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A TABULATION OF THE FRESNEL INTEGRALS



By

Robert Turner
and
Anne E. Downey

March 15, 1953

Technical Report No. 173

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

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A Tabulation of the Fresnel Integrals

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Robert Turner and Anne F. Downey

Cruft Laboratory, Harvard University

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Abstract

A tabulation of the Fresnel integrals

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \cos t \frac{dt}{\sqrt{t}}$$

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \sin t \frac{dt}{\sqrt{t}}$$

is given for $0 \leq x \leq 1.00$ by steps of 0.01, and for $1.0 \leq x \leq 30.0$ by steps of 0.10. In addition, values are given for $x = n \frac{\pi}{2}$, $n = 1-20$. Differences are tabulated, to facilitate interpolation. Some applications of the integrals are listed. Alternative forms and asymptotic expansions valid for large x are given.

* * * * *

The Fresnel integrals

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt,$$

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt,$$

were first obtained (although not in this form) by Augustin Jean Fresnel in 1818 in the course of the development of his theory of

diffraction of light. Since then, their chief application has been in diffraction problems. However, other applications occur. In hydrodynamics, for example, the velocity potential for surface waves generated by an impulsive pressure is readily evaluated in terms of the Fresnel integrals.*

The work that initiated this tabulation was concerned with the scattering of electromagnetic radiation by cylindrical mirrors. (The Fresnel integrals arose from the contributions to the scattered field from the singularity in the surface-current density at the edge of the mirror.) When the actual evaluation of certain integrals in the formulation was begun, it was found that they could easily be cast into forms yielding the Fresnel integrals, but that the arguments were such that interpolation would frequently be required. However, it was found that no tables existed for which the tabulation was complete enough to permit satisfactory interpolation. The best previous tabulation is probably that of Watson,** which is the basis for the present work.

There are two principal forms of the integrals. The first is given above, and is used principally for convenience in tabulation. The second form is that originally used by Fresnel:

$$C\left(\frac{\pi}{2} u^2\right) = \int_0^u \cos\left(\frac{\pi}{2} t^2\right) dt$$

$$S\left(\frac{\pi}{2} u^2\right) = \int_0^u \sin\left(\frac{\pi}{2} t^2\right) dt.$$

Certain other forms also exist:

$$C(x) = \frac{1}{2} \int_0^x J_{-\frac{1}{2}}(t) dt$$

*"Fourier Transforms," Ian N. Sneddon, McGraw-Hill, New York, 1948, pp. 278-282.

**"The Theory of Bessel Functions," G. N. Watson, Cambridge, London, 1922.

$$S(x) = \frac{1}{2} \int_0^x J_{\frac{1}{2}}(t) dt$$

when $J_{-\frac{1}{2}}$ and $J_{\frac{1}{2}}$ are Bessel functions of order $-\frac{1}{2}$ and $\frac{1}{2}$ respectively.

Further,

$$\frac{1}{2} \int_0^x H_{-\frac{1}{2}}^{(2)}(t) dt = C(x) - iS(x)$$

where $H_{-\frac{1}{2}}^{(2)}$ is the Hankel function of the second kind of order $-\frac{1}{2}$,

$$\frac{\mathfrak{E}(\sqrt{x})}{\sqrt{2i}} = C(x) - iS(x),$$

where $\mathfrak{E}$ is the error function. Also

$$\int_0^u 1-t^2 dt = C\left(\frac{\pi}{2} u^2\right) - iS\left(\frac{\pi}{2} u^2\right).$$

Integrals of half-order Bessel functions of order higher than $\frac{1}{2}$ may be computed by a simple integration by parts. For example,

$$\frac{1}{2} \int_0^x J_{\frac{3}{2}}(x) dx = C(x) - J_{\frac{1}{2}}(x).$$

This result may also be obtained from the recursion formula for the Bessel functions.

