TI-2137

3 Sep 53

WADC TECHNICAL REPORT 52-290

DOUBLETS AT SUPERSONIC SPEED

LEE S. WASSERMAN
AERONAUTICAL RESEARCH LABORATORY

NOVEMBER 1952

WRIGHT AIR DEVELOPMENT CENTER

AF-WP-(B)-O-26 AUG 53 100

6-74

NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

The information furnished herewith is made available for study upon the understanding that the Government's proprietary interests in and relating thereto shall not be impaired. It is desired that the Judge Advocate (WCJ), Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, be promptly notified of any apparent conflict between the Government's proprietary interests and those of others.

WADC TECHNICAL REPORT 52-290

DOUBLETS AT SUPERSONIC SPEED

Lee S. Wasserman
Aeronautical Research Laboratory

November 1952

RDO No. 465-1

Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

FOREWORD

The work covered in this report was accomplished by the Aerodynamics Research Branch of the Aeronautical Research Laboratory Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, under RDO No. 465-1, Aerodynamics of Perfect Fluids, with Mr. Lee S . Wasserman serving as project engineer.

ABSTRACT

The mechanism of the doublet at supersonic speeds is described using elementary physical reasoning insofar as is possible. The cause of the infinities introduced into the equations by the differentiation process across the Mach cone is discussed. In addition the physical process involved in passing from the fixed to the moving doublet is explained. Finally some applications of doublets to supersonic aerodynamic problems are included.

PUBLICATION REVIEW

The publication of this report does not constitute approval by the Air Force of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDER:

Leslie B. Williams

LESLIE B. WILLIAMS, Colonel, USAF
Chief, Flight Research Laboratory
Directorate of Research

TABLE OF CONTENTS

TABLE OF SYMBOLS

x, y, z, t	=	space and time coordinates (ft) (secs.)
$p(x,y,z,t)$	=	pressure at x, y, z, at time t lbs/sq ft
ϵ	=	source strength lbs/ft
τ	=	time of emission of source pulse striking the point x, y, z, at time t
W	=	vertical velocity (ft/sec)
U	=	horizontal velocity (ft/sec)
β	=	Mach angle
ϵ'	=	doublet strength lbs
Φ	=	potential (ft^2/sec)
c	=	velocity of sound (ft/sec)
t_1	=	time of emission of fixed source pulse striking the point x, y, z, at time t.
M	=	Mach number
$\xi\, \eta$	=	running x, y, coordinates (ft)
L	=	lift per unit area (#/sq ft)
θ	=	angle used as integration variable
h	=	y/x
ϕ	=	polar angle of curve surrounding a singularity
δ	=	radius of curve surrounding a singularity
$\bar{c}$	=	y/x value corresponding to edge of delta wing

INTRODUCTION

The use of sources and doublets for the analysis of flow fields has led to many useful results in aerodynamics. In the developments concerning incompressible flow, the physical pictures of sources and doublets were retained while for supersonic flow the approach has been more mathematical in the sense that the source and doublet flows have been referred to as particular solutions of the differential equation involved rather than physical flows. This mathematical approach has the disadvantage that the development of intuition by designers and engineers regarding supersonic flows is impeded.

In reference 1, the development of supersonic theory utilizing the source i.e., a point in the fluid at which fluid is added or withdrawn at a given velocity. The effect of moving the source is then derived following the superposition method of Prandtl given in reference 2. This process enables one to follow the physical mechanism involved in the fluid motion around bodies moving at speeds faster than sound. Another concept described in reference I is the pressure source in which fluid is introduced with a certain acceleration. For many problems, the pressure source seems to give a simpler picture of the flow than the velocity source.

Some problems in fluid mechanics can be readily solved with the use of the double source or doublet. In this case a source and sink are brought close to each other but the strength times the distance is held constant. Mathematically this situation can be represented as the derivative of the source along the line joining the source and sink. The pressure doublet then corresponds to the derivative of the pressure source. The stream lines can be readily visualized i. e.: one half of the fluid goes directly from the source to the sink and the remainder flows from the opposite side of the source to the corresponding side of the sink.

It has been found in supersonic flows that the pressure of both the source and doublet is infinite along the Mach cone. This situation causes little trouble in the case of the source but has required special mathematical technique in the case of the doublet. Another characteristic of the doublet is the difficulty of passing from the fixed to the moving doublet. It is shown that Prandtl's approach for the source as modified in reference 1 does not seem to be applicable to the doublet. One of the purposes of this report is to clarify the physics involved in the two difficulties listed above. Another purpose is to present some of the known applications of the doublet without the use of the special mathematical techniques normally applied.

PART I

SINGULARITY AT THE MACH CONE:

The singularity at the Mach cone for the doublet is clearly tied up with the fact that one takes derivative of the source pressure which jumps from zero to infinity on the Mach cone. From reference 1, we have for the pressure of the moving source the following expression:

$$\text{I.1} \qquad p(x,y,z,t) = \frac{\epsilon\, f(\tau)}{4\pi\left[x^2-(M^2-1)(y^2+z^2)\right]^{\frac{1}{2}}}$$

where p = (pressure) lbs/sq.ft.

ϵ = strength of source (lbs./ft.)

x, y, z = coordinates in the moving system (ft.)

