1}{2},\eta)\,d\eta$$

$$M_4 = M_4^\circ + \mu N_4 = 2\int_{1/4}^{3/4} M_x^\circ(1,\eta)\,d\eta + 2\mu\int_{1/4}^{3/4} M_y^\circ(1,\eta)\,d\eta$$

(28)

$$M_5 = M_5^\circ + \mu N_5 = 4\int_0^{1/4} M_y^\circ(\xi,\tfrac{1}{2})\,d\xi + 4\mu\int_0^{1/4} M_x^\circ(\xi,\tfrac{1}{2})\,d\xi$$

$$M_6 = M_6^\circ + \mu N_6 = 4\int_0^{1/4} M_y^\circ(\xi,1)\,d\xi + 4\mu\int_0^{1/4} M_x^\circ(\xi,1)\,d\xi$$

$$M_7 = M_7^\circ + \mu N_7 = 2\int_{1/4}^{3/4} M_y^\circ(\xi,\tfrac{1}{2})\,d\xi + 2\mu\int_{1/4}^{3/4} M_x^\circ(\xi,\tfrac{1}{2})\,d\xi$$

$$M_8 = M_8^\circ + \mu N_8 = 2\int_{1/4}^{3/4} M_y^\circ(\xi,1)\,d\xi + 2\mu\int_{1/4}^{3/4} M_x^\circ(\xi,1)\,d\xi$$

The numerical values of the integrals encountered after substituting equations (27) into equations (28) are given in Table 2. Since the deflection surface is symmetrical and has zero slope at the panel boundaries, the terms (N_1+N_3), (N_2+N_4), (N_5+N_7) and (N_6+N_8) must each be zero. The average moments on a full central or edge section are therefore independent of Poisson's ratio. The locations of the lines along which the moments are averaged are shown in Figure 1.

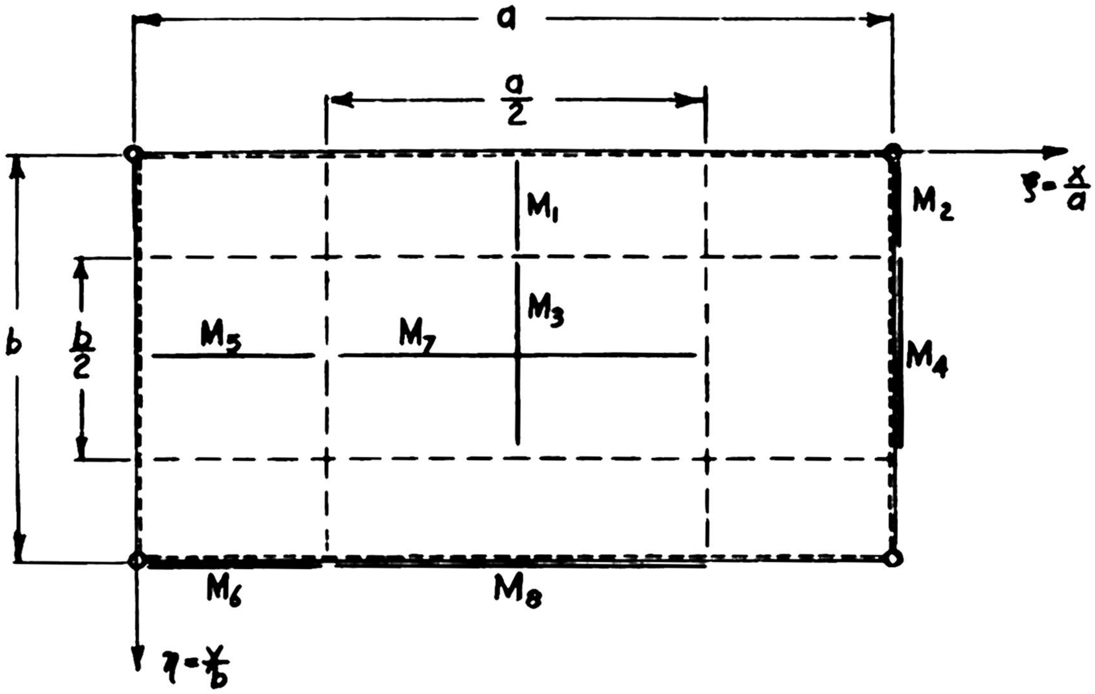

FIG. 1 LOCATION OF AVERAGE MOMENTS

The moments in the beams can be determined from the coefficients β_m and γ_n or they can be obtained directly from the plate moments:

$$M_a = -qa^3 \lambda_a \sum_1^{\infty} \beta_m S_m''(\xi) = a \lambda_a M_x^o(\xi, 0)$$

$$M_b = -\frac{qa^3 \lambda_b}{c} \sum_1^{\infty} \gamma_n S_n''(\eta) = a c \lambda_b M_y^o(0, \eta) \quad . \qquad (29)$$

10. Reduction to an Exact Solution when $\lambda_a \lambda_b = 1$.

It is evident from Table 3 that the coefficients α_{mn} will all be zero if

$$\lambda_a \lambda_b = 1 \quad , \qquad (30)$$

consequently, it follows from equations (25) that the only non-zero coefficients are β_1 and γ_1. On substituting the values of the coefficients into equation (20), the deflection is found to be

$$w = \frac{\sqrt{5}}{60}\frac{qa^4}{D}\left[\frac{S_1(\xi)}{1+\lambda_a/c} + \frac{c^4 S_1(\eta)}{1+c\lambda_b}\right] , \tag{31}$$

or, replacing S_1 by its polynomial expansion given in equations (9),

$$w = \frac{qa^4}{24D}\left[\frac{\xi^2(1-\xi)^2}{1+\lambda_a/c} + \frac{c^4\eta^2(1-\eta)^2}{1+c\lambda_b}\right] . \tag{32}$$

For Poisson's ratio equal to zero, the moments in the plate are

$$M_x^o = \frac{qa^2}{12(1+\lambda_a/c)}(6\xi - 6\xi^2 - 1)$$

and,

$$M_y^o = \frac{qa^2c^2}{12(1+c\lambda_b)}(6\eta - 6\eta^2 - 1) . \tag{33}$$

It is interesting that, for this case, the moment M_x^o is equal to the moment in a clamped-ended beam of unit width and of length equal to a, under a uniformly distributed load of $\frac{q}{(1+\lambda_a/c)}$ per unit length; M_y^o is equal to the moment in a similar beam but of length equal to b, and under a load of $\frac{q}{(1+c\lambda_b)}$ per unit of length. Therefore, plates supported on beams, with relative rigidities such that $\lambda_a\lambda_b = 1$, function much like independent prismatic beam elements; the load is distributed in the ratios $1:(1+c\lambda_b)$ and $1:(1+\lambda_a/c)$ to the short and long elements respectively. There are no twisting moments M_{xy} in such plates.

The shearing forces in the plate are

$$Q_x = \frac{qa}{2}\left[\frac{1-2\xi}{1+\lambda_a/c}\right]$$

and, (34)

$$\frac{qac}{2}\left[\frac{1-2\eta}{1+c\lambda_b}\right]$$

The supporting beams parallel to the x-axis (long beams) are therefore subjected to a uniform load of $\frac{qa\lambda_a}{1+\lambda_a/c}$ per unit of length while the load on those parallel to the y-axis is $\frac{qac\lambda_b}{1+c\lambda_b}$ per unit of length.

V. TABLES AND GRAPHS OF NUMERICAL SOLUTIONS

11. Tables of Results

Numerical solutions have been made for about forty cases, covering three different panel side-ratios, $c = 1.0$, $c = 0.8$, and $c = 0.5$, and embracing a wide range of beam-plate rigidities, λ_a and λ_b. In the case of square panels, the rigidities, λ_a and λ_b, were chosen so as to represent all combinations of values from zero to infinity. For the other panel side-ratios, it was assumed either that all beams were of equal rigidity (i.e. $\lambda_b = \lambda_a$) or that all beams had equal moments of inertia (i.e. $I_b = I_a$ or $\lambda_b = \lambda_a/c$). A tabulated summary of the solutions is presented in Table 4.

In general, the results presented in Table 4 are those obtained using nine of the coefficients α_{mn}; that is, m and n were taken over the range from 1 to 3. Each case thus involved the solution of a set of nine simultaneous equations. In addition, each case was solved using only $m = n = 1$ and again using $m = 1,2$ and $n = 1,2$. A sequence of approximate solutions was thus obtained; the rate of convergence of this sequence could be estimated by inspection. The accuracy of the results presented in Table 4 is indicated by the underlining of digits which differ by more than five units in comparison with a solution using fewer terms in the series. For example, solutions obtained using nine coefficients α_{mn} were compared to the solutions obtained using only four such coefficients. The solutions for square symmetrical panels were obtained using $m = 1,2,3,4$ and $n = 1,2,3,4$; these cases required the solution of sets of ten simultaneous equations. The results for the limiting case of a plate supported on columns without connecting beams were obtained from the exact infinite series solution which is summarized in the Appendix.

12. Graphs of Results

The results are presented in graphical form in Figures 2 to 42. In order to plot the relative beam rigidity, λ_a, over its full range from zero to infinity, a modified abscissa scale has been used. The abscissa scales are divided in the ratio $\lambda_a/(2+\lambda_a)$, therefore $\lambda_a = 2.0$ falls at the center of the scale, $\lambda_a = 1.0$ falls at the 1/3 point and so on. Figures 2 to 15 give deflections, moments and average moments for a square panel for all combinations of beam-plate rigidities. Figures 16 to 42 give these deflections and moments for panels with side ratios of c = 1.0, 0.8 and 0.5, for cases in which $\lambda_b = \lambda_a$ or $I_b = I_a$.

VI. ACCURACY OF THE SOLUTIONS

13. Convergence of the Sequence of Approximate Solutions

One of the most undesirable features of the Ritz procedure is that it does not include a method or general principal for determining rigorously the accuracy of the results (10,11). However, some indication of the accuracy can be obtained by inspecting the convergence of solutions in which increasingly larger numbers of terms of the approximating series were used. As has been previously described, a sequence of three such solutions was generally obtained for each case considered in the present work. Because of the simplifications implied by symmetry, it was feasible to obtain a sequence of four approximate solutions for cases of square panels in which $\lambda_b = \lambda_a$. These solutions are summarized in Table 5 and in the first page of Table 6.

The convergence of the sequence of solutions for the clamped plate ($\lambda_a = \lambda_b = \infty$) is very good, however, the convergence becomes even better as λ_a and λ_b approach unity. In fact, as has been previously discussed, the solution becomes exact when $\lambda_a \lambda_b = 1$. As λ_a and λ_b are decreased further, the convergence of the solutions becomes poorer. The solutions show the poorest convergence when λ_a and λ_b are both zero; the results of these cases are presented in Table 6. From this table, it is apparent that the convergence is better when the panels are square than when they are long and narrow.

14. Limiting Cases; Plate Supported on Columns Only and the Clamped Plate

The bending of a uniformly loaded plate supported by rows of equidistant columns has been discussed by several authors [3,12]. The analytical solution of this problem leads to the deflection function expressed as an infinite series of trigonometric and hyperbolic functions. This formula for the deflection, w, as well as formulae for moments and various average moments are summarized in the Appendix. Numerical results obtained using these formulae are summarized in Table 6 in the columns headed "Exact Solution".

The agreement between the energy solutions and the exact solutions is, in general, fairly good. Because of the assumption that the columns provide only concentrated point reactions, the exact solutions shows infinite plate moments at the panel corners. Therefore, the moments obtained using the energy method with a finite number of terms cannot agree with those of the exact solution at these points. Aside from this, the least accurate moments obtained using the energy method are the negative moments at the middle of the long sides of the panels; these moments, however, are generally fairly small.

Another important solution obtained as a limiting case in the present work is the solution to the problem of a uniformly loaded clamped plate. An exact analytic solution has not been obtained for this problem so that exact results are not available for comparison. However, certain deflections and moments have been computed by other investigators using various approximate methods (see for instance references (3) and (13) to (21)). Usually the method of Ritz (described herein) or the methods developed by Hencky [13] or Timoshenko [3] are used.

Some results obtained by investigators using Hencky's analysis are presented in Table 7; these are virtually in agreement with results from the present investigation which are also presented in this table. In Hencky's analysis, the deflection function consists of a parabolic term plus a correction function in the form of a series of trigonometric and hyperbolic functions. This deflection function satisfies the differential equation ($D \nabla^4 w = q$) exactly; the coefficients in the series are adjusted so that the boundary conditions are approximately satisfied. The numerical values given in Table 7 for Hencky's analysis were computed by Siess and Newmark [16] from coefficients tabulated by Wojtaszak [14] for the terms $n = 1, 3, 5 \ldots 27$ of the series. Average moments on an edge were computed by integration, however, average moments on interior sections were obtained by the application of Simpson's one-third rule to moments at the one-tenth points.

15. Finite Difference Solutions

The calculus of finite differences offers an alternate approximate numerical procedure for solving the general problem of a plate continuous over flexible beams. A number of such solutions have been made at the University of Illinois by various persons working under the direction of Professors N. M. Newmark and C. P. Siess. These solutions were made using a difference net which divided each span into eight equal parts. Each panel was therefore divided into 64 rectangular elements. With this difference net and noting the simplifications resulting from symmetry, it is found that each case requires the solution of 24 simultaneous linear equations. For square plates supported on beams which are all of the same rigidity, the number of equations reduces to 14.

Some of the deflections and moments obtained by these difference equation solutions are presented in Tables 7 and 8. It is interesting to note that the "S-function" solutions using only four coefficients α_{mn} (four simultaneous equations) are generally more accurate than the difference solutions using 24 simultaneous equations. It is also interesting that the difference equations determine the moments more accurately than they do the deflections.

VII. CONCLUSION

16. Summary and Concluding Remarks

The present investigation has resulted in a relatively simple approximate analytical solution to the problem of an interior panel of a uniformly loaded plate which is continuous over a rectangular grid of flexible beams supported at their intersections by columns. The relative simplicity of the analysis made it possible to present fairly accurate and complete results for panels with various side ratios and various ratios of beam rigidities to plate rigidities. These results are given in Table 4 and Figures 2 to 42.

Although the results presented are those for a plate uniformly loaded over its full area, they can be used to obtain solutions for several other cases of loading. In particular, solutions can be obtained for the case in which only alternate panels are uniformly loaded and for the case in which only alternate rows of panels are uniformly loaded. The alternate panel loading can be obtained by superposing a uniform load q/2 with another uniform load q/2 which is alternately positive and negative, changing sign at each panel boundary. Under this latter loading, and assuming that the torsional restraint of the beams can be neglected, each panel is in the same condition as a simply supported plate of panel size. Therefore, when alternate panels are loaded, the solution can be obtained by superposing the present solution with that for a simply supported plate. Similarly, the uniform loading of alternate rows of panels is equivalent to superposing a uniform load q/2 with a uniform load q/2 which changes sign from row to row of panels but which has the same sign in any one row. Under this latter loading each row of panels is essentially a simply supported strip. One representative panel from this strip can be solved by Levy's method [22,3] and can be superposed with the solution given herein to obtain the final solution.

One of the most important types of structures similar to the plate-beam system described is the type of reinforced concrete floor known as the two-way slab. The foregoing solution is based on a number of assumptions which are not completely satisfied by this type of structure. The results presented therefore require some modification before they can be applied in the design or analysis of such two-way slabs. The purpose of this investigation was not the actual development of an analysis of reinforced concrete slabs. However, analytical solutions, such as those presented, are a basic step toward the ultimate objective of a rational design procedure.

BIBLIOGRAPHY

(1) W. J. Crawford, "The Elastic Strength of Flat Plates: An Experimental Research", Edinburgh Royal Society Proc., Vol. 32, 1912, p. 348.

(2) B. C. Laws, "Distribution of Stress in Thin Mild-Steel Plates of Rectangular Shape, Fixed along their Edges, and subject to Uniformly-Distributed Loads", Min. of Proc. of the Inst. of Civil Eng., London, Vol. CCXIII, 1922, pp. 358-384.

(3) S. Timoshenko, "Theory of Plates and Shells," 1st Ed., McGraw-Hill, New York, 1940

(4) Lord Rayleigh, "The Theory of Sound," 1st Amer. Ed., Dover Pub., New York, 1945.

(5) W. Ritz, "Über eine neue Methode zur Lösung gewissen Variation-probleme der mathematischen Physik," Jour. für reine und ang. Math., Bd. 135, 1909, pp. 1-61.

(6) H. Langhaar, "Note on Energy of Bending of Plates," Jour. App. Mech., Vol. 19, No. 2, 1952, p. 228.

(7) W. J. Duncan, "Normalized Orthogonal Deflexion Functions for Beams," Areo. Res. Council, R. and M. 2281, 1950, pp. 23.

(8) Mathematical Tables, Part-Volume A, "Legendre Polynomials," Pub. for Brit. Assoc. at Univ. Press, Cambridge, 1946.

(9) T. M. MacRobert, "Sperical Harmonics," 2nd Ed., Dover Pub., New York, 1948, Ch. V.

(10) R. Courant, "Variational Methods for the Solution of Problems of Equilibrium and Vibrations," Amer. Math. Soc. Bull., Vol. 49, 1943, pp. 1-23.

(11) E. Trefftz, "Konvergenz und Fehlerabschaetzung beim Ritz'schen Verfahren," Math. Ann., Vol. 100, 1928, p. 503.

(12) A. Nadai, "Elastischen Platten," Julius Springer, Berlin, 1925.

(13) H. Hencky, "Der Spannungszustand in rechteckigen Platten," (Diss.) Darmstadt, pub. by R. Oldenbourg, München u. Berlin, 1913.

(14) I. A. Wojtaszak, "The Calculation of Maximum Deflection, Moment, and Shear for Uniformly Loaded Rectangular Plates with Clamped Edges," Jour. App. Mech., Vol. 4, No. 4, 1937, pp. 173-176.

(15) Dana Young, "Analyses of Clamped Rectangular Plates," Jour. App. Mech., Vol. 7, No. 4, 1940, pp. 139-142.

(16) C. P. Siess and N. M. Newmark, "Moments in Two-Way Concrete Floor Slabs," Bull. No. 385, Eng. Expt. Station, Univ. of Illinois, Urbana, Ill., 1950

(17) H. Leitz, "Berechnung der eingespannten rechteckigen Platte," Zeits. für Math. u. Phys., Bd. 64, 1916, pp. 262-272.

(18) T. H. Evans, "Tables of Moments and Deflections for a Rectangular Plate Fixed on All Edges and Carrying a Uniformly Distributed Load," Jour. App. Mech., Vol. 6, No. 1, 1939, pp. 7-11.

(19) G. Pickett, "Solution of Rectangular Clamped Plate With Lateral Load, by Generalized Energy Method," Jour. App. Mech., Vol. 6, No. 4, pp. 168-170.

(20) W. B. Stiles, "Bending of Clamped Plates," Jour. App. Mech., Vol. 14, 1947, pp. 55-62.

(21) R. Ohlig, "Die eingespannte Rechteckplatte," Ingen. Arch., Vol. XVII, 1949, pp. 243-263.

(22) M. Lévy, "Sur l'équilibre élastique d'une plaque rectangulaire," Comptes rendus, Vol. 129, 1899, pp. 535-539.

APPENDIX

Summary of the Exact Solution for the Case $\lambda_a = \lambda_b = 0$

The following summary of formulae for deflections, moments, and average moments in a uniformly loaded plate supported by rows of equidistant columns is based on the analytical solution presented in section 46 of S. Timoshenko's "Theory of Plates and Shells" (3). The notation used follows that of the previous part of this work, however, the origin of coordinates is taken at the center of a panel. The deflection of the panel of the plate is

$$\frac{w}{qa^4/D} = \frac{c^4}{384}\left(1 - \frac{4y_1^2}{b^2}\right)^2 + \frac{c}{2\pi^3}\sum_{m=2,4\ldots}^{\infty}\frac{1}{m^3}\left(\frac{1}{\tanh\alpha_m} + \frac{\alpha_m}{\sinh^2\alpha_m}\right)$$
$$+\frac{c}{2\pi^3}\sum_{m=2,4\ldots}^{\infty}\frac{(-1)^{\frac{m}{2}}\cos\frac{m\pi x_1}{a}}{m^3\sinh\alpha_m\tanh\alpha_m}\left[\frac{m\pi y_1}{a}\tanh\alpha_m\sinh\frac{m\pi y_1}{a} - (\alpha_m + \tanh\alpha_m)\cosh\frac{m\pi y_1}{a}\right] \qquad (35)$$

where $\alpha_m = \frac{m\pi c}{2}$. An alternate form of the deflection function is obtained by interchanging x_1 with y_1 and a with b.