More complicated integrals may be evaluated in terms of the Fresnel integrals. For example,

$$\begin{aligned} \int_{\beta}^{2\pi-\beta} e^{-i\rho \cos(\alpha-\theta)} \frac{d\theta}{\sqrt{\theta-\beta}} &= 2 J_0(\rho) \sqrt{2(\pi-\beta)} \\ &+ 2 \sum_{n=1}^{\infty} (-1)^n J_n(\rho) \sqrt{2\pi} [\cos n(\alpha+\beta) C\left(\frac{2}{n}(\pi-\beta)\right) \\ &+ \sin n(\alpha+\beta) S\left(\frac{2}{n}(\pi-\beta)\right)]. \end{aligned}$$

For $x \geq 30.0$, the following asymptotic formulas give values accurate to six decimal places:

$$C(x) \approx \frac{1}{2} + \sqrt{\frac{x}{2\pi}} \left[\sin x \left(\frac{1}{2x} - \frac{3}{8x^3} \right) - \cos x \left(\frac{1}{4x^2} - \frac{15}{16x^4} \right) \right]$$

$$S(x) \approx \frac{1}{2} - \sqrt{\frac{x}{2\pi}} \left[\cos x \left(\frac{1}{2x} - \frac{3}{8x^3} \right) + \sin x \left(\frac{1}{4x^2} - \frac{15}{16x^4} \right) \right].$$

The values of $S(x)$ at $x = n\pi$ and $C(x)$ at $x = \left(\frac{2n-1}{2}\right)\pi$ were taken from Watson.

X	S(X)	Δ	C(X)	Δ	5.
.01	.000266	.000436	.079788	.033045	
.02	.000752	.000630	.112833	.025352	
.03	.001382	.000745	.138135	.021366	
.04	.002127	.000846	.159551	.018817	
.05	.002973	.000935	.178368	.017003	
.06	.003908	.001016	.195371	.015626	
.07	.004924	.001091	.210997	.014534	
.08	.006015	.001162	.225531	.013641	
.09	.007177	.001227	.239172	.012889	
.10	.008404	.001291	.252061	.012247	
.11	.009695	.001349	.264308	.011690	
.12	.011044	.001407	.275998	.011198	
.13	.012451	.001461	.287196	.010761	
.14	.013912	.001514	.297957	.010368	
.15	.015426	.001564	.308325	.010013	
.16	.016990	.001614	.318338	.009689	
.17	.018604	.001660	.328027	.009392	
.18	.020264	.001706	.337419	.009117	
.19	.021970	.001750	.346536	.008864	
.20	.023720	.001794	.355400	.008628	
.21	.025514	.001836	.364028	.008406	
.22	.027350	.001876	.372434	.008199	
.23	.029226	.001916	.380633	.008004	
.24	.031142	.001955	.388637	.007819	
.25	.033097	.001993	.396456	.007645	
.26	.035090	.002030	.404101	.007480	
.27	.037120	.002065	.411581	.007322	
.28	.039135	.002101	.418903	.007172	
.29	.041236	.002136	.426075	.007028	
.30	.043422	.002169	.433103	.006890	

X	S(X)	Δ	C(X)	Δ	G.
.31	.045591	.002202	.439993	.006759	
.32	.047793	.002235	.446752	.006632	
.33	.050028	.002266	.453384	.006509	
.34	.052294	.002297	.459893	.006392	
.35	.054591	.002327	.466285	.006279	
.36	.056918	.002357	.472564	.006168	
.37	.059275	.002386	.478732	.006062	
.38	.061661	.002415	.484794	.005959	
.39	.064076	.002443	.490753	.005859	
.40	.066519	.002470	.496612	.005762	
.41	.068989	.002496	.502374	.005667	
.42	.071485	.002524	.508041	.005575	
.43	.074009	.002549	.513616	.005486	
.44	.076558	.002574	.519102	.005398	
.45	.079132	.002599	.524500	.005313	
.46	.081731	.002623	.529813	.005229	
.47	.084354	.002648	.535042	.005148	
.48	.087002	.002670	.540190	.005067	
.49	.089672	.002694	.545257	.004990	
.50	.092366	.002716	.550247	.004913	
.51	.095082	.002738	.555160	.004839	
.52	.097820	.002760	.559999	.004764	
.53	.100580	.002780	.564763	.004692	
.54	.103360	.002802	.569455	.004621	
.55	.106162	.002822	.574076	.004552	
.56	.108984	.002841	.578628	.004487	
.57	.111825	.002861	.583115	.004410	
.58	.114686	.002881	.587525	.004349	
.59	.117567	.002898	.591874	.004283	
.60	.120465	.002918	.596157	.004219	

