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LIMIT ANALYSIS AND DESIGN

by

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LIMIT ANALYSIS AND DESIGN*

By William Prager**

Synopsis. Many problems concerning limit analysis and limit design of reinforced concrete beams and frames can be treated geometrically in terms of the safe domain in load space. The procedure is illustrated by a typical example.

INTRODUCTION

The conventional analysis of indeterminate structures is restricted to the elastic range. Structures with ductile members may remain serviceable far beyond this range, so that the limits of their usefulness cannot be explored by the methods of elastic analysis. Limit analysis is concerned with estimating the load intensity at which a given indeterminate structure ceases to be serviceable. Limit design, on the other hand, is concerned with allocating local yield strength to the members or cross sections of an indeterminate structure in such a manner that this structure remains serviceable under given conditions of loading.

The basic concepts of limit analysis were developed more than thirty years ago (1). Early applications were restricted to continuous beams (see, for instance, (2)), but later on frames were also treated successfully (see, for instance, (3)). General principles were established by Greenberg and Prager (4), and

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Hill (5), and a very effective method of analysis was developed
by Symonds and Neal (6). Contrary to limit analysis, limit
design in the sense defined above constitutes a practically un-
explored field. Heyman (7) has studied certain problems in the
limit design of continuous beams and frames, and Foulkes (8)
has pointed out the relation between limit design and linear
programming. The following discussion is concerned with the
limit design of a particular frame (Fig. 1). It is felt that,
in the absence of general results, much is to be learned from
the discussion of such a specific example.

BASIC CONCEPTS

The elastic analysis of indeterminate beams and frames
is based on a linear relation between the bending moment M and
the curvature $\varkappa$ (dotted line in Fig. 2). As Hill (5) pointed
out, limit analysis may be based on the relation between M and $\varkappa$
which is represented by the full line in Fig. 2. According to
this relation, bending can take place only if the bending moment
attains the limiting values M' or - M". Since these extreme
values will be reached only at discrete cross sections, bending
will be localized in "plastic hinges".

Admittedly, the full-line graph in Fig. 2 represents an
oversimplification of the actual relation between M and $\varkappa$. One
important feature of the mechanical behavior of reinforced con-
crete beams is more adequately reflected by this graph, however,
than by the dotted line in Fig. 2: at certain (more or less well

defined) values of the positive or negative bending moment the curve M cersus x turns rather sharply and becomes fairly flat compared to its steep ascent in the elastic range.

EXAMPLE FRAME

The frame shown in Fig. 1 is built-in at 1 and pin-supported at 5. The loads P and Q are supposed to vary independently, and it is required to find all "safe states of loading", i.e., all combinations of P and Q (including negative values of these loads) which will not cause plastic failure of the frame. If a state of loading is represented by the point with the rectangular coordinates P, Q in a two-dimensional "load space", the points representing safe states of loading form the "safe domain" whose properties will be discussed in the following.

Bending moments will be considered as positive, if they produce tension on the inner side of the frame. Accordingly, the angle change at a plastic hinge will be considered positive, if it represents an increase of the interior angle.

It will be assumed that the limiting moments of column and beam match at 2 and 4, and that the limiting moments of each of the segments 1-2, ..., 4-5 vary linearly along this segment. Since the bending moments caused by the loads also vary linearly along each of these segments, plastic hinges need to be considered at the critical sections 1 to 4 only. The limiting moments at these sections will be written in the form

$$M_i' = M_i^* + M_i^O, \quad -M_i'' = M_i^* - M_i^O, \quad (i = 1, \ldots, 4). \quad [1]$$

If the limiting moments M_1' and $-M_1''$ are considered as the endpoints of the "safe range" of the considered cross section, M_1^* represents the center of this range and $2M_1^O$ its width.

