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ON THE MOTION OF ANCHOR CHAINS

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PREFACE

The non-linear character of the problem stems from the fact that the structure of the moving system changes with time: as the anchor chain is unwound from the windlass, the links change from rotary to rectilinear motion. Similar changes of the structure of a moving one-dimensional continuum occur in a wide variety of problems concerned with the behavior of plastic-rigid beams and frames under dynamic loading. The mathematical techniques developed in the present paper have, in fact, proved valuable in the treatment of such problems. It is for this reason that the present paper is issued as a technical report under Contract N7onr-35801 even though the subject matter of this paper does not fall within the scope of this contract.

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ON THE MOTION OF ANCHOR CHAINS

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1. Introduction. In an earlier paper the problem of the motion of an anchor chain was treated under some idealizing conditions such as, for instance, neglecting the friction of the anchor chain, chain holder and windlass and the moment of inertia of the windlass with respect to the rotating axis [1]. The anchor chain, however, also moves on inclined planes with different angles and friction occurs at each point where the chain is changing its direction. It may be shown that it is possible to take all these influences into account without making use of mathematical tools other than those in the paper cited above. In other words: it is possible by means of simple substitutions to consider the motion of anchor chains under more real physical conditions and to get solutions similar to those obtained earlier. The results may be interpreted in the same way as before.

Before giving the mathematical treatment of the problem let us consider the different parts of the anchor mechanism and the conditions which this mechanism must satisfy in order to be in agreement with the solution given in this paper. The links of the chain are moving from the chain locker, through the chain locker pipe, over the chain holder of the windlass and through the hawse pipe (Fig. 1)[2]. The chains should have a clear lead to the hawse pipe. The chain holder particularly and the windlass generally should be kept well lubricated in order to keep the friction moment small. The position of the chain locker is very

important if the windlass gear is to work in a satisfactory manner. In (Fig. 2) is shown the amount of grip of the chain by the chain holder: to insure this grip it is essential to have the tail end weight of the chain from the windlass down to the chain locker well below. It is also important for the locker to be below the main deck, with a chain locker pipe (Fig. 1) for controlling the lashing cable when the chain is dropped. On merchant marine ships anchor chains are made up in 90 foot lengths, usually coupled together by a bolt type of shackle necessitating extreme-end links of special size on either side of the shackle. The design of the windlass cable holder must accommodate the pitch of common links as well as the different pitches over the shackle connections as shown in (Fig. 2).

The transition from the main deck to the hawse pipe as shown in (Fig. 3) is to eliminate the wear at the hawse pipe; to reduce the frictional force, which has the effect of decreasing the effective force in the chain between the ship and the anchor, the hawse pipe should be given such a shape as to provide a smooth and continuous race for the chain links.

2. Notations. In this paper the same notations as those of Timoshenko and Young will be used [3]; some more mathematical notations are in agreement with those used in textbooks on elliptic integrals and functions [4].

a = acceleration (cm sec^{-2}),

C_0 = constant of integration

e_1, e_2, e_3 = roots of the equation $4s^3 - g_2s - g_3 = 0$.

F = force of friction (g),

g = acceleration of gravity (cm sec^{-2}),

g_2, g_3 = invariants of the Weierstrass elliptic function

I = moment of inertia of windlass (g cm sec^2),

k = modulus of the Jacobian elliptic function,

l_1 = length of a section of the chain (cm),

m = mass of the system in motion ($\text{g cm}^{-1} \text{sec}^2$)

$m_{\text{red}} = J/r^2$ = mass of the windlass reduced to the periphery of
the windlass ($\text{g cm}^{-1} \text{sec}^2$),

q = weight per unit length of the chain (g cm^{-1}),

r = average value of the windlass radius (cm),

S_1 = tension in the chain for an arbitrary section (g),

t = time (sec),

u = parameter of the elliptic function ,

v = velocity (cm sec^{-1}),

W_0 = weight of the anchor (g)