X	S(X)	Δ	C(X)	Δ	7.
.61	.123383	.002935	.600376	.004155	
.62	.126318	.002952	.604531	.004092	
.63	.129270	.002970	.608623	.004031	
.64	.132240	.002986	.612654	.003969	
.65	.135226	.003003	.616623	.003909	
.66	.138229	.003019	.620532	.003850	
.67	.141248	.003034	.624382	.003791	
.68	.144282	.003050	.628173	.003733	
.69	.147332	.003064	.631906	.003676	
.70	.150396	.003079	.635582	.003618	
.71	.153475	.003093	.639200	.003563	
.72	.156568	.003107	.642763	.003507	
.73	.159675	.003121	.646270	.003452	
.74	.162796	.003133	.649722	.003397	
.75	.165929	.003147	.653119	.003344	
.76	.169076	.003159	.656463	.003291	
.77	.172235	.003170	.659754	.003237	
.78	.175405	.003183	.662991	.003185	
.79	.178588	.003194	.666176	.003134	
.80	.181782	.003205	.669310	.003082	
.81	.184987	.003216	.672392	.003030	
.82	.188203	.003226	.675422	.002981	
.83	.191429	.003237	.678403	.002927	
.84	.194666	.003246	.681330	.002884	
.85	.197912	.003255	.684214	.002831	
.86	.201167	.003265	.687045	.002782	
.87	.204432	.003274	.689827	.002734	
.88	.207706	.003281	.692561	.002686	
.89	.210987	.003291	.695247	.002637	
.90	.214278	.003297	.697884	.002591	

X	S(X)	Δ	C(X)	Δ	8.
.91	.217575	.003306	.700475	.002543	
.92	.220881	.003312	.703018	.002496	
.93	.224193	.003320	.705514	.002450	
.94	.227513	.003326	.707964	.002404	
.95	.230839	.003333	.710368	.002358	
.96	.234172	.003338	.712726	.002313	
.97	.237510	.003344	.715039	.002267	
.98	.240854	.003350	.717306	.002222	
.99	.244204	.003354	.719528	.002178	
1.00	.247558		.721706		

X	S(X)	Δ	C(X)	Δ	9.
1.0	.247558	+.033759	.721706	.019383	
1.1	.281317	.033945	.741089	.015206	
1.2	.315262	.033851	.756295	.011260	
1.3	.349113	.033491	.767555	.007529	
1.4	.382604	.032879	.775084	.004000	
1.5	.415483	.032017	.779084	+.000679	
1.6	.447500	.030962	.779763	-.002453	
1.7	.478462	.029656	.777310	-.005362	
1.8	.508118	.028198	.771948	-.008079	
1.9	.536316	.026533	.763869	-.010567	
2.0	.562849	.024720	.753302	-.012837	
2.1	.587569	.022775	.740465	-.014888	
2.2	.610344	.020679	.725577	-.016692	
2.3	.631023	.018523	.708885	-.018283	
2.4	.649546	.016241	.690602	-.019616	
2.5	.665787	.013930	.670986	-.020728	
2.6	.679717	.011568	.650258	-.021594	
2.7	.691285	.009181	.628664	-.022227	
2.8	.700466	.006795	.606437	-.022624	
2.9	.707261	.004424	.585813	-.022793	
3.0	.711685	+.002091	.561020	-.022738	
3.1	.713776	-.000185	.538282	-.022469	
3.2	.713591	-.002391	.515813	-.021991	
3.3	.711200	-.004505	.493822	-.021316	
3.4	.706695	-.006515	.472506	-.020459	
3.5	.700180	-.008403	.452047	-.019425	
3.6	.691777	-.010159	.432622	-.018235	
3.7	.681618	-.011768	.414387	-.016899	
3.8	.669850	-.013222	.397488	-.015436	
3.9	.656628	-.014509	.382052	-.013859	