As was pointed out by Symonds and Neal (6), it is convenient to consider any plastic deformation (made possible by the appearance of a sufficient number of plastic hinges) as resulting from the cooperation of certain elementary mechanisms. In the present case, there are only two such elementary mechanisms: the frame mechanism of Fig. 3(a), and the beam mechanism of Fig. 3(b). The numbers on the inner side of the frame in Fig. 3 indicate the angle changes corresponding to unit linear displacements of the points of application of P and Q, respectively. A generic plastic deformation of the frame will be specified by the horizontal displacement p of 2 and the vertical displacement q of 3 (Fig. 4).

SAFE SEGMENTS OF EXAMPLE FRAME

To construct the safe domain of the frame, consider first the hypothetical case where $M_2^* = 0$ and $M_1^* = M_1^O = 0$ for i = 1, 3, 4. Since a section with vanishing limit moments acts as a perfect hinge, the frame would, in this case, have hinges at 1, 3, 4, and 5, and hence be capable of deformation even in the absence of a plastic hinge at 2. This type of deformation is obtained by combining the elementary deformations shown in Figs. 3(a) and (b) in such a manner that the resulting angle change at 2 vanishes. This condition of vanishing angle change

at 2 requires that $p = 2q$. The principle of virtual work shows then that the frame with perfect hinges at 1, 3, 4, and 5 can be in equilibrium only if

$$Q/P = -2. \qquad\qquad [2]$$

Even if this ratio between the loads is maintained, the frame will eventually fail because a plastic hinge will form at 2 when the loads are sufficiently large. Thus, the safe domain in this hypothetical case is a finite segment of the line with the equation [2]. The endpoints A and B of this segment shown in Fig. 5(a) are readily determined by the kinematic method of Greenberg and Prager (4); their coordinates are found to be

$$
\left.
\begin{array}{lll}
\text{A:} & P = M_2^o/6a, & Q = -M_2^o/3a, \\
\\
\text{B:} & P = -M_2^o/6a, & Q = M_2^o/3a.
\end{array}
\right\} \qquad [3]
$$

It is worth noting that A and B are symmetric with respect to the origin and that the coordinates of A are obtained by multiplying the angle changes at the joint 2 in Fig. 3(a) and (b) by M_2^o.

Next, consider the case which differs from the previous one only by the fact that $M_2^* \neq 0$. It is found that, in this case, the safe domain shown in Fig. 5(b) is a segment which has the same length and slope as before but is centered at the point C with the coordinates

$$\text{C:} \qquad P = M_2^*/6a, \qquad Q = -M_2^*/3a. \qquad\qquad [4]$$

The coordinates of C are obtained by multiplying the angle changes at the joint 2 in Fig. 3(a) and (b) by M_2^*.

Three other hypothetical cases have to be considered. In each of them the limiting moments vanish at all but one of the critical sections. The corresponding safe domains are readily determined by the method outlined above. For example, Fig. 6(a) shows the safe segment (for the case where $M_1^* = M_2^* = M_3^* = M_4^* = 0$, $M_1^O = M$, $M_2^O = M_3^O = M_4^O = 0$, the quantity $M/6a$ being taken as the unit of force. Each of these safe segments takes account of the yield strength of one critical section only assuming the other sections to have **vanishing** yield strength.

SAFE DOMAIN OF EXAMPLE FRAME

The actual safe domain of the considered frame can be obtained from the safe segments by applying the following superposition principle: a point S of the P, Q plane is in the safe domain of the considered frame if and only if the position vector of S can be obtained by selecting one point in each of the four safe segments and adding the position vectors of these four points.

In accordance with this superposition principle the desired safe domain is obtained by the following steps (Fig. 7):

1) let the safe segment of Fig. 6(a) undergo a translation such that its center moves along the safe segment of Fig. 6(b); the (dotted) parallelogram swept in this motion is the safe domain which takes account of the yield strength of the sections 1 and 2;

2) let this parallelogram undergo a translation such that its center moves along the safe segment of Fig. 6(c); the (dashed-line) hexagon swept in this motion is the safe domain which takes account of the yield strength of the sections 1, 2, and 3;

3) let this hexagon undergo a translation such that its center moves along the safe segment of Fig. 6(d); the (full-line) octagon swept in this motion is the desired safe domain of the frame, i.e., any combination of P and Q represented by a point inside this octagon will not cause plastic failure of the frame specified by $M_1^* = M_2^* = M_3^* = M_4^* = 0$, $M_1^O = M_2^O = M_3^O = M_4^O = M$.