The bending moments are:

$$\frac{M_x^o}{qa^2} = \frac{1}{24}\left(1 - \frac{12x_1^2}{a^2}\right) + \frac{1}{4}\sum_{m=2,4\ldots}^{\infty}\frac{(-1)^{\frac{m}{2}}\cos\frac{m\pi y_1}{b}}{\frac{\alpha_m}{c^2}\sinh\frac{\alpha_m}{c^2}\tanh\frac{\alpha_m}{c^2}}\left[\frac{\alpha_m}{c^2}\cosh\frac{m\pi x_1}{b} - \frac{m\pi x_1}{b}\tanh\frac{\alpha_m}{c^2}\sinh\frac{m\pi x_1}{b} - \tanh\frac{\alpha_m}{c^2}\cosh\frac{m\pi x_1}{b}\right] \qquad (36)$$

$$\frac{M_y^o}{qa^2} = \frac{c^2}{24}\left(1-\frac{12y_1^2}{b^2}\right) - \frac{c^2}{4}\sum_{m=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{m}{2}} \cos \frac{m\pi x_1}{a}}{\alpha_m \sinh \alpha_m \tanh \alpha_m} \left[\tanh \alpha_m \cosh \frac{m\pi y_1}{a} + \frac{m\pi y_1}{a} \tanh \alpha_m \sinh \frac{m\pi y_1}{a} - \alpha_m \cosh \frac{m\pi y_1}{a}\right] .$$

The average moments in the column and middle strips are:

$$\frac{M_1^o}{qa^2} = \frac{1}{24} - \frac{1}{2c}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2/c^4}\left[\frac{\alpha_m/c^2 - \tanh \alpha_m/c^2}{\sinh \alpha_m/c^2 \tanh \alpha_m/c^2}\right]$$

$$\frac{M_2}{qa^2} = -\frac{1}{12} + \frac{1}{2c}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2/c^4}\left[\frac{1}{\tanh \alpha_m/c^2} - \frac{\alpha_m/c^2}{\sinh^2 \alpha_m/c^2}\right]$$

$$\frac{M_3^o}{qa^2} = \frac{1}{24} + \frac{1}{2c}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2/c^4}\left[\frac{\alpha_m/c^2 - \tanh \alpha_m/c^2}{\sinh \alpha_m/c^2 \tanh \alpha_m/c^2}\right]$$

$$\frac{M_4^o}{qa^2} = -\frac{1}{12} - \frac{1}{2c}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2/c^4}\left[\frac{1}{\tanh \alpha_m/c^2} - \frac{\alpha_m/c^2}{\sinh^2 \alpha_m/c^2}\right] \tag{37}$$

$$\frac{M_5^o}{qa^2} = \frac{c^2}{24} - \frac{c^3}{2}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2}\left[\frac{\alpha_m - \tanh \alpha_m}{\sinh \alpha_m \tanh \alpha_m}\right]$$

$$\frac{M_6^o}{qa^2} = -\frac{c^2}{12} + \frac{c^3}{2}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2}\left[\frac{1}{\tanh \alpha_m} - \frac{\alpha_m}{\sinh^2 \alpha_m}\right]$$

$$\frac{M_7^o}{qa^2} = \frac{c^2}{24} + \frac{c^3}{2}\sum_{m=2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2}\left[\frac{\alpha_m - \tanh \alpha_m}{\sinh \alpha_m \tanh \alpha_m}\right]$$

$$\frac{M_8^o}{qa^2} = -\frac{c^2}{12} - \frac{c^3}{2}\sum_{2,6,10\ldots}^{\infty} \frac{(-1)^{\frac{m+2}{4}}}{\alpha_m^2}\left[\frac{1}{\tanh \alpha_m} - \frac{\alpha_m}{\sinh^2 \alpha_m}\right]$$

and the moment coefficients for correcting these average moments for Poissons ratio are:

$$-\frac{N_1}{qa^2} = \frac{N_3}{qa^2} = \frac{c^2}{32} + \frac{c^2}{8}\sum_{m=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{m}{2}} \tanh \alpha_m/2}{\alpha_m \cosh \alpha_m/2}$$

$$-\frac{N_2}{qa^2} = \frac{N_4}{qa^2} = \frac{c^2}{32} + \frac{c^2}{8}\sum_{m=2,4,\ldots}^{\infty} \frac{\tanh \alpha_m/2}{\alpha_m \cosh \alpha_m/2}$$

(38)

$$-\frac{N_5}{qa^2} = \frac{N_7}{qa^2} = \frac{1}{32} + \frac{1}{8}\sum_{m=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{m}{2}} \tanh \alpha_m/2c^2}{\alpha_m/c^2 \cosh \alpha_m/2c^2}$$

$$-\frac{N_6}{qa^2} = \frac{N_8}{qa^2} = \frac{1}{32} + \frac{1}{8}\sum_{m=2,4,\ldots}^{\infty} \frac{\tanh \alpha_m/2c^2}{\alpha_m/c^2 \cosh \alpha_m/2c^2} .$$

TABLE 1. THE FIRST FOUR S - FUNCTIONS AND THEIR SECOND DERIVATIVES

ξ	$S_1(\xi)$	$S_1''(\xi)$	$S_2(\xi)$	$S_2''(\xi)$	ξ
0.00	+0.000000	+2.236068	+0.000000	+3.000000	1.00
0.05	+0.002523	+1.598789	+0.002634	+0.623813	0.95
0.10	+0.009056	+1.028591	+0.007047	-0.699000	0.90
0.15	+0.018175	+0.525476	+0.009876	-1.236187	0.85
0.20	+0.028622	+0.089443	+0.009728	-1.224000	0.80
0.25	+0.039306	-0.279508	+0.006592	-0.867187	0.75
0.30	+0.049305	-0.581378	+0.001323	-0.339000	0.70
0.35	+0.057865	-0.816165	-0.004787	+0.218813	0.65
0.40	+0.064399	-0.983870	-0.010368	+0.696000	0.60
0.45	+0.068487	-1.084493	-0.014242	+1.013813	0.55
0.50	+0.069877	-1.118034	-0.015625	+1.125000	0.50

ξ	$S_3(\xi)$	$S_3''(\xi)$	$S_4(\xi)$	$S_4''(\xi)$	ξ
0.00	+0.000000	+3.605551	+0.000000	+4.123106	1.00
0.05	+0.002052	-0.869530	+0.001149	-1.689178	0.95
0.10	+0.002735	-1.412641	-0.000504	-0.068672	0.90
0.15	+0.000189	-0.451726	-0.002451	+1.264574	0.85
0.20	-0.003471	+0.620501	-0.001618	+0.875497	0.80
0.25	-0.005694	+1.165466	+0.001239	-0.303621	0.75
0.30	-0.005144	+1.055114	+0.003423	-1.100866	0.70
0.35	-0.002057	+0.465769	+0.003054	-0.985730	0.65
0.40	+0.002160	-0.290521	+0.000372	-0.163130	0.60
0.45	+0.005683	-0.897167	-0.002700	+0.743481	0.55
0.50	+0.007042	-1.126735	-0.004026	+1.127412	0.50

TABLE 2. NUMERICAL VALUES OF VARIOUS INTEGRALS INVOLVING THE S - FUNCTIONS AND THEIR DERIVATIVES

(a) Values of $(10)^6 \int_0^1 S_m(\xi) S_n(\xi) d\xi$

m \ n	1	2	3	4
1	+1984.1270	-161.3325	+11.18639	0
2		+74.92508	-20.01083	+2.355630
3			+14.85443	-5.364217
4				+4.819742

(b) Values of $(10)^3 \int_0^1 S_m''(\xi) S_n(\xi) d\xi$

m \ n	1	2	3	4
1	-23.80952	+5.323971	0	0
2	+ 5.323971	-6.493506	+2.101137	0
3	0	+2.101137	-3.030303	+1.121121
4	0	0	+1.121121	-1.754386

(c) Mean values of $S_m(\xi)$ and $S_m''(\xi)$ per quarter span

m	$4\int_0^{1/4} S_m(\xi) d\xi$	$2\int_{1/4}^{3/4} S_m(\xi) d\xi$	$4\int_0^{1/4} S_m''(\xi) d\xi$	$2\int_{1/4}^{3/4} S_m''(\xi) d\xi$
1	+0.01543120	+0.05910440	+0.8385255	-0.8385255
2	+0.00659180	-0.00659180	-0.3515625	+0.3515625
3	-0.00024757	+0.00024757	-0.0739420	+0.0739420
4	-0.00060161	+0.00060161	+0.2381909	-0.2381909

TABLE 3. MATRIX FOR THE TOTAL POTENTIAL ENERGY $\frac{V}{q^2 a^6 c (10)^{-6}/D}$

α_{mn} \ $\frac{\alpha_{rs}}{2}$	α_{11}	α_{12}	α_{21}	α_{22}	α_{13}	α_{31}	α_{23}	α_{32}	α_{33}	2
$\frac{\alpha_{11}}{2}$	$+1984.1270 - \frac{12,500}{9(1+\lambda_a/c)}$ $+1133.787/c^2$ $+\frac{1984.1270}{c^4} - \frac{12,500}{9c^4(1+c\lambda_b)}$	-161.3325 $-253.5224/c^2$ 0	0 $-253.5224/c^2$ $-161.3325/c^4$	0 $+56.68934/c^2$ 0	+11.18639 0 0	0 0 $+11.18639/c^4$	0 0 0	0 0 0	0 0 0	$\frac{+12,500(1-\lambda_a\lambda_b)}{9(1+\lambda_a/c)(1+c\lambda_b)}$
$\frac{\alpha_{12}}{2}$		+74.92508 $+309.2146/c^2$ $+1984.1270/c^4$ $-\frac{12,500}{9c^4(1+c\lambda_b)}$	0 $+56.68934/c^2$ 0	0 $-69.14249/c^2$ $-161.3325/c^4$	-20.01083 $-100.05415/c^2$ 0	0 0 0	0 $+22.37279/c^2$ 0	0 0 $+11.18639/c^4$	0 0 0	0
$\frac{\alpha_{21}}{2}$			+1984.1270 $-\frac{12,500}{9(1+\lambda_a/c)}$ $+309.2146/c^2$ $+74.92508/c^4$	-161.3325 $-69.14249/c^2$ 0	0 0 0	0 $-100.05415/c^2$ $-20.01083/c^4$	+11.18639 0 0	0 $+22.37279/c^2$ 0	0 0 0	0
$\frac{\alpha_{22}}{2}$				+74.92508 $+84.33125/c^2$ $+74.92508/c^4$	0 $+22.37279/c^2$ 0	0 $+22.37279/c^2$ 0	-20.01083 $-27.28750/c^2$ 0	0 $-27.28750/c^2$ $-20.01083/c^4$	0 $+8.82955/c^2$ 0	0
$\frac{\alpha_{13}}{2}$					+14.85443 $+144.3001/c^2$ $+1984.1270/c^4$ $-\frac{12,500}{9c^4(1+c\lambda_b)}$	0 0 0	0 $-32.26649/c^2$ $-161.3325/c^4$	0 0 0	0 0 $+11.18639/c^4$	0
$\frac{\alpha_{31}}{2}$						+1984.1270 $-\frac{12,500}{1+\lambda_a/c}$ $+144.3001/c^2$ $+14.85443/c^4$	0 0 0	-161.3325 $-32.26649/c^2$ 0	+11.18639 0 0	0
$\frac{\alpha_{23}}{2}$							+14.85443 $+39.35458/c^2$ $+74.92508/c^4$	0 $+8.82955/c^2$ 0	0 $-12.73416/c^2$ $-20.01083/c^4$	0
$\frac{\alpha_{32}}{2}$								+74.92508 $+39.35458/c^2$ $+14.85443/c^4$	-20.01083 $-12.73416/c^2$ 0	0
$\frac{\alpha_{33}}{2}$									-14.85443 $+18.36547/c^2$ $+14.85443/c^4$	0

TABLE 4. DEFLECTIONS AND MOMENTS IN THE PLATE AND BEAMS FOR VARIOUS BEAM STIFFNESSES AND VARIOUS PANEL SIDE RATIOS

c	1.0				Units
λ_a	∞	∞	∞	∞	
λ_b	∞	5.0	2.0	1.0	
α_{11}	See Table 5	+0.2453540	+0.2069246	+0.1642342	
α_{12}		+0.0502558	+0.0484327	+0.0448687	
α_{21}		+0.0452006	+0.0386811	+0.0313020	
α_{22}		+0.0347526	+0.0369009	+0.0374465	
α_{13}		+0.0014119	+0.0020701	+0.0028536	
α_{31}		+0.0009518	+0.0010036	+0.0010158	
α_{23}		+0.0006044	+0.0040019	+0.0074132	
α_{32}		+0.0003340	+0.0031922	+0.0057173	
α_{33}		−0.0062936	−0.0025925	+0.0011657	
$w(\frac{1}{2},\frac{1}{2})$	0.001265	0.001435	0.001622	0.001829	qa^4/D
$w(0,\frac{1}{2})$	0	0.0003324	0.0006977	0.001101	qa^4/D
$w(\frac{1}{2},0)$	0	0	0	0	qa^4/D
$M^c_x(\frac{1}{2},\frac{1}{2})$	0.01762	0.01537	0.01292	0.01020	qa^2
$M^c_y(\frac{1}{2},\frac{1}{2})$	0.01762	0.02068	0.02405	0.02777	qa^2
$-M^c_x(0,0)$	0	0	0	0	qa^2
$-M^c_y(0,0)$	0	0.009513	0.02013	0.03212	qa^2
$M^c_x(\frac{1}{2},0)$	0	0	0	0	qa^2
$-M^c_y(\frac{1}{2},0)$	0.05139	0.05536	0.05985	0.06481	qa^2
$-M^c_x(0,\frac{1}{2})$	0.05139	0.04452	0.03714	0.02906	qa^2
$M^c_y(0,\frac{1}{2})$	0	0.005582	0.01166	0.01829	qa^2
M^c_1	0.004107	0.003581	0.003024	0.002404	qa^2
$-M^c_2$	0.01328	0.01205	0.01050	0.008702	qa^2
M^c_3	0.01505	0.01315	0.01106	0.008743	qa^2
$-M^c_4$	0.04492	0.03920	0.03290	0.02594	qa^2
M^c_5	0.004107	0.009048	0.01447	0.02041	qa^2
$-M^c_6$	0.01328	0.02176	0.03102	0.04140	qa^2
M^c_7	0.01505	0.01845	0.02222	0.02634	qa^2
$-M^c_8$	0.04492	0.04979	0.05511	0.06102	qa^2
$-N_1 = N_3$	0.01467	0.01678	0.01910	0.02166	qa^2
$-N_2 = N_4$	0	0.004041	0.008475	0.01336	qa^2
$-N_5 = N_7$	0.01467	0.01280	0.01074	0.008471	qa^2
$-N_6 = N_8$	0	0	0	0	qa^2
$M_a(\frac{1}{2})$	0.03209	0.03330	0.03462	0.03609	qa^3
$-M_a(0)$	0.05423	0.05771	0.06162	0.06601	qa^3
$M_b(\frac{1}{2})$	0.03209	0.02791	0.02333	0.01829	qa^3
$-M_b(0)$	0.05423	0.04756	0.04026	0.03212	qa^3

TABLE 4. (continued)

c	1.0				Units
λ_a	∞	∞	5.0	5.0	
λ_b	0.5	0	5.0	2.0	
α_{11}	+0.1164395	0	See Table 5	+0.1640118	
α_{12}	+0.0382533	0		+0.0388454	
α_{21}	+0.0228039	0		+0.0344938	
α_{22}	+0.0348641	0		+0.0341673	
α_{13}	+0.0036594	0		+0.0018214	
α_{31}	+0.0009488	0		+0.0012804	
α_{23}	+0.0103083	0		+0.0051238	
α_{32}	+0.0073828	0		+0.0046578	
α_{33}	+0.0045744	0		−0.0003345	
$w(\frac{1}{2},\frac{1}{2})$	0.002059	0.002604	0.001622	0.001827	qa^4/D
$w(0,\frac{1}{2})$	0.001549	0.002604	0.0003484	0.0007331	qa^4/D
$w(\frac{1}{2},0)$	0	0	0.0003484	0.0003661	qa^4/D
$M_x^o(\frac{1}{2},\frac{1}{2})$	0.007177	0	0.01849	0.01605	qa^2
$M_y^o(\frac{1}{2},\frac{1}{2})$	0.03191	0.04167	0.01849	0.02191	qa^2
$-M_x^o(0,0)$	0	0	0.01019	0.01094	qa^2
$-M_y^o(0,0)$	0.04591	0.08333	0.01019	0.02169	qa^2
$M_x^o(\frac{1}{2},0)$	0	0	0.005797	0.006038	qa^2
$-M_y^o(\frac{1}{2},0)$	0.07032	0.08333	0.04857	0.05294	qa^2
$-M_x^o(0,\frac{1}{2})$	0.02022	0	0.04857	0.04103	qa^2
$M_y^o(0,\frac{1}{2})$	0.02551	0.04167	0.005797	0.01213	qa^2
M_1^o	0.001708	0	0.008732	0.008359	qa^2
$-M_2^o$	0.006541	0	0.02072	0.01962	qa^2
M_3^o	0.006162	0	0.01663	0.01460	qa^2
$-M_4^o$	0.01822	0	0.04400	0.03765	qa^2
M_5^o	0.02691	0.04167	0.008732	0.01436	qa^2
$-M_6^o$	0.05316	0.08333	0.02072	0.03055	qa^2
M_7^o	0.03092	0.04167	0.01663	0.02046	qa^2
$-M_8^o$	0.06760	0.08333	0.04400	0.04934	qa^2
$-N_1 = N_3$	0.02450	0.03125	0.01492	0.01726	qa^2
$-N_2 = N_4$	0.01875	0.03125	0.004226	0.008880	qa^2
$-N_5 = N_7$	0.005950	0	0.01492	0.01287	qa^2
$-N_6 = N_8$	0	0	0.004226	0.004430	qa^2
$M_a(\frac{1}{2})$	0.03773	0.04167	0.02899	0.03019	qa^3
$-M_a(0)$	0.07095	0.08333	0.05097	0.05470	qa^3
$M_b(\frac{1}{2})$	0.01276	0	0.02899	0.02426	qa^3
$-M_b(0)$	0.02295	0	0.05097	0.04339	qa^3

TABLE 4. (continued)

c	1.0				Units
λ_a	5.0	5.0	5.0	2.0	
λ_b	1.0	0.5	0	2.0	
α_{11}	+0.1161059	+0.0619847	-0.0727797	See Table 5	
α_{12}	+0.0321086	+0.0206235	-0.0418028		
α_{21}	+0.0249051	+0.0136689	-0.0180433		
α_{22}	+0.0301154	+0.0206320	-0.0487697		
α_{13}	+0.0021838	+0.0020724	-0.0116478		
α_{31}	+0.0010926	+0.0007295	-0.0017080		
α_{23}	+0.0067431	+0.0063941	-0.0304619		
α_{32}	+0.0056569	+0.0048797	-0.0166756		
α_{33}	+0.0021525	+0.0032376	-0.0200419		
$w(\frac{1}{2},\frac{1}{2})$	0.002058	0.002315	0.002932	0.002058	qa^4/D
$w(0,\frac{1}{2})$	0.001160	0.001636	0.002772	0.0007726	qa^4/D
$w(\frac{1}{2},0)$	0.0003860	0.0004084	0.0004639	0.0007726	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.01335	0.01034	0.003091	0.01954	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.02572	0.02996	0.04007	0.01954	qa^2
$-M_x^\circ(0,0)$	0.01179	0.01276	0.01527	0.02345	qa^2
$-M_y^\circ(0,0)$	0.03489	0.05039	0.09564	0.02345	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.006305	0.006604	0.007336	0.01265	qa^2
$-M_y^\circ(\frac{1}{2},0)$	0.05784	0.06331	0.07631	0.04534	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.03280	0.02378	0.003040	0.04534	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.01904	0.02657	0.04344	0.01265	qa^2
M_1°	0.007950	0.007485	0.006294	0.01422	qa^2
$-M_2^\circ$	0.01820	0.01639	0.009903	0.02992	qa^2
M_3°	0.01234	0.009805	0.003682	0.01852	qa^2
$-M_4^\circ$	0.03059	0.02271	0.004022	0.04293	qa^2
M_5°	0.02054	0.02732	0.04271	0.01422	qa^2
$-M_6^\circ$	0.04162	0.05435	0.08862	0.02992	qa^2
M_7°	0.02471	0.02944	0.04063	0.01852	qa^2
$-M_8^\circ$	0.05527	0.06192	0.07805	0.04293	qa^2
$-N_1 = N_3$	0.01987	0.02277	0.02974	0.01523	qa^2
$-N_2 = N_4$	0.01402	0.01973	0.03294	0.009332	qa^2
$-N_5 = N_7$	0.01059	0.008052	0.002012	0.01523	qa^2
$-N_6 = N_8$	0.004658	0.004916	0.005547	0.009332	qa^2
$M_a(\frac{1}{2})$	0.03152	0.03302	0.03668	0.02529	qa^3
$-M_a(0)$	0.05894	0.06379	0.07637	0.04691	qa^3
$M_b(\frac{1}{2})$	0.01904	0.01329	0	0.02529	qa^3
$-M_b(0)$	0.03489	0.02520	0	0.04691	qa^3

TABLE 4. (continued)

c	1.0				Units
λ_a	2.0	2.0	2.0	1.0	
λ_b	1.0	0.5	0	1.0	
α_{11}	+0.0618319	0	−0.1578685	0	
α_{12}	+0.0173684	0	−0.0924806	0	
α_{21}	+0.0151713	0	−0.0449433	0	
α_{22}	+0.0185623	0	−0.1149632	0	
α_{13}	+0.0012795	0	−0.0266033	0	
α_{31}	+0.0008993	0	−0.0050624	0	
α_{23}	+0.0046452	0	−0.0715104	0	
α_{32}	+0.0042376	0	−0.0412662	0	
α_{33}	+0.0021341	0	−0.0483329	0	
$w(\frac{1}{2},\frac{1}{2})$	0.002315	0.002604	0.003311	0.002604	qa^4/D
$w(0,\frac{1}{2})$	0.001226	0.001736	0.002968	0.001302	qa^4/D
$w(\frac{1}{2},0)$	0.0008173	0.0008681	0.0009968	0.001302	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.01688	0.01389	0.006620	0.02083	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.02342	0.02778	0.03825	0.02083	qa^2
$-M_x^\circ(0,0)$	0.02546	0.02778	0.03406	0.04167	qa^2
$-M_y^\circ(0,0)$	0.03802	0.05556	0.1104	0.04167	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.01323	0.01389	0.01552	0.02083	qa^2
$-M_y^\circ(\frac{1}{2},0)$	0.05015	0.05556	0.06846	0.04167	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.03698	0.02778	0.006489	0.04167	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.01988	0.02778	0.04548	0.02083	qa^2
M_1°	0.01407	0.01389	0.01336	0.02083	qa^2
$-M_2^\circ$	0.02906	0.02778	0.02175	0.04167	qa^2
M_3°	0.01634	0.01389	0.007878	0.02083	qa^2
$-M_4^\circ$	0.03579	0.02778	0.008659	0.04167	qa^2
M_5°	0.02068	0.02778	0.04390	0.02083	qa^2
$-M_6^\circ$	0.04174	0.05556	0.09414	0.04167	qa^2
M_7°	0.02289	0.02778	0.03944	0.02083	qa^2
$-M_8^\circ$	0.04886	0.05556	0.07203	0.04167	qa^2
$-N_1 = N_3$	0.01787	0.02083	0.02083	0.01562	qa^2
$-N_2 = N_4$	0.01477	0.02083	0.03490	0.01562	qa^2
$-N_5 = N_7$	0.01296	0.01042	0.004318	0.01562	qa^2
$-N_6 = N_8$	0.009840	0.01042	0.01186	0.01562	qa^2
$M_a(\frac{1}{2})$	0.02646	0.02778	0.03102	0.02083	qa^3
$-M_a(0)$	0.05091	0.05556	0.06813	0.04167	qa^3
$M_b(\frac{1}{2})$	0.01988	0.01389	0	0.02083	qa^3
$-M_b(0)$	0.03802	0.02778	0	0.04167	qa^3