M = Mach number of source moving along the x axis in the negative direction

τ = starting time of source striking the point x, y, z, at time t

$$\tau = t - \frac{\sqrt{(x-u(t-\tau))^2+y^2+z^2}}{c}$$

For the case where the source strength is steady i.e., $f(\tau) = 1$, there will be two waves striking the point x, y, z at each value of time i.e., the backward moving part of one wave and the forward moving part of another wave emitted at an earlier time. The steady state expression is therefore:

$$\text{I.2} \qquad p(x,y,z) = \frac{\epsilon}{2\pi\left[x^2-(M^2-1)(y^2+z^2)\right]^{\frac{1}{2}}}$$

This wave will accelerate the air particles according to Newton's Law i.e., proportional to the pressure gradient. Application of this formula demonstrates the cause of the mathematical difficulties in handling doublets at supersonic speeds i.e.:

$$\text{I.3} \qquad \frac{\partial p}{\partial Z} = \frac{\epsilon (M^2-1) Z}{2\pi [X^2 - (M^2-1)(y^2+Z^2)]^{3/2}} = -\rho \frac{dw}{dt} = -\rho U \frac{\partial w}{\partial x}$$

Considering this expression, we see that $\frac{\partial p}{\partial Z}$ always has the same sign behind the Mach cone and all particles would be accelerated down. However as the particle enters the Mach cone it will be accelerated up since the pressure jumps from zero to infinity. This upward acceleration which is neglected in the usual computation cancels the infinite downward acceleration which gives rise to the infinite velocity obtained in the mathematical procedure as follows:

$$\text{I.4} \qquad W = -\frac{\epsilon (M^2-1) Z}{2\pi \rho U} \int_{x^*}^{x} \frac{dx}{[X^2 - (M^2-1)(y^2+Z^2)]^{3/2}}$$

where the * refers to the value of x which will cancel the denominator i.e., the point where the particle is at the Mach cone. The integral yields the expression:

$$\text{I.5} \qquad W = \frac{\epsilon}{2\pi \rho U} \left| \frac{xZ}{(y^2+Z^2)\sqrt{X^2 - (M^2-1)(y^2+Z^2)}} \right|_{x^*}^{x}$$

so that substitution of the limit x^* gives an infinite velocity everywhere.

From simple acoustic theory, the velocity due to a plane wave is $p/\rho c$. Since the Mach cone appears plane to a particle very close, this formula can be used to evaluate the axially symmetric velocity caused by the positive pressure jump just ahead of the Mach cone. The vertical velocity caused by this jump is:

$$\text{I.6} \qquad W = \frac{p}{\rho c} \frac{Z \cos\beta}{\sqrt{y^2+Z^2}} = \frac{\epsilon}{2\pi\rho} \frac{\cos\beta}{(c)} \frac{Z}{\sqrt{y^2+z^2}\sqrt{x^2-(M^2-1)(y^2+z^2)}}$$

where β is the Mach angle.

Now $\frac{\cos\beta}{c} = \frac{\sqrt{M^2-1}}{U}$ and $\frac{x}{(\sqrt{y^2+z^2})U} = \frac{\sqrt{M^2-1}}{U}$ so that the infinite velocity due to the acceleration just behind the Mach cone is cancelled out by the infinite velocity due to the pressure jump just in front.

The above derivation gives a physical explanation of the infinities which arise in supersonic theory and which have required various mathematical schemes such as Hadamard's method.

The velocity caused by a pressure source at the origin is therefore given by the expression:

$$\text{I.7} \qquad W = \frac{\epsilon}{2\pi\rho U} \frac{xZ}{(y^2+z^2)\sqrt{x^2-(M^2-1)(y^2+z^2)}}$$

The velocity caused by a doublet is obtained by taking the derivative i.e.,

$$\text{I.8} \qquad W = \frac{\epsilon'}{2\pi\rho U}\left\{\frac{xZ\,(M^2-1)\,Z}{(y^2+z^2)\left(x^2-(M^2-1)(y^2+z^2)\right)^{3/2}} + \frac{x}{y^2+Z^2\sqrt{x^2-(M^2-1)(y^2+Z^2)}} - \frac{2xZ^2}{(y^2+z^2)^2\sqrt{x^2-(M^2-1)(y^2+Z^2)}}\right\}$$

We can obtain equation I.8 directly by considering the expression for the pressure doublet.

$$\text{I.9} \qquad p = \frac{\epsilon'(M^2-1)\,Z}{2\pi\left[x^2-(M^2-1)(y^2+Z^2)\right]^{3/2}}$$

The vertical acceleration is:

$$\text{I.10} \qquad \frac{\partial p}{\partial Z} = -\rho U\frac{\partial W}{\partial x} = \frac{\epsilon'}{2\pi}\left\{\frac{M^2-1}{\left(x^2-(M^2-1)(y^2+Z^2)\right)^{3/2}}+\frac{3(M^2-1)^2Z^2}{\left(x^2-(M^2-1)(y^2+Z^2)\right)^{5/2}}\right\}$$

Integration with respect to x yields for the velocity:

$$\text{I.11} \qquad W = \frac{\epsilon'}{2\pi\rho U}\left\{\left|_{x^*}^{x}\frac{x}{(y^2+Z^2)\sqrt{x^2-(M^2-1)(y^2+Z^2)}} - \frac{3Z^2}{(y^2+Z^2)^2}\left|_{x^*}^{x}\frac{x}{\sqrt{x^2-(M^2-1)(y^2+Z^2)}}-\frac{1}{3}\frac{x^3}{\left(x^2-(M^2-1)(y^2+Z^2)\right)^{3/2}}\right.\right.\right\}$$

If we neglect the lower limit, the result is:

$$\text{I.12} \qquad W = \frac{\epsilon'}{2\pi\rho U}\left\{\frac{x}{(y^2+Z^2)\sqrt{x^2-(M^2-1)(y^2+Z^2)}} - \frac{3Z^2x}{(y^2+Z^2)^2\sqrt{x^2-(M^2-1)(y^2+Z^2)}}+\frac{x^3Z^2}{(y^2+Z^2)\left(x^2-(M^2-1)(y^2+Z^2)\right)^{3/2}}\right\}$$

Comparison of I.12 with I.8 shows that the first term of I.12 equals the second term of I.8. Combining the other terms of I.8 yields:

$$\text{I.13} \qquad \frac{3(M^2-1)xZ^2(y^2+Z^2)-2x^3Z^2}{\left(x^2-(M^2-1)(y^2+Z^2)\right)^{3/2}(y^2+Z^2)^2}$$

An identical result is obtained from I.12 which shows the identity of the two methods but does not justify omitting the terms with X^* as the lower limit. This justification is obvious however from equation I.6, the derivative of which with respect to Z exactly cancelling out the infinite terms caused by the lower limit in I.11.