$W = W_0 + gx$ = weight of the anchor and a length x of the chain (g),

x = displacement, length of the chain from a fixed point (cm)

$x_0 = W_0/g$ = length of the chain, corresponding to the weight W_0 (cm),

α_1 = angle of an arbitrary inclined plane

γ = ratio of retarding and accelerating forces

Δ = discriminant of Weierstrass elliptic function,

δ = increment of a quantity ($\delta x, \delta q$),

ϵ = angular acceleration of the windlass (sec^{-2}),

μ = coefficient of friction,

$\xi = (W_0 + gx + g \sum l_1 + gm_{\text{red}})/W_0$,

φ = angle of rotation of the windlass,

ω = the real half period of the elliptic p-function

$p(u)$ = the elliptic p-function of Weierstrass,

$\zeta(u)$ = the elliptic ζ - function of Weierstrass.

3. Equation of motion and its integration. By using d'Alemberts principle and the principle of virtual displacements we get as equation of motion of the system by an increment δx of displacement x and $\delta\varphi$ of angle φ of rotation [5].

$$(W_0 + gx + g \sum_{i=1}^n l_i \sin \alpha_i) \delta x - F \delta x - (I \ddot{\varphi} + M_F) \delta \varphi - \frac{1}{g} \frac{d}{dt} \left[(W_0 + qx + q \sum_{i=1}^n l_i) v \right] \delta x = 0. \quad (1)$$

As will be shown at the end of this paper it is not necessary to make a more exact consideration of the friction than notating it by a constant force F .

Because of the fixed connection of anchor chain and windlass the kinematic condition yields

$$\delta x = r \delta \varphi \quad (2)$$

and for the velocities and accelerations

$$\dot{x} = v = r \dot{\varphi}, \quad \ddot{x} = a = r \ddot{\varphi}, \quad (2a)$$

respectively.

The term $I \ddot{\varphi} \delta \varphi$ may be written

$$I \frac{\ddot{x}}{r^2} \delta x = m_{red} \frac{dv}{dt} \delta x, \quad (3)$$

where $m_{\text{red}} = I/r^2$ is the moment of inertia reduced to the radius r of the windlass. This is added to the last term of Eq.(1).

In order to obtain a total differential the expression

$$g\Sigma l_1 (1 - \sin \alpha_1) + g m_{\text{red}}$$

is added to the two first terms of Eq.(1), giving the following equation

$$\begin{aligned} W_0 + qx + q\Sigma l_1 + g m_{\text{red}} - \left[F + \frac{M_F}{r} + q\Sigma l_1 (1 - \sin \alpha_1) + g m_{\text{red}} \right] \\ = \frac{1}{g} \frac{d}{dt} \left[(W_0 + qx + q\Sigma l_1 + g m_{\text{red}}) v \right]. \end{aligned} \quad (1a)$$

Multiplying by

$$(W_0 + qx + q\Sigma l_1 + g m_{\text{red}}) v dt$$

and integrating, we obtain

$$\begin{aligned} \frac{1}{3g} (W_0 + qx + q\Sigma l_1 + g m_{\text{red}})^3 - \frac{1}{2g} \left[F + \frac{M_F}{r} + q\Sigma l_1 (1 - \sin \alpha_1) \right. \\ \left. + g m_{\text{red}} \right] (W_0 + qx + q\Sigma l_1 + g m_{\text{red}})^2 + \frac{C_0}{2g} \\ = \frac{v^2}{2g} (W_0 + qx + q\Sigma l_1 + g m_{\text{red}})^2, \end{aligned} \quad (1b)$$

where the integration constant may be determined from the condition: $v = 0$ when $x = 0$, i.e.

$$\begin{aligned} C_0 = (W_0 + q\Sigma l_1 + g m_{\text{red}})^3 + \left[F + \frac{M_F}{r} + q\Sigma l_1 (1 - \sin \alpha_1) \right. \\ \left. + g m_{\text{red}} \right] (W_0 + q\Sigma l_1 + g m_{\text{red}})^2. \end{aligned}$$