It follows from this construction that the sides of the safe domain are parallel and equal to the safe segments. If the value of M^O at a critical section is doubled, for example, the corresponding safe segment and hence the corresponding side of the safe domain doubles in length but does not change its direction. Figure 8(a) shows how Fig. 7 changes when the value of M_1^O is doubled and that of M_3^O is halved. If the value of M^* at a critical section is changed, the corresponding safe segment slides along itself without changing its length (see Fig. 5); the safe domain of the frame therefore undergoes a translation in the direction of one side: Figure 8(b) shows how Fig. 8(a) changes when M_2^* is changed from 0 to $M/2$.

LIMIT DESIGN OF EXAMPLE FRAME

The loads in Fig. 1 may result from the action of structural weight, snow load, and wind pressure, positive values of P corresponding to wind pressure on the left wall and negative values to wind pressure on the right wall. If wind suction on the flat roof and the lee-side wall is taken into account, the possible states of combined loading are represented by the points of a "domain of loading" such as the hexagon ABCDEF in Fig. 9. This domain of loading will be assumed to incorporate the appropriate load factors. The octagon BCIKEFGH is circumscribed to this domain of loading and has sides of the appropriate directions. The manner in which this octagonal safe domain is built up from the safe segments is indicated in Fig. 9. By analyzing these segments, the values of M* and M^O for all critical sections are readily determined. One finds

$$\left.\begin{array}{ll} M^*_1 = -0.5M, & M^O_1 = 6.5M, \\[2mm] M^*_2 = -0.5M, & M^O_2 = M , \\[2mm] M^*_3 = M, & M^O_3 = 2M, \\[2mm] M^*_4 = -0.5M, & M^O_4 = M, \end{array}\right\} \qquad [5]$$

where M is the value of the limiting moment used in constructing Fig. 6. With the values [5], the limiting moments at the critical sections are easily found from [1] as follows:

$$6.0 \text{ M} \quad \text{and} \quad -7.0 \text{ M} \quad \text{for section 1,}$$

$$0.5 \text{ M} \quad \text{and} \quad -1.5 \text{ M} \quad \text{for section 2,}$$

$$3.0 \ M \quad \text{and} \quad -1.0 \ M \quad \text{for section 3,}$$
$$0.5 \ M \quad \text{and} \quad -1.5 \ M \quad \text{for section 4.}$$

There is, of course, more than one way of circumscribing an appropriate octagon to the domain of loading. For instance, the octagon BC'I'KF'G'H could be used; this leads to heavier sections 1 and 3 and a lighter section 4. It is likely, however, that the design for which safe domain and domain of loading have a maximum number of vertices in common represents the most economic use of materials.

CONCLUDING REMARKS

The method developed above is adequate whenever the plastic deformation of the structure can be described in terms of two elementary mechanisms. When there are three elementary mechanisms, as in the case of the frame shown in Fig. 9, a three-dimensional load space must be used (with P, Q, R as rectangular coordinates). The safe domain is then a polyhedron which can be constructed from safe segments very much in the same way as the polygonal safe domain was constructed above. Complications arise, however, when the three loads P, Q, R are not independent of each other. When P and Q result from wind pressure, for instance, the ratio P/Q will have a fixed value n. The safe domain for this case is then obtained as the intersection of the afore-mentioned polyhedron and the plane P-nQ = 0. When the polyhedron is not centered at the origin, this intersection can assume a rather irregular shape. It is believed that the influence of changes in cross section on the shape of this two-

dimensional safe domain can be properly understood only if this

domain is visualized as a plane intersection of the much more

regular three-dimensional safe domain.

