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WADC TECHNICAL REPORT 52-133

LAGRANGIAN INTERPOLATION

CLARENCE ROSS
FLIGHT RESEARCH LABORATORY

SEPTEMBER 1952

WRIGHT AIR DEVELOPMENT CENTER

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WADC TECHNICAL REPORT 52-133

LAGRANGIAN INTERPOLATION

Clarence Ross
Flight Research Laboratory

September 1952

RDO No. 468-1

Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

FOREWORD

This report was prepared by Dr. Clarence Ross, Project Scientist, Computation Branch, Flight Research Laboratory. Work was completed under RDO No. 468-1, Computation Services. The opportunity to write the report was due in part to the delay in delivery of OARAC from Syracuse, New York. Because the elements tabulated were left in fraction form it was necessary to carry out a considerable amount of hand computation. Valuable assistance was afforded by Mr. Carl S. Fluke. Mr. Frank M. Williams and Rice P. White, Jr., A/1c checked most of the results.

ABSTRACT

A systematic method of constructing formulae, together with error terms, is given for use in interpolation, extrapolation, differentiation, and integration. Both closed and open type formulae are developed, using ordinates, and based on Lagrange's interpolation formulae for equal intervals. The procedure was suggested by Professor H. H. Aiken several years ago at Harvard University. The whole procedure may be extended easily to obtain cubature formulae and formulae used for surface fitting.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDING GENERAL:

LESLIE B. WILLIAMS
Colonel, USAF
Chief, Flight Research Laboratory

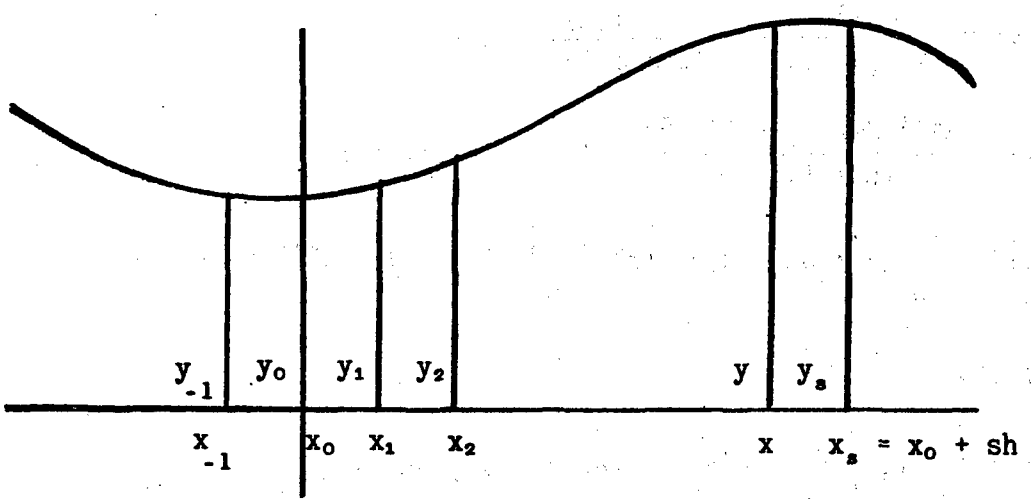
CONTENTS

LAGRANGIAN INTERPOLATION

The purpose of this paper is to apply Lagrange's interpolation formulae 1, 2, 3, 4 to derive polynomial approximating formulae (using ordinates) for interpolation, differentiation, and integration. Many of these formulae appear in the literature[5] in terms of differences. Of course these differences may be expressed in terms of ordinates and in a sense the formulae which appear here are a recapitulation of elementary formulae found in Numerical Analysis. However, the simple derivations given here admit easy extensions to new formulae of great accuracy (exclusive of round-off error) to be used in connection with large scale high speed digital calculators. The number of ordinates employed in the formulae are indicated in the tables and include as many as ten.

LAGRANGE'S INTERPOLATION FORMULA

The following assumptions are made and the notation is more or less standard.



(a) $x = x_0 + uh$, $y_s = y(x_0 + sh) = y(x_s)$.

(b) s, a, b are integers.

(c) $a \leq u \leq b$, where u may be integral or not.

(d) $n + 1 = b - a + 1$ is the number of points through which the approximating polynomial (1) passes.

(e) ξ lies between the smallest and the largest of the set x_s where $s - a = 0, 1, 2, \ldots, n$.

(f) $y^{(n+1)}(\xi)$ is familiar notation for the (n + 1)st derivative evaluated at $x = \xi$.

Lagrange's formula may be written

$$(1)\quad y(x) = \sum_{s=a}^{b} C_{s-a}(x)\, y(x) + R(x) \text{ where}$$

$$C_s(x) = \frac{\cdots \phi(x) \cdots}{(x - x_s)\phi'(x_s)} = \frac{\cdots \phi(u) \cdots}{(u - s)\phi'(s)}, \text{ and}$$

$$(2)\quad R(x) = \frac{h^{n+1}\, \phi(u)\, y^{(n+1)}(\xi)}{(n+1)!} \cdot \text{ Also by definition}$$

$$\phi(x) = \prod_{s=a}^{b} (x + ah - x_s) = h^{n+1} \prod_{s=0}^{n} (u - s) = h^{n+1}\, \phi(u)$$

Proof of (1).

Assume $y = \sum_{s=a}^{b} \psi_s \frac{\phi(x)}{x - x_s} + R(x)$. Then for any particular s (an integer) we have

$y_s = \psi_s \lim_{x \to x_s} \frac{\phi(x)}{x - x_s} = \psi_s \phi'(x_s)$. Hence

$\psi_s = \frac{ys}{\phi'(x_s)}$ which leads to (1).

Proof [6] of (2).

Write $f(z) = R(z) - R(x) \frac{\phi(z)}{\phi(x)}$. It follows that $f = 0$ at $z = x, x_s$

for $s - a = 0, 1, 2, 3, \ldots, n$. Hence by Rolle's theorem [7] $f^{(n+1)}(\xi) = 0$.

Also it is apparent that $\phi^{(n+1)}(z) = (n + 1)!$ Consequently

$0 = R^{(n+1)}(\xi) - R(x) \frac{(n+1)!}{\phi(x)}$ so that $R(x) = \frac{\phi(x)\, R^{(n+1)}(\xi)}{(n + 1)!}$

Now by (1)

$R^{(n+1)}(\xi) = y^{(n+1)}(\xi)$ which proves (2).

If $b < u$ or $u < a$ then (1) may be used to extrapolate y if we accept the error produced by taking ξ in (2) to lie in the interval between the least and greatest of the set x, x_s and again where $s - a = 0, 1, 2, \ldots, n$.

DIFFERENTIATION FORMULAE

By differentiating (1) and substituting $x = x_t$ where t is a particular s we get

(3) $y'(x_t) = \frac{1}{h} \sum_{s=a}^{b} C'_{s-a} y_s + R'(t)$ where

$$(4)\quad R'(t) = \frac{h^n \phi'(t)\, y^{(n+1)}(\xi)}{(n+1)!}.$$

In order to prove (4) consider (2) which may be written

$R(x) = \phi(x)\, y(x, x_s)$ since $\frac{y^{(n+1)}(\xi)}{(n+1)!}$ is a function of the set x, x_s

where $s - a = 0, 1, 2, \ldots, n$. Hence $R' = \phi' y + \phi y'$ and since

$\phi(x_t) = 0$ and $\phi'(x) = h^{n+1} \frac{d\phi}{du} \cdot \frac{du}{dx} = h^n \phi'(u)$ we have (4).

In order to simplify the remainder terms the derivative formulae are tabulated at the given ordinates only. Moreover not all these are tabulated since some may be implied. For example the 29 first derivative formulae imply 25 more. Indeed for $s \neq t$ we have

$$C'_s(t) = \frac{1}{\phi'_s} \lim_{u \to t} \frac{\phi(u)}{(u-t)(u-s)} = \frac{1}{t-s} \cdot \frac{\phi'_t}{\phi'_s}, \text{ and}$$

$$C'_{n-s}(n-t) = \lim_{u \to n-s} \frac{u-n+s}{\phi(u)} \cdot \lim_{u \to n-t} \frac{\phi(u)}{(u-n+t)(u-n+s)}$$

$$= \lim_{u \to n-s} \frac{u-n+s}{\phi(n-u)} \cdot \lim_{u \to n-t} \frac{\phi(n-u)}{(u-n+t)(u-n+s)}$$

$$= -\lim_{u \to s} \frac{u-s}{\phi(u)} \cdot \lim_{u \to t} \frac{\phi(u)}{(u-t)(u-s)} = -\frac{1}{t-s} \cdot \frac{\phi'_t}{\phi'_s}.$$

Hence $C'_s(t) = -C'_{n-s}(n-t)$. The diagonal elements obey this relation also since

$$\sum_{s=0}^{n} C'_s(t) + \sum_{s=0}^{n} C'_{n-s}(n-t) = 0.$$

In order to obtain higher order differentiation formulae we may differentiate $y' = \frac{1}{h} C'_{s-a}\, y_s$, omitting the remainder for a moment. Here y' and y_s are vectors while C'_{s-a} is a square matrix of order $n+1$. Hence in general

$$(5)\quad y^{(n)} = \frac{1}{h^n} C'^{n}_{s-a}\, y_s.$$

The remainder is obtained by differentiating $R(x) = \phi(x)\, y(x, x_s)$ as before. We have in general, using symbolic notation,

$$(6)\quad R^{(n)}(x) = [\phi(x) + y(x, x_s)]^{(n)}.$$

Only the first term of $R^{(n)}(x)$ is retained i.e. $R^{(n)}(x) = \phi^{(n)}(x)\, y(x, x_s)$ unless this term vanishes. If $\phi^{(n)}(x) = 0$ then obviously the order of $R^{(n)}(x)$ is multiplied by h and is so indicated.