TABLE 4. (continued)

c	1.0				Units
λ_a	1.0	1.0	0.5	0.5	
λ_b	0.5	0	0.5	0	
α_{11}	-0.0714177	-0.2592456	See Table 5	-0.3834098	
α_{12}	-0.0246824	-0.1559401		-0.2394708	
α_{21}	-0.0210902	-0.0867370		-0.1558476	
α_{22}	-0.0311203	-0.2095022		-0.3548763	
α_{13}	-0.0028365	-0.0467405		-0.0758723	
α_{31}	-0.0019368	-0.0119897		-0.0276712	
α_{23}	-0.0106518	-0.1298752		-0.2196882	
α_{32}	-0.0095379	-0.0803875		-0.1498967	
α_{33}	-0.0069255	-0.0910728		-0.1624646	
$w(\frac{1}{2},\frac{1}{2})$	0.002933	0.003754	0.003312	0.004284	qa^4/D
$w(0,\frac{1}{2})$	0.001851	0.003201	0.001985	0.003483	qa^4/D
$w(\frac{1}{2},0)$	0.001389	0.001616	0.001985	0.002346	qa^4/D
$M_x^c(\frac{1}{2},\frac{1}{2})$	0.01790	0.01068	0.02247	0.01540	qa^2
$M_y^c(\frac{1}{2},\frac{1}{2})$	0.02530	0.03616	0.02247	0.03372	qa^2
$-M_x^c(0,0)$	0.04595	0.05812	0.06904	0.09095	qa^2
$-M_y^c(0,0)$	0.06162	0.1287	0.06904	0.1523	qa^2
$M_x^c(\frac{1}{2},0)$	0.02192	0.02467	0.03072	0.03485	qa^2
$-M_y^c(\frac{1}{2},0)$	0.04696	0.05965	0.03759	0.04976	qa^2
$-M_x^c(0,\frac{1}{2})$	0.03230	0.01042	0.03759	0.01489	qa^2
$M_y^c(0,\frac{1}{2})$	0.02915	0.04789	0.03072	0.05079	qa^2
M_1^c	0.02100	0.02130	0.02890	0.03021	qa^2
$-M_2^c$	0.04121	0.03635	0.05742	0.05521	qa^2
M_3^c	0.01850	0.01270	0.02371	0.01827	qa^2
$-M_4^c$	0.03355	0.01407	0.04021	0.02051	qa^2
M_5^c	0.02829	0.04527	0.02890	0.04687	qa^2
$-M_6^c$	0.05669	0.1015	0.05742	0.1095	qa^2
M_7^c	0.02589	0.03806	0.02371	0.03646	qa^2
$-M_8^c$	0.04836	0.06514	0.04021	0.05722	qa^2
$-N_1 = N_3$	0.01865	0.02607	0.01616	0.02383	qa^2
$-N_2 = N_4$	0.02210	0.03718	0.02357	0.03988	qa^2
$-N_5 = N_7$	0.01310	0.006986	0.01616	0.01011	qa^2
$-N_6 = N_8$	0.01660	0.01909	0.02357	0.02741	qa^2
$M_a(\frac{1}{2})$	0.02192	0.02467	0.01536	0.01742	qa^3
$-M_a(0)$	0.04595	0.05812	0.03452	0.04548	qa^3
$M_b(\frac{1}{2})$	0.01457	0	0.01536	0	qa^3
$-M_b(0)$	0.03081	0	0.03452	0	qa^3

TABLE 4. (continued)

c	1.0	0.8			Units
λ_a	0	∞	5.0	2.0	
λ_b	0	∞	5.0	2.0	
α_{11}		+0.1672068	+0.1241461	+0.0700498	
α_{12}		+0.0183744	+0.0162397	+0.0111216	
α_{21}		+0.0515578	+0.0419296	+0.0263703	
α_{22}	See	+0.0210508	+0.0225457	+0.0179534	
α_{13}		+0.0000877	+0.0003087	+0.0004259	
α_{31}	Table 6	+0.0023380	+0.0025288	+0.0021278	
α_{23}		−0.0007190	+0.0014330	+0.0024542	
α_{32}		−0.0003757	+0.0035654	+0.0050842	
α_{33}		−0.0048035	−0.0013400	+0.0011089	
$w(\frac{1}{2},\frac{1}{2})$	0.005800	0.0007463	0.001017	0.001345	qa^4/D
$w(0,\frac{1}{2})$	0.004350	0	0.0001506	0.0003425	qa^4/D
$w(\frac{1}{2},0)$	0.004350	0	0.0003179	0.0006962	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.02758	0.009218	0.01189	0.01495	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.02758	0.01733	0.01653	0.01569	qa^2
$-M_x(0,0)$	∞	0	0.009374	0.02122	qa^2
$-M_y(0,0)$	∞	0	0.006853	0.01622	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.05733	0	0.005261	0.01136	qa^2
$-M_y^\circ(\frac{1}{2},0)$	0.02975	0.04244	0.03866	0.03443	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.02975	0.03578	0.03637	0.03680	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.05733	0	0.003926	0.008772	qa^2
M_1°	0.05111	0.002008	0.006699	0.01214	qa^2
$-M_2^\circ$	0.1287	0.008969	0.01651	0.02561	qa^2
M_3°	0.03223	0.007799	0.01087	0.01439	qa^2
$-M_4^\circ$	0.03793	0.03106	0.03298	0.03497	qa^2
M_5°	0.05111	0.004354	0.007079	0.01049	qa^2
$-M_6^\circ$	0.1287	0.01217	0.01646	0.02208	qa^2
M_7°	0.03223	0.01502	0.01485	0.01477	qa^2
$-M_8^\circ$	0.03793	0.03783	0.03538	0.03267	qa^2
$-M_1 = M_3$	0.01821	0.01375	0.01294	0.01207	qa^2
$-M_2 = M_4$	0.04779	0	0.002854	0.006465	qa^2
$-M_5 = M_7$	0.01821	0.008398	0.009997	0.01180	qa^2
$-M_6 = M_8$	0.04779	0	0.003852	0.008404	qa^2
$M_a(\frac{1}{2})$	0	0.02941	0.02631	0.02272	qa^3
$-M_a(0)$	0	0.05066	0.04687	0.04244	qa^3
$M_b(\frac{1}{2})$	0	0.01698	0.01570	0.01403	qa^3
$-M_b(0)$	0	0.02833	0.02741	0.02596	qa^3

TABLE 4. (continued)

c	0.8				Units
λ_a	1.0	0.5	0.2	0	
λ_b	1.0	0.5	0.2	0	
α_{11}	0	-0.0948797	-0.2337528	See Table 6	
α_{12}	0	-0.0244703	-0.0844420		
α_{21}	0	-0.0483141	-0.1499828		
α_{22}	0	-0.0497286	-0.1891866		
α_{13}	0	-0.0026621	-0.0155721		
α_{31}	0	-0.0075401	-0.0357348		
α_{23}	0	-0.0133305	-0.0673447		
α_{32}	0	-0.0238455	-0.1112755		
α_{33}	0	-0.0128421	-0.0706093		
$w(\frac{1}{2},\frac{1}{2})$	0.001750	0.002270	0.002973	0.004052	qa^4/D
$w(0,\frac{1}{2})$	0.0005926	0.0009287	0.001405	0.002185	qa^4/D
$w(\frac{1}{2},0)$	0.001157	0.001739	0.002508	0.003654	qa^4/D
$M_x^c(\frac{1}{2},\frac{1}{2})$	0.01852	0.02277	0.02797	0.03448	qa^2
$M_y^c(\frac{1}{2},\frac{1}{2})$	0.01482	0.01390	0.01295	0.01216	qa^2
$-M_x^c(0,0)$	0.03704	0.06010	0.09951	∞	qa^2
$-M_y^c(0,0)$	0.02963	0.05037	0.08775	∞	qa^2
$M_x^c(\frac{1}{2},0)$	0.01852	0.02702	0.03729	0.04919	qa^2
$-M_y^c(\frac{1}{2},0)$	0.02963	0.02412	0.01751	0.01260	qa^2
$-M_x^c(0,\frac{1}{2})$	0.03704	0.03712	0.03716	0.03954	qa^2
$M_y^c(0,\frac{1}{2})$	0.01482	0.02244	0.03222	0.04456	qa^2
M_1^c	0.01852	0.02610	0.03520	0.04635	qa^2
$-M_2^c$	0.03704	0.05228	0.07500	0.1202	qa^2
M_3^c	0.01852	0.02345	0.02948	0.03699	qa^2
$-M_4^c$	0.03704	0.03927	0.04191	0.04643	qa^2
M_5^c	0.01482	0.02034	0.02747	0.03686	qa^2
$-M_6^c$	0.02963	0.04016	0.05614	0.08828	qa^2
M_7^c	0.01482	0.01504	0.01556	0.01648	qa^2
$-M_8^c$	0.02963	0.02621	0.02244	0.01838	qa^2
$-N_1 = N_3$	0.01111	0.01007	0.008982	0.007876	qa^2
$-N_2 = N_4$	0.01111	0.01723	0.02553	0.03740	qa^2
$-N_5 = N_7$	0.01389	0.01636	0.01942	0.02340	qa^2
$-N_6 = N_8$	0.01389	0.02065	0.02919	0.04037	qa^2
$M_a(\frac{1}{2})$	0.01852	0.01351	0.007459	0	qa^3
$-M_a(0)$	0.03704	0.03005	0.01990	0	qa^3
$M_b(\frac{1}{2})$	0.01185	0.008976	0.005155	0	qa^3
$-M_b(0)$	0.02370	0.02015	0.01404	0	qa^3

TABLE 4. (continued)

c	0.8				Units
λ_a	5.0	2.0	1.0	0.89443	
λ_b	6.25	2.5	1.25	1.11803	
α_{11}	+0.1284045	+0.0787999	+0.0131284	0	
α_{12}	+0.0163288	+0.0118853	+0.0024315	0	
α_{21}	+0.0433052	+0.0295710	+0.0056153	0	
α_{22}	+0.0225577	+0.0191622	+0.0044660	0	
α_{13}	+0.0002773	+0.0004007	+0.0001412	0	
α_{31}	+0.0025828	+0.0023419	+0.0006040	0	
α_{23}	+0.0012468	+0.0023937	+0.0008341	0	
α_{32}	+0.0032919	+0.0051678	+0.0016686	0	
α_{33}	−0.0016361	+0.0008683	+0.0006478	0	
$w(\frac{1}{2},\frac{1}{2})$	0.001008	0.001324	0.001717	0.001793	qa^4/D
$w(0,\frac{1}{2})$	0.0001236	0.0002894	0.0005169	0.0005631	qa^4/D
$w(\frac{1}{2},0)$	0.0003165	0.0006902	0.001144	0.001230	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.01211	0.01534	0.01901	0.01967	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.01630	0.01527	0.01424	0.01408	qa^2
$-M_x^\circ(0,0)$	0.009303	0.02090	0.03624	0.03935	qa^2
$-M_y^\circ(0,0)$	0.005617	0.01364	0.02558	0.02815	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.005245	0.01129	0.01837	0.01967	qa^2
$-M_y^\circ(\frac{1}{2},0)$	0.03840	0.03396	0.02903	0.02815	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.03727	0.03847	0.03926	0.03935	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.003226	0.007430	0.01298	0.01408	qa^2
M_1°	0.006731	0.01216	0.01851	0.01967	qa^2
$-M_2^\circ$	0.01665	0.02579	0.03714	0.03935	qa^2
M_3°	0.01104	0.01471	0.01891	0.01967	qa^2
$-M_4^\circ$	0.03374	0.03638	0.03893	0.03935	qa^2
M_5°	0.006505	0.009389	0.01330	0.01408	qa^2
$-M_6^\circ$	0.01548	0.02013	0.02678	0.02815	qa^2
M_7°	0.01456	0.01423	0.01408	0.01408	qa^2
$-M_8^\circ$	0.03501	0.03198	0.02872	0.02815	qa^2
$-N_1 = N_3$	0.01279	0.01179	0.01074	0.01056	qa^2
$-N_2 = N_4$	0.002344	0.005466	0.009705	0.01056	qa^2
$-N_5 = N_7$	0.01020	0.01217	0.01436	0.01475	qa^2
$-N_6 = N_8$	0.003836	0.008338	0.01374	0.01475	qa^2
$M_a(\frac{1}{2})$	0.02622	0.02259	0.01837	0.01760	qa^3
$-M_a(0)$	0.04651	0.04180	0.03624	0.03519	qa^3
$M_b(\frac{1}{2})$	0.01613	0.01486	0.01298	0.01259	qa^3
$-M_b(0)$	0.02809	0.02728	0.02558	0.02518	qa^3

TABLE 4. (continued)

c	0.8	0.5			Units
λ_a	0.5	∞	5.0	2.0	
λ_b	0.625	∞	5.0	2.0	
α_{11}	-0.0783882	+0.0394424	+0.0302503	+0.0176893	
α_{12}	-0.0187947	+0.0017901	+0.0018314	+0.0014293	
α_{21}	-0.0396280	+0.0336760	+0.0267743	+0.0163853	
α_{22}	-0.0385983	+0.0073822	+0.0070560	+0.0052186	
α_{13}	-0.0018237	+0.0000087	+0.0000264	+0.0000425	
α_{31}	-0.0060379	+0.0062732	+0.0053878	+0.0036287	
α_{23}	-0.0097162	+0.0002331	+0.0003675	+0.0003871	
α_{32}	-0.0182439	+0.0035093	+0.0040001	+0.0033857	
α_{33}	-0.0093493	+0.0000093	+0.0003240	+0.0004643	
$w(\frac{1}{2},\frac{1}{2})$	0.002226	0.0001584	0.0003756	0.0006417	qa^4/D
$w(0,\frac{1}{2})$	0.0008402	0	0.0000243	0.0000588	qa^4/D
$w(\frac{1}{2},0)$	0.001715	0	0.0002309	0.0005133	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.02326	0.0009630	0.004459	0.008681	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.01328	0.01000	0.009161	0.008159	qa^2
$-M_x^\circ(0,0)$	0.05853	0	0.007009	0.01591	qa^2
$-M_y^\circ(0,0)$	0.04480	0	0.002833	0.007137	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.02678	0	0.003755	0.008293	qa^2
$-M_y^\circ(\frac{1}{2},0)$	0.02352	0.02071	0.01887	0.01664	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.03947	0.01420	0.01810	0.02256	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.02044	0	0.001622	0.003851	qa^2
M_1°	0.02603	0.0001895	0.003896	0.008374	qa^2
$-M_2^\circ$	0.05220	0.003525	0.009853	0.01768	qa^2
M_3°	0.02382	0.0008004	0.004341	0.008617	qa^2
$-M_4^\circ$	0.04131	0.01227	0.01664	0.02172	qa^2
M_5°	0.01866	0.003376	0.004174	0.005314	qa^2
$-M_6^\circ$	0.03663	0.008345	0.009500	0.01125	qa^2
M_7°	0.01423	0.009174	0.008549	0.007818	qa^2
$-M_8^\circ$	0.02523	0.01956	0.01800	0.01614	qa^2
$-N_1 = N_3$	0.009668	0.007580	0.006929	0.006152	qa^2
$-N_2 = N_4$	0.01562	0	0.001179	0.002838	qa^2
$-N_5 = N_7$	0.01685	0.001472	0.003894	0.006812	qa^2
$-N_6 = N_8$	0.02041	0	0.002788	0.006184	qa^2
$M_a(\frac{1}{2})$	0.01339	0.02059	0.01877	0.01659	qa^3
$-M_a(0)$	0.02926	0.03772	0.03504	0.03182	qa^3
$M_b(\frac{1}{2})$	0.01022	0.004142	0.004055	0.003851	qa^3
$-M_b(0)$	0.02240	0.006881	0.007083	0.007137	qa^3

TABLE 4. (continued)

c	0.5				Units
λ_a	1.0	0.5	0	5	
λ_b	1.0	0.5	0	10	
α_{11}	0	−0.0261229	See Table 6	+0.0329700	
α_{12}	0	−0.0039854		+0.0017675	
α_{21}	0	−0.0278774		+0.0291420	
α_{22}	0	−0.0134618		+0.0071600	
α_{13}	0	−0.0003172		+0.0000198	
α_{31}	0	−0.0082243		+0.0058336	
α_{23}	0	−0.0018024		+0.0003236	
α_{32}	0	−0.0107920		+0.0039343	
α_{33}	0	−0.0024858		+0.0002378	
$w(\frac{1}{2},\frac{1}{2})$	0.0009766	0.001413	0.002914	0.0003747	qa^4/D
$w(0,\frac{1}{2})$	0.0001085	0.0001828	0.0005301	0.0000130	qa^4/D
$w(\frac{1}{2},0)$	0.0008681	0.001329	0.002900	0.0002303	qa^4/D
$M_x^o(\frac{1}{2},\frac{1}{2})$	0.01389	0.02048	0.04088	0.004521	qa^2
$M_y^o(\frac{1}{2},\frac{1}{2})$	0.006944	0.005447	0.001081	0.009134	qa^2
$-M_x^o(0,0)$	0.02778	0.04487	∞	0.006959	qa^2
$-M_y^o(0,0)$	0.01389	0.02520	∞	0.001507	qa^2
$M_x^o(\frac{1}{2},0)$	0.01389	0.02097	0.04245	0.003752	qa^2
$-M_y^o(\frac{1}{2},0)$	0.01389	0.01035	0.001084	0.01885	qa^2
$-M_x^o(0,\frac{1}{2})$	0.02778	0.03414	0.05576	0.01908	qa^2
$M_y^o(0,\frac{1}{2})$	0.006944	0.01132	0.02758	0.0008691	qa^2
M_1^o	0.01389	0.02085	0.04217	0.003905	qa^2
$-M_2^o$	0.02778	0.04164	0.1065	0.01003	qa^2
M_3^o	0.01389	0.02055	0.04117	0.004392	qa^2
$-M_4^o$	0.02778	0.03530	0.06013	0.01747	qa^2
M_5^o	0.006944	0.009300	0.01808	0.003677	qa^2
$-M_6^o$	0.01389	0.01796	0.03881	0.008679	qa^2
M_7^o	0.006944	0.005871	0.002755	0.008465	qa^2
$-M_8^o$	0.01389	0.01111	0.002853	0.01792	qa^2
$-N_1 = N_3$	0.005208	0.004043	0.0006890	0.006914	qa^2
$-N_2 = N_4$	0.005208	0.008692	0.02320	0.0006307	qa^2
$-N_5 = N_7$	0.01042	0.01501	0.02958	0.003989	qa^2
$-N_6 = N_8$	0.01042	0.01584	0.03300	0.002783	qa^2
$M_a(\frac{1}{2})$	0.01389	0.01048	0	0.01876	qa^3
$-M_a(0)$	0.02778	0.02243	0	0.03479	qa^3
$M_b(\frac{1}{2})$	0.003472	0.002831	0	0.004346	qa^3
$-M_b(0)$	0.006944	0.006299	0	0.007535	qa^3

TABLE 4. (continued)

c	0.5				Units
λ_a	2.0	1.0	0.70711	0.5	
λ_b	4.0	2.0	1.41421	1.0	
α_{11}	+0.0237609	+0.0099808	0	-0.0122056	
α_{12}	+0.0015529	+0.0008326	0	-0.0013975	
α_{21}	+0.0219426	+0.0097622	0	-0.0129257	
α_{22}	+0.0061546	+0.0032094	0	-0.0051911	
α_{13}	+0.0000311	+0.0000276	0	-0.0000766	
α_{31}	+0.0048069	+0.0024038	0	-0.0037317	
α_{23}	+0.0003744	+0.0002569	0	-0.0005528	
α_{32}	+0.0038573	+0.0022732	0	-0.0041327	
α_{33}	+0.0004169	+0.0003418	0	-0.0007962	
$w(\frac{1}{2},\frac{1}{2})$	0.0006385	0.0009686	0.001174	0.001397	qa^4/D
$w(0,\frac{1}{2})$	0.0000339	0.0000686	0.0000953	0.0001292	qa^4/D
$w(\frac{1}{2},0)$	0.0005108	0.0008611	0.001079	0.001315	qa^4/D
$M^\circ_x(\frac{1}{2},\frac{1}{2})$	0.008803	0.01406	0.01726	0.02067	qa^2
$M^\circ_y(\frac{1}{2},\frac{1}{2})$	0.008099	0.006850	0.006102	0.005319	qa^2
$-M^\circ_x(0,0)$	0.01565	0.02703	0.03452	0.04315	qa^2
$-M^\circ_y(0,0)$	0.004067	0.008560	0.01220	0.01705	qa^2
$M^\circ_x(\frac{1}{2},0)$	0.008279	0.01385	0.01726	0.02090	qa^2
$-M^\circ_y(\frac{1}{2},0)$	0.01662	0.01388	0.01220	0.01041	qa^2
$-M^\circ_x(0,\frac{1}{2})$	0.02463	0.03095	0.03452	0.03808	qa^2
$M^\circ_y(0,\frac{1}{2})$	0.002237	0.004444	0.006102	0.008153	qa^2
M°_1	0.008386	0.01390	0.01726	0.02084	qa^2
$-M^\circ_2$	0.01799	0.02809	0.03452	0.04170	qa^2
M°_3	0.008716	0.01402	0.01726	0.02070	qa^2
$-M^\circ_4$	0.02347	0.03046	0.03452	0.03867	qa^2
M°_5	0.004245	0.005277	0.006102	0.007159	qa^2
$-M^\circ_6$	0.009434	0.01094	0.01220	0.01388	qa^2
M°_7	0.007639	0.006668	0.006102	0.005521	qa^2
$-M^\circ_8$	0.01596	0.01361	0.01220	0.01074	qa^2
$-N_1 = N_3$	0.006119	0.005155	0.004576	0.003970	qa^2
$-N_2 = N_4$	0.001641	0.003304	0.004576	0.006176	qa^2
$-N_5 = N_7$	0.007007	0.01070	0.01294	0.01533	qa^2
$-N_6 = N_8$	0.006162	0.01036	0.01294	0.01573	qa^2
$M_a(\frac{1}{2})$	0.01656	0.01385	0.01220	0.01045	qa^3
$-M_a(0)$	0.03130	0.02703	0.02441	0.02157	qa^3
$M_b(\frac{1}{2})$	0.004475	0.004444	0.004315	0.004076	qa^3
$-M_b(0)$	0.008135	0.008560	0.008630	0.008523	qa^3

TABLE 5. CONVERGENCE OF A SEQUENCE OF APPROXIMATE SOLUTIONS FOR VARIOUS SYMMETRICAL SQUARE PANELS

$\lambda_b = \lambda_a = \infty$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	m = 1,2,3,4 n = 1,2,3,4	Units
α_{11}	+0.2722222	+0.2801583	+0.2801638	+0.2801659	
β_1	0	0	0	0	
α_{12}		+0.0510295	+0.0510287	+0.0510045	
β_2		0	0	0	
α_{22}		+0.0326245	+0.0317670	+0.0312770	
α_{13}			+0.0008797	+0.0007522	
β_3			0	0	
α_{23}			-0.0026147	-0.0046450	
α_{33}			-0.0098057	-0.0168698	
α_{14}				-0.0003257	
β_4				0	
α_{24}				-0.0043477	
α_{34}				-0.0114604	
α_{44}				-0.0095108	
$w(\frac{1}{2},\frac{1}{2})$	0.0013292	0.0012645	0.0012653	0.0012653	qa^4/D
$w(0,\frac{1}{2})$	0	0	0	0	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.02127	0.01756	0.01761	0.01762	qa^2
$-M_x^\circ(0,0)$	0	0	0	0	qa^2
$M_x^\circ(\frac{1}{2},0)$	0	0	0	0	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.04254	0.05116	0.05128	0.05139	qa^2
M_1°	0.004696	0.004082	0.004086	0.004107	qa^2
$-M_2^\circ$	0.009393	0.01343	0.01341	0.01328	qa^2
M_3°	0.01799	0.01499	0.01506	0.01505	qa^2
$-M_4^\circ$	0.03600	0.04468	0.04493	0.04492	qa^2
$-N_1 = N_3$	0.01595	0.01467	0.01468	0.01467	qa^2
$-N_2 = N_4$	0	0	0	0	qa^2
$M_a(\frac{1}{2})$	0.03032	0.03213	0.03210	0.03209	qa^3
$-M_a(0)$	0.06065	0.05428	0.05416	0.05423	qa^3

TABLE 5. (continued)