PART II

THE MOVING DOUBLET

In reference 1, the effects of motion on the pressure or velocity source is demonstrated in a very simple manner. A moving source is represented as a series of fixed sources emitting fluid for short intervals of time in successive positions corresponding to the location of the moving source. The pressure change here is caused by overlapping of the waves at a given point in space. This treatment essentially follows that of Prandtl given in reference 2.

The usual treatment which yields the moving doublet is to take the derivative of the moving source although it is possible that a clearer picture could be obtained by directly passing from the fixed doublet to the moving doublet. An attempt to use the Prandtl method for this purpose proved unsuccessful as explained in the derivations of this part of the report.

We consider first the potential caused by a fixed source.

II.1 $$\phi = \frac{\epsilon \quad f(t - r/c)}{4\pi r}$$

The fixed doublet then follows from the expression for the derivative

II.2 $$\Phi_D \doteq \frac{\partial \Phi}{\partial Z} = -\frac{\epsilon'}{4\pi}\left\{\frac{f(t - r/c)Z}{r^3} + \frac{f'(t - r/c)Z}{r^2 C}\right\}$$

This is the same expression as the velocity of fluid from the source. The first term represents the velocity one would obtain from an incompressible source. The second term represents the velocity due to the acoustic wave which is emitted whenever the source changes its strength. For steady flow behind the wave front, we can neglect the second term. It was noted in Part I that the velocity caused by this second term cancels the infinity occuring in the mathematics for velocity calculation.

If we calculate the potential of a moving source using Prandtl's method, we obtain the following relation (see reference 1):

$$\text{II.3} \qquad \phi(x,y,z,t) = \frac{\epsilon}{4\pi r}\left|\frac{cr}{cr-u(x-ut_1)}\right| f(t-r/c)$$

Where the term on the right hand side between the bars represents the contraction of a wavelet due to its motion. In the fixed system the wave is contained between two spheres having the same center while in the moving system the center of the two spheres is different. The potential increases in proportion to the contraction ratio. We can visualize this as follows: each wavelets is made up of a number of elementary wavelets each of which represents a certain value of the potential. When the wavelet is contracted as a result of motion, the elementary wavelets overlap so the potentials add.

The value ut_1 in II.3 represents the position of the elementary wavelets striking the point x, y, z, at time t. Now if we express t_1 in terms of x, y, z, and t.ie., $c^2(t-t_1) - u(x-ut_1) = c\sqrt{(x-ut)^2-(M^2-1)(y^2+z^2)}$ the expression for the potential in the moving system $X = x-ut$ is:

$$\text{II.4} \qquad \phi = \frac{\epsilon}{4\pi\sqrt{X^2-(M^2-1)(y^2+z^2)}}$$

which is the well known result.

When we take the derivative of Φ with respect to Z, the term outside the brackets in II.3 turns out to be the velocity of the fixed source however inside the brackets both r and t_1 are functions of Z so that these terms must be included in the derivative expression. This means that the velocities add because of the overlap as is the case for the potential but additional gradients in the potential exists because of the contraction and because of the change in t_1 with Z which result in additional velocities. This means that we cannot add the velocities at a point caused by a succession of fixed sources and obtain the velocity of the moving source but we must also add the incremental velocities caused by the motion. A similar condition exists in the case of the pressures due to a moving doublet.

If we substitute $c(t-t_1)$ for r which is its equivalent we obtain for from II.3:

II.5 $$\Phi = \frac{c\epsilon}{c^2(t-t_1) - u(x-ut_1)}$$

The derivative then yields:

II.6 $$\frac{\partial\Phi}{\partial Z} = \frac{-c\epsilon(u^2-c^2)\frac{\partial t_1}{\partial Z}}{\left[c^2(t-t_1) - u(x-ut_1)\right]^2}$$

Using the relation mentioned above i.e.:

II.7 $$r^2 = (x-ut_1)^2 + y^2 + z^2 = c^2(t-t_1)^2$$

which gives after differentiation:

II.8 $$\frac{\partial t_1}{\partial z} = \frac{z}{c^2(t-t_1) - v(x-vt_1)}$$

so that:

II.9 $$\frac{\partial \Phi}{\partial z} = \frac{\epsilon z (M^2-1)}{4\pi \left[(x-vt)^2 - (M^2-1)(y^2+z^2)\right]^{3/2}}$$

where $x-vt = X$ in the moving system of coordinates.

This part of the report indicates that the superposition method of Prandtl's does not seem to be applicable for the velocity due to a source or the pressure due to a doublet. However there is no question that a moving doublet is the equivalent of a series of fixed doublets each emitting successive pulses from the position of the moving doublet. The problem here seems to obtain the correct value for the summation of the velocities and pressure pulses. For a more detailed understanding of the details of such a summation, it appears that more research is needed.

PART III

APPLICATIONS OF DOUBLETS TO SUPERSONIC PROBLEMS

The first step in the application of the doublet to the solution of aerodynamic problems at supersonic speeds is to relate the doublet strength to a quantity at the surface of the airfoil. It will be shown by analogy with the velocity source that the strength of a pressure doublet is equal to the lift on an element $dx\,dy$ of the wing surface. The pressure of a doublet corresponds to the velocity of source since the first is obtained by the partial derivative of the pressure source and the second by the partial derivative of an ordinary fluid source. In a fluid source the potential and velocity are given respectively by:

III.1 $$\Phi = \frac{\epsilon}{4\pi r}$$

III.2 $$\frac{\partial \Phi}{\partial z} = -\frac{\epsilon}{4\pi r^2}\,\frac{z}{r}$$

Equation III.1 and III.2 correspond to a sink since the vertical velocity is negative for positive z and positive for negative z. In order to evaluate ϵ, we assume a small area $dx\,dy$ is represented by a distribution of sinks of uniform strength per unit area i.e., ϵ is replaced by $\bar{\epsilon}\,dx\,dy$. Very close to the surface the flow will be plane so that the velocity is given by the volume flow per second divided by the area i.e.