The implied differentiation formulae are obtained by using the relation $C_s^{(n)}(t) = (-1)^{n+1}\ C_{n-s}^{(n)}(n-t)$.

In the tabulation below $r^{(n)} = \dfrac{\phi^{(n)}}{(n+1)!}$.

No.	t	LCD	C_0'	C_1'	C_2'	C_3'	C_4'	C_5'	C_6'	C_7'	C_8'	C_9'	r'
						2 ordinates							
1	0	1	-1	1									$-\frac{1}{2}$
						3 ordinates							
2	0	2	-3	4	-1								$\frac{1}{3}$
3	1	2	-1	0	1								$-\frac{1}{6}$
						4 ordinates							
4	0	6	-11	18	-9	2							$-\frac{1}{4}$
5	1	6	-2	-3	6	-1							$\frac{1}{12}$
						5 ordinates							
6	0	12	-25	48	-36	16	-3						$\frac{1}{5}$
7	1	12	-3	-10	18	-6	1						$-\frac{1}{20}$

No.	t	LCD	C'_0	C'_1	C'_2	C'_3	C'_4	C'_5	C'_6	C'_7	C'_8	C'_9	r'
8	2	12	1	-8	0	8	-1						$-\frac{1}{30}$
						6 ordinates							
9	0	60	-137	12	-12	200	-75	12					$-\frac{1}{6}$
10	1	60	-12	-65	120	-60	20	-3					$\frac{1}{30}$
11	2	60	3	-30	-20	60	-15	2					$-\frac{1}{60}$
						7 ordinates							
12	0	60	-147	360	-450	400	-225	72	-10				$\frac{1}{7}$
13	1	60	-10	77	150	-100	50	-15	2				$-\frac{1}{42}$
14	2	60	2	-24	-35	80	-30	8	-1				$\frac{1}{105}$
15	3	60	-1	9	-45	0	45	-9	1				$-\frac{1}{140}$

No.	t	LCD	C_0'	C_1'	C_2'	C_3'	C_4'	C_5'	C_6'	C_7'	C_8'	C_9'	r'
						8 ordinates							
16	0	420	-1089	2940	-1410	4900	-3675	1764	-490	60			$-\frac{1}{8}$
17	1	420	-60	-609	1260	-1050	700	-315	84	-10			$\frac{1}{56}$
18	2	420	10	-140	-329	700	-350	140	-35	4			$-\frac{1}{168}$
19	3	420	-4	42	-252	-105	420	-126	28	-3			$\frac{1}{280}$
						9 ordinates							
20	0	840	-2283	6720	-11760	15680	-14700	9408	-3920	960	-105		$\frac{1}{9}$
21	1	840	-105	-1338	2940	-2940	2450	-1470	588	-140	15		$-\frac{1}{72}$
22	2	840	15	-240	-798	1680	-1050	560	-210	48	-5		$\frac{1}{252}$
23	3	840	-5	60	-420	-378	1050	-420	140	-30	3		$-\frac{1}{504}$

No.	t	LCD	C_0'	C_1'	C_2'	C_3'	C_4'	C_5'	C_6'	C_7'	C_8'	C_9'	r
24	4	840	3	-32	168	-672	0	672	-168	32	-3		$\frac{1}{630}$
						10 ordinates							
25	0	2520	-7129	22680	-45360	70560	-79380	63504	-35280	12960	-2835	280	$-\frac{1}{10}$
26	1	2520	-280	-4329	10080	-11760	11760	-8820	4704	-1680	360	-35	$\frac{1}{90}$
27	2	2520	35	-630	-2754	5880	-4410	2940	-1470	504	-105	10	$-\frac{1}{360}$
28	3	2520	-10	135	-1080	-1554	3780	-1890	840	-270	54	-5	$\frac{1}{840}$
29	4	2520	5	-60	360	-1680	-504	2520	-840	240	-45	4	$-\frac{1}{1260}$

No.	t	LCD	C_0''	C_1''	C_2''	C_3''	C_4''	C_5''	C_6''	C_7''	C_8''	C_9''	r''
						3 ordinates							
30	0	1	1	-2	1								-1
31	1	1	1	-2	1								$R'' = O(h^3)$
						4 ordinates							
32	0	1	2	-5	4	-1							$\frac{11}{12}$
33	1	1	1	-2	1	0							$-\frac{1}{12}$
						5 ordinates							
34	0	12	35	-104	114	-56	11						$-\frac{5}{6}$
35	1	12	11	-20	6	4	-1						$\frac{1}{12}$
36	2	12	-1	16	-30	16	-1						$R'' = O(h^5)$

NO.	t	LCD	C_0''	C_1''	C_2''	C_3''	C_4''	C_5''	C_6''	C_7''	C_8''	C_9''	r''
						6 ordinates							
37	0	12	45	-154	214	-156	61	-10					$\frac{137}{180}$
38	1	12	10	-15	-4	14	-6	1					$-\frac{13}{180}$
39	2	12	-1	16	-30	16-	-1	0					$\frac{1}{90}$
						7 ordinates							
40	0	180	812	-3132	5265	-5080	2970	-972	137				$-\frac{7}{10}$
41	1	180	137	-147	-255	470	-285	93	-13				$\frac{11}{180}$
42	2	180	-13	228	-420	200	15	-12	2				$-\frac{1}{90}$
43	3	180	2	-27	270	-490	270	-27	2				$R'' = O(h^7)$

No.	t	LCD	C_0''	C_1''	C_2''	C_3''	C_4''	C_5''	C_6''	C_7''	C_8''	C_9''	r''
						8 ordinates							
44	0	180	988	-4014	7911	-9490	7380	-3618	1019	-126			$\frac{363}{560}$
45	1	180	126	-70	-486	855	-670	324	-90	11			$-\frac{87}{1680}$
46	2	180	-11	214	-378	130	85	-54	16	-2			$\frac{47}{5040}$
47	3	180	2	-27	270	-490	270	-27	2	0			$-\frac{1}{560}$

No.	t	LCD	C_0''	C_1''	C_2''	C_3''	C_4''	C_5''	C_6''	C_7''	C_8''	C_9''	
						9 ordinates							
48	0	5040	29531	-138528	312984	-448672	435330	-284256	120008	-29664	3267		$-\frac{761}{1260}$
49	1	5040	3267	128	-20916	38556	-37030	23688	-9828	2396	-261		$\frac{223}{5040}$
50	2	5040	-261	5616	-9268	1008	5670	-4144	1764	-432	47		$-\frac{19}{2520}$
51	3	5040	47	-684	7308	-13216	6930	-252	-196	72	-9		$\frac{1}{560}$
52	4	5040	-9	128	-1008	8064	-14350	8064	-1008	128	-9		$R'' = O(h^9)$
						10 ordinates							
53	0	5040	32575	-165924	422568	-704368	818874	-667800	375704	-139248	30663	-3044	$\frac{7129}{12600}$
54	1	5040	3044	2135	-28944	57288	-65128	51786	-28560	10424	-2268	223	$-\frac{481}{12600}$
55	2	5040	-223	5274	-7900	-2184	10458	-8932	4956	-1800	389	-38	$\frac{153}{25200}$
56	3	5040	38	-603	6984	-12460	5796	882	-952	396	-90	9	$-\frac{37}{25200}$
57	4	5040	-9	128	-1008	8064	-14350	8064	-1008	128	-9	0	$\frac{1}{3150}$

No.	t	LCD	C'''_0	C'''_1	C'''_2	C'''_3	C'''_4	C'''_5	C'''_6	C'''_7	C'''_8	C'''_9	r'''
						4 ordinates							
58	0	1	-1	3	-3	1							$-\frac{3}{2}$
59	1	1	-1	3	-3	1							$-\frac{1}{2}$
						5 ordinates							
60	0	6	-15	54	-72	42	-9						$\frac{7}{4}$
61	1	6	-9	30	-36	18	-3						$\frac{1}{4}$
62	2	6	-3	6	0	-6	3						$-\frac{1}{4}$
						6 ordinates							
63	0	4	-17	71	-118	98	-41	7					$-\frac{15}{8}$
64	1	4	-7	25	-34	22	-7	1					$-\frac{1}{8}$
65	2	4	-1	-1	10	-14	7	-1					$\frac{1}{8}$

No.	t	LCD	C_0'''	C_1'''	C_2'''	C_3'''	C_4'''	C_5'''	C_6'''	C_7'''	C_8'''	C_9'''	r'''
						7 ordinates							
66	0	8	-49	232	-461	496	-307	104	-15				$\frac{29}{15}$
67	1	8	-15	56	-83	64	-29	8	-1				$\frac{7}{120}$
68	2	8	-1	-8	35	-48	29	-8	1				$-\frac{1}{15}$
69	3	8	1	-8	13	0	-13	8	-1				$\frac{7}{120}$
						8 ordinates							
70	0	120	-967	5104	-11787	15560	-12725	6432	-1849	232			$-\frac{469}{240}$
71	1	120	-232	889	-1392	1205	-680	267	-64	7			$-\frac{1}{48}$
72	2	120	-7	-176	693	-1000	715	-288	71	-8			$\frac{3}{80}$
73	3	120	8	-71	48	245	-440	267	-64	7			$-\frac{7}{240}$