It is convenient to introduce the following abbreviations

$$(W_0 + qx + q\Sigma l_1 + g m_{red})/W_0 = \xi, \quad (4)$$

$$(W_0 + q\Sigma l_1 + g m_{red})/W_0 = \xi_0, \quad (4a)$$

$$[F + \frac{M_F}{F} + q\Sigma l_1(1 - \sin \alpha_1) + g m_{red}] / W_0 = \gamma. \quad (4b)$$

The velocity may now be written as

$$v^2 = \frac{W_0}{q} g \left\{ \frac{2}{3} (\xi^3 - \xi_0^3) + \gamma (\xi_0^2 - \xi^2) \right\} / \xi^2. \quad (1c)$$

With the notation

$$W_0/g = x_0, \text{ i.e. } \frac{dx}{x_0} = d\xi,$$

we obtain after some simple calculations the time

$$t = \sqrt{6 \frac{x_0}{g}} \int_{\xi_0}^{\xi} \frac{\xi d\xi}{\{4(\xi^3 - \xi_0^3) + 6\gamma(\xi_0^2 - \xi^2)\}^{1/2}}, \quad (5)$$

when it is assumed that $\xi = \xi_0 (x = 0)$ for $t = 0$.

Equation (1) is now integrated, but it is more convenient to introduce a new independent variable, let us say u , and present the kinematic quantities (displacement, velocity, acceleration) and also the time as dependent on this new variable u .

4. Introduction of a new independent variable. From Eq.(1b) the time t may always be expressed by the integral

$$t = \int_{x_0}^x \frac{dx}{x} = \int_{\xi_0}^{\xi} \frac{d\xi}{\xi}, \quad (6)$$

where $x = x_0$ correspond to $t = 0$. In the second integral we have introduced ζ instead of x .

Instead of expressing the time as a function of the displacement let us introduce a new independent variable

$$u = \int \frac{d\zeta}{\dot{\zeta}} = \int \frac{d(\ln \zeta)}{\dot{\zeta}}, \quad (7)$$

the inverse function being

$$\zeta = p(u). \quad (8)$$

From Eq.(6) we have

$$dt = \frac{d\zeta}{\dot{\zeta}}, \quad (6a)$$

and from Eq.(7)

$$du = \frac{d\zeta}{\zeta \dot{\zeta}} = \frac{dt}{\zeta} = \frac{dt}{p(u)}. \quad (7a)$$

Consequently we find, by integrating,

$$t = \int_{u_0}^u p(u) du. \quad (9)$$

The velocity is according to Eqs.(7a) and (8)

$$\dot{\zeta} = \zeta^{-1} \frac{d\zeta}{du} = \frac{p'(u)}{p(u)}. \quad (10)$$

The acceleration expressed by $p(u)$ is

$$\ddot{\zeta} = \frac{d\dot{\zeta}}{dt} = \frac{d}{dt} \left(\frac{p'(u)}{p(u)} \right) = \frac{d}{du} \frac{p'(u)}{p(u)} \frac{du}{dt} = \frac{p(u)p''(u) - p'(u)^2}{p(u)^3}.$$

As $\ddot{\zeta}$ is given as a function $F(\zeta) = F[p(u)]$ we get the following differential equation for $p(u)$:

$$\frac{p'(u)}{p(u)} \left[\frac{p''(u)}{p'(u)} - \frac{p'(u)}{p(u)} \right] = F[p(u)] p(u). \quad (11)$$

This equation may serve as a check at the end of the calculation.

In our case we have to substitute

$$u = \int_{\xi}^{\infty} \frac{d\xi}{\{4(\xi^3 - \xi_0^3) + 6\gamma(\xi^2 - \xi_0^2)\}^{1/2}} \quad (7b)$$

which is an elliptic integral, and the inverse function

$$\xi = p(u) \quad (8a)$$

is an elliptic function of u .