III.3 $$w = \frac{\bar{\epsilon}/2\; dx\,dy}{dx\,dy}$$

Since W corresponds to the pressure $\bar{\epsilon}$ is the twice the pressure on the upper surface and ϵ is the lift acting on the element $dxdy$. In corresponding fashion ϵ for a velocity doublet can be shown to be equal to the difference in potential times the area.

From the above, we have a relation between the lift on an element on the surface of a wing and the pressure in the field. Using Newton's Law we can compute the acceleration of the fluid from the pressure gradient and compute the velocity by integration with respect to time.

We will consider as the first example of the method, the derivation of the formulas for the downwash due to a lifting doublet at both subsonic and supersonic speeds. The pressure caused by a doublet is given by the expression derived in Part I with $L d\xi d\eta$ substituted for ϵ i.e.

III.4 $$p(x,y,z,t) = \frac{L(\xi,\eta)\, d\xi\, d\eta\, f(\tau)(M^2-1)Z}{4\pi\left((x-\xi)^2-(M^2-1)\left((y-\eta)^2+z^2\right)\right)^{3/2}} + \frac{L(\xi,\eta)\, d\xi\, d\eta\; F'(\tau)\; d\tau/dZ}{4\pi\left((x-\xi)^2-(M^2-1)\left((y-\eta)^2+z^2\right)\right)^{1/2}}$$

Where $L\, d\xi\, d\eta$ = lift on the element $d\xi\, d\eta$ (lbs)

$f(\tau)$ = function showing lift variation with time

τ = time of emission of pressure pulse striking the point x, y, z at time t.

Considering first the steady state case $f(\tau)$ is the constant unity and τ is given by the relation (Reference 1.)

III.5 $$C^2(t-\tau)^2 = \left[(x-\xi)-u(t-\tau)\right]^2 + (y-\eta)^2 + Z^2$$

The left hand side of III.5 equals the distance the wave travels in the elapsed time $(t-\tau)$ and the right side shows the space distance from ξ, η to the point x, y, z. Solution of this equation shows only one real root for subsonic speeds but two for supersonic speeds. This means that two pressure out waves reach each point at supersonic speeds where as only one occurs at subsonic speeds. So $f(\tau) = 2$ for $M > 1$ and $f(\tau) = 1$ for $M < 1$

Newtons Law gives $\rho \frac{dW}{dt} = -\frac{\partial p}{\partial z}$ or for the steady case $\frac{dW}{dt} = U\frac{\partial W}{\partial x}$ Writing this equation for the subsonic case give for W, the following expression at $z = 0$ i.e., the plane of the airfoil.

III.6 $$W(x,y) = \frac{L\, d\xi\, d\eta\, (M^2-1)}{4\pi\rho U}\int_{-\infty}^{x} \frac{dx}{\left((x-\xi)^2-(M^2-1)(y-\eta)^2\right)^{3/2}}$$

and after integration

III.7 $$W = -\frac{L\, d\xi\, d\eta}{4\pi\rho U}\left[\frac{(x-\xi)}{(y-\eta)^2\left((x-\xi)^2-(M^2-1)(y-\eta)^2\right)^{1/2}} + \frac{1}{(y-\eta)^2}\right]$$

Far behind the doublet we obtain:

III.8 $$W = -\frac{L\, d\xi\, d\eta}{2\pi\rho U}\left[\frac{1}{(y-\eta)^2}\right]$$

An if we integrate this equation for $-\eta$ to $+\eta$ we note immediately the similarity between a row of doublets and a horsehoe vortex extending from $-\eta$ to $+\eta$.

For the supersonic case we use 2 for the value of $f(\tau)$ and use the Mach line as the lower limit. This yields:

$$\text{III.9} \qquad W(x,y) = \frac{L\,d\xi\,d\eta}{2\pi\rho U}\int_{x^*}^{x}\frac{(M^2-1)\,dx}{\left((x-\xi)^2-(M^2-1)(y-\eta)^2\right)^{3/2}}$$

It is shown in Part I, that the lower limit of this integral gives a downwash of zero so that:

$$\text{III.10} \qquad W(x,y) = \frac{L\,d\xi\,d\eta}{2\pi\rho U}\left[\frac{(x-\xi)}{(y-\eta)^2\left((x-\xi)^2-(M^2-1)(y-\eta)^2\right)^{1/2}}\right]$$

If $(x-\xi)$ is large compared to $(y-\eta)$, III.10 equals III.8, showing the downwash at far distances from both a supersonic and subsonic doublet is the same.