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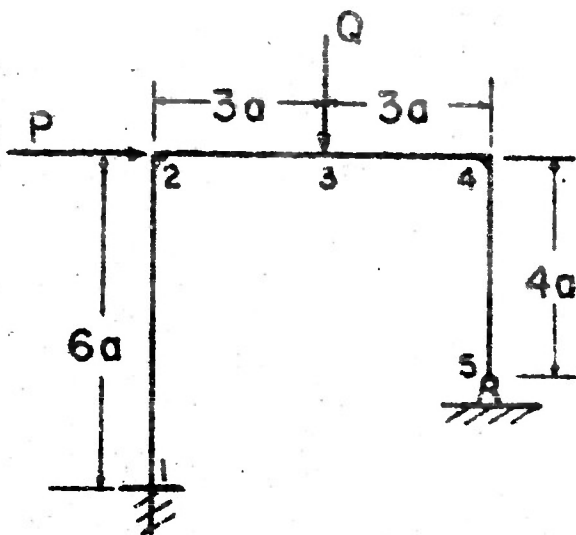
Fig. 1. Example frame.

Fig. 2. Bending moment versus curvature.

......elastic

____ ideal plastic
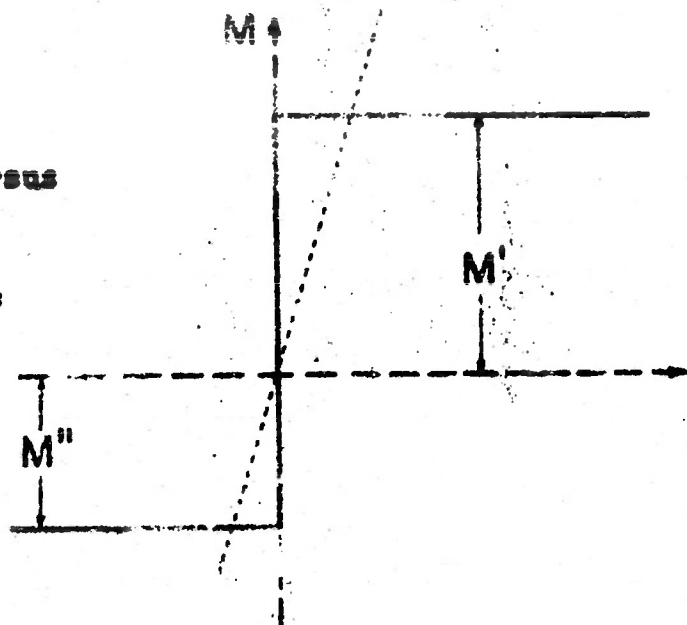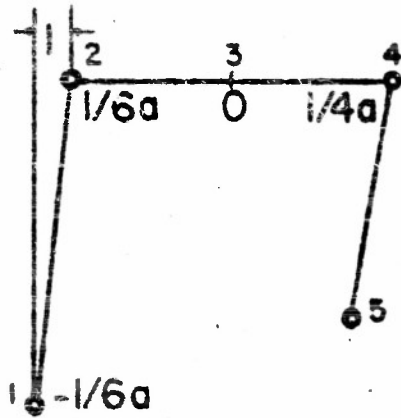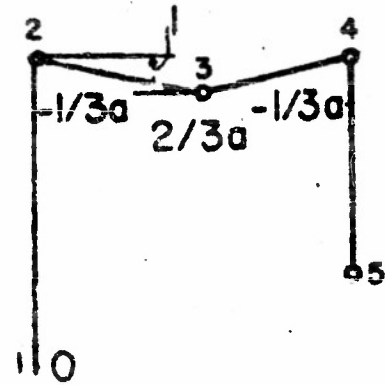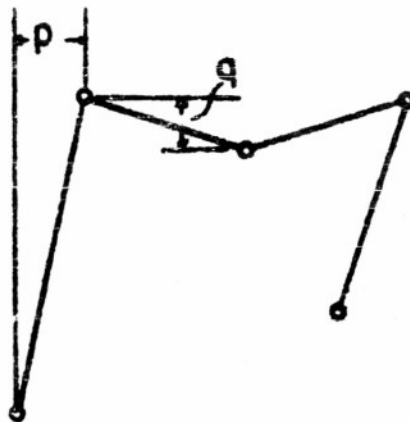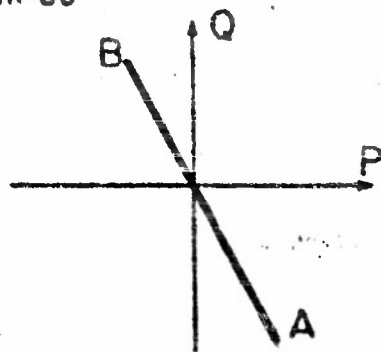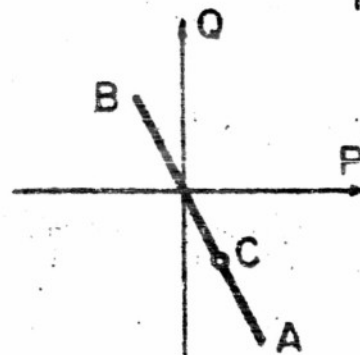
Fig. 3. Elementary mechanisms.