By means of the substitution

$$\xi = (s + \frac{\gamma}{2}), \quad d\xi = ds, \quad (12)$$

we transform the integral (7b) into

$$u = \int_s^{\infty} \frac{ds}{\sqrt{4s^3 - g_2s - g_3}}, \quad (7c)$$

where the invariants of the elliptic integral

$$g_2 = 3\gamma^2, \quad g_3 = 4\xi_0^2(\xi_0 - \frac{3}{2}\gamma) + \gamma^3 \quad (13)$$

and the inverse function

$$s = p(u) \quad (8b)$$

is the Weierstrass p -function if we choose the indicated limits of integration in (7b) and (7c).

From Eq.(5) we obtain the time

$$t = \left(\frac{6x_0}{g}\right)^{1/2} \int_{s_0}^s \frac{(s + \frac{\gamma}{2})ds}{\sqrt{4s^3 - g_2s - g_3}} \quad (5a)$$

the integral expressing the time

$$\int_{s_0}^s \frac{sds}{\sqrt{4s^3 - g_2s - g_3}} = \int_{u_0}^u p(u)du = \zeta(u_0) - \zeta(u) \quad (5b)$$

where $\zeta(u)$ is the Weierstrass ζ -function.

For the time we finally get

$$t = \left(\frac{6x_0}{g}\right)^{1/2} \left[\zeta(u_0) - \zeta(u) + \frac{\gamma}{2} (u - u_0) \right]. \quad (5c)$$

From Eqs.(4), (12) and (8b) we have

$$x = x_0 \left[\left(p(u) + \frac{\gamma}{2} \right) - 1 \right] - \left(\sum_1 \lambda_1 + \frac{g}{q} m_{\text{red}} \right) \quad (8c)$$

and the velocity from Eqs. (1c), (8b) and (12)

$$\dot{x} = \sqrt{\frac{x_0 g}{6}} \frac{p'(u)}{p(u) + \frac{\gamma}{2}}. \quad (10a)$$

The acceleration can be calculated from Eq.(1a), which may be written

$$W_0 (\xi - \gamma) = \frac{1}{g} W_0 \frac{d}{dt} (\xi v). \quad (11a)$$

Recognizing that

$$\frac{d}{dt} (W_0 \xi) v = q v^2,$$

we obtain from Eq. (11a)

$$W_0(\xi - \gamma) = \frac{W_0}{g} \xi \frac{dv}{dt} + \frac{g}{g} v^2. \quad (11b)$$

By means of Eq.(8b) and (12) we get

$$a = \frac{dv}{dt} = \frac{g}{p(u) + \frac{\gamma}{2}} \left[p(u) - \frac{1}{2} \gamma - \frac{1}{6} \frac{p'(u)^2}{(p(u) + \frac{\gamma}{2})^2} \right]. \quad (11c)$$

There is an asymptotic value of acceleration equal to $g/3$. This constant value of acceleration corresponds to a straight line for velocity and a parabolic curve for displacement.

By specializing the results obtained above to the case $\gamma = 0$ we get the same expressions as in the earlier paper. Therefore we don't need consider this special case now.

5. Investigation of the discriminant of the Weierstrass p-function. From a numerical calculating point of view it is important to consider the discriminant of the Weierstrass p-function, i.e. [6]

$$\Delta = g_2^3 - 27g_3^2, \quad (14)$$

where g_2, g_3 are the invariants, introduced in Eq.(7c). Inserting g_2 and g_3 from Eq.(13) in (14) we obtain after some elementary calculations

$$\Delta = -216 \xi_0^2 (\xi_0 - \frac{3}{2} \gamma) (\xi^3 + 2 \xi_0^3 - 3\gamma \xi_0^2), \quad (14a)$$

where ξ_0 and γ are given by (4a) and (4b). From Eq.(1a), however, we see that motion is possible only if $\xi_0 > \gamma$ and from (4a) and (4b) that $\xi_0 > 0, \gamma \geq 0$, where $\gamma = 0$ corresponds to the case investigated earlier. Writing the discriminant

$$\Delta = 432 \zeta_0^6 \left(\frac{3}{2} \frac{\gamma}{\zeta_0} - 1\right) \left(\frac{\gamma}{\zeta_0} - 1\right)^2 \left(\frac{\gamma}{2\zeta_0} + 1\right). \quad (14b)$$