No.	t	LCD	C'''_0	C'''_1	C'''_2	C'''_3	C'''_4	C'''_5	C'''_6	C'''_7	C'''_8	C'''_9	
						9 ordinates							
74	0	240	-2403	13960	-36706	57384	-58280	39128	-16830	4216	-469		$\frac{29531}{15120}$
75	1	240	-469	1818	-2924	2690	-1710	814	-268	54	-5		$-\frac{1}{945}$
76	2	240	-5	-424	1638	-2504	2060	-1080	394	-88	9		$-\frac{331}{15120}$
77	3	240	9	-86	-100	882	-1370	926	-324	70	-7		$\frac{59}{3780}$
78	4	240	-7	72	-338	488	0	-488	338	-72	7		$-\frac{41}{3024}$
						10 ordinates							
79	0	1512	-180220	1145259	-3375594	6095796	-7392546	6185970	-3540894	1328724	-295326	29531	$-\frac{1303}{672}$
80	1	1512	-29531	114390	-183636	168126	-105714	49266	-15540	2826	-171	-16	$\frac{61}{4320}$
81	2	1512	16	-29691	115110	-185556	171486	-109746	52626	-17460	3546	-331	$\frac{79}{6048}$

No.	t	LCD	C_0'''	C_1'''	C_2'''	C_3'''	C_4'''	C_5'''	C_6'''	C_7'''	C_8'''	C_9'''	r'''
82	3	1512	331	-3294	-14796	75390	-116046	88074	-40236	12906	-2565	236	$-\frac{89}{10080}$
83	4	1512	-236	2691	-13914	13524	25830	-56574	38514	-11916	2286	-205	$\frac{41}{6048}$

No.	t	LCD	$C_0^{(4)}$	$C_1^{(4)}$	$C_2^{(4)}$	$C_3^{(4)}$	$C_4^{(4)}$	$C_5^{(4)}$	$C_6^{(4)}$	$C_7^{(4)}$	$C_8^{(4)}$	$C_9^{(4)}$	$r^{(4)}$
						5 ordinates							
84	0	1	1	-4	6	-4	1						-2
85	1	1	1	-4	6	-4	1						-1
86	2	1	1	-4	6	-4	1						$R^{(4)} = O(h^5)$
						6 ordinates							
87	0	1	3	-14	26	-24	11	-2					$\frac{17}{6}$
88	1	1	2	-9	16	-14	6	-1					$\frac{5}{6}$
89	2	1	1	-4	6	-4	1	0					$-\frac{1}{6}$

No.	t	LCD	$C_0^{(4)}$	$C_1^{(4)}$	$C_2^{(4)}$	$C_3^{(4)}$	$C_4^{(4)}$	$C_5^{(4)}$	$C_6^{(4)}$	$C_7^{(4)}$	$C_8^{(4)}$	$C_9^{(4)}$	$\Gamma^{(4)}$
						7 ordinates							
90	0	6	35	-186	411	-484	321	-114	17				$-\frac{7}{2}$
91	1	6	17	-84	171	-184	111	-36	5				$-\frac{2}{3}$
92	2	6	5	-18	21	-4	-9	6	-1				$\frac{1}{6}$
93	3	6	-1	12	-39	56	-39	12	-1			$R^{(4)} = O(h^7)$	
						8 ordinates							
94	0	6	56	-333	852	-1219	1056	-555	164	-21			$\frac{967}{240}$
95	1	6	21	-112	255	-324	251	-120	33	-4			$\frac{127}{240}$
96	2	6	4	-11	0	31	-44	27	-8	1			$-\frac{11}{80}$
97	3	6	-1	12	-39	56	-39	12	-1	0			$\frac{7}{240}$

No.	t	LCD	$C_0^{(4)}$	$C_1^{(4)}$	$C_2^{(4)}$	$C_3^{(4)}$	$C_4^{(4)}$	$C_5^{(4)}$	$C_6^{(4)}$	$C_7^{(4)}$	$C_8^{(4)}$	$C_9^{(4)}$	$r^{(4)}$
						9 ordinates							
98	0	240	3207	-21056	61156	-102912	109980	-76352	33636	-8576	967		$-\frac{89}{20}$
99	1	240	967	-5496	13756	-20072	18980	-11912	4876	-1176	127		$-\frac{101}{240}$
100	2	240	127	-176	-924	3088	-4070	2928	-1244	304	-33		$\frac{13}{120}$
101	3	240	-33	424	-1364	1848	-1070	88	156	-56	7		$-\frac{7}{240}$
102	4	240	7	-96	676	-1952	2730	-1952	676	-96	7	$R^{(4)}$	$= O(h^9)$
						10 ordinates							
103	0	24	4275	-30668	99604	-192624	244498	-210920	123348	-47024	10579	-1068	$\frac{4523}{945}$
104	1	24	1068	-6405	17392	-28556	31656	-24638	13360	-4812	1036	-101	$\frac{1271}{3780}$
105	2	24	101	58	-1860	5272	-7346	6204	-3428	1240	-267	26	$-\frac{1279}{15120}$
106	3	24	-26	361	-1112	1260	-188	-794	744	-308	70	-7	$\frac{359}{15120}$
107	4	24	7	-96	676	-1952	2730	-1952	676	-96	7	0	$-\frac{41}{7560}$

No.	t	LCD	$C_0^{(5)}$	$C_1^{(5)}$	$C_2^{(5)}$	$C_3^{(5)}$	$C_4^{(5)}$	$C_5^{(5)}$	$C_6^{(5)}$	$C_7^{(5)}$	$C_8^{(5)}$	$C_9^{(5)}$	$r^{(5)}$
						6 ordinates							
108	0	1	-1	5	-10	10	-5	1					$-\frac{5}{2}$
109	1	1	-1	5	-10	10	-5	1					$-\frac{3}{2}$
110	2	1	-1	5	-10	10	-5	1					$-\frac{1}{2}$
						7 ordinates							
111	0	2	-7	40	-95	120	-85	32	-5				$\frac{25}{6}$
112	1	2	-5	28	-65	80	-55	20	-3				$\frac{5}{3}$
113	2	2	-3	16	-35	40	-25	8	-1				$\frac{1}{6}$
114	3	2	-1	4	-5	0	5	-4	1				$-\frac{1}{3}$

No.	t	LCD	$C_0^{(5)}$	$C_1^{(5)}$	$C_2^{(5)}$	$C_3^{(5)}$	$C_4^{(5)}$	$C_5^{(5)}$	$C_6^{(5)}$	$C_7^{(5)}$	$C_8^{(5)}$	$C_9^{(5)}$	$r^{(5)}$
						8 ordinates							
115	0	6	-46	295	-810	1235	-1130	621	-190	25			$-\frac{35}{6}$
116	1	6	-25	154	-405	590	-515	270	-79	10			$-\frac{5}{3}$
117	2	6	-10	55	-126	155	-110	45	-10	1			$O(h^8)$
118	3	6	-1	-2	27	-70	85	-54	17	-2			$\frac{1}{6}$
						9 ordinates							
119	0	6	-81	575	-1790	3195	-3580	2581	-1170	305	-35		$\frac{1069}{144}$
120	1	6	-35	234	-685	1150	-1215	830	-359	90	-10		$\frac{229}{144}$
121	2	6	-10	55	-126	155	-110	45	-10	1	0		$-\frac{11}{144}$
122	3	6	0	-10	55	-126	155	-110	45	-10	1		$-\frac{11}{144}$
123	4	6	1	-9	26	-29	0	29	-26	9	-1		$\frac{13}{144}$

No.	t	LCD	$C_0^{(5)}$	$C_1^{(5)}$	$C_2^{(5)}$	$C_3^{(5)}$	$C_4^{(5)}$	$C_5^{(5)}$	$C_6^{(5)}$	$C_7^{(5)}$	$C_8^{(5)}$	$C_9^{(5)}$	$r^{(5)}$
						10 ordinates							
124	0	144	-3013	23421	-81444	166476	-220614	196638	-117876	45804	-10461	1069	$-\frac{285}{32}$
125	1	144	-1069	7677	-24684	46836	-58014	48774	-27852	10404	-2301	229	$-\frac{427}{288}$
126	2	144	-229	1221	-2628	2796	-1254	-306	684	-372	99	-11	$\frac{31}{288}$
127	3	144	11	-339	1716	-3948	5106	-4026	2004	-636	123	-11	$\frac{1}{32}$
128	4	144	11	-99	156	396	-1638	2334	-1716	684	-141	13	$-\frac{13}{288}$

No.	t	LCD	$C_0^{(6)}$	$C_1^{(6)}$	$C_2^{(6)}$	$C_3^{(6)}$	$C_4^{(6)}$	$C_5^{(6)}$	$C_6^{(6)}$	$C_7^{(6)}$	$C_8^{(6)}$	$C_9^{(6)}$	$r^{(6)}$
						7 ordinates							
129	0	1	1	-6	15	-20	15	-6	1				-3
130	1	1	1	-6	15	-20	15	-6	1				-2

No.	t	LCD	$C_0^{(6)}$	$C_1^{(6)}$	$C_2^{(6)}$	$C_3^{(6)}$	$C_4^{(6)}$	$C_5^{(6)}$	$C_6^{(6)}$	$C_7^{(6)}$	$C_8^{(6)}$	$C_9^{(6)}$	$r^{(6)}$
131	2	1	1	-6	15	-20	15	-6	1				-1
132	3	1	1	-6	15	-20	15	-6	1				$R^{(6)} = O(h^7)$
						8 ordinates							
133	0	1	4	-27	78	-125	120	-69	22	-3			$\frac{23}{4}$
134	1	1	3	-20	57	-90	85	-48	15	-2			$\frac{11}{4}$
135	2	1	2	-13	36	-55	50	-27	8	-1			$\frac{3}{4}$
136	3	1	1	-6	15	-20	15	-6	1	0			$-\frac{1}{4}$
						9 ordinates							
137	0	4	39	-292	956	-1788	2090	-1564	732	-196	23		-9
138	1	4	23	-168	536	-976	1110	-808	368	-96	11		$-\frac{13}{4}$
139	2	4	11	-76	228	-388	410	-276	116	-28	3		$-\frac{1}{2}$