$c = 1.0$ $\lambda_b = \lambda_a = 5.0$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	m = 1,2,3,4 n = 1,2,3,4	Units
α_{11}	+0.1995927	+0.2068115	+0.2068197	+0.2068207	
β_1	+0.0049716	+0.0049267	+0.0049267	+0.0049267	
α_{12}		+0.0426816	+0.0428564	+0.0428407	
β_2		-0.0002651	-0.0002662	-0.0002661	
α_{22}		+0.0339484	+0.0352927	+0.0350170	
α_{13}			+0.0013736	+0.0012926	
β_3			-0.0000085	-0.0000080	
α_{23}			+0.0028258	+0.0016943	
α_{33}			-0.0032718	-0.0072868	
α_{14}				-0.0002065	
β_4				+0.0000013	
α_{24}				-0.0022682	
α_{34}				-0.0065512	
α_{44}				-0.0058765	
$w(\frac{1}{2},\frac{1}{2})$	0.0016694	0.0016217	0.0016222	0.0016222	qa^4/D
$w(0,\frac{1}{2})$	0.0003474	0.0003484	0.0003484	0.0003484	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.021152	0.018460	0.018479	0.018487	qa^2
$-M_x^\circ(0,0)$	0.011117	0.010221	0.010187	0.010194	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.0055584	0.0058065	0.0057981	0.0057971	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.042303	0.048400	0.048520	0.048572	qa^2
M_1°	0.0090018	0.0086964	0.0087226	0.0087325	qa^2
$-M_2^\circ$	0.018004	0.020634	0.020780	0.020720	qa^2
M_3°	0.018748	0.016572	0.016630	0.016630	qa^2
$-M_4^\circ$	0.037496	0.043821	0.044016	0.044003	qa^2
$-N_1 = N_3$	0.015864	0.014921	0.014922	0.014920	qa^2
$-N_2 = N_4$	0.0041688	0.0042244	0.0042254	0.0042256	qa^2
$M_a(\frac{1}{2})$	0.027792	0.029032	0.028990	0.028985	qa^3
$-M_a(0)$	0.055584	0.051106	0.050935	0.050972	qa^3

TABLE 5. (continued)

$c = 1.0 \qquad \lambda_b = \lambda_a = 2.0$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	m = 1,2,3,4 n = 1,2,3,4	Units
α_{11}	+0.1108597	+0.1159889	+0.1160046	+0.1160048	
β_1	+0.0110454	+0.0109817	+0.0109815	+0.0109815	
α_{12}		+0.0276114	+0.0278949	+0.0278906	
β_2		−0.0003430	−0.0003465	−0.0003465	
α_{22}		+0.0262708	+0.0286896	+0.0286240	
α_{13}			+0.0014561	+0.0014329	
β_3			−0.0000181	−0.0000178	
α_{23}			+0.0054417	+0.0051624	
α_{33}			+0.0014661	+0.0003244	
α_{14}				−0.0000587	
β_4				+0.0000007	
α_{24}				−0.0004054	
α_{34}				−0.0019356	
α_{44}				−0.0022095	
$w(\frac{1}{2},\frac{1}{2})$	0.0020850	0.0020579	0.0020581	0.0020581	qa^4/D
$w(0,\frac{1}{2})$	0.0007718	0.0007727	0.0007726	0.0007726	qa^4/D
$M_x^{\circ}(\frac{1}{2},\frac{1}{2})$	0.021010	0.019534	0.019533	0.019538	qa^2
$-M_x^{\circ}(0,0)$	0.024698	0.023527	0.023451	0.023455	qa^2
$M_x^{\circ}(\frac{1}{2},0)$	0.012349	0.012664	0.012647	0.012647	qa^2
$-M_x^{\circ}(0,\frac{1}{2})$	0.042020	0.045242	0.045340	0.045344	qa^2
M_1°	0.0014262	0.014194	0,014224	0.014224	qa^2
$-M_2^{\circ}$	0.028524	0.029734	0.029928	0.029921	qa^2
M_3°	0.019675	0.018484	0.018521	0.018522	qa^2
$-M_4^{\circ}$	0.039350	0.042826	0.042936	0.042926	qa^2
$-M_1 = M_3$	0.015758	0.015230	0.015229	0.015228	qa^2
$-M_2 = M_4$	0.0092619	0.0093290	0.0093315	0.0093316	qa^2
$M_a(\frac{1}{2})$	0.024698	0.025328	0.025294	0.025293	qa^3
$M_a^{\circ}(0)$	0.049396	0.047054	0.046902	0.046910	qa^3

TABLE 5. (continued)

$c = 1.0$ $\lambda_b = \lambda_a = 0.5$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	m = 1,2,3,4 n = 1,2,3,4	Units
α_{11}	−0.1424419	−0.1548472	−0.1550907	−0.1550979	
β_1	+0.0283842	+0.0286924	+0,0286985	+0.0286986	
α_{12}		−0,0531865	−0.0552652	−0,0553585	
β_2		+0.0013214	+0.0013731	+0.0013754	
α_{22}		−0.0672048	−0.0809059	−0.0817265	
α_{13}			−0.0069943	−0.0073321	
β_3			+0,0001738	+0,0001822	
α_{23}			−0.0292747	−0.0319844	
α_{33}			−0.0217654	−0.0288477	
α_{14}				−0.0007988	
β_4				+0.0000198	
α_{24}				−0.0061798	
α_{34}				−0.0102947	
α_{44}				−0.0056857	
$w(\frac{1}{2},\frac{1}{2})$	0.0032713	0.0033122	0,0033124	0.0033124	qa^4/D
$w(0,\frac{1}{2})$	0.0019834	0.0019843	0.0019851	0.0019851	qa^4/D
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	0.020606	0.022424	0,022477	0.022472	qa^2
$-M_y^\circ(0,0)$	0.063469	0.068122	0.068917	0.069037	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.031734	0.030592	0.030737	0.030722	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.041212	0.037787	0.037428	0.037589	qa^2
M_1°	0.029277	0.028950	0.028874	0.028897	qa^2
$-M_2^\circ$	0.058554	0,058204	0.057554	0.057416	qa^2
M_3°	0.022322	0.023790	0.023722	0.023714	qa^2
$-M_4^\circ$	0.044644	0.040340	0.040196	0.040214	qa^2
$-N_1 = N_3$	0.015455	0.016156	0.016164	0.016161	qa^2
$-N_2 = N_4$	0.023801	0.023595	0.023569	0.023572	qa^2
$M_a(\frac{1}{2})$	0.015867	0.015296	0.015368	0.015361	qa^3
$-M_a(0)$	0.031734	0.034061	0.034459	0.034518	qa^3

TABLE 6. COMPARISON OF APPROXIMATE AND EXACT SOLUTIONS FOR A PLATE SUPPORTED ONLY BY COLUMNS

$c = 1.0$ $\lambda_b = \lambda_a = 0$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	m = 1,2,3,4 n = 1,2,3,4	Exact Sol'n	Units
α_{11}	-0.597561	-0.743593	-0.7702079	-0.7781116		
β_1	+0.059538	+0.064980	+0.0659718	+0.0662663		
α_{12}		-0.458432	-0.5691109	-0.6035976		
β_2		+0.017085	+0.0212095	+0.0224948		
α_{22}		-0.722347	-1.1627929	-1.3126828		
α_{13}			-0.2235778	-0.3016386		
β_3			+0.0083322	+0.0112414		
α_{23}			-0.7367847	-1.0697145		
α_{33}			-0.6860838	-1.3620243		
α_{14}				-0.1247104		
β_4				+0.0046477		
α_{24}				-0.5051813		
α_{34}				-0.8239495		
α_{44}				-0.6025994		
$w(\frac{1}{2},\frac{1}{2})$	0.005403	0.005741	0.005781	0.005792	0.005800	qa^4/D
$w(0,\frac{1}{2})$	0.004160	0.004274	0.004337	0.004339	0.004350	qa^4/D
$M_x^c(\frac{1}{2},\frac{1}{2})$	0.01988	0.02669	0.02736	0.02746	0.02758	qa^2
$-M_x^c(0,0)$	0.1331	0.1966	0.2412	0.2754	∞	qa^2
$M_x^c(\frac{1}{2},0)$	0.06656	0.05343	0.05929	0.05621	0.05733	qa^2
$-M_x^c(0,\frac{1}{2})$	0.03976	0.03414	0.02460	0.03408	0.02975	qa^2
M_1^c	0.05626	0.05054	0.05100	0.05111	0.05111	qa^2
$-M_2^c$	0.1125	0.1286	0.1282	0.1292	0.1287	qa^2
M_3^c	0.02708	0.03280	0.03234	0.03222	0.03223	qa^2
$-M_4^c$	0.05416	0.03804	0.03845	0.03745	0.03793	qa^2
$-M_1 = M_3$	0.01491	0.01821	0.01833	0.01823	0.01821	qa^2
$-M_2 = M_4$	0.04992	0.04848	0.04725	0.04793	0.04779	qa^2
$M_a(\frac{1}{2})$	0	0	0	0	0	qa^3
$-M_a(0)$	0	0	0	0	0	qa^3

TABLE 6. (continued)

$c = 0.8$ $\lambda_b = \lambda_a = 0$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	Exact Sol'n	Units
α_{11}	−0.3635837	−0.4610980	−0.4753682		
α_{12}		−0.2189531	−0.2671234		
α_{21}		−0.3651994	−0.4311212		
α_{22}		−0.4297204	−0.6482044		
α_{13}			−0.0259602		
α_{31}			−0.1748941		
α_{23}			−0.1884084		
α_{32}			−0.4717248		
α_{33}			−0.2527159		
$w(\frac{1}{2},\frac{1}{2})$	0.003789	0.004014	0.004037	0.004052	qa^4/D
$w(0,\frac{1}{2})$	0.002014	0.002140	0.002156	0.002185	qa^4/D
$w(\frac{1}{2},0)$	0.003551	0.003592	0.003637	0.003654	qa^4/D
$M_x^c(\frac{1}{2},\frac{1}{2})$	0.02841	0.03452	0.03459	0.03448	qa^2
$M_y^c(\frac{1}{2},\frac{1}{2})$	0.005955	0.01112	0.01311	0.01216	qa^2
$-M_x^c(0,0)$	0.1136	0.1626	0.1946	∞	qa^2
$-M_y^c(0,0)$	0.1007	0.1516	0.1673	∞	qa^2
$M_x^c(\frac{1}{2},0)$	0.05682	0.04557	0.05074	0.04919	qa^2
$-M_y^c(\frac{1}{2},0)$	0.01191	0.01874	0.01127	0.01260	qa^2
$-M_x^c(0,\frac{1}{2})$	0.05682	0.04178	0.04142	0.03955	qa^2
$M_y^c(0,\frac{1}{2})$	0.05034	0.04234	0.04182	0.04456	qa^2
M_1^c	0.05054	0.04553	0.04634	0.04635	qa^2
$-M_2^c$	0.1011	0.1180	0.1210	0.1202	qa^2
M_3^c	0.03279	0.03781	0.03699	0.03699	qa^2
$-M_4^c$	0.06558	0.04862	0.04570	0.04643	qa^2
M_5^c	0.04054	0.03663	0.03589	0.03686	qa^2
$-M_6^c$	0.08107	0.08924	0.08423	0.08828	qa^2
M_7^c	0.01280	0.01671	0.01745	0.01648	qa^2
$-M_8^c$	0.02559	0.01743	0.02243	0.01838	qa^2
$-\overline{M}_1 = \overline{M}_3$	0.004466	0.008010	0.007912	0.007876	qa^2
$-\overline{M}_2 = \overline{M}_4$	0.03775	0.03803	0.03763	0.03740	qa^2
$-\overline{M}_5 = \overline{M}_7$	0.02131	0.02334	0.02345	0.02340	qa^2
$-\overline{M}_6 = \overline{M}_8$	0.04261	0.04087	0.03998	C.04037	qa^2

TABLE 6. (continued)

$c = 0.5 \qquad \lambda_b = \lambda_a = 0$

Parameter	m = 1 n = 1	m = 1,2 n = 1,2	m = 1,2,3 n = 1,2,3	Exact Sol'n	Units
α_{11}	−0.0947776	−0.1393460	−0.1500302		
α_{12}		−0.0354567	−0.0584519		
α_{21}		−0.1758046	−0.2241403		
α_{22}		−0.0910702	−0.1892505		
α_{13}			−0.0134650		
α_{31}			−0.1219897		
α_{23}			−0.0526255		
α_{32}			−0.1897833		
α_{33}			−0.0691311		
$w(\frac{1}{2},\frac{1}{2})$	0.002798	0.002898	0.002906	0.002914	qa^4/D
$w(0,\frac{1}{2})$	0.0004096	0.0005050	0.0005230	0.0005301	qa^4/D
$w(\frac{1}{2},0)$	0.002851	0.002865	0.002896	0.002900	qa^4/D
$M_x^\circ(\frac{1}{2},\frac{1}{2})$	0.03821	0.04205	0.04074	0.04088	qa^2
$M_y^\circ(\frac{1}{2},\frac{1}{2})$	−0.003405	0.001180	0.001175	0.001081	qa^2
$-M_x^\circ(0,0)$	0.09123	0.1146	0.1373	∞	qa^2
$-M_y^\circ(0,0)$	0.05243	0.08314	0.1042	∞	qa^2
$M_x^\circ(\frac{1}{2},0)$	0.04562	0.04010	0.04364	0.04245	qa^2
$-M_y^\circ(\frac{1}{2},0)$	−0.006810	0.007961	−0.004205	0.001084	qa^2
$-M_x^\circ(0,\frac{1}{2})$	0.07642	0.06148	0.05465	0.05576	qa^2
$M_y^\circ(0,\frac{1}{2})$	0.02621	0.02769	0.02788	0.02758	qa^2
M_1°	0.04398	0.04116	0.04240	0.04217	qa^2
$-M_2^\circ$	0.08796	0.09933	0.10594	0.10653	qa^2
M_3°	0.03935	0.04217	0.04094	0.04117	qa^2
$-M_4^\circ$	0.07871	0.06734	0.06073	0.06013	qa^2
M_5°	0.01967	0.01806	0.01809	0.01808	qa^2
$-M_6^\circ$	0.03934	0.03977	0.03759	0.03881	qa^2
M_7°	0.001161	0.002774	0.002740	0.002755	qa^2
$-M_8^\circ$	0.002322	0.001895	0.004081	0.002853	qa^2
$-N_1 = N_3$	−0.002554	0.001410	0.0007004	0.0006890	qa^2
$-N_2 = N_4$	0.01966	0.02337	0.02335	0.02320	qa^2
$-N_5 = N_7$	0.02866	0.02942	0.02961	0.02958	qa^2
$-N_6 = N_8$	0.03421	0.03330	0.03267	0.03300	qa^2

TABLE 7. COMPARISON OF SOLUTIONS FOR PLATES FIXED ON ALL FOUR BOUNDARIES

(All Units qa^2)

c	Mom. or Av. Mom.	S - Function Analysis	Hencky's Analysis*	Finite Difference Solution**
1.0	$M_x^o(\frac{1}{2},\frac{1}{2})$	0.0176	0.0175	0.0180
	$-M_x(0,\frac{1}{2})$	0.0514	0.0513	0.0474
	$\frac{1}{2}(M_1 + M_3)$	0.0096	0.0096	0.0102
	$-\frac{1}{2}(M_2 + M_4)$	0.0291	0.0290	0.0278
0.8	$M_x^o(\frac{1}{2},\frac{1}{2})$	0.0092		0.0094
	$M_y^o(\frac{1}{2},\frac{1}{2})$	0.0173		0.0178
	$-M_x(0,\frac{1}{2})$	0.0358	0.0358	0.0314
	$-M_y(\frac{1}{2},0)$	0.0424	0.0425	0.0404
	$\frac{1}{2}(M_1 + M_3)$	0.0049		0.0051
	$-\frac{1}{2}(M_2 + M_4)$	0.0200	0.0199	0.0179
	$\frac{1}{2}(M_5 + M_7)$	0.0097		0.0103
	$-\frac{1}{2}(M_6 + M_8)$	0.0250	0.0249	0.0244
0.5	$M_x^o(\frac{1}{2},\frac{1}{2})$	0.00096	0.00098	0.00101
	$M_y^o(\frac{1}{2},\frac{1}{2})$	0.0100	0.0101	0.0102
	$-M_x(0,\frac{1}{2})$	0.0142	0.0143	0.0099
	$-M_y(\frac{1}{2},0)$	0.0207	0.0207	0.0201
	$\frac{1}{2}(M_1 + M_3)$	0.00049	0.0005	0.00053
	$-\frac{1}{2}(M_2 + M_4)$	0.00789	0.00785	0.00558
	$\frac{1}{2}(M_5 + M_7)$	0.00628	0.00637	0.00670
	$-\frac{1}{2}(M_6 + M_8)$	0.01395	0.0139	0.0139

* Numerical values are those computed by Siess and Newmark (16) from coefficients tabulated by Wojtaszak (14).

** From solutions made at the University of Illinois by various persons working under the direction of N. M. Newmark and C. P. Siess. Each span was divided into eight equal parts; each panel consisted of 64 rectangular elements.

TABLE 8. FINITE DIFFERENCE SOLUTIONS FOR

VARIOUS SYMMETRICAL SQUARE PANELS*

$\lambda_a = \lambda_b$	0	0.5	1.0	2.0	5.0	∞	Units
$w(\frac{1}{2},\frac{1}{2})$	0.0067	0.0037	0.0029	0.0023	0.0018	0.0014	qa^4/D
$w(0,\frac{1}{2})$	0.0051	0.0023	0.0015	0.0009	0.0004	0	qa^4/D
$M_x^c(\frac{1}{2},\frac{1}{2})$	0.0288	0.0233	0.0215	0.0201	0.0190	0.0180	qa^2
$-M_x^c(0,0)$	0.1897	0.0665	0.0410	0.0233	0.0102	0	qa^2
$M_x(\frac{1}{2},0)$	0.0602	0.0318	0.0215	0.0130	0.0059	0	qa^2
$-M_x(0,\frac{1}{2})$	0.0335	0.0383	0.0410	0.0434	0.0455	0.0474	qa^2
M_1^c	0.0526	0.0297	0.0215	0.0148	0.0094	0.0047	qa^2
$-M_2^c$	0.1174	0.0563	0.0410	0.0297	0.0209	0.0139	qa^2
M_3^c	0.0332	0.0244	0.0215	0.0191	0.0172	0.0156	qa^2
$-M_4^c$	0.0430	n.0409	0.0410	0.0412	0.0414	0.0417	qa^2
$M_a(\frac{1}{2})$	0	0.0159	0.0215	0.0260	0.0297	-	qa^3
$-M_a(0)$	0	0.0332	0.0410	0.0467	0.0510	-	qa^3

* From solutions made at the University of Illinois by various persons working under the direction of N. M. Newmark and C. P. Siess. Each span was divided into eight equal parts; each panel consisted of 64 square elements.