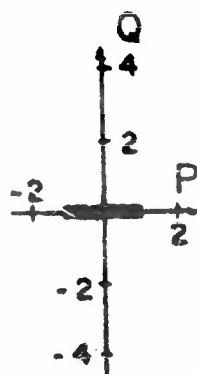Fig. 4. Specification of deformation by displacements p and q.
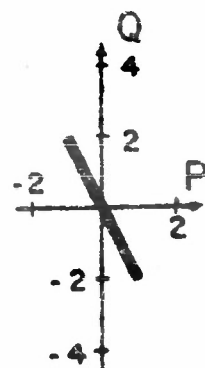
(a) $M_2{}^O \neq 0$

(b) $M_2{}^* \neq 0$, $M_2{}^O \neq 0$
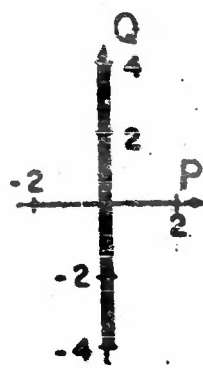
Fig. 5. Safe segments (all $M_1{}^*$ and $M_1{}^O$ except those indicated on figure are assumed to vanish).
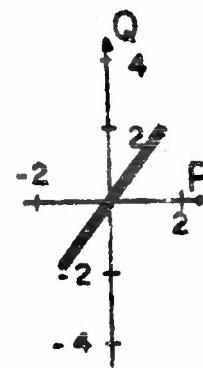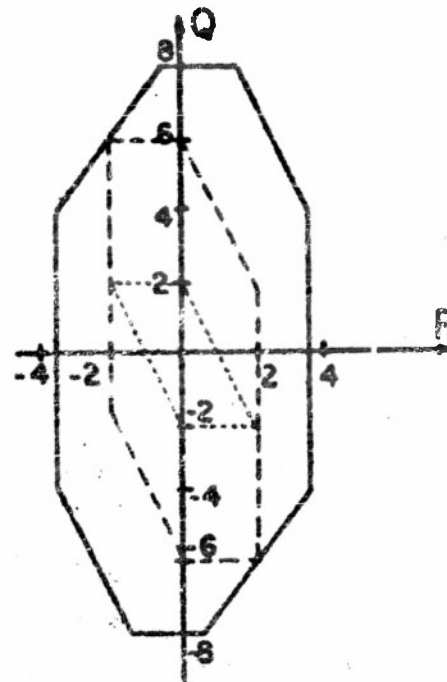

(a) $M_1{}^O = M$ (b) $M_2{}^O = M$ (c) $M_3{}^O = M$ (d) $M_4{}^O = M$

Fig. 6. Safe segments (all $M_1{}^*$ and $M_1{}^O$ except those indicated on figure are assumed to vanish; unit of force = 6M/a)

Fig. 7. Safe domain ($M_1{}^* = M_2{}^*$ = $M_3{}^* = M_4{}^* = 0$; $M_1{}^O$ = $M_2{}^O = M_3{}^O = M_4{}^O = M$; unit of force = 6M/a).