X	S(X)	Δ	C(X)	Δ	10.
4.0	.642119	-.014625	.368193	-.012189	
4.1	.626494	-.016558	.356004	-.010439	
4.2	.609936	-.017313	.345565	-.008631	
4.3	.592623	-.017876	.336934	-.006780	
4.4	.574747	-.018258	.330154	-.004905	
4.5	.556489	-.018448	.325249	-.004415	
4.6	.538041	-.018458	.320834	+.000236	
4.7	.519583	-.018284	.321070	.000687	
4.8	.501299	-.017937	.321757	.002482	
4.9	.483362	-.017420	.324239	.004218	
5.0	.465942	-.016744	.328457	.005876	
5.1	.449198	-.015917	.334333	.007446	
5.2	.433281	-.014950	.341770	.008911	
5.3	.418331	-.013854	.350690	.010262	
5.4	.404477	-.012643	.360952	.011487	
5.5	.391834	-.011330	.372439	.012577	
5.6	.380504	-.009927	.385016	.013524	
5.7	.370577	-.008455	.398540	.014321	
5.8	.362122	-.006921	.412861	.014965	
5.9	.355201	-.005349	.427826	.015448	
6.0	.349852	-.003748	.443274	.015774	
6.1	.346104	-.002136	.459048	.015937	
6.2	.343968	-.000531	.474985	.015943	
6.3	.343437	+.001056	.490928	.015789	
6.4	.344493	.002606	.506717	.015485	
6.5	.347099	.004109	.522202	.015030	
6.6	.351208	.005543	.537232	.014436	
6.7	.356751	.006909	.551668	.013707	
6.8	.363660	.008178	.565375	.012855	
6.9	.371838	.009357	.578230	.011886	

X	S(X)	Δ	C(X)	Δ	11.
7.0	.381195	.010419	.590116	.010816	
7.1	.391614	.011369	.600932	.009654	
7.2	.402983	.012188	.610586	.008411	
7.3	.415171	.012882	.618997	.007105	
7.4	.428053	.013432	.626102	.005743	
7.5	.441485	.013848	.631845	.004345	
7.6	.455333	.014118	.636190	.002921	
7.7	.469451	.014248	.639111	+.001488	
7.8	.483699	.014232	.640599	+.000057	
7.9	.497931	.014079	.640656	-.001355	
8.0	.512010	.013785	.639301	-.002736	
8.1	.525795	.013362	.636565	-.004075	
8.2	.539157	.012809	.632490	-.005354	
8.3	.551966	.012138	.627136	-.006568	
8.4	.564104	.011353	.620568	-.007700	
8.5	.575457	.010467	.612868	-.008743	
8.6	.585824	.009485	.604125	-.009689	
8.7	.595409	.008423	.594436	-.010526	
8.8	.603832	.007288	.583910	-.011251	
8.9	.611120	.006094	.572659	-.011855	
9.0	.617214	.004853	.560804	-.012336	
9.1	.622067	.003578	.548468	-.012688	
9.2	.625645	.002280	.535780	-.012912	
9.3	.627925	+.000975	.522868	-.013005	
9.4	.628900	-.000327	.509863	-.012968	
9.5	.628573	-.001612	.496895	-.012803	
9.6	.626961	-.002865	.484092	-.012513	
9.7	.624096	-.004081	.471579	-.012102	
9.8	.620015	-.005240	.459477	-.011575	
9.9	.614775	-.006339	.447902	-.010938	