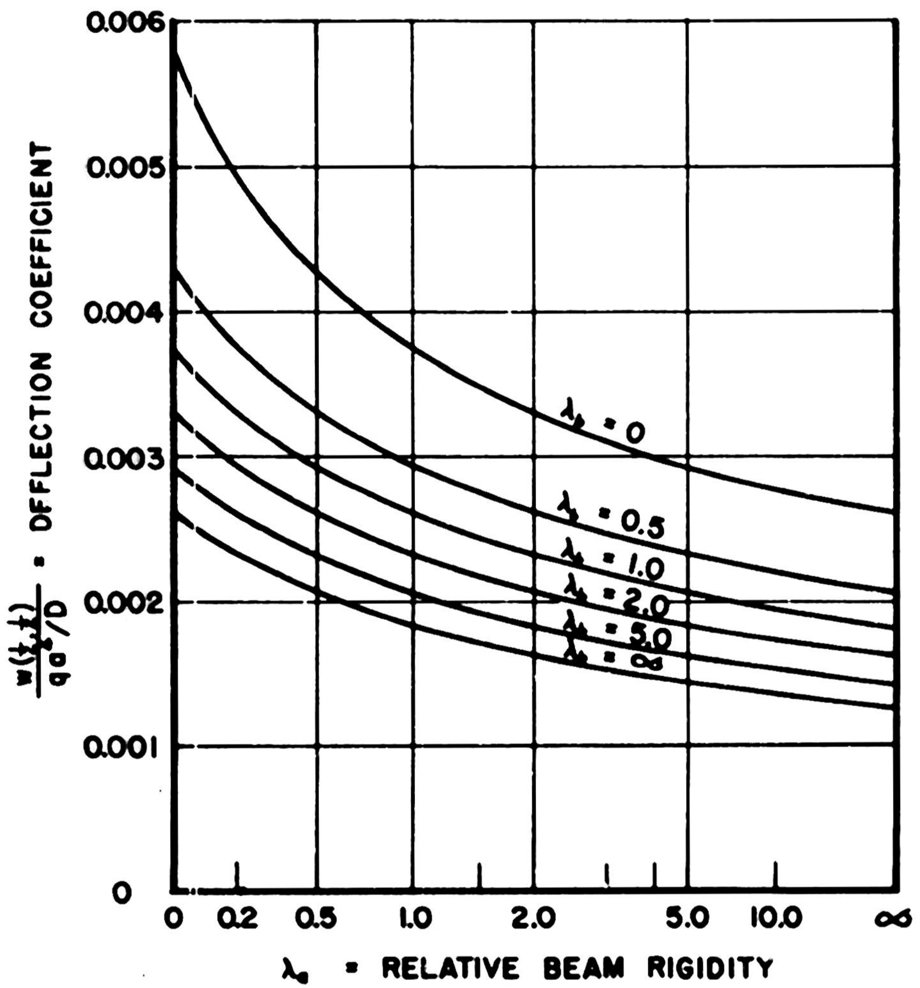

FIG. 2 DEFLECTION AT THE CENTER OF A SQUARE PANEL

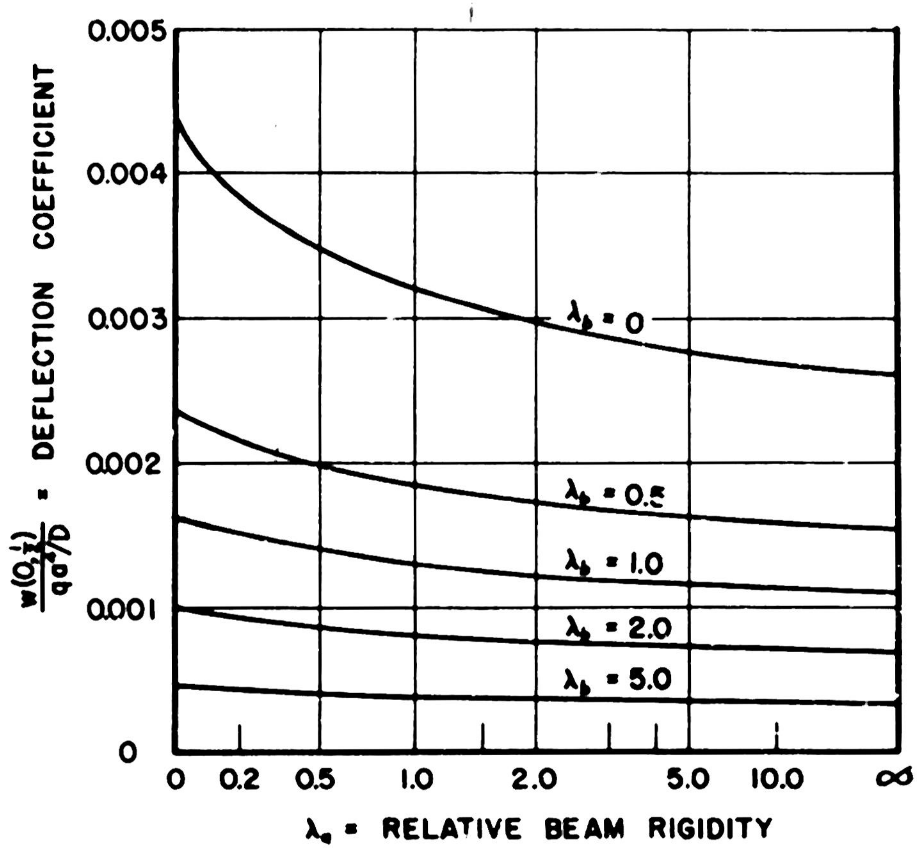

FIG. 3 DEFLECTION AT THE CENTERS OF THE BEAMS OF A SQUARE PANEL

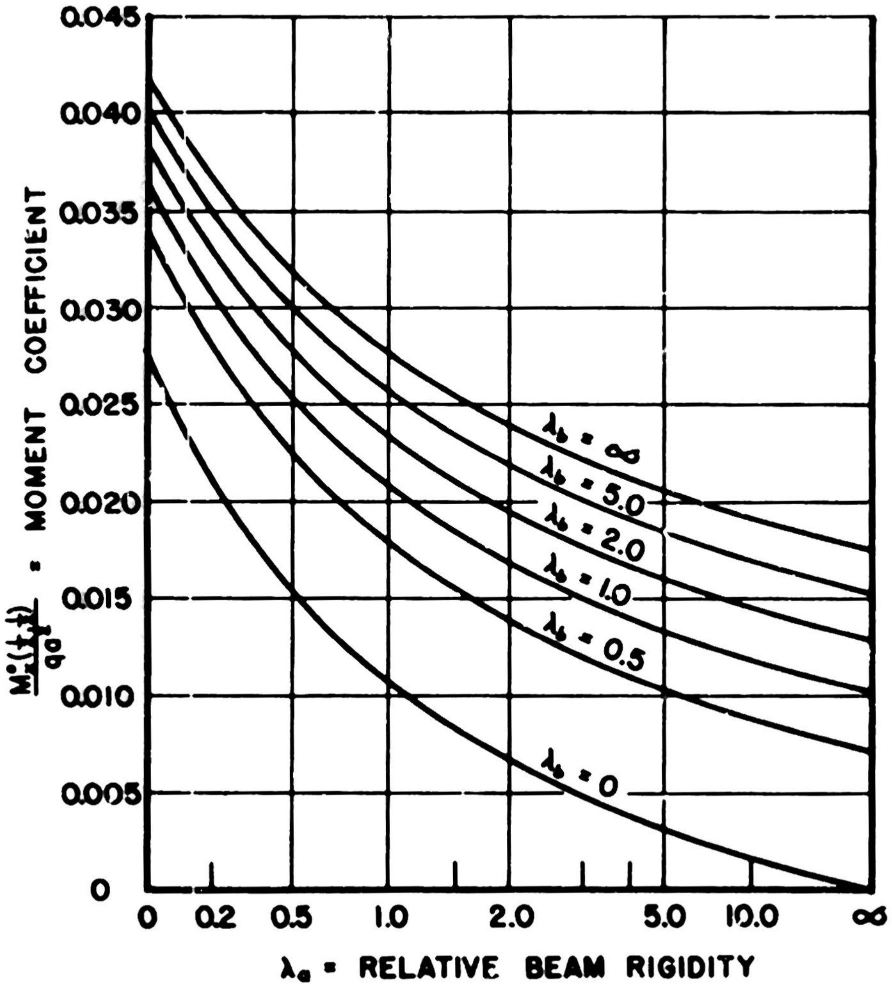

FIG. 4 MOMENT AT THE CENTER OF A SQUARE PANEL

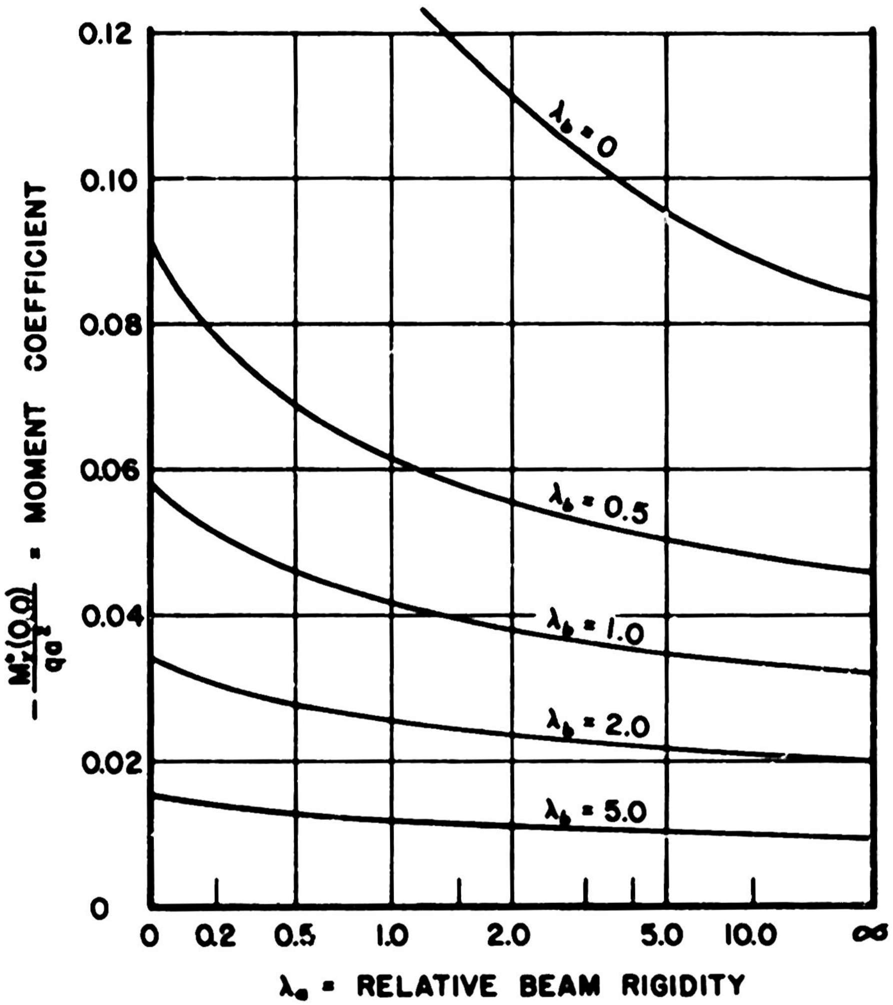

FIG. 5 PLATE MOMENT AT THE CORNER OF A SQUARE PANEL

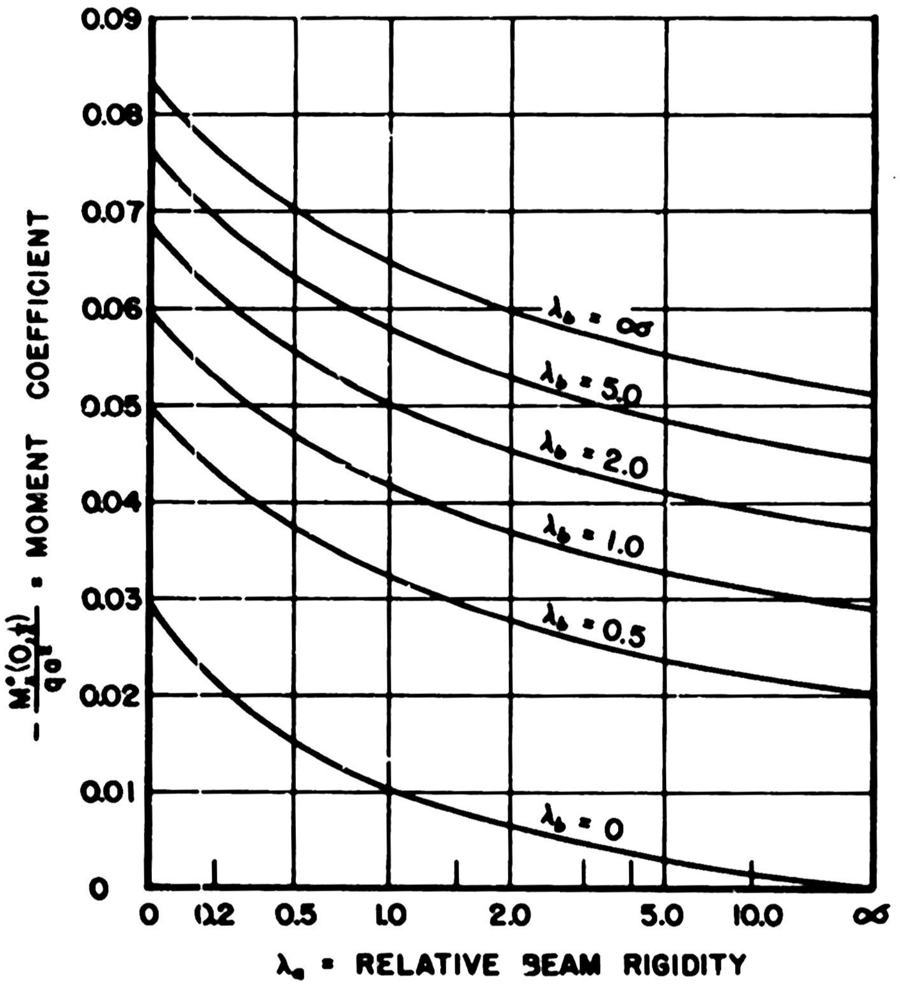

FIG. 6 NEGATIVE PLATE MOMENT AT THE MIDDLE OF A SIDE OF A SQUARE PANEL

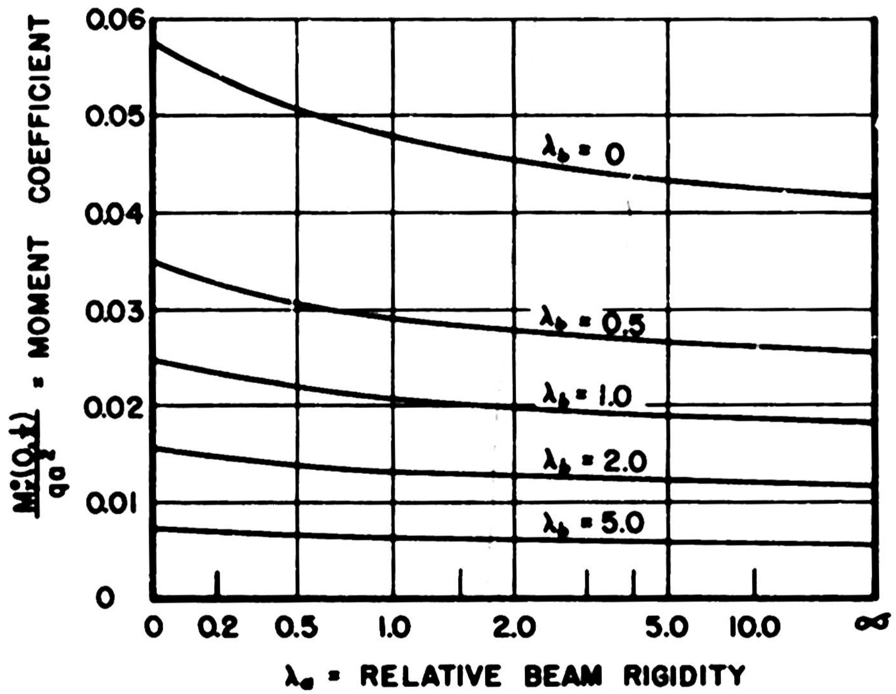

FIG. 7 POSITIVE PLATE MOMENT AT THE MIDDLE OF A SIDE OF A SQUARE PANEL

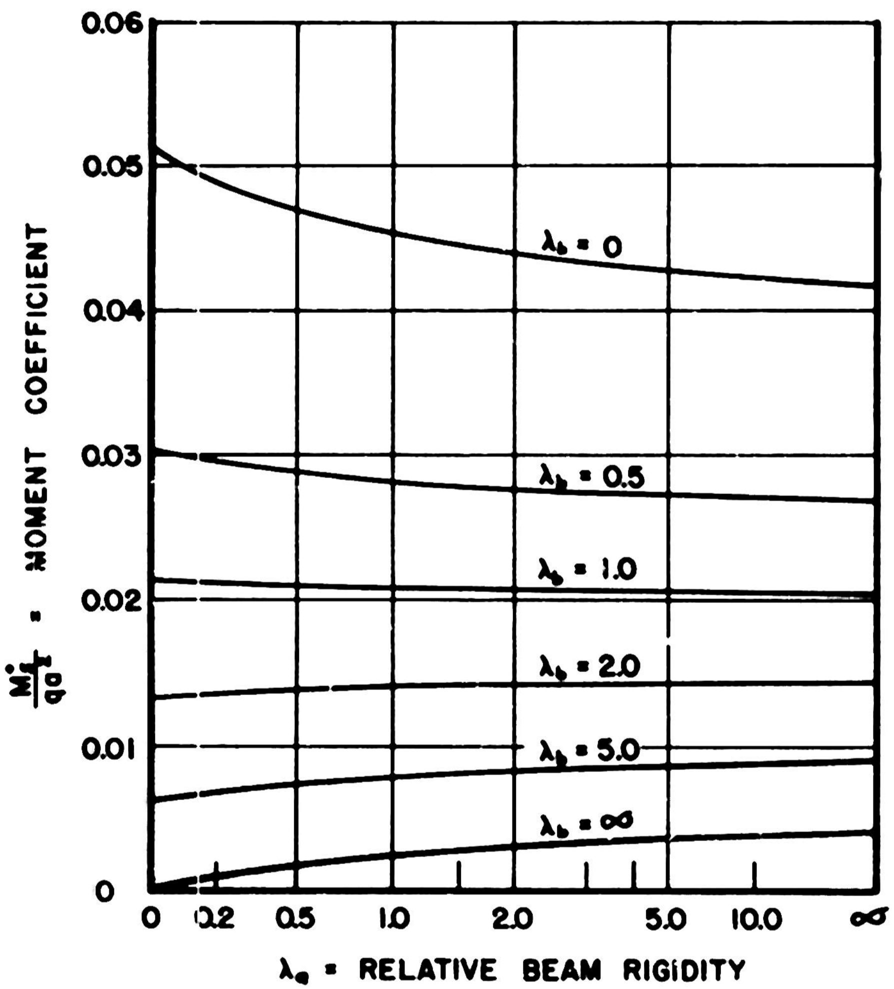

FIG. 8 AVERAGE POSITIVE MOMENT IN THE COLUMN STRIP FOR A SQUARE PANEL

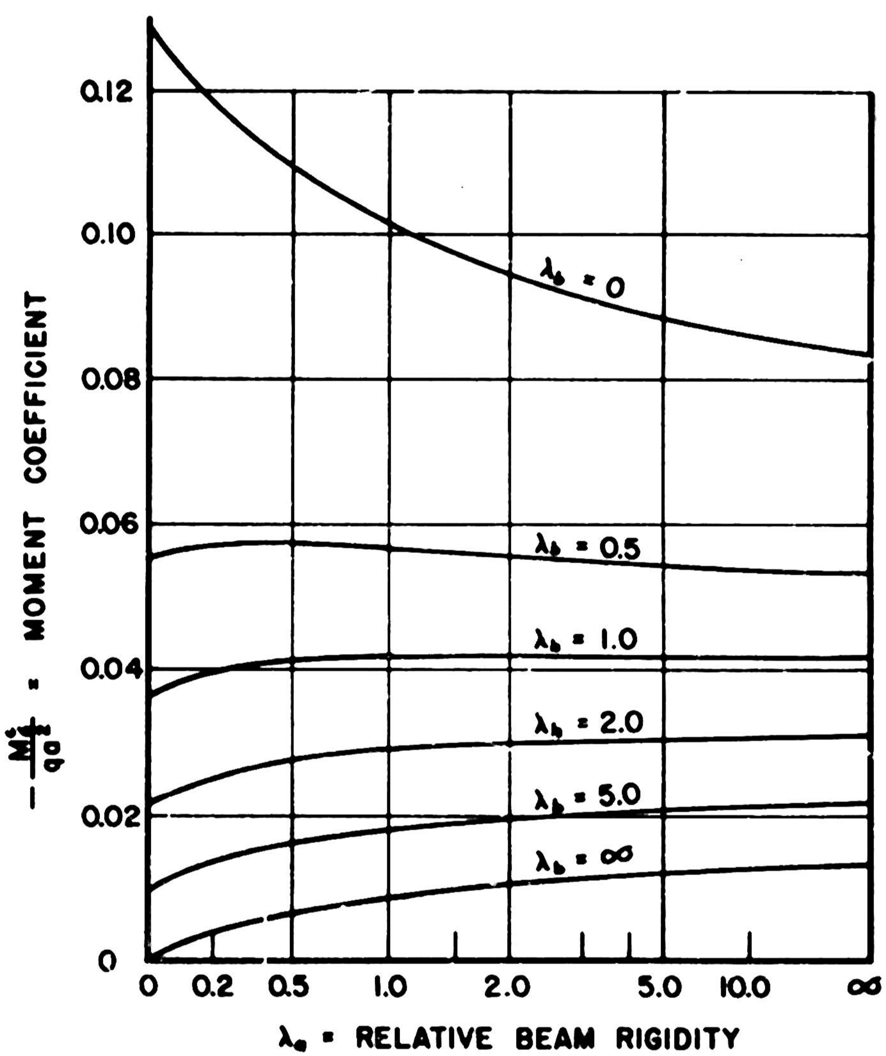

FIG. 9 AVERAGE NEGATIVE MOMENT IN THE COLUMN STRIP FOR A SQUARE PANEL

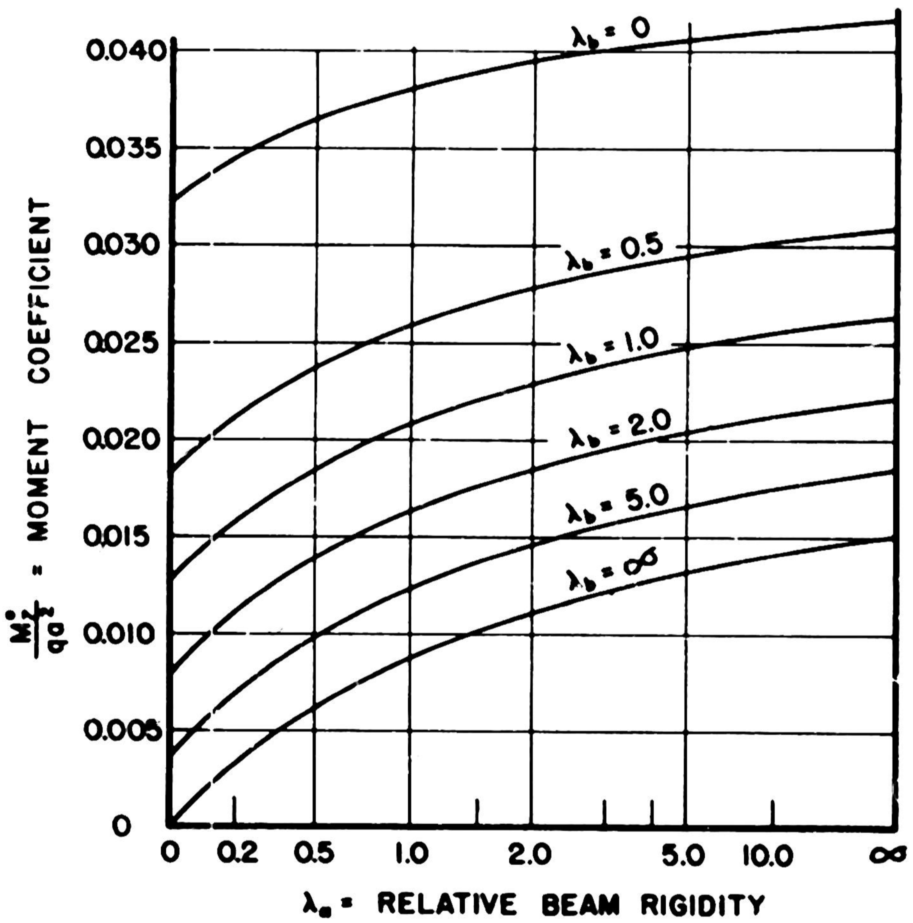