The value of the discriminant for $\gamma = 0$ is

$$\Delta_0 = -432 \zeta_0^6. \quad (14c)$$

In order to obtain a convenient representation of the discriminant Δ we write

$$\Delta / \Delta_0 = - \left(\frac{3}{2} \frac{\gamma}{\zeta_0} - 1\right) \left(\frac{\gamma}{\zeta_0} - 1\right)^2 \left(\frac{\gamma}{2\zeta_0} + 1\right). \quad (14d)$$

This expression considered as a function of γ/ζ_0 has a maximum for $\gamma/\zeta_0 = (\sqrt{17} - 1)/4 = 0.781$ and is equal to zero for $\gamma/\zeta_0 = 2/3$ and $\gamma/\zeta_0 = 1$. The curve Δ/Δ_0 is shown in Fig. 5.

If we write the polynomial under the square root of Eq.(7c)

$$4s^3 - g_2s - g_3 = 4(s - e_1)(s - e_2)(s - e_3), \quad (15)$$

e_1, e_2, e_3 being the roots of the equation

$$4s^3 - g_2s - g_3 = 0 \quad (15a)$$

the negative value of the discriminant Δ means that two of the roots, let us say, e_1 and e_3 are complex, the third root e_2 being real.

Introducing the quantities H and k^2 defined by

$$\left. \begin{aligned} H^2 &= (e_2 - e_1)(e_2 - e_3) = 2e_2^2 + \frac{g_3}{4e_2} \\ k^2 &= \frac{1}{2} - \frac{3e_2}{4H} \end{aligned} \right\} \quad (16)$$

where k is the modulus of the Jacobian elliptic functions, Eqs. (16) establish the connections between the quantities g_3 , e_2 , H and k^2 , where $g_3 = 3\gamma^2$ and e_2 may be found as the real root of Eq.(15a). To get a survey over the numerical calculations it is, however, more convenient to consider the values H and k^2 as given and from these calculate g_3 and e_2 , getting

$$\left. \begin{aligned} e_2 &= \frac{2}{3} H(1 - 2k^2), \\ g_3 &= \left(\frac{2H}{3}\right)^3 (1 - 2k^2) [9 - 8(1 - 2k^2)^2]. \end{aligned} \right\} \quad (16a)$$

The connection between the invariant g_2 and g_3 , on the other hand, are given by

$$g_2 = 4e_2^2 - \frac{g_3}{e_2} = 3\gamma^2. \quad (17)$$

Introducing e_2 and g_3 from Eqs.(16a), we obtain

$$g_2 = 4H^2 \left[\frac{4}{3} (1 - 2k^2)^2 - 1 \right]. \quad (17a)$$

Because of $g_2 = 3\gamma^2$, we must have $g_2 \geq 0$ or

$$\frac{4}{3} (1 - 2k^2)^2 - 1 > 0,$$

which means $k^2 \leq 0.06699$, $k \leq 0.259$, the sign equal corresponding to the equianharmonic case ($g_2 = 0$) considered earlier. Because of $g_2 > 0$ the value of k^2 is small and therefore as a good approximation we may assume $k^2 = 0$, giving

$$\left. \begin{aligned} e_2 &= \frac{2}{3} H = \frac{1}{3} g_2 = \gamma, \\ g_3 &= \left(\frac{2}{3} H\right)^3 = \gamma^3. \end{aligned} \right\} \quad (16b)$$

On the other hand we have from Eq.(13)

$$g_3 = 4\xi_0^2 (\xi_0 - \frac{3}{2} \gamma) + \gamma^3 \quad (13a)$$

and therefore the following condition between ξ_0 and γ holds for the case $k^2 = 0$:

$$\gamma/\xi_0 = 2/3,$$

all quantities being expressed by γ .