No.	t	LCD	$C_0^{(6)}$	$C_1^{(6)}$	$C_2^{(6)}$	$C_3^{(6)}$	$C_4^{(6)}$	$C_5^{(6)}$	$C_6^{(6)}$	$C_7^{(6)}$	$C_8^{(6)}$	$C_9^{(6)}$	$r^{(6)}$
140	3	4	3	-16	32	-24	-10	32	-24	8	-1		$\frac{1}{4}$
141	4	4	-1	12	-52	116	-150	116	-52	12	-1		$R^{(6)} = O(h^9)$
						10 ordinates							
142	0	4	75	-616	2252	-4812	6626	-6100	3756	-1492	347	-36	$\frac{3013}{240}$
143	1	4	36	-285	1004	-2068	2748	-2446	1460	-564	128	-13	$\frac{853}{240}$
144	2	4	13	-94	300	-556	662	-528	284	-100	21	-2	$\frac{73}{240}$
145	3	4	2	-7	-4	60	-136	158	-108	44	-10	1	$\frac{47}{240}$
146	4	4	-1	12	-52	116	-150	116	-52	12	-1	0	$\frac{13}{240}$

No.	t	LCD	$C_0^{(7)}$	$C_1^{(7)}$	$C_2^{(7)}$	$C_3^{(7)}$	$C_4^{(7)}$	$C_5^{(7)}$	$C_6^{(7)}$	$C_7^{(7)}$	$C_8^{(7)}$	$C_9^{(7)}$	$r^{(7)}$
						8 ordinates							
147	0	1	-1	7	-21	35	-35	21	-7	1			$-\frac{7}{2}$
148	1	1	-1	7	-21	35	-35	21	-7	1			$-\frac{5}{2}$
149	2	1	-1	7	-21	35	-35	21	-7	1			$-\frac{3}{2}$
150	3	1	-1	7	-21	35	-35	21	-7	1			$-\frac{1}{2}$
						9 ordinates							
151	0	2	-9	70	-238	462	-560	434	-210	58	-7		$\frac{91}{12}$
152	1	2	-7	54	-182	350	-420	322	-154	42	-5		$\frac{49}{12}$
153	2	2	-5	38	-126	238	-280	210	-98	26	-3		$\frac{19}{12}$
154	3	2	-3	22	-70	126	-140	98	-42	10	-1		$\frac{1}{12}$
155	4	2	-1	6	-14	14	0	-14	14	-6	1		$-\frac{5}{12}$

No.	t	LCD	$C_0^{(7)}$	$C_1^{(7)}$	$C_2^{(7)}$	$C_3^{(7)}$	$C_4^{(7)}$	$C_5^{(7)}$	$C_6^{(7)}$	$C_7^{(7)}$	$C_8^{(7)}$	$C_9^{(7)}$	$r^{(7)}$
						10 ordinates							
156	0	12	-145	1239	-4704	10416	-14826	14070	-8904	3624	-861	91	$-\frac{105}{8}$
157	1	12	-91	765	-2856	6216	-8694	8106	-5040	2016	-471	49	$-\frac{133}{24}$
158	2	12	-49	399	-1440	3024	-4074	3654	-2184	840	-189	19	$-\frac{35}{24}$
159	3	12	-19	141	-456	840	-966	714	-336	96	-15	1	$-\frac{1}{8}$
160	4	12	-1	-9	96	-336	630	-714	504	-216	51	-5	$\frac{5}{24}$

No.	t	LCD	$C_0^{(8)}$	$C_1^{(8)}$	$C_2^{(8)}$	$C_3^{(8)}$	$C_4^{(8)}$	$C_5^{(8)}$	$C_6^{(8)}$	$C_7^{(8)}$	$C_8^{(8)}$	$C_9^{(8)}$	$r^{(8)}$
						9 ordinates							
161	0	1	1	-8	28	-56	70	-56	28	-8	1		-4
162	1	1	1	-8	28	-56	70	-56	28	-8	1		-3
163	2	1	1	-8	28	-56	70	-56	28	-8	1		-2
164	3	1	1	-8	28	-56	70	-56	28	-8	1		-1
165	4	1	1	-8	28	-56	70	-56	28	-8	1	$R^{(8)} = O(h^9)$	
						10 ordinates							
166	0	1	5	-44	172	-392	574	-560	364	-152	37	-4	$\frac{29}{3}$
167	1	1	4	-35	136	-308	448	-434	280	-116	28	-3	$\frac{17}{3}$
168	2	1	3	-26	100	-224	322	-308	196	-80	19	-2	$\frac{8}{3}$
169	3	1	2	-17	64	-140	196	-182	112	-44	10	-1	$\frac{2}{3}$
170	4	1	1	-8	28	-56	70	-56	28	-8	1	0	$-\frac{1}{3}$

No.	t	LCD	$C_0^{(9)}$	$C_1^{(9)}$	$C_2^{(9)}$	$C_3^{(9)}$	$C_4^{(9)}$	$C_5^{(9)}$	$C_6^{(9)}$	$C_7^{(9)}$	$C_8^{(9)}$	$C_9^{(9)}$	$r^{(9)}$
						10 ordinates							
171	0	1	-1	9	-36	84	-126	126	-84	36	-9	1	$-\frac{9}{2}$
172	1	1	-1	9	-36	84	-126	126	-84	36	-9	1	$-\frac{7}{2}$
173	2	1	-1	9	-36	84	-126	126	-84	36	-9	1	$-\frac{5}{2}$
174	3	1	-1	9	-36	84	-126	126	-84	36	-9	1	$-\frac{3}{2}$
175	4	1	-1	9	-36	84	-126	126	-84	36	-9	1	$-\frac{1}{2}$

QUADRATURE FORMULAE

We may obtain quadrature formulae by integrating (1) to get

$$(7)\quad \int_{b-k}^{b+1} y\,dx = h \sum_{s=a}^{b} C^*_{s-a}\, y_s + R^* \quad \text{where}$$

$$(8)\quad R^* \leq \frac{h^{n+2}\, y^{(n+1)}\,(\xi)}{(n+1)!} \int_{-k}^{1} \phi(u+n)\,du$$

(unless the last integral vanishes) and where

$$C^*_s = \int_{-k}^{1} C_s(u+n)\,du = \frac{1}{\phi'_s} \int_{n-k}^{n+1} \frac{\phi(u)\,du}{u-s}\,.$$

In order to prove (8) consider again

$R(x) = \phi(x)\, y(x, x'_s)$ from which

$$R^* = h^{n+2} \int_{u=-k}^{1} \phi(u+n)\, y(x, x_s)\,du$$

Now $\phi(u+n) = \prod_{s=0}^{n} (u+s)$ which is monotonic increasing for $u > 0$.

By the Mean Value Theorem we may therefore write

$$\int_0^1 \phi(u+n)\, y(x, x_s)\,du = \frac{y^{(n+1)}\,(\xi_1)}{(n+1)!} \int_0^1 \phi(u+n)\,du \quad \text{where}$$

$0 < \xi_1 < 1$. Again by Rolle's Theorem since $\phi(u+n)$ is continuous and vanishes at $u = 0, -1$, there is a point $-1 < \eta < 0$ such that $\phi'(u+n) = 0$.

Consequently in the interval $\eta \leq u \leq 0$, $\phi(u + n)$ is monotonic increasing so that we may write

$$\int_{\eta}^{0} \phi(u + n)\, y(x, x_s)\, du = \frac{y^{(n+1)}(\xi_2)}{(n + 1)!} \int_{\eta}^{0} \phi(u + n)\, du \text{ where}$$

$\eta < \xi_2 < 0$. If $y^{(n+1)}(\xi)$ is the larger of $y^{(n+1)}(\xi_1)$, $y^{(n+1)}(\xi_2)$ then we may write

$$\int_{\eta}^{1} \phi(u + n)\, y(x, x_s)\, du \leq \frac{y^{(n+1)}(\xi)}{(n +)1!} \int_{\eta}^{1} \phi(u + n)\, du \text{ where}$$

$\eta < \xi < 1$.

Continuing for the interval $-1 \leq u < \eta$ we find that $\phi(u + n)$ is monotonic decreasing so that we may write

$$\int_{-1}^{\eta} \phi(u + n)\, y(x, x_s)\, du = \frac{y^{(n+1)}(\xi_3)}{(n + 1)!} \int_{-1}^{\eta} \phi(u + n)\, du \text{ where}$$

$-1 < \xi_3 < \eta$. Therefore we may write

$$\int_{-1}^{1} \phi(u + n)\, y(x, x_s)\, du = \frac{1}{(n+1)!} \left[y^{(n+1)}(\xi_3) \int_{-1}^{\eta} + \right.$$

$$\left. y^{(n+1)}(\xi_2) \int_{\eta}^{0} + y^{(n+1)}(\xi_1) \int_{0}^{1} \right] \leq \frac{y^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{1} \phi(u + n)\, du$$

where $-1 < \xi < 1$ and $y^{(n+1)}(\xi)$ is the largest of $y^{(n+1)}(\xi_1)$, $y^{(n+1)}(\xi_2)$, $y^{(n+1)}(\xi_3)$. Since the polynomial ϕ vanishes at $-u = 0, 1, 2, \ldots, n$ we find (8) by continuing in this way.