The next problem which will be considered here is the conically loaded delta wing. It is convenient to use for this problem the vertical acceleration due to a line of doublets rather than the vertical velocity. We obtain from formula III.9, the result

$$\text{III.11} \qquad \frac{\partial w}{\partial x} = \frac{L(\xi,\eta)\,d\xi\,d\eta \quad (M^2-1)}{2\pi\rho U\;\left((x-\xi)^2-(M^2-1)(y-\eta)^2\right)^{3/2}}$$

We use the coordinates: h equal the tangent of the angle from the centerline and ξ equal the distance to the point along the x axis. The vertex of the delta wing is at the origin and its centerline is on the x axis. Therefore $\eta = h\xi$ and $d\xi\,d\eta = \xi\,d\xi\,dh$ so that:

$$\text{III.12} \qquad \frac{\partial w}{\partial x} = \frac{L(h)\,dh}{2\pi\rho U}\int_{0}^{\xi^*}\frac{(M^2-1)\,\xi\,d\xi}{\left((x-\xi)^2-(M^2-1)(y-h\xi)^2\right)^{3/2}}$$

The integral here has the form $\int d\xi / X\sqrt{X}$

$$X = a + b\xi + c\xi^2$$

This yields:

$$\text{III.13} \qquad \frac{\partial w}{\partial x} = -\frac{L(h)\,dh}{2\pi\rho U}\left\{\frac{\sqrt{x^2-(M^2-1)y^2}}{(hx-y)^2}\right\}$$

The infinity along the Mach Cone of the last doublet to influence $\frac{\partial w}{\partial x}$ is thrown out since it was proved in Part I, that the downwash is finite from a finite doublet so that a single distributed doublet will give zero. This is not the case for $\frac{\partial w}{\partial x}$ but while this derivative has infinities at the Mach Cone, the integrated effect of these is zero.

In order to obtain the pressure distribution $L(h)$, we must fill the surface of the wing with lines of doublets at various angles $\tan^{-1} h$ from $h = -\bar{c}$ to $h = +\bar{c}$ where $\bar{c}$ is the tangent of the half angle of the wing. The integral equation is :

$$\text{III.14} \qquad \frac{\partial w}{\partial x} = -\frac{\sqrt{x^2-(M^2-1)y^2}}{2\pi\rho U}\int_{-\bar{c}}^{+\bar{c}}\frac{L(h)\,dh}{(hx-y)^2}$$

We interpret III.14 in the same way we consider a vortex line in subsonic flow because of the similarities of the infinite velocities occurin in the two types of flows. For example a line vortex in subsonic flow located at x gives a downwash.

$$\text{III.15} \qquad W = \frac{\Gamma}{2\pi(x_1-x)} \quad \text{and} \quad \frac{\partial W}{\partial x_1} = -\frac{\Gamma}{2\pi(x_1-x)^2}$$

The W in III.14 due a single line of doublets must therefore exhibit an infinite velocity of the form $(hx-y)$

To solve the integral equation III.14 i.e., we need the function $L(h)$ which will give $\frac{\partial w}{\partial x}=0$ for $h<|c|$ and which is symmetrical with respect to the center-line, we use $L(h)=\frac{K}{\sqrt{c^2-h^2}}$. For other values of $\frac{\partial w}{\partial x}$, we can utilize the results of the lifting-line integral equation of Reference 3.

In order to evaluate K, we calculate $\frac{\partial w}{\partial x}$ for $|y/x|>\bar{c}$ and integrate from the Mach line to the wing through the singularity at the wing tip. Performing this integration yields:

$$\text{III.16} \qquad \frac{\partial w}{\partial x} = -\frac{K}{2\rho U}\,\frac{\sqrt{x^2-(M^2-1)\,y^2}\;\;(y/x)}{x^2\,\left((y/x)^2-\bar{c}^2\right)^{3/2}}$$

This integration, with respect to x for W, can be most easily performed by letting $\lambda=\frac{y}{x}$ to yield:

$$\text{III.17} \qquad W = U\alpha = -\frac{K}{2\rho U}\int_{\frac{1}{\sqrt{M^2-1}}}^{\bar{c}} \frac{\sqrt{1-(M^2-1)\,\lambda^2}\;d\lambda}{(\lambda^2-\bar{c}^2)^{3/2}}$$

We integrate III.17 by parts to obtain:

$$\text{III.18} \qquad U\alpha = -\frac{K}{2\rho U}\left\{-\left|\frac{\sqrt{1-(M^2-1)\lambda^2}\;\lambda}{c^2\sqrt{\lambda^2-c^2}}\right|_{\frac{1}{\sqrt{M^2-1}}}^{\bar{c}} + \frac{1}{c^2}\int_{\frac{1}{\sqrt{M^2-1}}}^{c}\frac{\lambda^2\,d\lambda}{\sqrt{\lambda^2-c^2}\,\sqrt{1-(M^2-1)\lambda^2}}\right\}$$

The first integral is zero on the Mach cone and infinite at $\lambda=\bar{c}$. This infinity however can be set equal to zero since W must be $U\alpha$ on the airfoil according to the boundary condition. An explanation of this singularity is given in the discussion of leading edge suction contained in this report.

Let us now consider the second integral and let $t = \frac{1}{\lambda}$ to yield:

III.19
$$U\alpha = \frac{K}{2\rho U \bar{c}^2} \int_{\sqrt{M^2-1}}^{1/c} \frac{dt}{t^2 \sqrt{\frac{1}{c^2} - t^2} \sqrt{t^2 - (M^2-1)}}$$

This integral is known from previous work to be an elliptic integral of the second kind, Reference 4, P. 135. Thus we obtain:

III.20
$$K = \frac{2\bar{c}^2 U^2 \alpha \rho}{E\left(\sqrt{1-(M^2-1)\bar{c}^2}\right)}$$

and the lift distribution is given by the expression:

III.21
$$L(h) = \frac{2\bar{c}^2 U^2 \alpha \rho}{\sqrt{\bar{c}^2 - h^2}\; E\left(\sqrt{1-(M^2-1)\bar{c}^2}\right)}$$

This well-known result is contained in many previous references.

NON-STEADY PRESSURE DOUBLETS

The non-steady pressure doublet can be derived from the non-steady pressure source by differentiation. The expression for the non-steady pressure source is given in reference 1 as follows:

$$\text{III.22} \qquad p(x,y,z,t) = \frac{1}{4\pi} \frac{\epsilon f(\tau)}{\sqrt{x^2 - (M^2-1)(y^2+z^2)}}$$

where $\epsilon f(\tau)$ represents the strength of the source as a function of τ which is related to x, y, z and t by the following:

$$\text{III.23} \qquad \tau = t - \frac{Mx}{c(M^2-1)} \pm \frac{\sqrt{x^2 - (M^2-1)(y^2+z^2)}}{c(M^2-1)}$$

This expression means that each point of the fluid x, y, z receives pressure impulses at time t generated by the sources at time τ. In the supersonic case there are two pulses received simultaneously, a backward moving pulse corresponding to the larger value of τ and a forward moving pulse corresponding to the smaller value of τ.