As $\gamma/\xi_0 = 2/3$ corresponds to the special case $\Delta = 0$, where

$$4s^3 - g_2 s - g_3 = 4s^3 - 3\gamma^2 s - \gamma^3 = 0$$

with all roots real $e_1 = \gamma$, $e_2 = e_3 = -\gamma/2$, we have the following equation for the Weierstrass p-function

$$p(u) = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(u \sqrt{e_1 - e_3})}, \quad (18)$$

the real half period being

$$\omega_1 = \frac{k}{\sqrt{e_1 - e_3}} = \frac{\pi}{\sqrt{6\gamma}} = \frac{\pi}{\sqrt{6e_1}}. \quad (19)$$

6. Representation of the kinematic quantities in the case

$$\underline{\gamma/\xi_0 = 2/3.}$$

From Eq.(18) we get, inserting for e_1 and e_3

$$p(u) + \frac{\gamma}{2} = \frac{\frac{3}{2} \gamma}{\sin^2(u \sqrt{\frac{3}{2}} \gamma)}.$$

The value of u corresponding to the argument $\pi/2$ is $u_0 = \frac{\frac{\pi}{2}}{\sqrt{\frac{3}{2}} \gamma}$

and therefore we may write

$$p(u) + \frac{\gamma}{2} = \frac{\frac{3}{2} \gamma}{\sin^2(\frac{\pi}{2} \frac{u}{u_0})}.$$

Introducing this in Eq.(8c) we get, remembering from Eq.(4a) that

$$\Sigma l_1 + \frac{g}{q} m_{\text{red}} = \frac{W_0}{q} (\xi_0 - 1) = x_0 (\frac{3}{2} \gamma - 1), \quad (8d)$$

$$x = x_0 \frac{3}{2} \gamma \left[\frac{1}{\sin^2(u \sqrt{\frac{3}{2}} \gamma)} - 1 \right]$$

From this we see that the boundary condition $x = 0$ for $t = 0$ ($u = u_0$) is satisfied. Equation (8d) may be written

$$x = \frac{3}{2} \gamma x_0 \cot^2(u \sqrt{\frac{3}{2}} \gamma). \quad (8e)$$

The time may be written as

$$t = \left(\frac{6x_0}{g}\right)^{1/2} \int_0^1 \left[p(u) + \frac{\gamma}{2} \right] du = \left(\frac{6x_0}{g}\right)^{1/2} \frac{3}{2} \gamma \int \frac{du}{\sin^2(\sqrt{\frac{3}{2}} \gamma u)} \quad (5d)$$

which, after integration, yields

$$t = - \left(\frac{6x_0}{g} \right)^{1/2} \left(\frac{3}{2} \gamma \right)^{1/2} \cot \left(\sqrt{\frac{3}{2}} \gamma u \right). \quad (5e)$$

From this we obtain

$$\frac{3}{2} \gamma \cot^2 \left(\sqrt{\frac{3}{2}} \gamma u \right) = \frac{gt^2}{6x_0} \quad (5f)$$

which inserted in Eq. (8e) yields the simple result

$$x = \frac{g}{6} t^2. \quad (8f)$$

From Eq. (10a) for the velocity we get

$$\begin{aligned} \dot{x} &= -2 \sqrt{\frac{x_0 g}{6}} \left(\frac{3}{2} \gamma \right)^{1/2} \cot \sqrt{\frac{3}{2}} \gamma u = \\ &= 2 \left(\frac{6x_0}{6} \right)^{1/2} \left(\frac{6x_0}{g} \right)^{-1/2} t \end{aligned} \quad (10b)$$