The condition $\int_{-k}^{1} \phi(u+n)\,du \neq 0$ cannot be relaxed. However because of this condition no formula will be lost. For consider

$$\phi_{s-1,\,n-1}\, C^*_{s-1,\,n-1} = \int_{n-k-1}^{n+1-1} \frac{\phi(u)\,du}{(u-s+1)\,(u-n)} = \int_{n-k}^{n+1} \frac{\phi(u-1)\,du}{(u-s)\,(u-n-1)}$$

$$= \int_{n-k}^{n+1} \frac{\phi(u)\,du}{u(u-s)} \,.$$ Hence we have

$$\phi'_s\, C^*_s - s\,\phi'_{s-1,\,n-1}\, C^*_{s-1,\,n-1} = \int_{n-k}^{n+1} \left[\frac{\phi(u)}{u-s} - \frac{s\,\phi(u)}{u(u-s)} \right] du$$

$$= \int_{n-k}^{n+1} \frac{\phi(u)\,du}{u}\; \phi'_o\, C^*_o$$

and since $s\,\phi'_{s-1,\,n-1} = \phi'_s$ we have

(9) $$C^*_s = \frac{\phi'_o}{\phi'_s}\, C^*_o + C^*_{s-1,\,n-1}.$$

The $C^*_{s-1,\,n-1}$ refers to that value immediately above C^*_{s-1} in the tables below. Of course

$\phi'_o\, C^*_o = \int_{-k}^{1} \phi_{n-1}(u+n)\,du$ where the integral is on the line immediately above C^*_o in the tables. If this integral vanishes we have $C^*_s = C^*_{s-1,\,n-1}$ by (9), for $s \neq 0$, and therefore the formula is repeated.

It is easy to see that this integral cannot vanish twice in succession.

By taking $l = 1$ we obtain open quadrature formulae and by taking $l \leq 0$ we obtain closed quadrature formulae. Because of symmetry the implied quadrature formulae double to number of formulae listed below. For example if C_s^* is replaced by C_{n-s}^* in a formula for extrapolating ahead we shall replace it by one which extrapolates backward. The remainders may be different because the intervals are different for ξ.