FIG. 10 AVERAGE POSITIVE MOMENT IN THE MIDDLE STRIP FOR A SQUARE PANEL

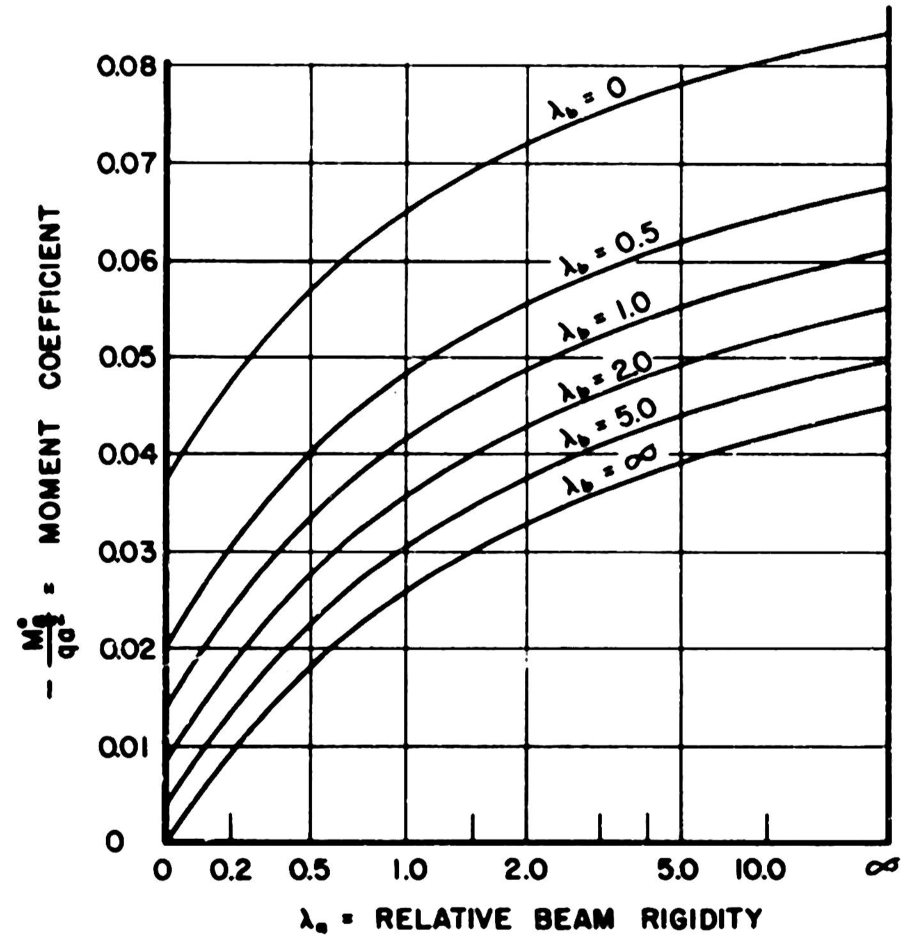

FIG. 11 AVERAGE NEGATIVE MOMENT IN THE MIDDLE STRIP FOR A SQUARE PANEL

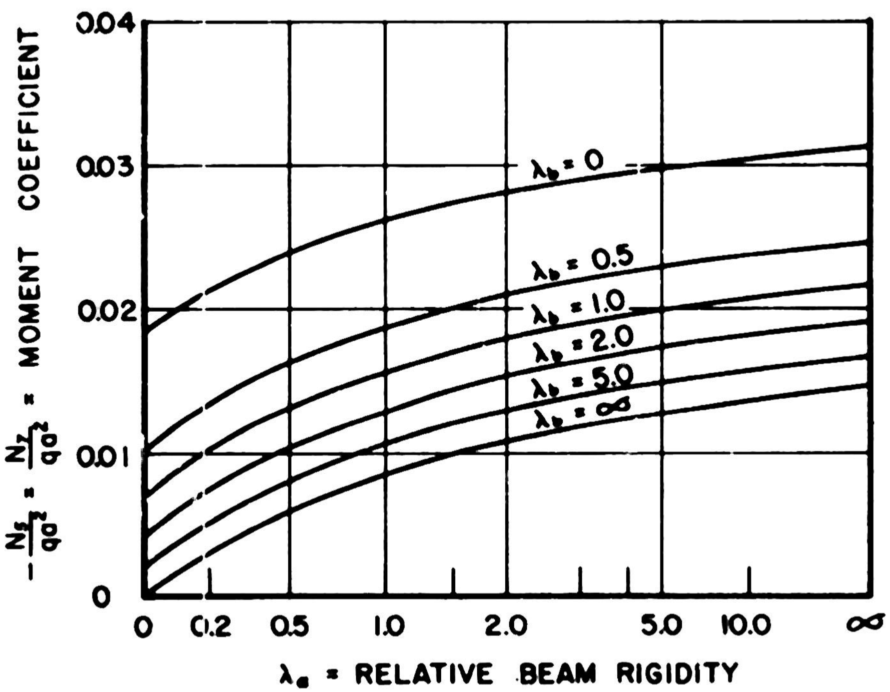

FIG. 12 POISSON'S RATIO CORRECTION TO AVERAGE POSITIVE MOMENT FOR A SQUARE PANEL

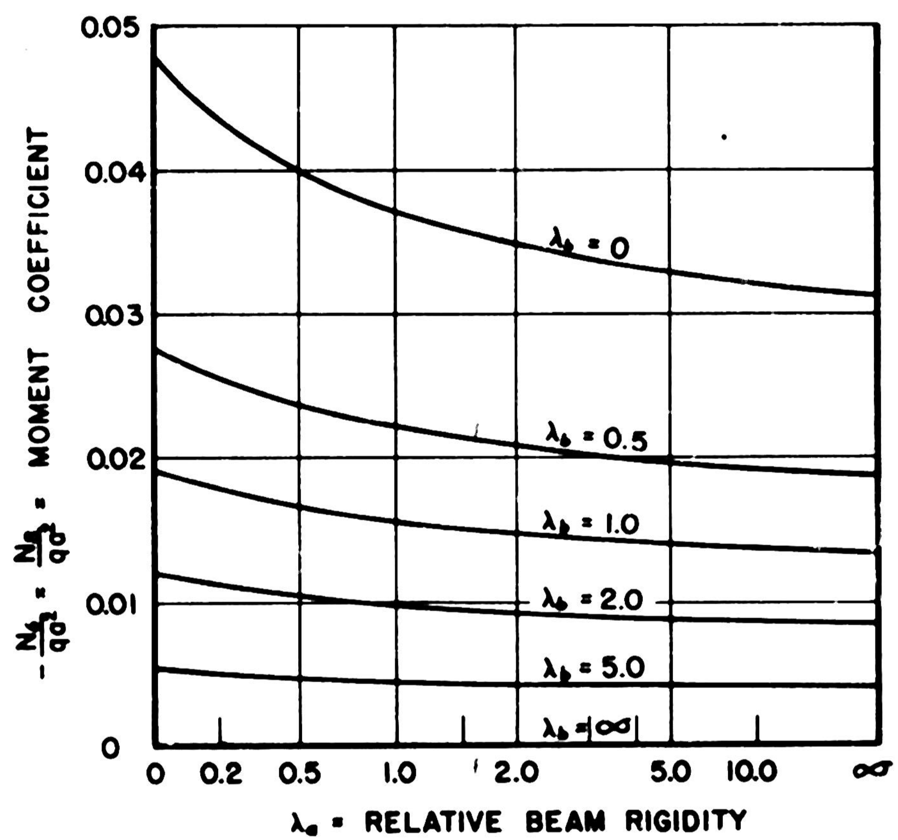

FIG. 13 POISSON'S RATIO CORRECTION TO AVERAGE NEGATIVE MOMENT FOR A SQUARE PANEL

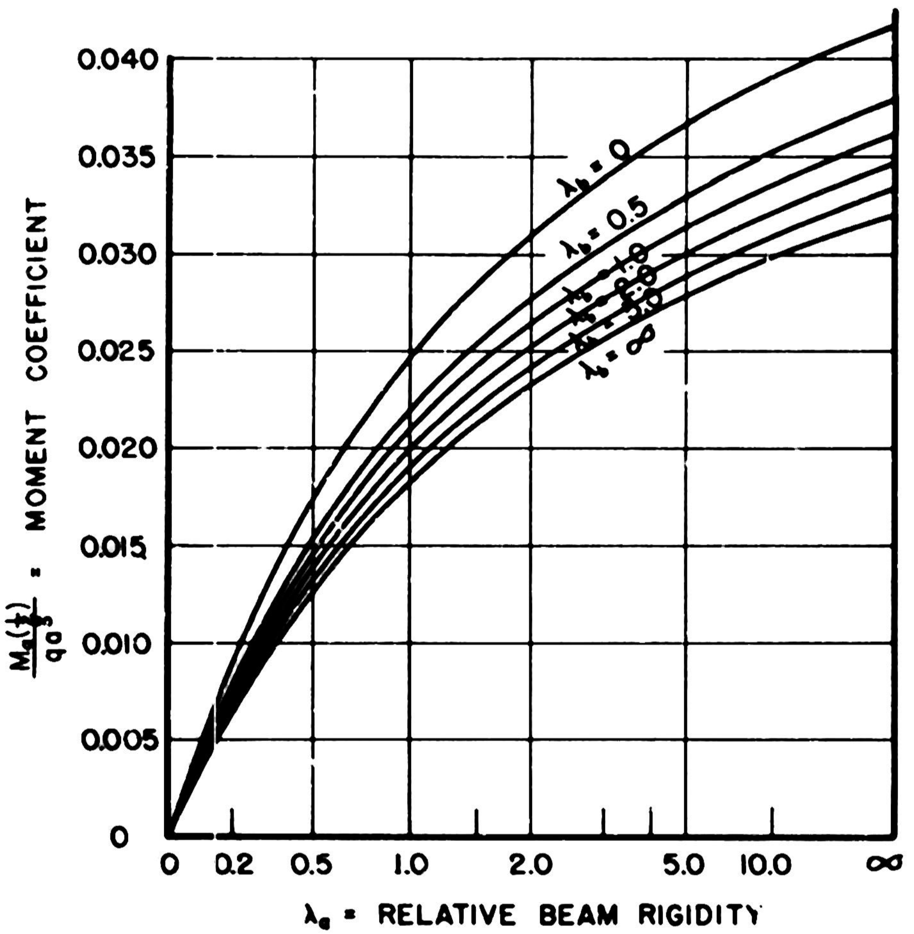

FIG. 14 MAXIMUM POSITIVE BEAM MOMENT FOR A SQUARE PANEL

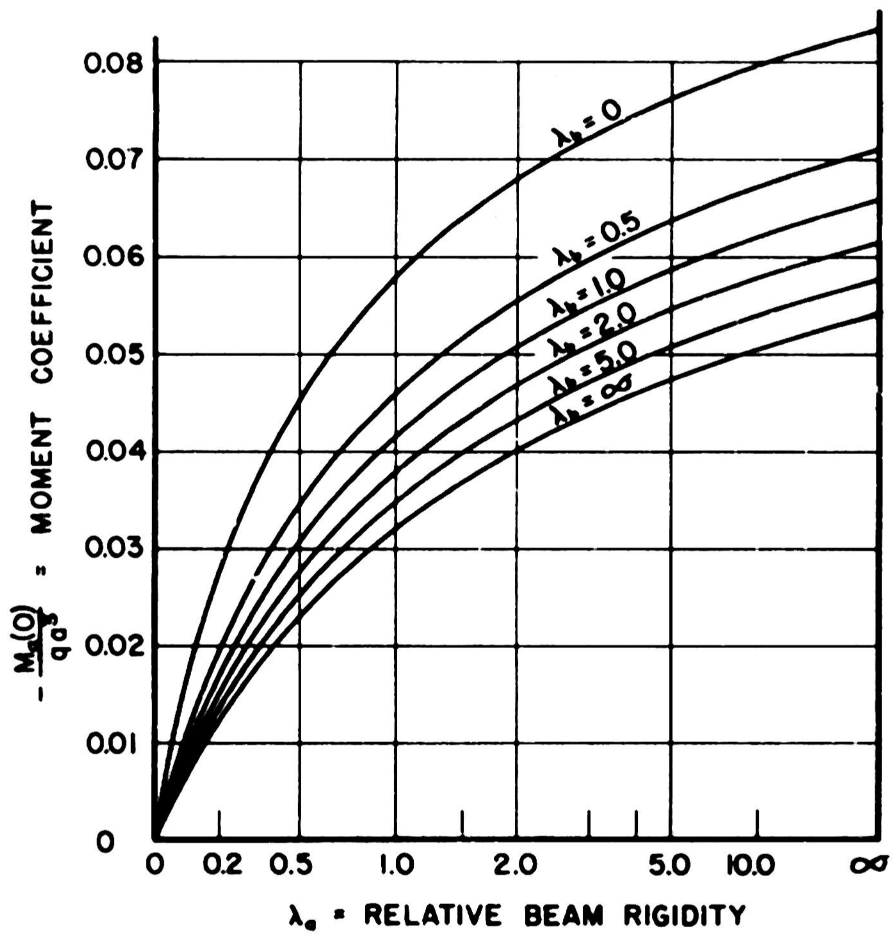

FIG. 15 MAXIMUM NEGATIVE BEAM MOMENT FOR A SQUARE PANEL

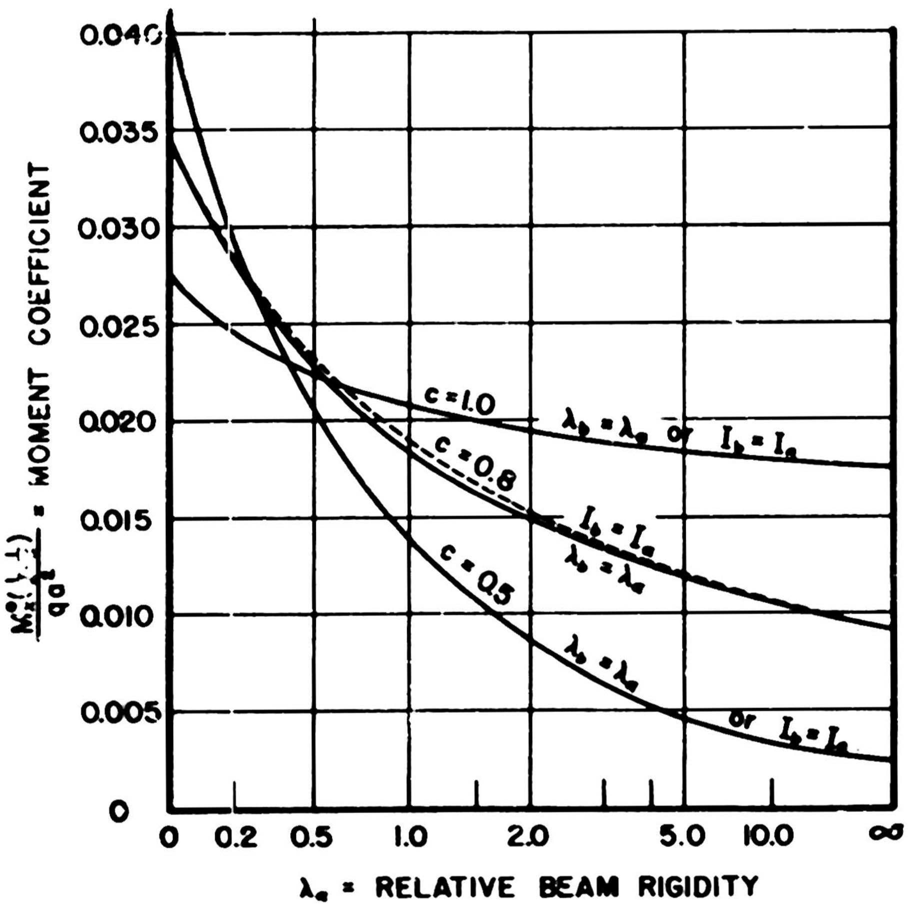

FIG. 19 MOMENT M_x° AT THE CENTER OF A PANEL

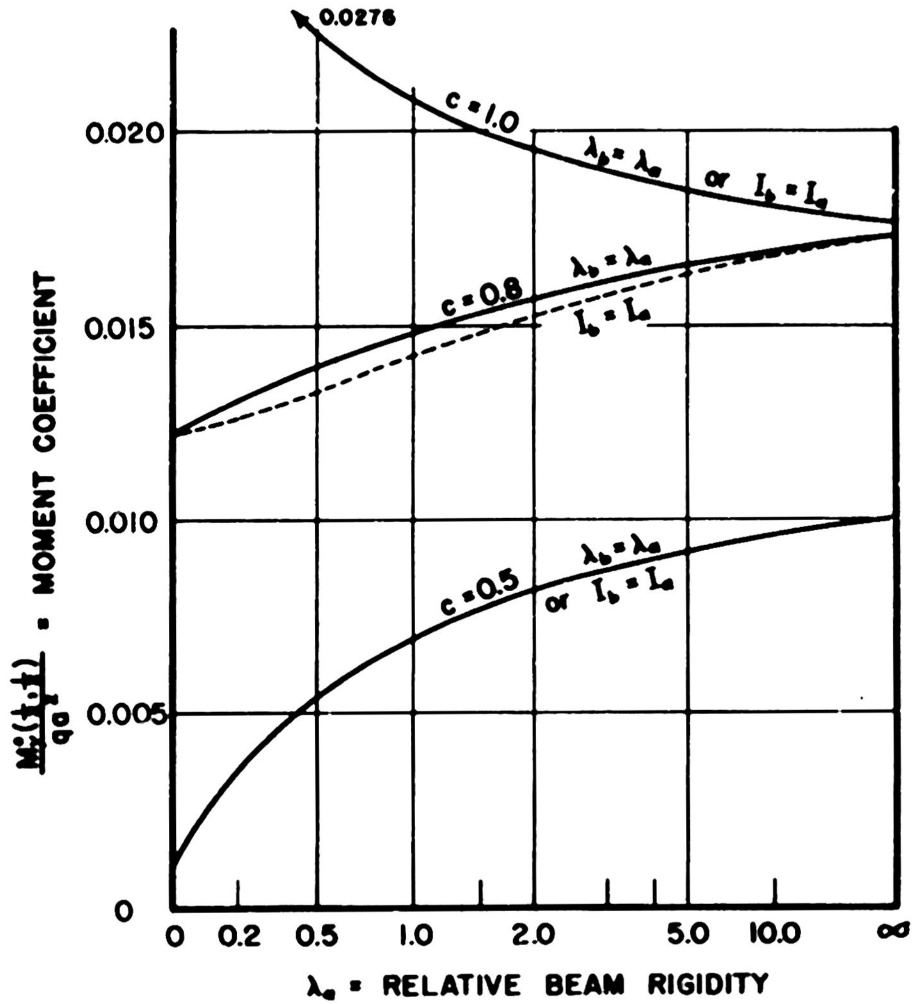

FIG. 20 MOMENT M_y° AT THE CENTER OF A PANEL

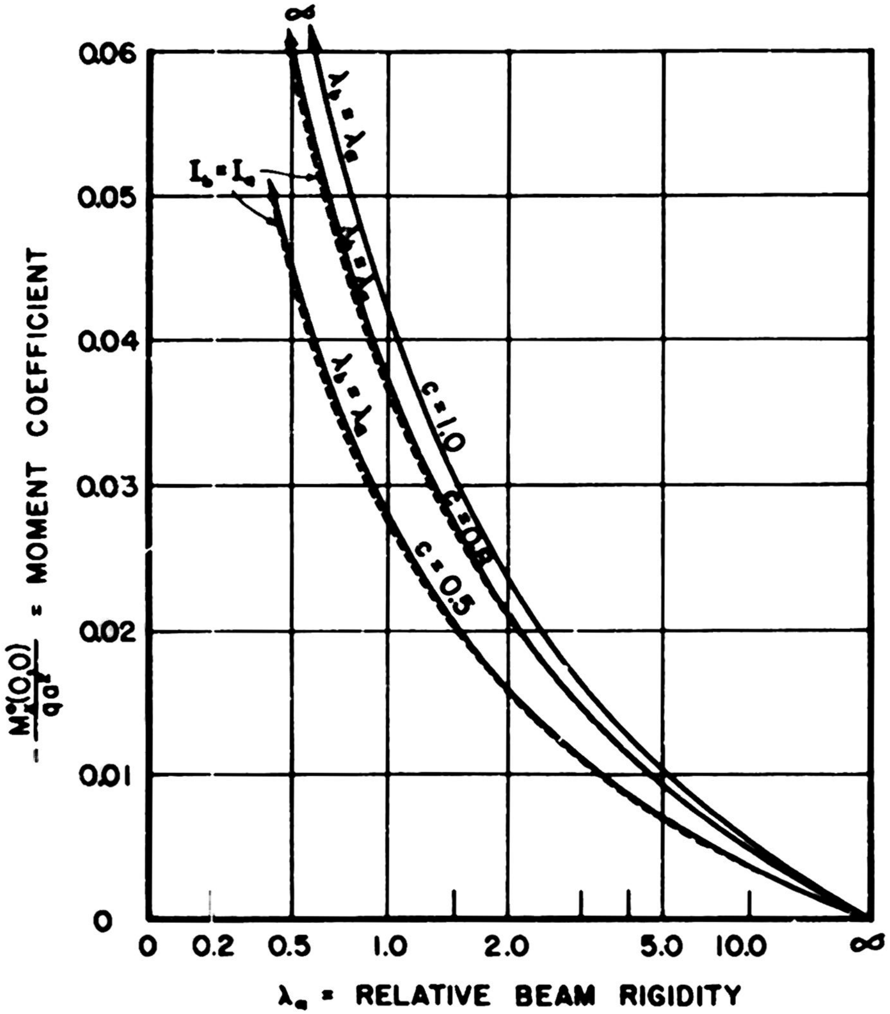

FIG. 21 MOMENT M_x° AT THE CORNER OF A PANEL

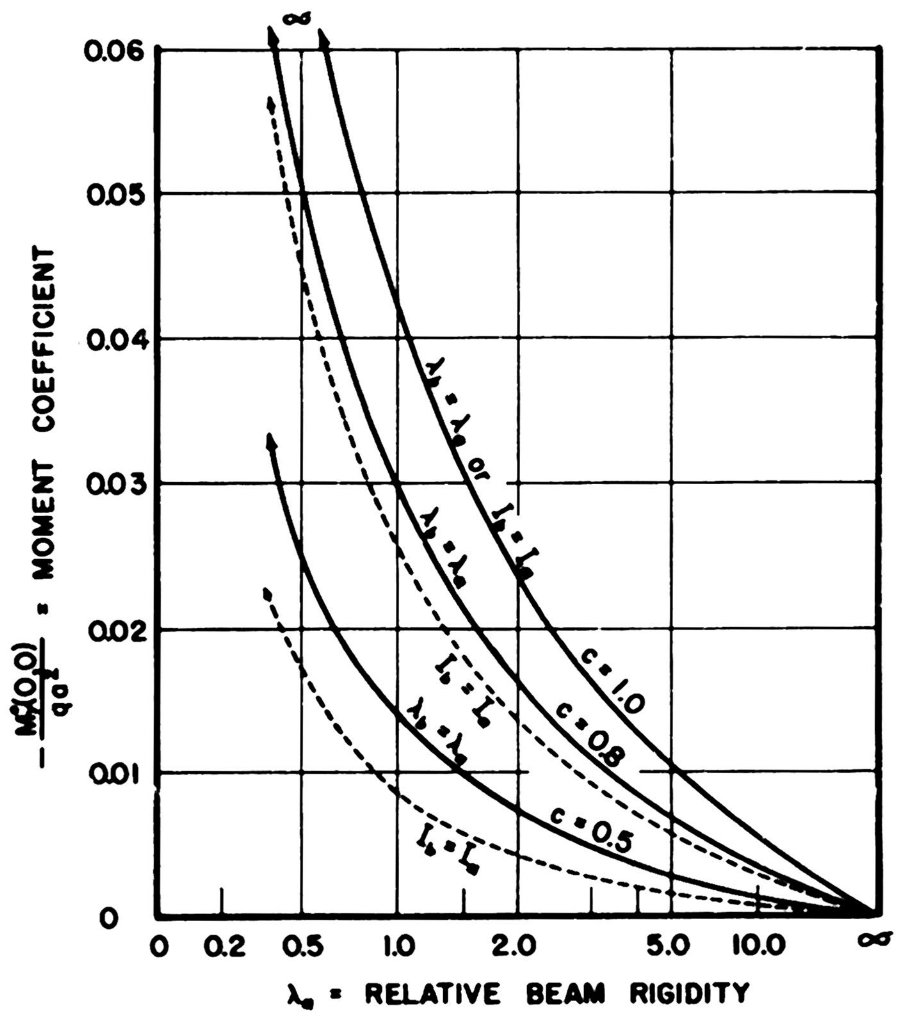