X	S(X)	Δ	C(X)	Δ	12.
10.0	.608436	-.007362	.436964	-.010199	
10.1	.601074	-.008302	.426765	-.009368	
10.2	.592772	-.009151	.417397	-.008450	
10.3	.583621	-.009900	.408947	-.007458	
10.4	.573721	-.010545	.401489	-.006402	
10.5	.563176	-.011080	.395087	-.005294	
10.6	.552096	-.011495	.389793	-.004137	
10.7	.540601	-.011800	.385656	-.002960	
10.8	.528801	-.011976	.382696	-.001754	
10.9	.516825	-.012041	.380942	-.000552	
11.0	.504784	-.011978	.380390	+.000651	
11.1	.492806	-.011801	.381041	.001838	
11.2	.481005	-.011506	.382879	.002990	
11.3	.469499	-.011102	.385869	.004110	
11.4	.458397	-.010588	.389979	.005170	
11.5	.447809	-.009976	.395149	.006180	
11.6	.437833	-.009270	.401329	.007108	
11.7	.428563	-.008477	.408437	.007971	
11.8	.420086	-.007607	.416408	.008733	
11.9	.412479	-.006669	.425141	.009416	
12.0	.405810	-.006671	.434557	.009990	
12.1	.400139	-.005629	.444547	.010463	
12.2	.395510	-.003544	.455010	.010828	
12.3	.391966	-.002438	.465838	.011083	
12.4	.389528	-.001311	.476921	.011225	
12.5	.388217	-.000184	.488146	.011256	
12.6	.388033	+.000936	.499402	.011172	
12.7	.388969	.002039	.510574	.010981	
12.8	.391008	.003113	.521555	.010679	
12.9	.394121	.004147	.532234	.010277	

X	S(X)	Δ	C(X)	Δ	13.
12.0	.398268	.005133	.542511	.009773	
13.1	.403401	.006059	.552284	.009177	
13.2	.409460	.006920	.561461	.008493	
13.3	.416380	.007703	.569954	.007732	
13.4	.424083	.008406	.577686	.006897	
13.5	.432489	.009018	.584583	.006002	
13.6	.441507	.009538	.590585	.005053	
13.7	.451045	.009958	.595638	.004061	
13.8	.461003	.010277	.599699	.003035	
13.9	.471280	.010490	.602734	.001987	
14.0	.481770	.010598	.604721	+.006057	
14.1	.492368	.010601	.610778	-.005266	
14.2	.502969	.010496	.605512	-.001190	
14.3	.513465	.010290	.604322	-.002222	
14.4	.523755	.009981	.602100	-.003229	
14.5	.533736	.009576	.598871	-.004195	
14.6	.543312	.009079	.594676	-.005111	
14.7	.552391	.008493	.589565	-.005973	
14.8	.560884	.007830	.583592	-.006766	
14.9	.568714	.007089	.576826	-.007491	
15.0	.575803	.006285	.569335	-.008132	
15.1	.582088	.005426	.561203	-.008692	
15.2	.587514	.004514	.552511	-.009159	
15.3	.592028	.003567	.543352	-.009533	
15.4	.595595	.002588	.533819	-.009810	
15.5	.598183	.001592	.524009	-.009986	
15.6	.599775	+.000583	.514023	-.010064	
15.7	.600358	-.000421	.503959	-.010039	
15.8	.599937	-.001419	.493920	-.009916	
15.9	.598518	-.002392	.484004	-.009694	