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_0^1 \varphi(u+n)du$
						1-strip						
1.	2	$-\frac{1}{2}$	$\frac{3}{2}$									$\frac{5}{6}$
2.	3	$\frac{5}{12}$	$-\frac{4}{3}$	$\frac{23}{12}$								$\frac{9}{4}$
3.	4	$-\frac{3}{8}$	$\frac{37}{24}$	$-\frac{59}{24}$	$\frac{55}{24}$							$\frac{251}{30}$
4.	5	$\frac{251}{720}$	$-\frac{637}{360}$	$\frac{109}{30}$	$-\frac{1387}{360}$	$\frac{1901}{720}$						$\frac{475}{12}$
5.	6	$-\frac{95}{288}$	$\frac{959}{480}$	$-\frac{3649}{720}$	$\frac{4991}{720}$	$\frac{2641}{480}$	$\frac{4277}{1440}$					$\frac{19087}{84}$
6.	7	$\frac{19087}{60480}$	$-\frac{5603}{2520}$	$\frac{135713}{20160}$	$-\frac{10754}{945}$	$\frac{235183}{20160}$	$-\frac{18637}{2520}$	$\frac{198721}{60480}$				$\frac{36799}{24}$
7.	8	$-\frac{5257}{17280}$	$\frac{32863}{13440}$	$-\frac{115747}{13440}$	$\frac{2102243}{120960}$	$-\frac{296053}{13440}$	$\frac{242653}{13440}$	$-\frac{1152169}{120960}$	$\frac{16083}{4480}$			$\frac{1070017}{90}$
8.	9	$-\frac{1070017}{3628800}$	$-\frac{4832053}{1814400}$	$\frac{19416743}{1814400}$	$-\frac{45586321}{1814400}$	$\frac{862303}{22680}$	$\frac{69927631}{1814400}$	$\frac{47738393}{1814400}$	$-\frac{21562603}{1814400}$	$\frac{14097247}{3628800}$		$\frac{2082753}{20}$
9.	10	$-\frac{231417}{806400}$	$\frac{20884811}{7257600}$	$-\frac{2357683}{181440}$	$\frac{15788639}{453600}$	$-\frac{222386081}{3628800}$	$\frac{269181919}{3628800}$	$-\frac{28416361}{453600}$	$\frac{6648317}{181440}$	$-\frac{104995189}{7257600}$	$\frac{4325321}{1036800}$	$\frac{134211265}{132}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-1}^{1} \varphi(u+n)du$
						2-strip						
10.	2	0	2	(Mid-ordinate Rule)								$\frac{2}{3}$
11.	3	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{7}{3}$								2
12.	4	$-\frac{1}{3}$	$\frac{4}{3}$	$-\frac{5}{3}$	$\frac{8}{3}$							$\frac{116}{15}$
13.	5	$\frac{29}{90}$	$-\frac{73}{45}$	$\frac{49}{15}$	$-\frac{133}{45}$	$\frac{269}{90}$						$\frac{112}{3}$
14.	6	$-\frac{14}{45}$	$\frac{169}{90}$	$-\frac{71}{15}$	$\frac{287}{45}$	$-\frac{203}{45}$	$\frac{33}{10}$					$\frac{4556}{21}$
15.	7	$\frac{1139}{3780}$	$-\frac{89}{42}$	$\frac{2687}{420}$	$-\frac{10168}{945}$	$\frac{4577}{420}$	$-\frac{1327}{210}$	$\frac{13613}{3780}$				1476
16.	8	$-\frac{41}{140}$	$\frac{2222}{945}$	$-\frac{3473}{420}$	$\frac{1748}{105}$	$-\frac{79417}{3780}$	$\frac{358}{21}$	$-\frac{703}{84}$	$\frac{736}{189}$			$\frac{518032}{45}$
17.	9	$\frac{32377}{113400}$	$-\frac{146113}{56700}$	$\frac{293299}{28350}$	$-\frac{1375411}{56700}$	$\frac{415423}{11340}$	$-\frac{2097811}{56700}$	$\frac{709939}{28350}$	$-\frac{604033}{56700}$	$\frac{67711}{16200}$		$\frac{506368}{5}$
18.	10	$-\frac{3956}{14175}$	$\frac{317209}{113400}$	$-\frac{715777}{56700}$	$\frac{957907}{28350}$	$-\frac{673847}{11340}$	$\frac{4070939}{56700}$	$\frac{3427027}{56700}$	$\frac{994771}{28350}$	$-\frac{746449}{56700}$	$\frac{20225}{4536}$	$\frac{32740208}{33}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-2}^{1} \varphi(u+n)du$
						3-strip						
19.	3	$\frac{3}{4}$	0	$\frac{9}{4}$	(Heun's Rule)							$\frac{9}{4}$
20.	4	$-\frac{3}{8}$	$\frac{15}{8}$	$-\frac{9}{8}$	$\frac{21}{8}$							$\frac{81}{10}$
21.	5	$\frac{27}{80}$	$-\frac{69}{40}$	$\frac{39}{10}$	$-\frac{99}{40}$	$\frac{237}{80}$						$\frac{153}{4}$
22.	6	$-\frac{51}{160}$	$\frac{309}{160}$	$-\frac{393}{80}$	$\frac{567}{80}$	$-\frac{651}{160}$	$\frac{525}{160}$					$\frac{6165}{28}$
23.	7	$\frac{137}{448}$	$-\frac{603}{280}$	$\frac{14601}{2240}$	$-\frac{386}{35}$	$\frac{26151}{2240}$	$-\frac{1653}{280}$	$\frac{1607}{448}$				$\frac{11925}{8}$
24.	8	$-\frac{265}{896}$	$\frac{2129}{896}$	$-\frac{37473}{4480}$	$\frac{75577}{4480}$	$-\frac{95783}{4480}$	$\frac{80127}{4480}$	$-\frac{35723}{4480}$	$\frac{3479}{896}$			$\frac{115929}{10}$
25.	9	$\frac{12881}{44800}$	$-\frac{8307}{3200}$	$\frac{233559}{22400}$	$-\frac{548033}{22400}$	$\frac{82872}{2240}$	$-\frac{839583}{22400}$	$\frac{580969}{22400}$	$-\frac{32877}{3200}$	$\frac{186831}{44800}$		$\frac{2036097}{20}$
26.	10	$-\frac{3591}{12800}$	$\frac{50399}{17920}$	$-\frac{20313}{1600}$	$\frac{190359}{5600}$	$-\frac{2679687}{44800}$	$\frac{3241071}{44800}$	$-\frac{136746}{2240}$	$\frac{403601}{11200}$	$-\frac{163827}{12800}$	$\frac{398799}{89600}$	$\frac{43830315}{44}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-3}^{1} \varphi(u+n)du$
						4-strip						
27.	4	0	$\frac{8}{3}$	$-\frac{4}{3}$	$\frac{8}{3}$	(Milne's Rule)						$\frac{112}{15}$
28.	5	$\frac{14}{45}$	$-\frac{56}{45}$	$\frac{68}{15}$	$-\frac{116}{45}$	$\frac{134}{45}$						$\frac{112}{3}$
29.	6	$-\frac{14}{45}$	$\frac{28}{15}$	$-\frac{196}{45}$	$\frac{344}{45}$	$-\frac{62}{15}$	$\frac{148}{45}$					$\frac{4576}{21}$
30.	7	$\frac{286}{945}$	$-\frac{134}{63}$	$\frac{2018}{315}$	$-\frac{9836}{945}$	$\frac{3838}{315}$	$-\frac{1874}{315}$	$\frac{3394}{945}$				$\frac{4448}{3}$
31.	8	$-\frac{278}{945}$	$\frac{248}{105}$	$-\frac{872}{105}$	$\frac{15784}{945}$	$-\frac{2174}{105}$	$\frac{1928}{105}$	$-\frac{7568}{945}$	$\frac{408}{105}$			$\frac{520064}{45}$
32.	9	$\frac{4063}{14175}$	$-\frac{36674}{14175}$	$\frac{147244}{14175}$	$-\frac{345248}{14175}$	$\frac{104234}{2835}$	$-\frac{521018}{14175}$	$\frac{374044}{14175}$	$-\frac{146024}{14175}$	$\frac{59143}{14175}$		$\frac{508032}{5}$
33.	10	$-\frac{7}{25}$	$\frac{39784}{14175}$	$-\frac{179558}{14175}$	$\frac{96128}{2835}$	$-\frac{845342}{14175}$	$\frac{1021264}{14175}$	$-\frac{854414}{14175}$	$\frac{516928}{14175}$	$-\frac{181745}{14175}$	$\frac{63112}{14175}$	$\frac{32829952}{33}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-4}^{1} \varphi(u+n)du$
						5-strip						
34.	5	$\frac{95}{144}$	$-\frac{25}{72}$	$\frac{25}{6}$	$-\frac{175}{72}$	$\frac{425}{144}$						$\frac{475}{12}$
35.	6	$-\frac{95}{288}$	$\frac{665}{288}$	$-\frac{175}{48}$	$\frac{1075}{144}$	$-\frac{1175}{288}$	$\frac{105}{32}$					$\frac{18575}{84}$
36.	7	$\frac{3715}{12096}$	$-\frac{365}{168}$	$\frac{9295}{1344}$	$-\frac{1850}{189}$	$\frac{16225}{1344}$	$-\frac{995}{168}$	$\frac{43405}{12096}$				$\frac{11925}{8}$
37.	8	$-\frac{265}{896}$	$\frac{57515}{24192}$	$-\frac{22535}{2688}$	$\frac{46415}{2688}$	$-\frac{487225}{24192}$	$\frac{49145}{2688}$	$-\frac{21485}{2688}$	$\frac{93965}{24192}$			$\frac{208525}{18}$
38.	9	$\frac{41705}{145152}$	$\frac{188285}{72576}$	$\frac{756415}{72576}$	$-\frac{1776185}{72576}$	$\frac{169555}{4536}$	$-\frac{2629415}{72576}$	$\frac{1910785}{72576}$	$-\frac{746915}{72576}$	$\frac{605495}{145152}$		$\frac{406925}{4}$
39.	10	$-\frac{81385}{290304}$	$\frac{815875}{290304}$	$-\frac{460375}{36288}$	$\frac{616375}{18144}$	$-\frac{8679625}{145152}$	$\frac{10553015}{145152}$	$\frac{1084625}{18144}$	$\frac{1321625}{36288}$	$-\frac{3720125}{290304}$	$\frac{184625}{41472}$	$\frac{131410625}{132}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-5}^{1} \varphi(u+n)du$
						6-strip						
40.	6	0	$\frac{33}{10}$	$-\frac{21}{5}$	$\frac{39}{5}$	$-\frac{21}{5}$	$\frac{33}{10}$					$\frac{1476}{7}$
41.	7	$\frac{41}{140}$	$-\frac{123}{70}$	$\frac{1077}{140}$	$-\frac{352}{35}$	$\frac{1707}{140}$	$-\frac{417}{70}$	$\frac{503}{140}$				1476
42.	8	$-\frac{41}{140}$	$\frac{82}{35}$	$-\frac{1107}{140}$	$\frac{628}{35}$	$-\frac{2843}{140}$	$\frac{642}{35}$	$-\frac{1121}{140}$	$\frac{136}{35}$			$\frac{57744}{5}$
43.	9	$\frac{401}{1400}$	$-\frac{1809}{700}$	$\frac{3627}{350}$	$-\frac{16763}{700}$	$\frac{5319}{140}$	$-\frac{25443}{700}$	$\frac{9227}{350}$	$-\frac{7209}{700}$	$\frac{5841}{1400}$		$\frac{508032}{5}$
44.	10	$-\frac{7}{25}$	$\frac{3929}{1400}$	$-\frac{1773}{140}$	$\frac{11859}{350}$	$-\frac{41459}{700}$	$\frac{51291}{700}$	$-\frac{41907}{700}$	$\frac{2551}{70}$	$-\frac{8973}{700}$	$\frac{6233}{1400}$	$\frac{10944720}{11}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-6}^{1} \varphi(u+n)du$
						7-strip						
45.	7	$\frac{5257}{8640}$	$-\frac{49}{72}$	$\frac{19943}{2880}$	$-\frac{1274}{135}$	$\frac{34153}{2880}$	$-\frac{2107}{360}$	$\frac{30919}{8640}$				$\frac{36799}{24}$
46.	8	$-\frac{5257}{17280}$	$\frac{5257}{1920}$	$-\frac{13573}{1920}$	$\frac{303653}{17280}$	$-\frac{38563}{1920}$	$\frac{35035}{1920}$	$-\frac{137935}{17280}$	$\frac{2485}{640}$			$\frac{1046689}{90}$
47.	9	$\frac{149527}{518400}$	$-\frac{676963}{259200}$	$\frac{2803073}{259200}$	$-\frac{6019111}{259200}$	$\frac{122353}{3240}$	$-\frac{9392761}{259200}$	$\frac{6823103}{259200}$	$\frac{2667133}{259200}$	$\frac{2162377}{518400}$		$\frac{2036097}{20}$
48.	10	$-\frac{3591}{12800}$	$\frac{2916893}{1036800}$	$-\frac{1647401}{129600}$	$\frac{2227841}{64800}$	$-\frac{6072619}{103680}$	$\frac{37901353}{518400}$	$-\frac{3875263}{64800}$	$\frac{4720471}{129600}$	$-\frac{13286371}{1036800}$	$\frac{184625}{41472}$	$\frac{131427457}{132}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-7}^{1} \varphi(u+n)du$
						8-strip						
49.	8	0	$\frac{736}{189}$	$-\frac{2544}{315}$	$\frac{1952}{105}$	$-\frac{19672}{945}$	$\frac{1952}{105}$	$-\frac{2544}{315}$	$\frac{736}{189}$			$\frac{506368}{45}$
50.	9	$\frac{3956}{14175}$	$-\frac{31648}{14175}$	$\frac{165968}{14175}$	$-\frac{336016}{14175}$	$\frac{108088}{2835}$	$-\frac{516616}{14175}$	$\frac{374288}{14175}$	$-\frac{146128}{14175}$	$\frac{59156}{14175}$		$\frac{506368}{5}$
51.	10	$-\frac{3956}{14175}$	$\frac{7912}{2835}$	$-\frac{174064}{14175}$	$\frac{498272}{14175}$	$-\frac{834472}{14175}$	$\frac{1038896}{14175}$	$-\frac{169784}{2835}$	$\frac{516704}{14175}$	$-\frac{181732}{14175}$	$\frac{9016}{2025}$	$\frac{32814080}{33}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-8}^{1} \varphi(u+n)du$
						9-strip						
52.	9	$\frac{25713}{44800}$	$-\frac{22437}{22400}$	$\frac{233847}{22400}$	$-\frac{496449}{22400}$	$\frac{10287}{280}$	$-\frac{796959}{22400}$	$\frac{583497}{22400}$	$-\frac{228987}{22400}$	$\frac{26649}{6400}$		$\frac{2082753}{20}$
53.	10	$-\frac{25713}{89600}$	$\frac{282843}{89600}$	$-\frac{126927}{11200}$	$\frac{38691}{1120}$	$-\frac{2812817}{44800}$	$\frac{3265839}{44800}$	$-\frac{334233}{5600}$	$\frac{407457}{11200}$	$-\frac{1147365}{89600}$	$\frac{398799}{89600}$	$\frac{43928811}{44}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-9}^{1} \varphi(u+n)du$
						10-strip						
54.	10	0	$\frac{40450}{9072}$	$-\frac{29225}{2268}$	$\frac{41675}{1134}$	$-\frac{137675}{2268}$	$\frac{169555}{2268}$	$-\frac{137675}{2268}$	$\frac{41675}{1134}$	$-\frac{29225}{2268}$	$\frac{40450}{9072}$	$\frac{32134000}{33}$
	No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-1}^{0} \varphi(u+n)du$
							1-strip					
55.	2	$\frac{1}{2}$	$\frac{1}{2}$	(Trapezoidal Rule)								$-\frac{1}{6}$
56.	3	$-\frac{1}{12}$	$\frac{2}{3}$	$\frac{5}{12}$								$-\frac{1}{4}$
57.	4	$\frac{1}{24}$	$-\frac{5}{24}$	$\frac{19}{24}$	$\frac{3}{8}$							$-\frac{19}{30}$
58.	5	$-\frac{19}{720}$	$\frac{53}{360}$	$-\frac{11}{30}$	$\frac{323}{360}$	$\frac{251}{720}$						$-\frac{9}{4}$
59.	6	$\frac{3}{160}$	$-\frac{173}{1440}$	$\frac{241}{720}$	$-\frac{133}{240}$	$\frac{1427}{1440}$	$\frac{95}{288}$					$-\frac{863}{84}$
60.	7	$-\frac{863}{60480}$	$\frac{263}{2520}$	$-\frac{6737}{20160}$	$\frac{586}{945}$	$-\frac{15487}{20160}$	$\frac{2713}{2520}$	$\frac{19087}{60480}$				$-\frac{1375}{24}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-2}^{0} \varphi(u+n)du$
61.	8	$\frac{275}{24192}$	$-\frac{11351}{120960}$	$\frac{1537}{4480}$	$-\frac{88547}{120960}$	$\frac{123133}{120960}$	$-\frac{4511}{4480}$	$\frac{139849}{120960}$	$\frac{5257}{17280}$			$-\frac{33953}{90}$
62.	9	$-\frac{33953}{3628800}$	$\frac{156437}{1814400}$	$-\frac{645607}{1814400}$	$\frac{1573169}{1814400}$	$-\frac{31457}{22680}$	$\frac{2797679}{1814400}$	$-\frac{2302297}{1814400}$	$\frac{2233547}{1814400}$	$\frac{1070017}{3628800}$		$-\frac{57281}{20}$
63.	10	$\frac{8183}{1036800}$	$-\frac{116687}{1451520}$	$\frac{335983}{907200}$	$-\frac{462127}{453600}$	$\frac{6755041}{3628800}$	$-\frac{8641823}{3628800}$	$\frac{200029}{90720}$	$-\frac{1408913}{907200}$	$\frac{9449717}{7257600}$	$\frac{25713}{89600}$	$-\frac{3250433}{132}$