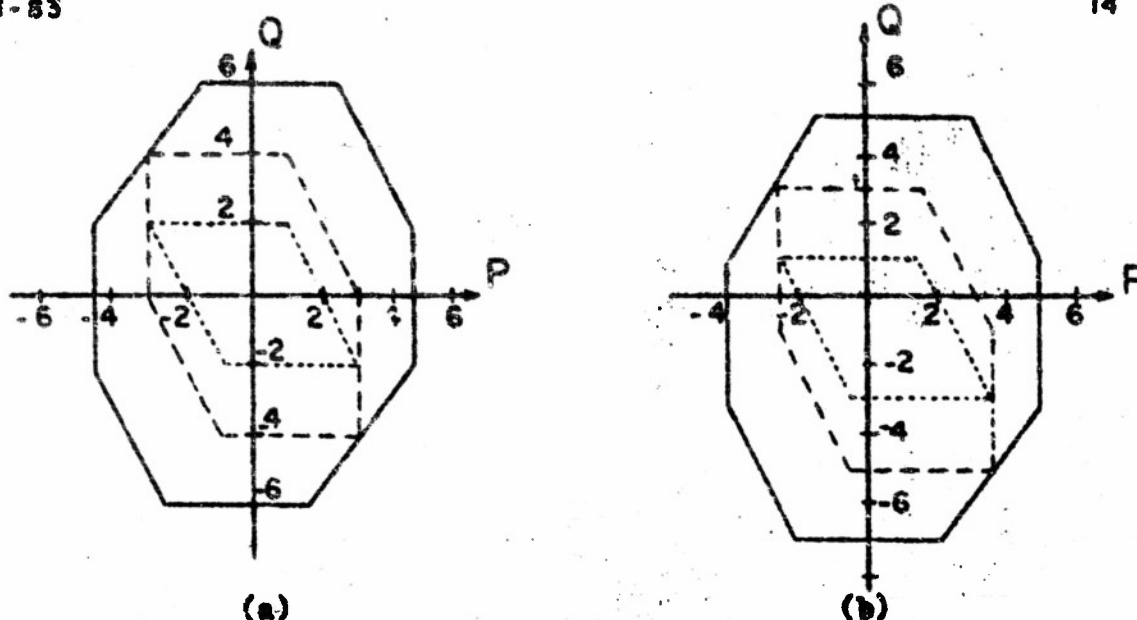**Fig. 8.** Safe domains (unit of force = $6M/a$).
 (a) $M_1^* = M_2^* = M_3^* = M_4^* = 0$
 (b) $M_1^* = M_3^* = M_4^* = 0$, $M_2^* = M/2$
 $\Big\}$ $M_1^o = 2M$, $M_2^o = M$, $M_3^o = M/2$, $M_4^o = M$