X	S(X)	Δ	C(X)	Δ	14.
16.0	.596126	-.003338	.474310	-.009377	
16.1	.592788	-.004245	.464933	-.008970	
16.2	.588543	-.005103	.455963	-.008474	
16.3	.583440	-.005906	.447489	-.007899	
16.4	.577534	-.006644	.439590	-.007247	
16.5	.570890	-.007312	.432343	-.006527	
16.6	.563578	-.007904	.425816	-.005749	
16.7	.555674	-.008413	.420067	-.004915	
16.8	.547261	-.008835	.415152	-.004040	
16.9	.538426	-.009167	.411112	-.003127	
17.0	.529259	-.009406	.407985	-.002011	
17.1	.519853	-.009550	.405794	-.001236	
17.2	.510303	-.009597	.404558	-.000277	
17.3	.500706	-.009550	.404281	+.000682	
17.4	.491156	-.009406	.404963	.001626	
17.5	.481750	-.009171	.406589	.002551	
17.6	.472579	-.008846	.409140	.003442	
17.7	.463733	-.008433	.412582	.004298	
17.8	.455300	-.007941	.416880	.005102	
17.9	.447359	-.007370	.421982	.005855	
18.0	.439989	-.006730	.427837	.006542	
18.1	.433259	-.006028	.434379	.007162	
18.2	.427231	-.005267	.441541	.007706	
18.3	.421964	-.004460	.449247	.008172	
18.4	.417504	-.003611	.457419	.008552	
18.5	.413893	-.002733	.465971	.008848	
18.6	.411160	-.001830	.474819	.009049	
18.7	.409330	-.000916	.483868	.009165	
18.8	.408414	+.000004	.493033	.009184	
18.9	.408418	.000918	.502217	.009115	

X	S(X)	Δ	C(X)	Δ	15.
19.0	.409336	.001819	.511332	.008952	
19.1	.411155	.002697	.520284	.008706	
19.2	.413852	.003543	.528990	.008369	
19.3	.417395	.004350	.537359	.007955	
19.4	.421745	.005108	.545314	.007460	
19.5	.426853	.005814	.552774	.006896	
19.6	.432667	.006455	.559670	.006265	
19.7	.439122	.007032	.565935	.005575	
19.8	.446154	.007533	.571510	.004833	
19.9	.453687	.007959	.576343	.004046	
20.0	.461646	.008303	.580389	.003225	
20.1	.469949	.008560	.583614	.002371	
20.2	.478509	.008734	.585985	.001504	
20.3	.487243	.008817	.587489	+.000620	
20.4	.496060	.008815	.588109	-.000260	
20.5	.504875	.008666	.587849	-.001137	
20.6	.513541	.008617	.586712	-.001998	
20.7	.522158	.008269	.584714	-.002833	
20.8	.530427	.007968	.581881	-.003638	
20.9	.538395	.007490	.578243	-.004401	
21.0	.545885	.007024	.573842	-.005118	
21.1	.552909	.006465	.568724	-.005781	
21.2	.559374	.005839	.562943	-.006381	
21.3	.565213	.005164	.556562	-.006917	
21.4	.570377	.004434	.549645	-.007379	
21.5	.574811	.003667	.542266	-.007768	
21.6	.578478	.002869	.534498	-.008076	
21.7	.581347	.002041	.526422	-.008304	
21.8	.583388	.001202	.518118	-.008445	
21.9	.584590	+.000349	.509673	-.008506	

X	S(X)	Δ	C(X)	Δ	16.
22.0	.584939	-.000499	.501167	-.008478	
22.1	.584440	-.001341	.492689	-.008366	
22.2	.583099	-.002163	.484323	-.008172	
22.3	.580936	-.002963	.476151	-.007898	
22.4	.577973	-.003727	.468253	-.007546	
22.5	.574246	-.004453	.460707	-.007321	
22.6	.569793	-.005129	.453386	-.006425	
22.7	.564664	-.005753	.446961	-.006067	
22.8	.558911	-.006316	.440894	-.005450	
22.9	.552595	-.006813	.435444	-.004782	
23.0	.545782	-.007241	.430662	-.004069	
23.1	.538541	-.007595	.426593	-.003319	
23.2	.530946	-.007871	.423274	-.002538	
23.3	.523075	-.008068	.420736	-.001737	
23.4	.515007	-.008183	.418999	-.000919	
23.5	.506824	-.008216	.418080	-.000099	
23.6	.498608	-.008169	.417981	+.000722	
23.7	.490439	-.008038	.418703	.001529	
23.8	.482401	-.007830	.420232	.002320	
23.9	.474571	-.007542	.422552	.003083	
24.0	.467029	-.007183	.425635	.003813	
24.1	.459846	-.006751	.429448	.004501	
24.2	.453095	-.006256	.433949	.005143	
24.3	.446839	-.005699	.439092	.005729	
24.4	.441140	-.005089	.444821	.006257	
24.5	.436051	-.004429	.451078	.006720	
24.6	.431622	-.003729	.457798	.007114	
24.7	.427893	-.002993	.464912	.007436	
24.8	.424900	-.002233	.472348	.007682	
24.9	.422667	-.001450	.480030	.007850	