2-strip

						(Simpson's $\frac{1}{3}$ Rule)						
64.	4	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$							$-\frac{4}{15}$
65.	5	$-\frac{1}{90}$	$\frac{2}{45}$	$\frac{4}{15}$	$\frac{62}{45}$	$\frac{29}{90}$						$-\frac{4}{3}$
66.	6	$\frac{1}{90}$	$-\frac{1}{15}$	$\frac{7}{45}$	$\frac{7}{45}$	$\frac{43}{30}$	$\frac{14}{45}$					$-\frac{148}{21}$
67.	7	$-\frac{37}{3780}$	$\frac{22}{315}$	$-\frac{269}{1260}$	$\frac{332}{945}$	$\frac{11}{1260}$	$\frac{94}{63}$	$\frac{1139}{3780}$				$-\frac{128}{3}$
68.	8	$\frac{8}{945}$	$-\frac{29}{420}$	$\frac{26}{105}$	$-\frac{1927}{3780}$	$\frac{68}{105}$	$-\frac{71}{420}$	$\frac{1466}{945}$	$\frac{41}{140}$			$-\frac{13328}{45}$

69.	9	$-\frac{119}{16200}$	$-\frac{953}{14175}$	$\frac{15577}{56700}$	$\frac{9341}{14175}$	$-\frac{2803}{2835}$	$\frac{15011}{14175}$	$-\frac{21247}{56700}$	$\frac{22823}{14175}$	$\frac{32377}{113400}$		$-\frac{11664}{5}$
70.	10	$\frac{9}{1400}$	$-\frac{3687}{56700}$	$\frac{8467}{28350}$	$-\frac{9239}{11340}$	$\frac{83291}{56700}$	$-\frac{103987}{56700}$	$\frac{45331}{28350}$	$-\frac{34369}{56700}$	$\frac{37829}{22680}$	$\frac{3956}{14175}$	$-\frac{680080}{33}$
	No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-3}^{0} \varphi(u+n)du$
						3-strip						
71.	4	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{3}{8}$	(Simpson's $\frac{3}{8}$ Rule)						$-\frac{9}{10}$
72.	5	$-\frac{3}{80}$	$\frac{21}{40}$	$\frac{9}{10}$	$\frac{51}{40}$	$\frac{27}{80}$						$-\frac{9}{4}$
73.	6	$\frac{3}{160}$	$-\frac{21}{160}$	$\frac{57}{80}$	$\frac{57}{80}$	$\frac{219}{160}$	$\frac{51}{160}$					$-\frac{261}{28}$
74.	7	$-\frac{29}{2240}$	$\frac{27}{280}$	$-\frac{729}{2240}$	$\frac{34}{35}$	$\frac{1161}{2240}$	$\frac{81}{56}$	$\frac{137}{448}$				$-\frac{405}{8}$
75.	8	$\frac{9}{896}$	$-\frac{373}{4480}$	$\frac{1377}{4480}$	$-\frac{3033}{4480}$	$\frac{5927}{4480}$	$\frac{1377}{4480}$	$\frac{1359}{896}$	$\frac{265}{896}$			$-\frac{3321}{10}$
76.	9	$-\frac{369}{44800}$	$\frac{243}{3200}$	$-\frac{7031}{22400}$	$\frac{17217}{22400}$	$-\frac{351}{280}$	$\frac{39967}{22400}$	$\frac{1719}{22400}$	$\frac{35451}{22400}$	$\frac{12881}{44800}$		$-\frac{10125}{4}$
77.	10	$\frac{75}{10752}$	$-\frac{909}{12800}$	$\frac{3663}{11200}$	$-\frac{5039}{5600}$	$\frac{73809}{44800}$	$-\frac{19107}{8960}$	$\frac{13273}{5600}$	$-\frac{279}{1600}$	$\frac{147429}{89600}$	$\frac{75411}{268800}$	$-\frac{963819}{44}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-4}^{0} \varphi(u+n)du$
						4-strip						
78.	6	0	$\frac{14}{45}$	$\frac{64}{45}$	$\frac{8}{15}$	$\frac{64}{45}$	$\frac{14}{45}$					$-\frac{128}{21}$
79.	7	$-\frac{8}{945}$	$\frac{16}{315}$	$\frac{58}{315}$	$\frac{1504}{945}$	$\frac{128}{315}$	$\frac{464}{315}$	$\frac{286}{945}$				$-\frac{128}{3}$
80.	8	$\frac{8}{945}$	$-\frac{64}{945}$	$\frac{8}{35}$	$-\frac{106}{945}$	$\frac{1784}{945}$	$\frac{8}{35}$	$\frac{1448}{945}$	$\frac{278}{945}$			$-\frac{13696}{45}$
81.	9	$-\frac{107}{14175}$	$\frac{976}{14175}$	$-\frac{3956}{14175}$	$\frac{9232}{14175}$	$-\frac{1816}{2835}$	$\frac{32752}{14175}$	$\frac{244}{14175}$	$\frac{22576}{14175}$	$\frac{4063}{14175}$		$-\frac{12032}{5}$
82.	10	$\frac{94}{14175}$	$-\frac{953}{14175}$	$\frac{872}{2835}$	$-\frac{11852}{14175}$	$\frac{21076}{14175}$	$-\frac{20924}{14175}$	$-\frac{40648}{14175}$	$-\frac{3140}{14175}$	$\frac{23422}{14175}$	$\frac{7}{25}$	$-\frac{700160}{33}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-5}^{0} \varphi(u+n)du$
						5-strip						
83.	6	$\frac{95}{288}$	$\frac{125}{96}$	$\frac{125}{144}$	$\frac{125}{144}$	$\frac{125}{96}$	$\frac{95}{288}$					$-\frac{1375}{84}$
84.	7	$-\frac{275}{12096}$	$\frac{235}{504}$	$\frac{3875}{4032}$	$\frac{250}{189}$	$\frac{2125}{4032}$	$\frac{725}{504}$	$\frac{3715}{12096}$				$-\frac{1375}{24}$