For an oscillating source represented by $\epsilon e^{j\omega\tau}$, the pressure source expression yields:

$$\text{III.24} \qquad p(x,y,z,t) = \frac{\epsilon e^{j\omega\left(t - \frac{Mx}{c(M^2-1)}\right)}}{4\pi\sqrt{x^2 - (M^2-1)(y^2+z^2)}} \left\{ e^{\frac{j\omega}{c(M^2-1)}\sqrt{x^2-(M^2-1)(y^2+z^2)}} + e^{\frac{-j\omega}{c(M^2-1)}\sqrt{x^2-(M^2-1)(y^2+z^2)}} \right\}$$

The pressure along the Mach Wave from both pulses has the phase lag of $e^{-\frac{j\omega Mx}{c(M^2-1)}}$ so that wave front has a sinusoidal spatial distribution of pressure as might be expected.

The pressure doublet is given by differentiation with respect to Z as mentioned above, and involves four terms i.e.,

III.25
$$p(x,y,z,t) = \left\{ \frac{\epsilon'(M^2-1)Z}{4\pi} \; e^{j\omega\left[t - \frac{Mx}{c(M^2-1)}\right]} \; e^{\pm \frac{j\omega}{c(M^2-1)}\sqrt{x^2-(M^2-1)(y^2+z^2)}} \right\}$$
$$\left\{ \frac{1}{\left(x^2-(M^2-1)(y^2+z^2)\right)^{3/2}} \mp \frac{j\omega}{c(M^2-1)\left(x^2-(M^2-1)(y^2+z^2)\right)} \right\}$$

where the $\pm$ sign means the sum of the plus and minus terms.

The vertical acceleration of the air particle caused by the doublet is given by Newton's Law $\rho \frac{dw}{dt} = -\frac{\partial p}{\partial Z}$. The expression for $\frac{\partial p}{\partial Z}$ at $Z=0$ is the same as equation III.25 except that we must divide by Z and set $Z=0$.

Suppose we desire to compute the w velocity of a particle at $x, y, Z=0$ at some value of time t. We can first consider the increment of downwash when the particle is at $x_1, y, Z=0$ of the moving system. The vertical acceleration is given by the pressure gradient at x_1 but with a time $t - \frac{x-x_1}{U}$. The increment of velocity is $\frac{dw}{dt}\frac{dx_1}{U}$ os that the expression for w is:

III.26
$$w(x, y, Z=0, t) = -\frac{1}{\rho}\int_{x^*}^{x} \frac{\partial p}{\partial Z}\left(x, y, Z=0, t - \frac{(x-x_1)}{U}\right)\frac{dx_1}{U}$$

were x^* refers to the value of x at the Mach cone. The final formula for the downwash caused by a doublet is then:

III.27
$$w(x,y,Z=0,t) = -\frac{\epsilon'(M^2-1)}{4\pi\rho U} \; e^{j\omega\left(t-\frac{x}{U}\right)} \int_{x^*}^{x} \left(e^{-\frac{j\omega x_1}{U(M^2-1)}} \; e^{\pm \frac{j\omega}{c(M^2-1)}\sqrt{x_1^2-(M^2-1)y^2}} \right)$$
$$\times \left(\frac{1}{\left(x_1^2-(M^2-1)(y^2+z^2)\right)^{3/2}} \mp \frac{j\omega}{c(M^2-1)} \frac{1}{\left(x_1^2-(M^2-1)y^2\right)} \right) dx_1$$

The solution for the above integral has not been obtained, but some idea of the functions involved can be obtained from the following approach which involves an integration with respect to y for the source before developing the doublet.

In this approach we will integrate equation III.24 with respect to y between the two values of y which make the denominator vanish. In reference 1, we use a substitution similar to the following:

$$\cos\theta = \sqrt{1 - \frac{(M^2-1)\,y^2}{X^2-(M^2-1)Z^2}}$$

$$\left(\sqrt{M^2-1}\right)y = \sqrt{X^2-(M^2-1)Z^2}\,\sin\theta$$

$$dy = \frac{\sqrt{X^2-(M^2-1)Z^2}}{\sqrt{M^2-1}}\cos\theta$$

so that

III.28 $$p(x,y,z,t) = \frac{\epsilon}{4\pi} e^{j\omega\left(t-\frac{Mx}{c(M^2-1)}\right)} \left\{ \int_{-\pi/2}^{\pi/2} e^{j\omega\frac{\sqrt{x^2-(M^2-1)z^2}}{c(M^2-1)}\cos\theta} + e^{-j\omega\frac{\sqrt{x^2-(M^2-1)z^2}}{c(M^2-1)}\cos\theta} \right\} d\theta$$

which yields

III.29 $$p(x,y,z,t) = \frac{\epsilon\, e^{j\omega\left(t-\frac{Mx}{c(M^2-1)}\right)}}{2\sqrt{M^2-1}} J_0\left(\frac{\omega\sqrt{x^2-(M^2-1)z^2}}{c(M^2-1)}\right)$$

The pressure caused by a line of doublets is obtained from III.29 by differentiation ie:

III.30 $$p(x,y,z,t) = \frac{\epsilon\, e^{j\omega\left(t-\frac{Mx}{c(M^2-1)}\right)}}{2\sqrt{M^2-1}} \left(\frac{\omega z}{c\sqrt{x^2-(M^2-1)z^2}}\right) J_1,$$

The pressure gradient at $z=0$ can be obtained from III.30 by dividing by z and setting $z=0$. Using Newton's Law as explained previously for the oscillating downwash, we obtain the following equation:

III.31 $$W = -\frac{\epsilon' \, e^{j\omega(t - x/U)}}{2\rho U\sqrt{M^2-1}} \int_0^x e^{\frac{-j\omega x_1}{U(M^2-1)}} \left(\frac{\omega}{c x_1}\right) J_1 \, dx_1$$

The function in the above equation could be tabulated to obtain the oscillating downwash behind a two-dimensional lifting line. For finite lifting-lines a similar expression is obtained except that J_0 is an incomplete integral. This relation shows that type of integrals which may be expected from equation III.27.