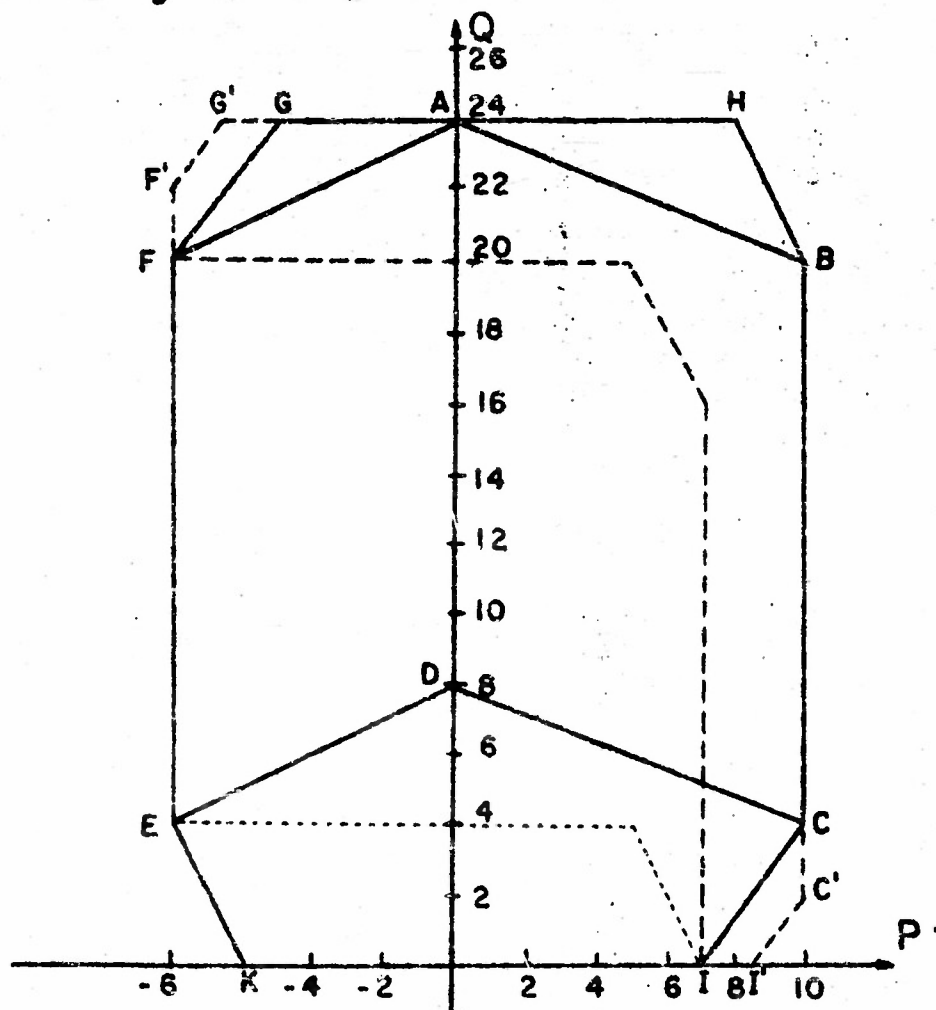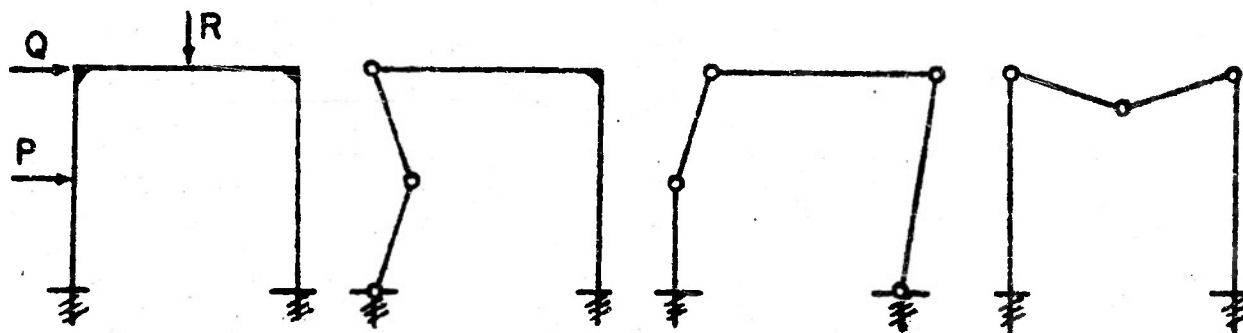**Fig. 9.** Limit design of example frame (unit of force = $6M/a$).

Fig. 10. Frame with three elementary mechanisms.