FIG. 22 MOMENT M_y° AT THE CORNER OF A PANEL

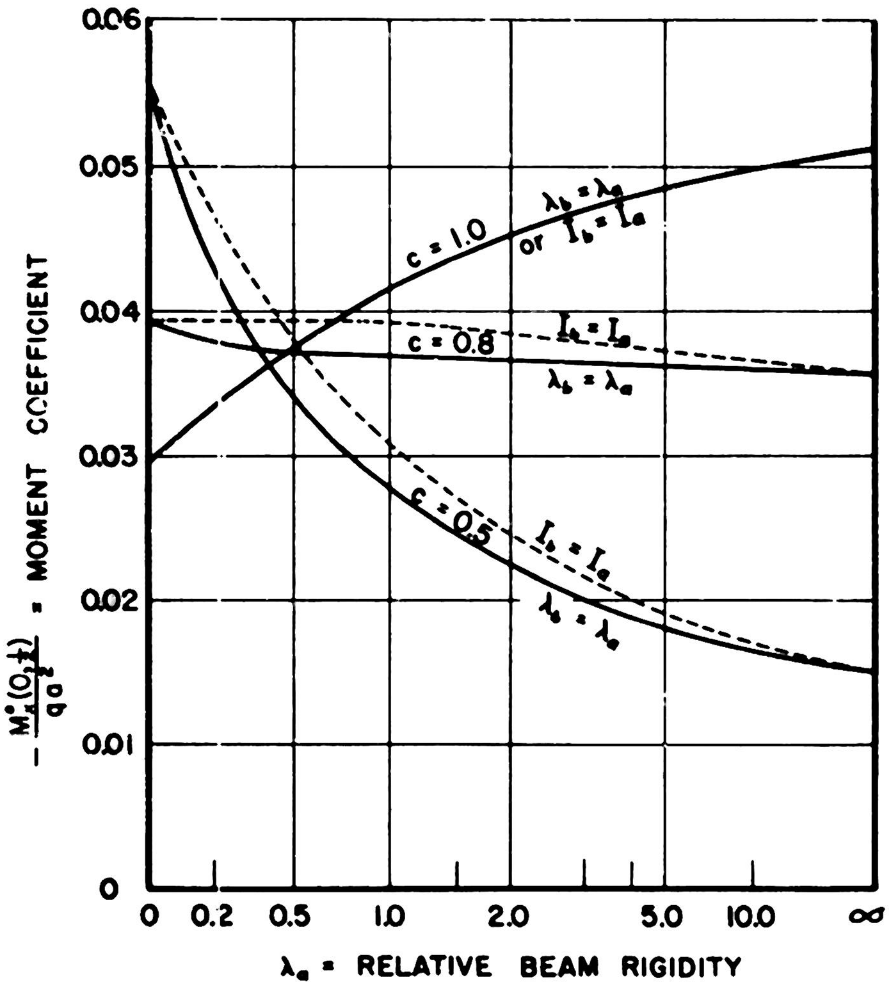

FIG. 25 NEGATIVE PLATE MOMENT AT THE CENTER OF THE SHORT SIDE

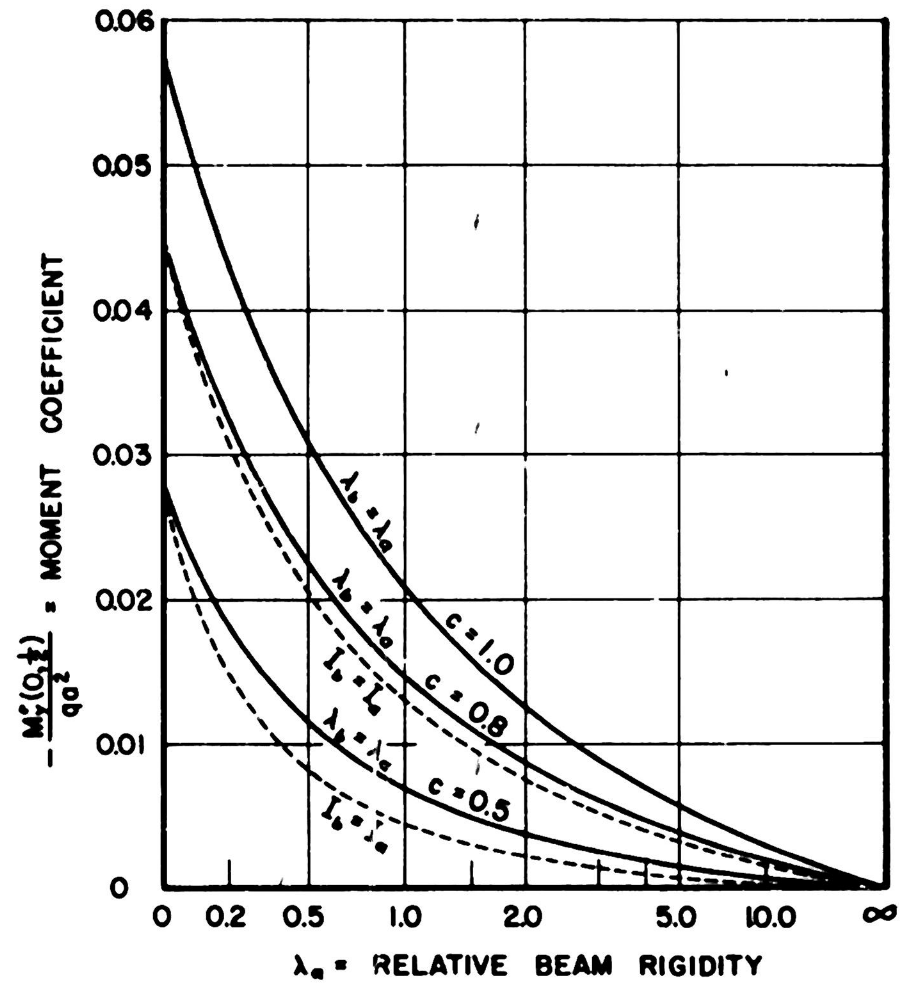

FIG. 26 POSITIVE PLATE MOMENT AT THE CENTER OF THE SHORT SIDE

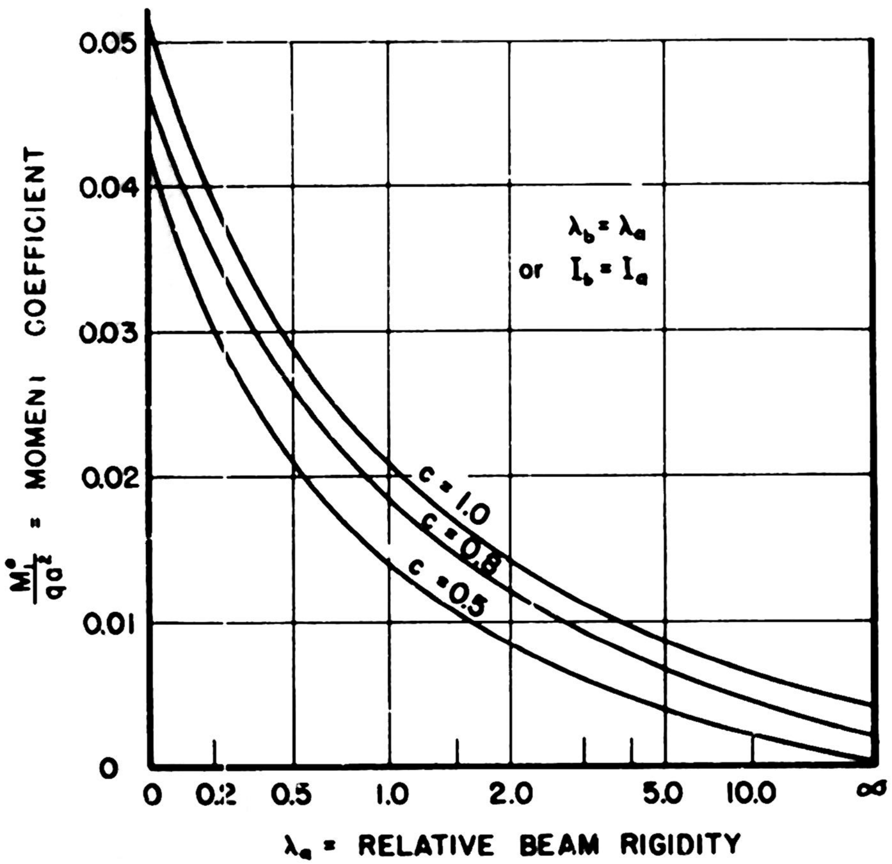

FIG. 27 AVERAGE POSITIVE MOMENT IN THE LONG COLUMN STRIP

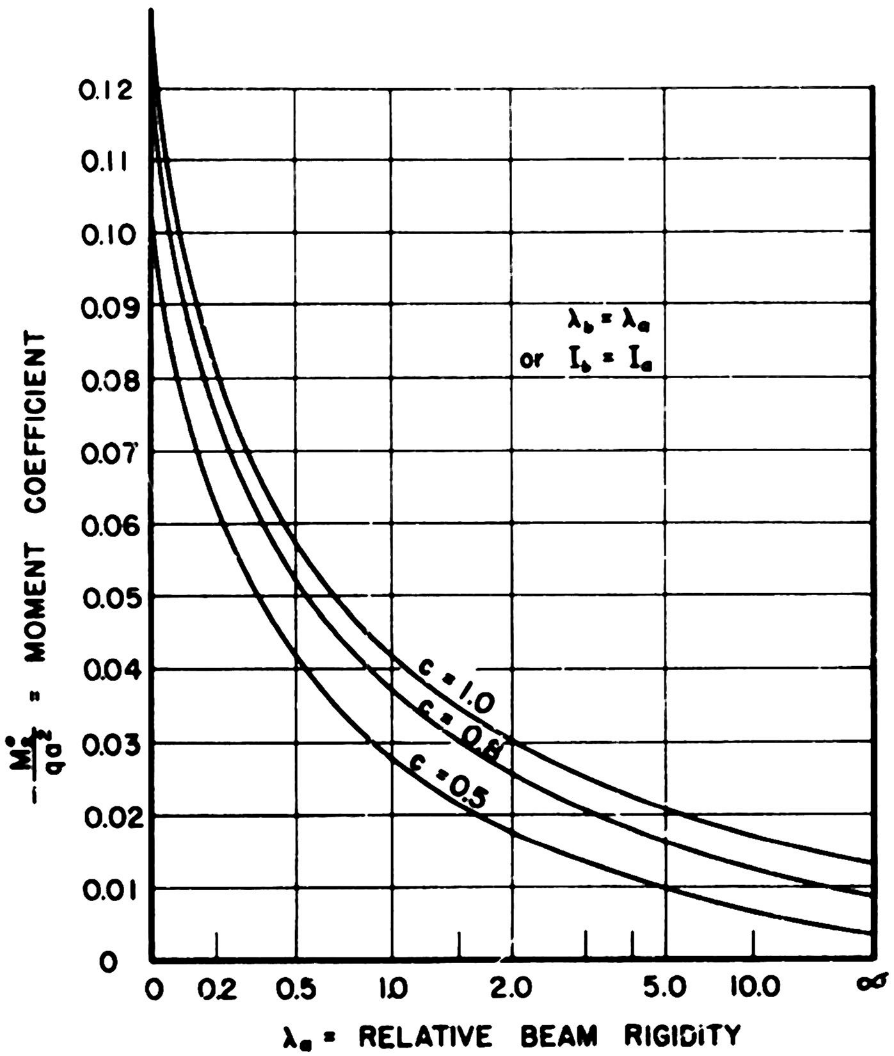

FIG. 28 AVERAGE NEGATIVE MOMENT IN THE LONG COLUMN STRIP

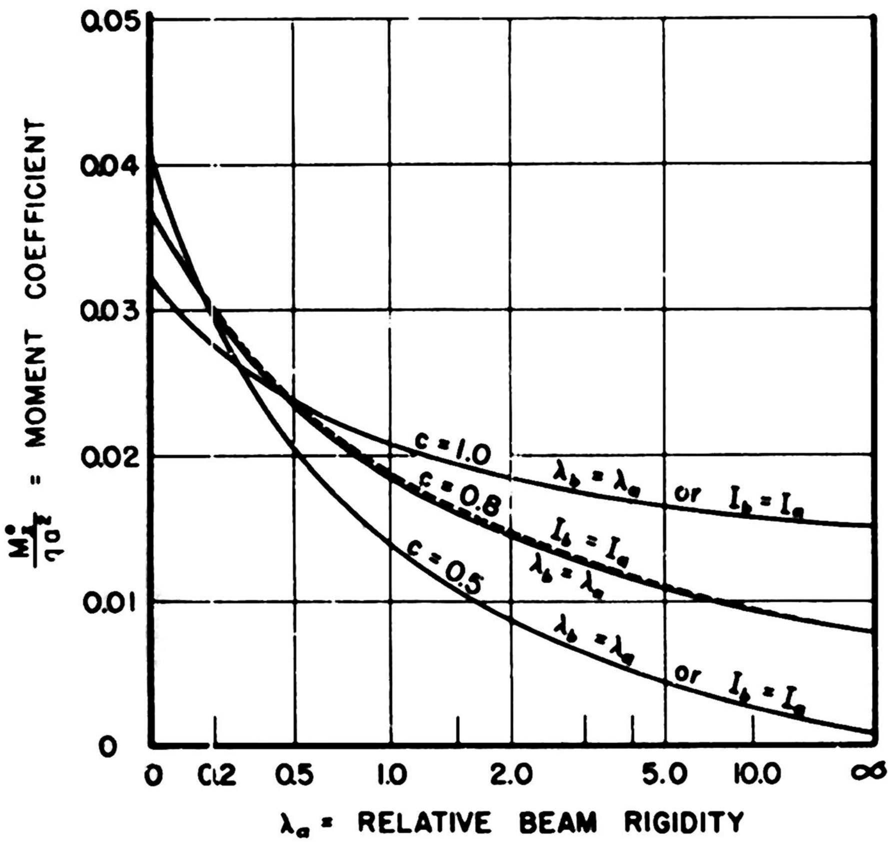

FIG. 29 AVERAGE POSITIVE MOMENT IN THE LONG MIDDLE STRIP

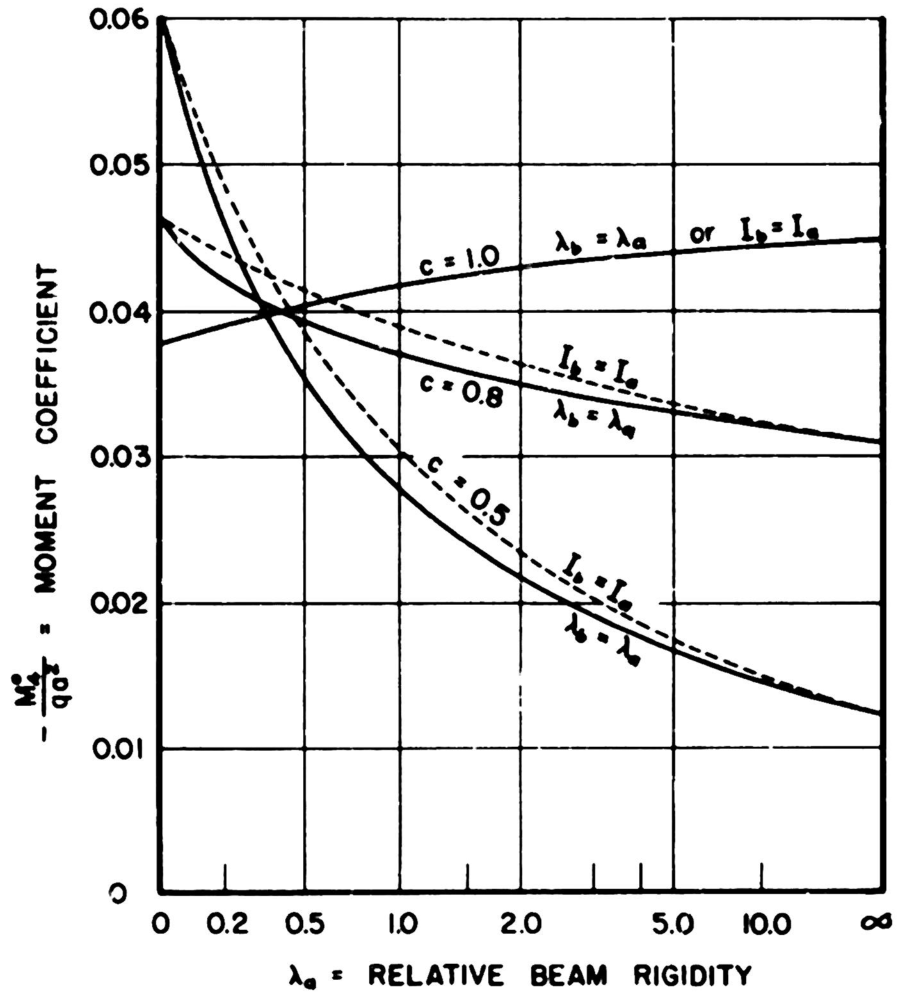

FIG. 30 AVERAGE NEGATIVE MOMENT IN THE LONG MIDDLE STRIP

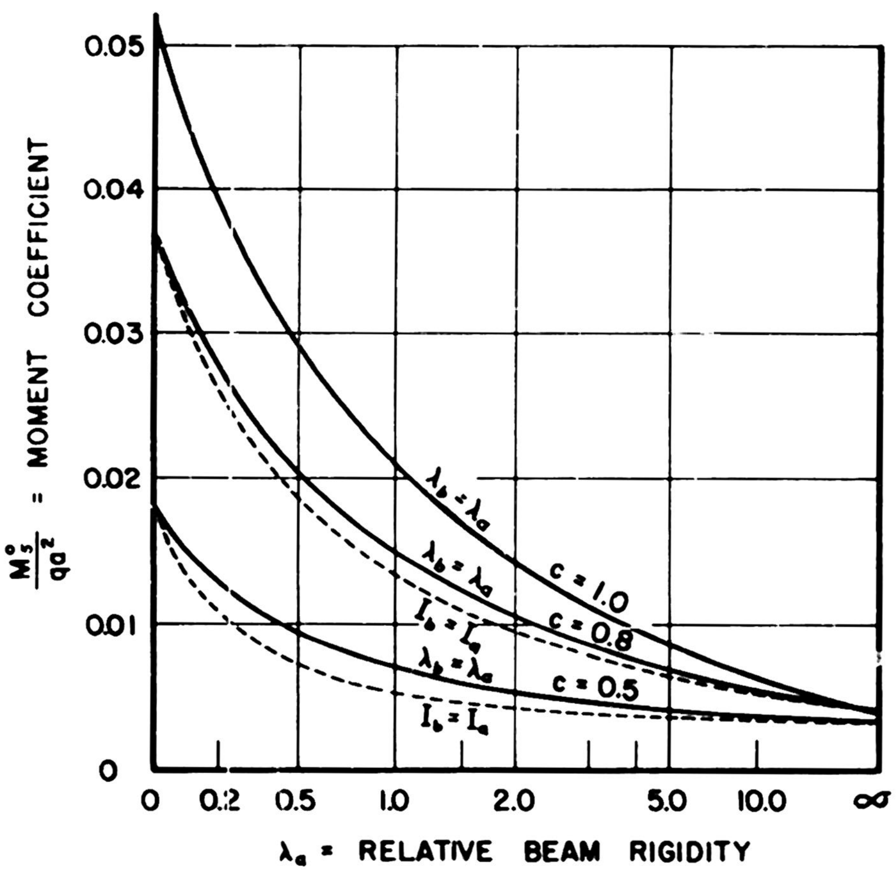

FIG. 31 AVERAGE POSITIVE MOMENT IN THE SHORT COLUMN STRIP

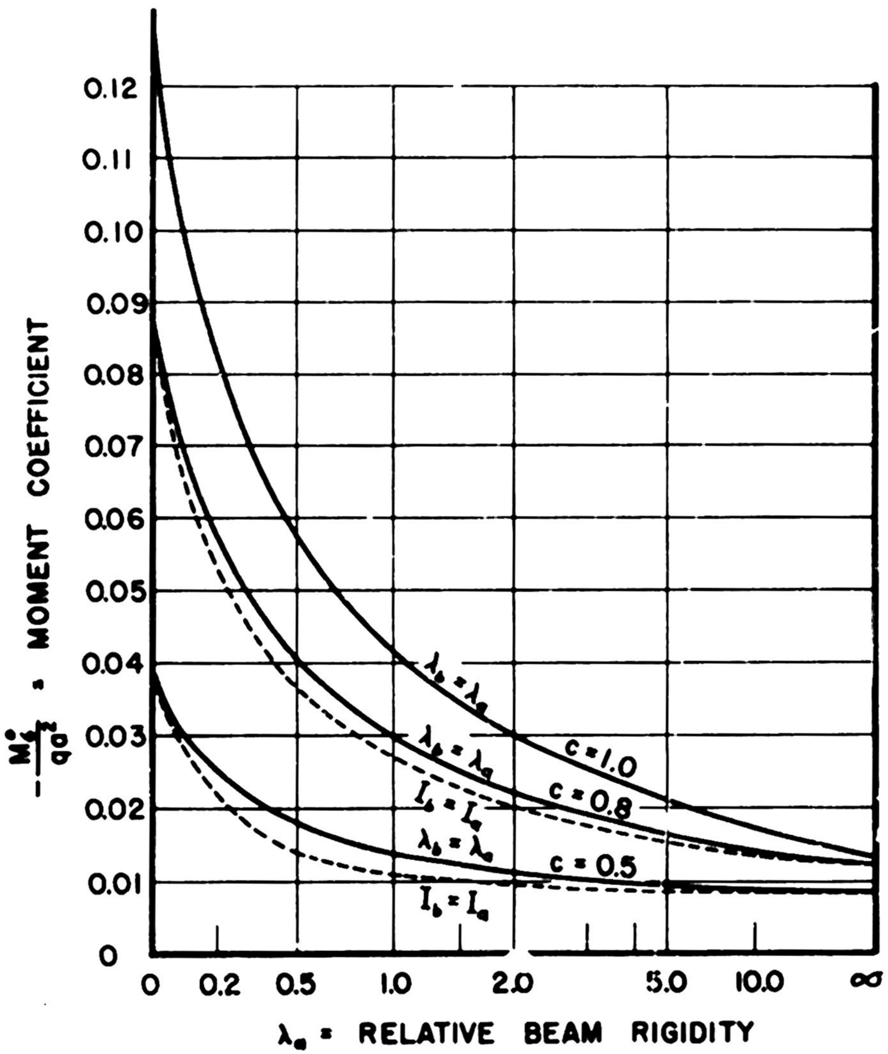

FIG. 32 AVERAGE NEGATIVE MOMENT IN THE SHORT COLUMN STRIP

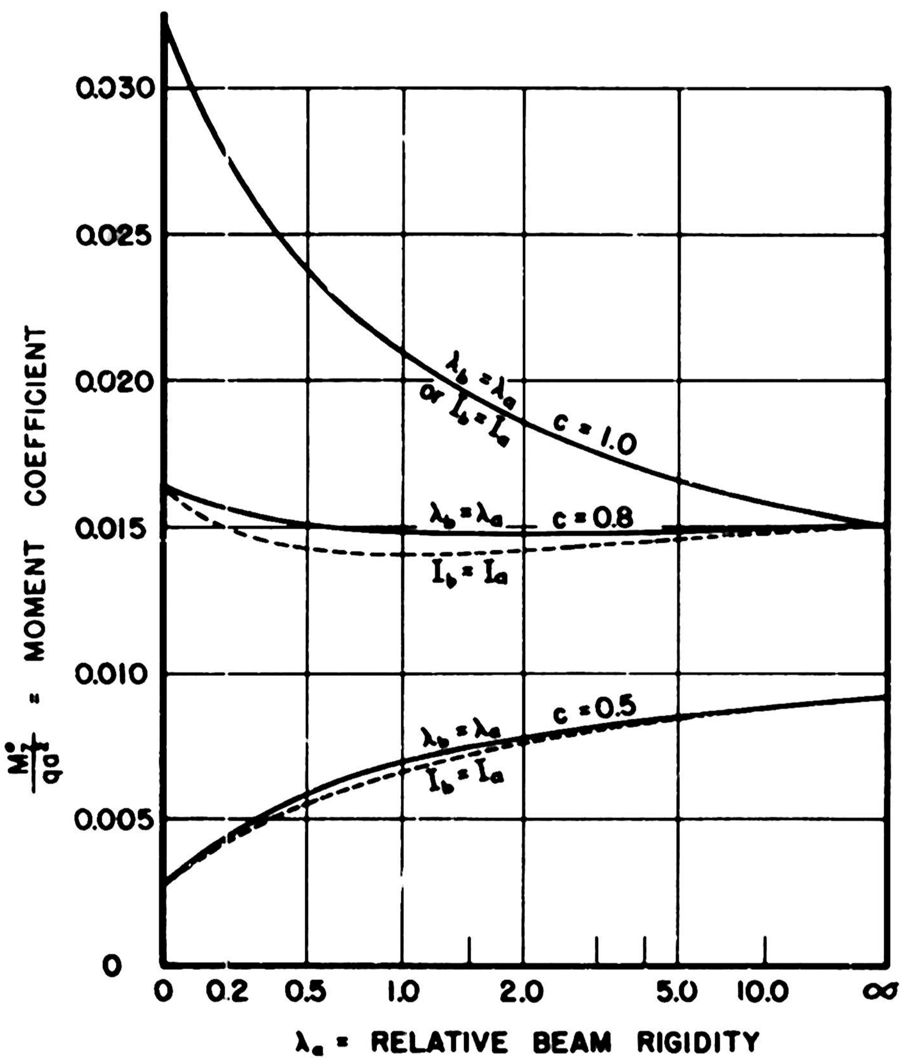