or, in agreement with (8f),

$$\dot{x} = \frac{g}{3} t. \quad (10c)$$

In the same manner we finally obtain for the acceleration

$$\begin{aligned} a &= \frac{2}{3} \frac{g}{\gamma} \sin^2 \left(\sqrt{\frac{3}{2}} \gamma u \right) \left[\frac{\frac{3}{2} \gamma}{\sin^2 \left(\sqrt{\frac{3}{2}} \gamma u \right)} - \gamma - \gamma \frac{\cos^2 \left(\sqrt{\frac{3}{2}} \gamma u \right)}{\sin^2 \left(\sqrt{\frac{3}{2}} \gamma u \right)} \right] \\ a &= \frac{2}{3} \frac{g}{\gamma} \sin^2 \left(\sqrt{\frac{3}{2}} \gamma u \right) \left[\frac{3}{2} \gamma - \gamma \right] \frac{1}{\sin^2 \left(\sqrt{\frac{3}{2}} \gamma u \right)} = \frac{g}{3}. \end{aligned}$$

We therefore have for $\xi_0 = \frac{2}{3} \gamma$ the very simple case of constant acceleration, velocity linearly dependent on time and displacement depending on the square of time. If we are starting with an acceleration equal to $g/3$, the motion will continue with this constant value of acceleration. In Fig. 6 is shown the displacement

as a function of time for the cases $\gamma/\xi_0 = 0$, $\gamma/\xi_0 = 1/3$ and $\gamma/\xi_0 = 2/3$ (constant acceleration); Fig. 7 and 8 present velocity resp. acceleration for the same values of γ/ξ_0 .

7. Simple proof that constant acceleration is equal to $g/3$. The equation of motion may be written

$$\frac{dm}{dt} v + m \frac{dv}{dt} = P, \quad (18)$$

where $dm = qdx/g$, $dm/dt = qv/g$,

$$\frac{dm}{dt} v = \frac{q}{g} v^2. \quad (19)$$

At the beginning of motion we have $v = 0$ and

$$\frac{dv}{dt} = \frac{dv}{dt}_{(t=0)} = a, \quad P_{t=0} = m_0 \frac{dv}{dt}_{(t=0)} = m_0 a. \quad (18a)$$

Subtracting this last equation from Eq.(18) we get

$$\frac{q}{g} v^2 + (m - m_0) a = P - P_{(t=0)} = qx \quad (18b)$$

because the difference of forces at an arbitrary time t and $t = 0$ is qx . Furthermore is $m - m_0 = \frac{q}{g} x$. By constant acceleration the square of velocity is $v^2 = 2ax$, which value inserted in Eq. (18b) yields

$$\frac{q}{g} 2ax + \frac{q}{g} ax = qx \text{ or } a = \frac{g}{3} = \text{const.}$$

This is the only value of constant acceleration which is valid for all values of x , that means during the whole motion from beginning to stop. From Eq.(18b) we see that a constant acceleration is possible by motion with variable mass if the mass in

motion and acting forces are increasing proportional to the displacement. If the motion takes place under the influence of gravity the constant value of acceleration is just equal to $g/3$.

8. Summary. The motion of an anchor chain under the influence of gravity and friction forces is considered. The equation of motion involves also the influence of the moment of inertia of the windlass and the frictional moment acting on the axis of the windlass. The solution of the equation can be expressed by the elliptic functions of Weierstrass, the time by the ζ -function of a parameter v , the kinematic quantities (displacement, velocity, acceleration) by the p -function and its derivative of the same parameter. In a special case these functions can be expressed by trigonometric functions, corresponding to a constant acceleration $a = g/3$, the velocity $v = gt/3$ and the displacement $s = gt^2/6$. The absence of frictional forces corresponds to the equianharmonic case of the Weierstrass function, tabulated by A. G. Greenhill [7]. All real cases of motion of the anchor chain are included between this case and the case of constant acceleration $a = g/3$ and can be found by linear interpolation with respect to the parameter γ , expressing the ratio of the friction forces and the accelerating forces.