85.	8	$\frac{275}{24192}$	$-\frac{275}{2688}$	$\frac{1895}{2688}$	$\frac{13625}{24192}$	$\frac{4625}{2688}$	$\frac{775}{2688}$	$\frac{36725}{24192}$	$\frac{265}{896}$			$-\frac{6125}{18}$
86.	9	$-\frac{175}{20736}$	$\frac{5725}{72576}$	$-\frac{24575}{72576}$	$\frac{85465}{72576}$	$-\frac{125}{4536}$	$\frac{159175}{72576}$	$\frac{3775}{72576}$	$\frac{115075}{72576}$	$\frac{41705}{145152}$		$-\frac{10125}{4}$
87.	10	$\frac{25}{3584}$	$-\frac{20675}{290304}$	$\frac{11975}{36288}$	$-\frac{16775}{18144}$	$\frac{298505}{145152}$	$-\frac{131575}{145152}$	$\frac{50425}{18144}$	$-\frac{7225}{36288}$	$\frac{478525}{290304}$	$\frac{81385}{290304}$	$-\frac{2874625}{132}$
	No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-6}^{0} \varphi(u+n)du$
						6-strip						
88.	8	0	$\frac{41}{140}$	$\frac{54}{35}$	$\frac{27}{140}$	$\frac{68}{35}$	$\frac{27}{140}$	$\frac{54}{35}$	$\frac{41}{140}$			$-\frac{1296}{5}$
89.	9	$-\frac{9}{1400}$	$\frac{9}{175}$	$\frac{79}{700}$	$\frac{333}{175}$	$-\frac{9}{35}$	$\frac{403}{175}$	$\frac{9}{700}$	$\frac{279}{175}$	$\frac{401}{1400}$		$-\frac{11664}{5}$
90.	10	$\frac{9}{1400}$	$-\frac{9}{140}$	$\frac{99}{350}$	$-\frac{299}{700}$	$\frac{1899}{700}$	$-\frac{747}{700}$	$\frac{199}{70}$	$-\frac{153}{700}$	$\frac{2313}{1400}$	$\frac{49}{175}$	$-\frac{231984}{11}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-7}^{0} \varphi(u+n)du$
						7-strip						
91.	8	$\frac{5257}{17280}$	$\frac{25039}{17280}$	$\frac{9261}{17280}$	$\frac{20923}{17280}$	$\frac{20923}{17280}$	$\frac{9261}{17280}$	$\frac{25039}{17280}$	$\frac{5257}{17280}$			$-\frac{57281}{90}$
92.	9	$-\frac{8183}{518400}$	$\frac{111587}{259200}$	$\frac{261023}{259200}$	$\frac{368039}{259200}$	$\frac{343}{3240}$	$\frac{542969}{259200}$	$\frac{24353}{259200}$	$\frac{408317}{259200}$	$\frac{149527}{518400}$		$-\frac{57281}{20}$
93.	10	$\frac{8183}{1036800}$	$-\frac{90013}{1036800}$	$\frac{92617}{129600}$	$\frac{4459}{12960}$	$\frac{1251607}{518400}$	$-\frac{460649}{518400}$	$\frac{178703}{64800}$	$-\frac{24647}{129600}$	$\frac{1706915}{1036800}$	$\frac{3591}{12800}$	$-\frac{2954945}{132}$
No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-8}^{0} \varphi(u+n)du$
						8-strip						
94.	10	0	$\frac{3956}{14175}$	$\frac{23552}{14175}$	$-\frac{3712}{14175}$	$\frac{41984}{14175}$	$-\frac{3632}{2835}$	$\frac{41984}{14175}$	$-\frac{3712}{14175}$	$\frac{23552}{14175}$	$\frac{3956}{14175}$	$-\frac{606208}{33}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-9}^{0} \varphi(u+n)du$
						9-strip						
95.	10	$\frac{25713}{89600}$	$\frac{141669}{89600}$	$\frac{243}{2240}$	$\frac{10881}{5600}$	$\frac{26001}{44800}$	$\frac{26001}{44800}$	$\frac{10881}{5600}$	$\frac{243}{2240}$	$\frac{141669}{89600}$	$\frac{25713}{89600}$	$-\frac{1891755}{44}$

The following formulae are self-explanatory.

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-2}^{-1} \phi(u+n)du$
96.	4	$-\frac{1}{24}$	$\frac{13}{24}$	$\frac{13}{24}$	$-\frac{1}{24}$							$\frac{11}{30}$
97.	5	$\frac{11}{720}$	$-\frac{37}{360}$	$\frac{19}{30}$	$\frac{173}{360}$	$-\frac{19}{720}$						$\frac{11}{12}$
98.	6	$-\frac{11}{1440}$	$\frac{77}{1440}$	$-\frac{43}{240}$	$\frac{511}{720}$	$\frac{637}{1440}$	$-\frac{3}{160}$					$\frac{271}{84}$
99.	7	$\frac{271}{60480}$	$-\frac{29}{840}$	$\frac{811}{6720}$	$-\frac{254}{945}$	$\frac{5221}{6720}$	$\frac{349}{840}$	$-\frac{863}{60480}$				$\frac{117}{8}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-3}^{-2} \phi(u+n)\,du$
100.	8	$-\frac{13}{4480}$	$\frac{2999}{120960}$	$-\frac{1283}{13440}$	$\frac{2987}{13440}$	$-\frac{44797}{120960}$	$\frac{11261}{13440}$	$\frac{5311}{13440}$	$-\frac{275}{24192}$			$\frac{7297}{90}$
101.	9	$\frac{7297}{3628800}$	$-\frac{34453}{1814400}$	$\frac{147143}{1814400}$	$-\frac{377521}{1814400}$	$\frac{8233}{22680}$	$-\frac{876271}{1814400}$	$\frac{1622393}{1814400}$	$\frac{687797}{1814400}$	$-\frac{33953}{3628800}$		$\frac{2125}{4}$
102.	10	$-\frac{425}{290304}$	$\frac{110219}{7257600}$	$-\frac{65039}{907200}$	$\frac{92567}{453600}$	$-\frac{1424417}{3628800}$	$\frac{397331}{725760}$	$-\frac{274849}{907200}$	$\frac{859009}{907200}$	$\frac{2655563}{7257600}$	$\frac{8183}{1036800}$	$\frac{530113}{132}$
103.	6	$\frac{11}{1440}$	$-\frac{31}{480}$	$\frac{401}{720}$	$\frac{401}{720}$	$-\frac{31}{480}$	$\frac{11}{1440}$					$-\frac{191}{84}$
104	7	$-\frac{191}{60480}$	$\frac{67}{2520}$	$-\frac{2257}{20160}$	$\frac{586}{945}$	$\frac{10273}{20160}$	$+\frac{23}{504}$	$\frac{271}{60480}$				$-\frac{191}{24}$
105.	8	$\frac{191}{120960}$	$-\frac{191}{13440}$	$\frac{803}{13440}$	$-\frac{20227}{120960}$	$\frac{9077}{13440}$	$\frac{6403}{13440}$	$-\frac{4183}{120960}$	$\frac{13}{4480}$			$-\frac{3233}{90}$
106.	9	$-\frac{3233}{3628800}$	$\frac{14797}{1814400}$	$\frac{71047}{1814400}$	$\frac{198929}{1814400}$	$-\frac{5207}{22680}$	$\frac{1315919}{1814400}$	$\frac{819143}{1814400}$	$-\frac{49813}{1814400}$	$\frac{7297}{3628800}$		$-\frac{3969}{20}$
107.	10	$\frac{7}{12800}$	$-\frac{42187}{7257600}$	$\frac{25759}{907200}$	$-\frac{38599}{453600}$	$\frac{129581}{725760}$	$-\frac{1083167}{3628800}$	$\frac{349817}{453600}$	$\frac{391711}{907200}$	$-\frac{163531}{7257600}$	$\frac{425}{290304}$	$-\frac{171137}{132}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-4}^{-3} \phi(u+n)dx$
108.	7	$\frac{271}{60480}$	$-\frac{23}{504}$	$\frac{10273}{20160}$	$\frac{586}{2945}$	$-\frac{2257}{20160}$	$\frac{67}{2520}$	$-\frac{191}{60480}$				$\frac{191}{24}$
109.	8	$-\frac{191}{120960}$	$\frac{1879}{120960}$	$-\frac{353}{4480}$	$\frac{68323}{120960}$	$\frac{68323}{120960}$	$-\frac{353}{4480}$	$\frac{1879}{120960}$	$\frac{191}{120960}$			$\frac{2497}{90}$
110.	9	$\frac{2497}{3628800}$	$-\frac{12853}{1814400}$	$\frac{63143}{1814400}$	$-\frac{212881}{1814400}$	$\frac{13903}{22680}$	$\frac{954929}{1814400}$	$-\frac{108007}{1814400}$	$\frac{18197}{1814400}$	$-\frac{3233}{3628800}$		$\frac{2497}{20}$
111.	10	$-\frac{2497}{7257600}$	$\frac{27467}{7257600}$	$-\frac{17663}{907200}$	$\frac{5779}{90720}$	$-\frac{583073}{3628800}$	$\frac{2381791}{3628800}$	$\frac{225623}{453600}$	$-\frac{42767}{907200}$	$\frac{50315}{7257600}$	$-\frac{21}{38400}$	$\frac{90817}{132}$

No.	n+1	C_0^*	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	$\int_{-5}^{-4} \phi(u+n)du$
112.	8	$\frac{13}{4480}$	$-\frac{4183}{120960}$	$\frac{6403}{13440}$	$\frac{9077}{13440}$	$-\frac{20227}{120960}$	$\frac{803}{13440}$	$-\frac{191}{13440}$	$\frac{191}{12095}$			$-\frac{3233}{90}$
113.	9	$-\frac{3233}{3628800}$	$\frac{18197}{1814400}$	$-\frac{108007}{1814400}$	$\frac{954929}{1814400}$	$\frac{13903}{22680}$	$-\frac{212881}{1814400}$	$\frac{63143}{1814400}$	$-\frac{12853}{1814400}$	$\frac{2497}{3628800}$		$-\frac{2497}{20}$
114.	10	$\frac{2497}{7257600}$	$-\frac{28939}{7257600}$	$\frac{581}{25920}$	$-\frac{40111}{453600}$	$\frac{2067169}{3628800}$	$-\frac{2067169}{3628800}$	$\frac{40111}{453600}$	$-\frac{581}{25920}$	$\frac{28939}{7257600}$	$-\frac{2497}{7257600}$	$-\frac{73985}{132}$

APPLICATION OF FORMULAE

Desirable closed formulae are easily seen to be No's 64, 78, 88, and 94. Some caution should be used when the 8-strip formula is employed for since some of its coefficients are negative the round-off error may appear excessive. Although some of the coefficients are negative, nevertheless, open formulae No's 27, 40, 49, and 54 show considerable appeal. For example No. 40 was employed in A.F. Problem No. 19 at Harvard Computation Laboratory.

Very little can be said in favor of increasing the error in order to simplify the coefficients when the numerical operations are performed on a large scale digital electronic calculator. Consequently Weddle's Rule, Hardy's Formula, Shovelton's Rule, etc. will not be introduced here.

Often it is necessary to change the mesh interval when integrating a system of differential equations. Formula (1) may be used to generate a set of formulae to cut such an interval. Obviously no special formulae are necessary to double an interval.

It is interesting to note that by decreasing the mesh size indefinitely and at the same time keeping all the ordinates (an infinite number in a finite interval) we are led to the cardinal interpolation function mentioned in AFTR No. 6581 by F. W. Bubb.

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