FIG. 33 AVERAGE POSITIVE MOMENT IN THE SHORT MIDDLE STRIP

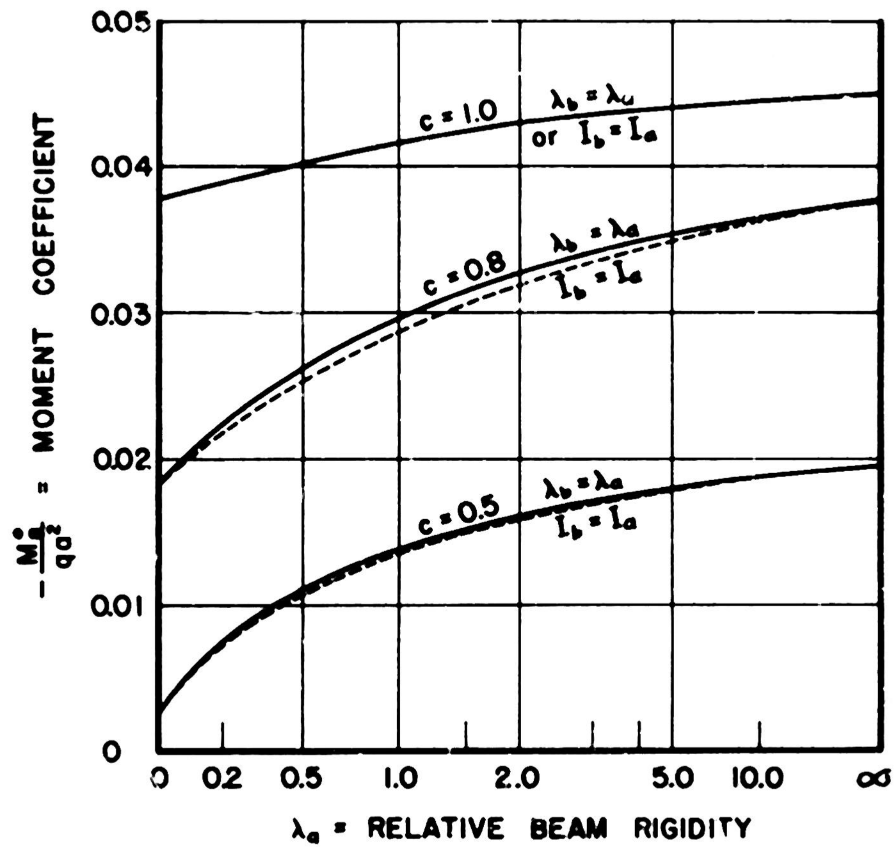

FIG. 34 AVERAGE NEGATIVE MOMENT IN THE SHORT MIDDLE STRIP

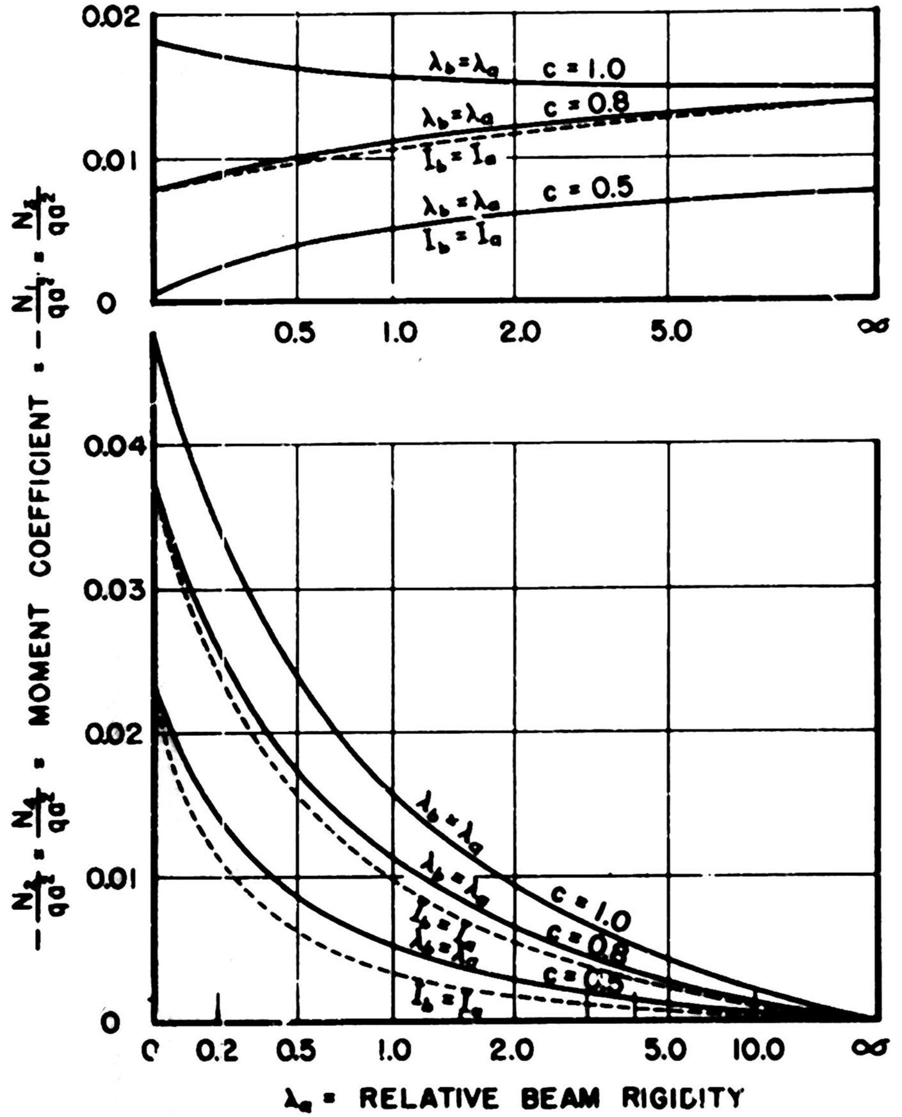

FIG. 35 POISSON'S RATIO CORRECTION TO AVERAGE POSITIVE MOMENT IN THE LONG SPAN

FIG. 36 POISSON'S RATIO CORRECTION TO AVERAGE NEGATIVE MOMENT IN THE LONG SPAN

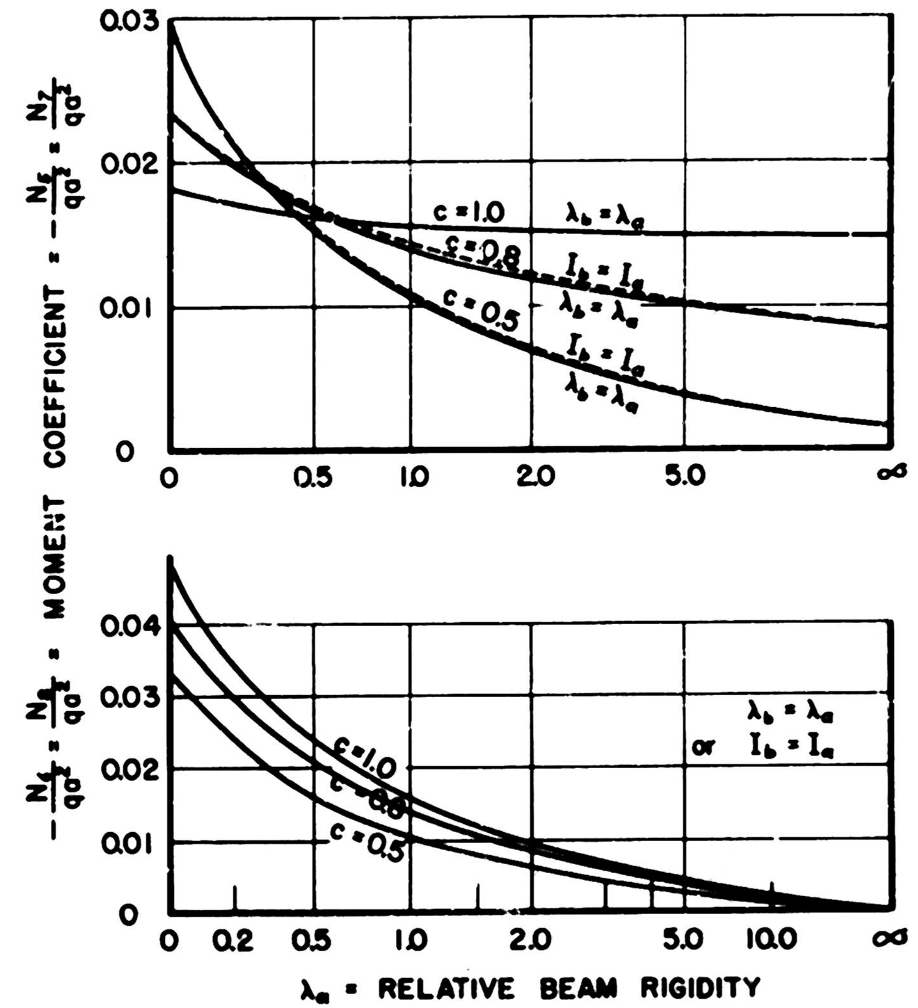

FIG. 37 POISSON'S RATIO CORRECTION TO AVERAGE POSITIVE MOMENT IN THE SHORT SPAN

FIG. 38 POISSON'S RATIO CORRECTION TO AVERAGE NEGATIVE MOMENT IN THE SHORT SPAN

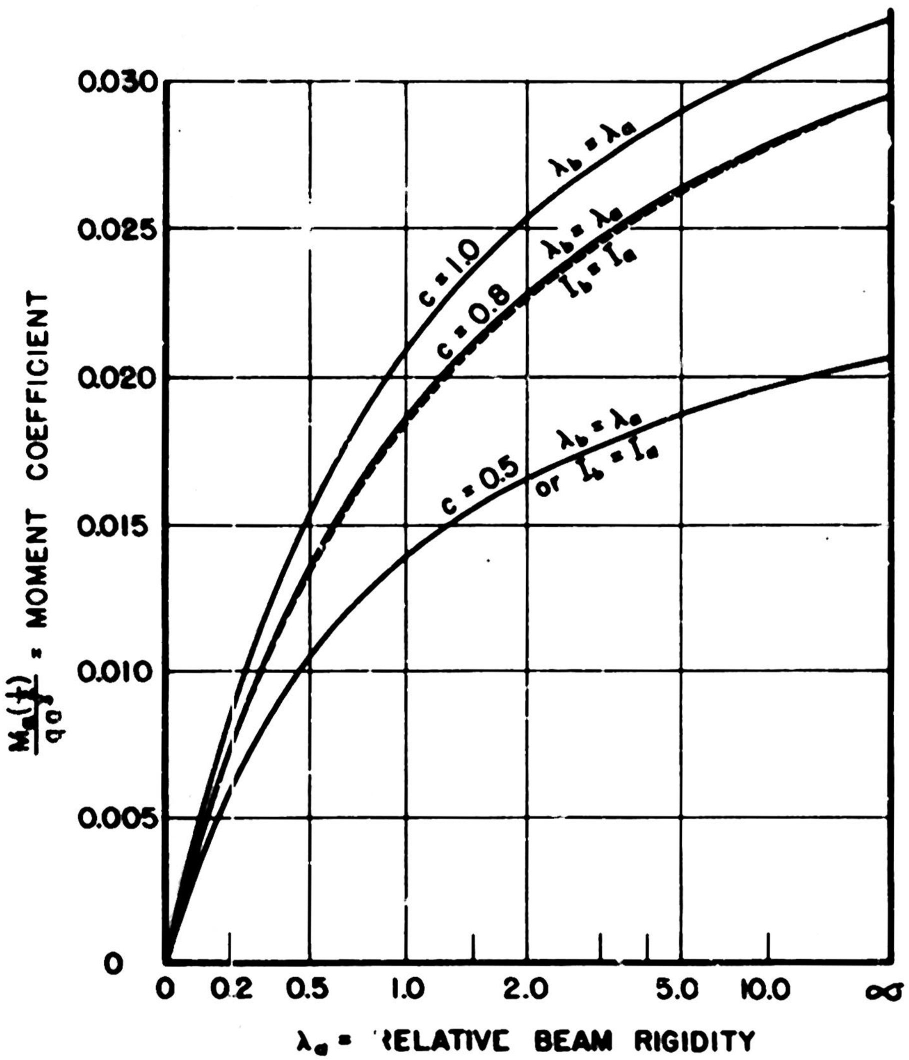

FIG. 39 MAXIMUM POSITIVE MOMENT IN THE LONG BEAMS

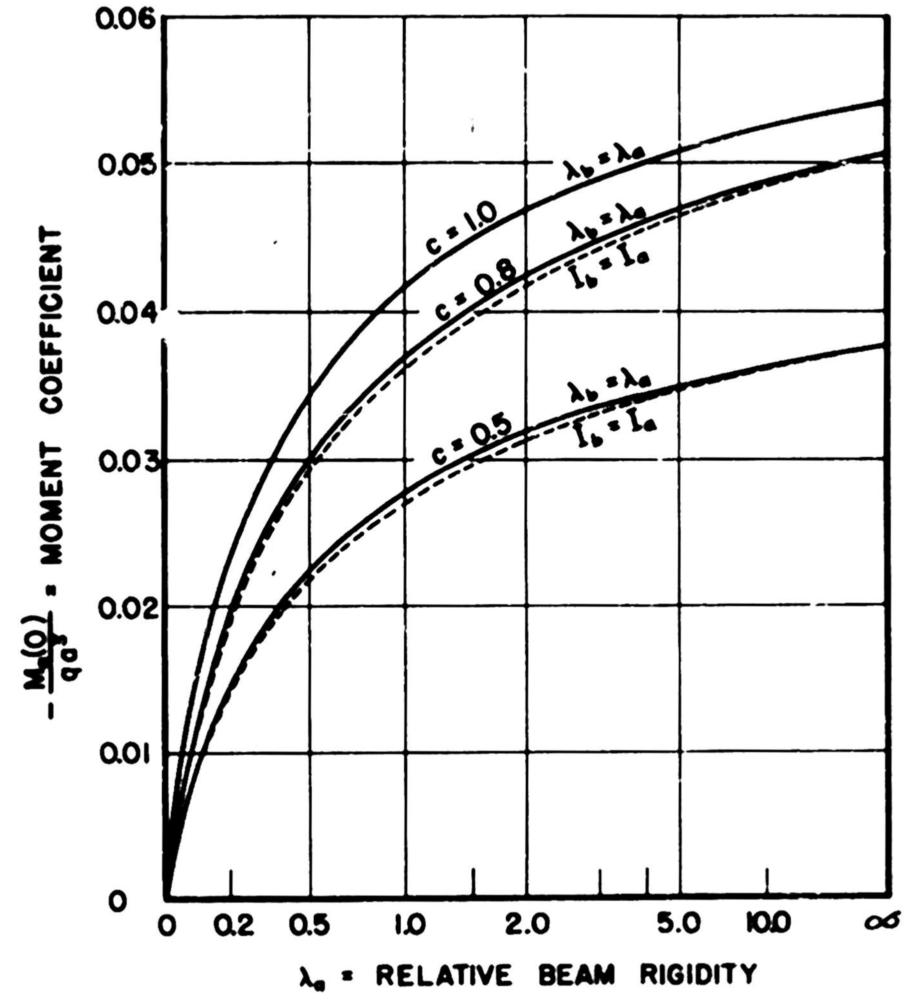

FIG. 40 MAXIMUM NEGATIVE MOMENT IN THE LONG BEAMS

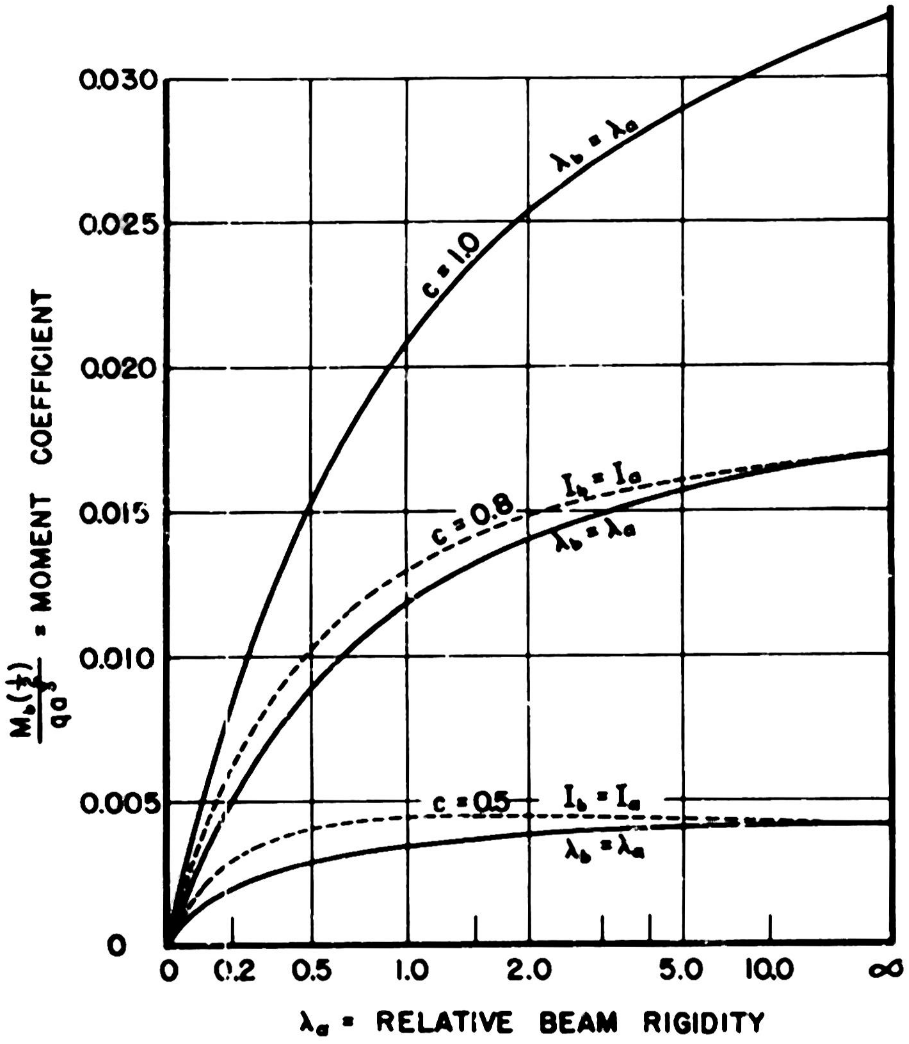

FIG. 41 MAXIMUM POSITIVE MOMENT IN THE SHORT BEAMS

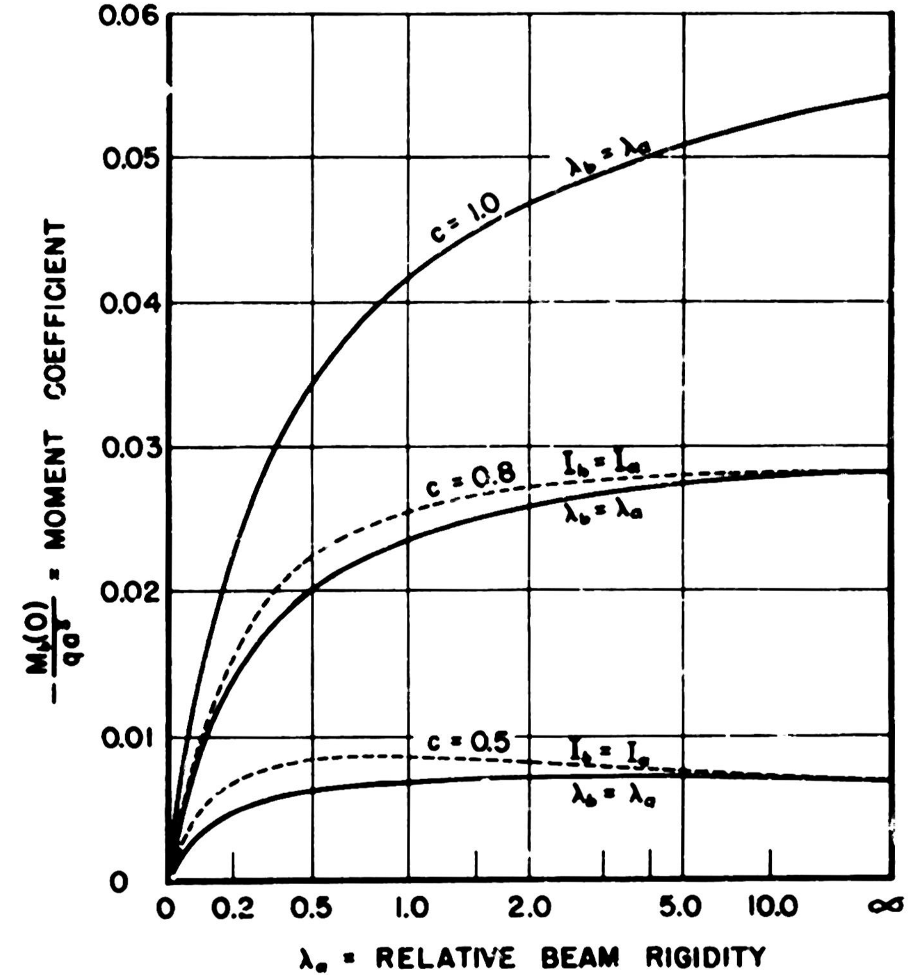

FIG. 42 MAXIMUM NEGATIVE MOMENT IN THE SHORT BEAMS