X	S(X)	Δ	C(X)	Δ	17.
25.0	.421217	-.000658	.487880	.007940	
25.1	.420559	+.000036	.495820	.007951	
25.2	.420695	.000929	.503771	.007881	
25.3	.421624	.001706	.511652	.007735	
25.4	.423330	.002467	.519387	.007509	
25.5	.425797	.003197	.526896	.007209	
25.6	.428994	.003894	.534105	.006863	
25.7	.432888	.004548	.540968	.006393	
25.8	.437436	.005156	.547361	.005926	
25.9	.442592	.005708	.553287	.005341	
26.0	.448300	.006203	.558628	.004751	
26.1	.454503	.006634	.563379	.004089	
26.2	.461137	.006996	.567478	.003411	
26.3	.468133	.007288	.570889	.002689	
26.4	.475421	.007506	.573578	.001946	
26.5	.482927	.007648	.575524	.001183	
26.6	.490575	.007714	.576707	+.000414	
26.7	.498289	.007702	.577121	-.000358	
26.8	.505991	.007613	.576763	-.001123	
26.9	.513604	.007450	.575640	-.001874	
27.0	.521054	.007211	.573766	-.002604	
27.1	.528265	.006904	.571162	-.003304	
27.2	.535169	.006526	.567858	-.003970	
27.3	.541695	.006087	.563888	-.004593	
27.4	.547782	.005587	.559295	-.005168	
27.5	.553369	.005035	.554127	-.005690	
27.6	.558404	.004433	.548437	-.006153	
27.7	.562837	.003790	.542284	-.006553	
27.8	.566627	.003111	.535731	-.006886	
27.9	.569738	.002404	.528845	-.007150	

X	S(X)	Δ	C(X)	Δ	18.
28.0	.572142	.001676	.521695	-.007341	
28.1	.573818	.000931	.514354	-.007458	
28.2	.574749	+.000184	.506896	-.007501	
28.3	.574933	-.000567	.499395	-.007468	
28.4	.574366	-.001306	.491927	-.007361	
28.5	.573060	-.002032	.484566	-.007181	
28.6	.571028	-.002733	.477385	-.006931	
28.7	.568295	-.003406	.470454	-.006611	
28.8	.564889	-.004041	.463843	-.006228	
28.9	.560848	-.004636	.457615	-.005783	
29.0	.556212	-.005180	.451832	-.005284	
29.1	.551032	-.005673	.446548	-.004729	
29.2	.545359	-.006105	.441819	-.004135	
29.3	.539254	-.006477	.437684	-.003496	
29.4	.532777	-.006782	.434188	-.002830	
29.5	.525995	-.007017	.431358	-.002131	
29.6	.518978	-.007186	.429227	-.001423	
29.7	.511792	-.007277	.427804	-.000693	
29.8	.504515	-.007300	.427111	+.000034	
29.9	.497215	-.007246	.427145	.000763	
30.0	.489969		.427908		

Values of $C(x)$
at $x = (2n-1)\frac{\pi}{2}$

N	C(x)
1	0.779893
2	0.321056
3	0.640807
4	0.380389
5	0.605721
6	0.404260
7	0.588128
8	0.417922
9	0.577121
10	0.427036

Values of $S(x)$
at $x = n\pi$

N	S(x)
1	0.713972
2	0.343415
3	0.628940
4	0.387969
5	0.600361
6	0.408301
7	0.584942
8	0.420516
9	0.574957
10	0.428877