LEADING EDGE SUCTION

A suction force is provided when air flows around a corner without separation. Suction occurs at the leading edge of a subsonic airfoil or the leading edge of a supersonic airfoil when the edge is swept behind the Mach cone and the airfoil develops lift. A physical picture of the suction force is important in the design of airfoils for subsonic and supersonic flight.

In this part of the report, a simple mathematical calculation of the suction force shall be given based on linear theory. The pressure at x,y,Z caused by a subsonic doublet at $x, \eta, Z=0$ is as follows:

III.32 $$p = -\frac{L(\delta,\eta)\,d\xi\,d\eta\,(1-M^2)Z}{4\pi\left(x^2+(1-M^2)\left((y-\eta)^2+Z^2\right)\right)^{3/2}}$$

To obtain the pressure due to a line of doublets extending from $\eta=-\infty$ to $\eta=+\infty$ we let $\lambda=(y-\eta)\sqrt{M^2-1}$ and obtain

III.33 $$p = +\frac{L\sqrt{1-M^2}\,Z}{4\pi}\int_{+\infty\sqrt{M^2-1}}^{-\infty\sqrt{M^2-1}} \frac{d\lambda}{\left(x^2+(1-M^2)Z^2+\lambda^2\right)^{3/2}}$$

which gives:

III.34 $$p = \frac{-L\left(\sqrt{1-M^2}\right)Z}{2\pi\left[x^2+(1-M^2)Z^2\right]}$$

whence: $$p_Z = -\frac{L\sqrt{1-M^2}}{2\pi}\,\frac{\left(x^2\ (1-M^2)Z^2\right)}{\left[x^2+(1-M^2)Z^2\right]^2}$$

and $$p_x = +\frac{L\sqrt{1-M^2}}{2\pi}\,\frac{2Zx}{\left[x^2+(1-M^2)Z^2\right]^2}$$

Using Newton's Law $\rho U \frac{\partial u}{\partial x} = -p_x$ and $\rho U \frac{\partial w}{\partial x} = -p_z$, we obtain for u and w, the following:

III.35 $$W = \frac{L d\xi \sqrt{1-M^2}}{2\pi\rho U} \frac{x}{(x^2+(1-M^2)z^2)}$$

III.36 $$U = \frac{L d\xi \sqrt{1-M^2}}{2\pi\rho U} \frac{z}{(x^2+(1-M^2)z^2)}$$

The relation $p = -\rho u U$ is seen to hold between III.34 and III.36.

The above relations for W and U are the same as these for a two-dimensionsl vortex in compressible flow where W in III.35 refers to the downward velocity so that sign was changed in the process of going from III.34 to III.35.

For a flat plate at angle α, we use III.35 with $z=0$ and $x-\xi$ substituted for x yielding the well-known integral equation:

III.37 $$W = U\alpha = \frac{\sqrt{1-M^2}}{2\pi\rho U} \int_{-b}^{+b} \frac{L(\xi)d\xi}{(x-\xi)}$$

From reference 3 $L(\xi)$ is given as $\frac{2\rho U^2 \alpha \cot \theta/2}{\sqrt{1-M^2}}$ where $x = -b\cos\theta$.

In order to calculate the suction force we enclose the leading edge with a curve given by the equations $\frac{x}{b} = -1-\delta\cos\phi$ $\frac{z}{b} = \frac{\delta \sin\phi}{\sqrt{1-M^2}}$. On this curve we calculate u and w and use the momentum equations to obtain the force caused by the leading edge. In this treatment δ is small so that the range of integration of θ can be small as long as $\theta \gg \delta$. The expression for W and U on this curve are as follows:

III.38 $$W = \frac{2U\alpha}{\pi}\int_0^{\theta} \frac{-[\delta\cos\theta + \theta^2/2]\, d\theta}{[\theta^2/2 + \delta\cos\phi]^2 + \delta^2\sin^2\phi}$$

III.39 $$u = \frac{2U_0\alpha}{\pi}\int_0^{\theta} \frac{\delta\sin\phi/\sqrt{1-M^2}\, d\theta}{[\theta^2/2 + \delta\cos\phi]^2 + \delta^2\sin^2\phi}$$

so that III.40 $$W - i\sqrt{1-M^2}\, u = -\frac{2U_0\alpha}{\pi}\int_0^{\theta}\frac{d\theta}{\theta^2/2 + \delta e^{-i\phi}}$$

III.40 $$W = -\frac{2U_0\alpha\cos\phi/2}{\sqrt{2\delta}} \qquad u = \frac{2U_0\alpha\sin\phi/2}{\sqrt{2\delta}\sqrt{1-M^2}}$$

In order to calculate the suction force from III.40, we need the pressure and momentum equations for linearized flow retaining terms up to the second order. These equations are derived as follows:

III.41 $$-\frac{dp}{\rho} = v\,dv + (U_0 + u)\,du$$

where ρ according to the linear theory is $\rho = \rho_0\left(1 - \frac{U_0 u}{a_0^2}\right)$

Substituting the expression for ρ an integration of III.41 yields:

III.42 . $$-\frac{p}{\rho_0} = U_0 u + \frac{u^2}{2}(1-M_0^2) + \frac{v^2}{2}$$

For the momentum equation, we obtain:

III.43 $$\frac{dM}{dt} = \int\rho(U_0+u)u\,dz + \int\rho u v\,dx$$

which after substitution for p yields:

III.44 $$\frac{1}{\rho_0}\frac{dM}{dt} = \int U_0 u\, dz + (1-M_0^2)\int u^2 dz + \int uv\, dx$$

The suction force with zero pressure round the control ... is

III.45 $$\frac{F}{\rho_0} = -(1-M_0^2)\oint u^2 dz - \oint uv\, dx$$

$$= \frac{2\pi\ b\, U_0^2 \alpha^2}{\sqrt{1-M^2}}$$

The momentum caused by the pressure is:

III.46 $$\frac{dM}{dt} = \int_{-\pi}^{+\pi} p\, dz = \int_{-\pi}^{+\pi} \left(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\right)\frac{\cos\phi\, d\phi}{\sqrt{M^2-1}} = 0$$

Therefore the expression for the drag force comes out as would be expected from the known fact that no pressure drag exists at subsonic speeds.

We will now consider the suction force on a supersonic delta wing. The lift distribution is given by formula III.21 . . . so that u is obtained by dividing this formula by $-2\rho U_0$

III.47 $$u = \pm \frac{\bar{c}^2\ U_0 \alpha}{\sqrt{\bar{c}^2-(y/x)^2}\ E\left(\sqrt{1-(M^2-1)\bar{c}^2}\right)}$$ + upper surface
− lower surface

V may be obtained by integration of the relation $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, or its Newton's Law equavalent $\rho U_0 \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}$. The expression for $\frac{\partial v}{\partial x}$ may be written as:

$$\text{III.48} \qquad \frac{\partial v}{\partial x} = \pm \frac{\bar{c}^2 U_0 \alpha \, y/x^2}{\left[\bar{c}^2 - (\frac{y}{x})^2\right]^{3/2} E} = \frac{\bar{c}^2 \, U_0 \alpha \, y x}{(\bar{c}^2 x^2 - y^2)^{3/2}}$$

Now $V = \infty$ at the leading edge at $Z = 0$ since $U = \infty$ at the leading edge. Integration of III.48 from cy to x yields:

$$\text{III.49} \qquad V = \mp \frac{U_0 \alpha \, y/x}{\sqrt{\bar{c}^2 - (y/x)^2} \; E}$$

The singular part of this integral is neglected because V goes to infinity like U at the leading edge. as mentioned above.

At the leading edge y/x equals $\bar{c}$ and $\frac{U}{V} = \bar{c}$ so that the resultant velocity is normal to the leading edge and has a constant value. This means that the flow is two-dimensional in a direction normal to the leading edge. The value of this resultant velocity near the leading edge is

$$\text{III.50} \qquad \sqrt{U^2 + V^2} = \frac{U_0 \alpha \, \bar{c} \sqrt{\bar{c}^2 + 1}}{\sqrt{\bar{c}^2 - (y/x)^2} \, (E)}$$

or letting $\frac{y}{x} = c - \delta$ $\qquad \sqrt{U^2 + V^2} = \dfrac{U_0 \alpha \sqrt{c} \sqrt{c^2 + 1}}{\sqrt{2\delta} \, (E)}$

Comparing with III.40, U and W should be given by the relations

III.51

$$v = \frac{U_0 \alpha}{\sqrt{2\delta}} \frac{\sqrt{c}\,\sqrt{\bar{c}^2+1}\ \sin \phi/2}{E}$$

$$w = \frac{U_0 \alpha}{\sqrt{2\delta}} \frac{\sqrt{\bar{c}}\,\sqrt{\bar{c}^2+1}\ \sqrt{1-M_n^2}\ \cos \phi/2}{E}$$

where M_n the Mach number normal to the leading edge.

A verification of this formula at $\phi = 0$ can be obtained by considering the singular term in equation 111.18 letting $y/x = c + \delta$

III.52 $$W = \frac{U_0 \alpha \sqrt{\bar{c}}\ \sqrt{1-(M^2-1)\bar{c}^2}}{E\ \sqrt{2\delta}}$$

and noting that $$(1 - M_n^2) = \frac{\sqrt{1-(M^2-1)\bar{c}^2}}{\sqrt{1+\bar{c}^2}}$$

The suction force can be obtained in the same way as in equation 111.45

$$\frac{dF}{dx} = \pi \rho \delta x \sqrt{1-M_n^2}\ (v^2)_{\phi=\pi}$$ or the total suction force considering the forward component is for both sides:

III.53 $$F = \frac{\pi \rho}{2} \left(\frac{U_0^2 \alpha^2 \bar{c}^2 \sqrt{1+\bar{c}^2}}{(E)^2} \right) x^2 \sqrt{1-M_n^2}$$

where x is the length of the maximum chord.

CONCLUSIONS

1) This report contains the derivation of the doublet flow on the basis of physical reasohing:

2) The physical picture of doublet flows permits one to comprehend the sigularities involved and to better appreciate some of the approximations of linear theory.

RECOMMENDATIONS

1) Numerical work should be carried out to obtain the effects of unsteady motions on supersonic downwash.

2) Further work should be done to clarify the mechanism involved in passing from the fixed to the moving doublet.

REFERENCES

1. Wasserman, Lee S.: "Acoustic Derivation of Supersonic Theory", Air Force Technical Report No. 6617, August 1951.

2. Prandtl, L.: "Theory of the Airplane Wing in Compressible Media", Luftfahrt-Forschug, Volume 12, October 1936.

3. Glauert, H.: "Aerofoil and Airscrew Theory".

4. Magnus, W., Obenhettinger, F.: "Integral Tables", 1948.

5. Ferri, A.: "Elements of Supersonic Aerodynamics".