9. Appendix. As mentioned in section 3 it is possible to give a more complete expression of the frictional force. This will be done in the following. Let us consider the forces acting on an inclined plane $(n - k)$ with the angle of the slope equal to $\alpha_{(n-k)}$, the length $l_{(n-k)}$ and the friction coefficient μ , which

may be assumed equal for all inclined planes (Fig.9). For the motion of this part of the chain we have the following equation

$$S_{n-k} - S'_{n-k} + g l_{n-k} (\sin \alpha_{n-k} - \mu \cos \alpha_{n-k}) = \frac{q}{g} l_{n-k} \frac{dv}{dt}. \quad (20)$$

At the corner of two adjacent slopes there exists the following relation between the forces in the chain on both sides of the corner

$$S'_{n-k+1} = S_{n-k} \exp [\mu (\alpha_{n-k+1} - \alpha_{n-k})]. \quad (21)$$

The prime index indicates the force in the chain just below the corner, the same notation without the prime index indicates the force on the same slope just above the corner. If the first inclined plane where the chain is moving after leaving the windlass has the number $(n - m)$, the force in the chain is

$$\left. \begin{aligned} S'_{n-m} &= \frac{d}{dt}(m_{\text{red}} v) + \frac{M_F}{r} + \frac{q}{g}(l_0 + l_1 + l_2) \frac{dv}{dt} \\ &- q(l_0 \sin \alpha_0 + l_1 \sin \alpha_1 + l_2 \sin \alpha_2), \end{aligned} \right\} \quad (22)$$

l_0, l_1, l_2 are the lengths of the chain of the three first sections, according to Fig. 4. On the other hand the force in the chain just below the last corner corresponding to the vertical plane $(n + 1)$, is given by

$$S'_{n+1} = (W_0 + qx) - \frac{1}{g} \frac{d}{dt} [(W_0 + qx)v]. \quad (23)$$

From the last four equations we can establish an equation of

motion similar to Eq.(1a), section 3. It is convenient to introduce

$$\begin{aligned}
 L &= (\ell_0 + \ell_1 + \ell_2) \exp \left[\left(\frac{\pi}{2} - \alpha_{n-m} \right) \right] \\
 &+ \sum_{k=0}^{k=m} \ell_{n-k} \exp \left[\mu \left(\frac{\pi}{2} - \alpha_{(n-k)} \right) \right], \\
 L' &= \sum_{k=0}^{k=m} \ell_{(n-k)} (\sin \alpha_{n-k} - \mu \cos \alpha_{n-k}) \exp \left[\mu \left(\frac{\pi}{2} - \alpha_{n-k} \right) \right].
 \end{aligned}
 \tag{24}$$

If we expand the equation (23) on both sides by $(g_{m_{red}} + qL)$, in order to have a total differential we obtain the equation of motion

$$\begin{aligned}
 (W_0 + qx + g_{m_{red}} + qL) - \frac{1}{3} \frac{d}{dt} [(W_0 + qx + g_{m_{red}} + qL) v] \\
 = g_{m_{red}} + gL - \left\{ \frac{M_F}{r} \exp \left[\mu \left(\frac{\pi}{2} - \alpha_{n-m} \right) \right] + qL' \right\}.
 \end{aligned}
 \tag{25}$$

The two last terms contain the influence of the friction forces on the system. If we now multiply the equation by

$$(W_0 + qx + g_{m_{red}} + qL) v dt = (W_0 + qx + g_{m_{red}} + qL) dx$$

and integrate, we get the same equation as Eq.(1b), section 3, apart a slight change of some of the constants. We thus have given a proof that it is allowed to introduce the friction as a constant, if we neglect the influence of centrifugal forces at the corners during motion.

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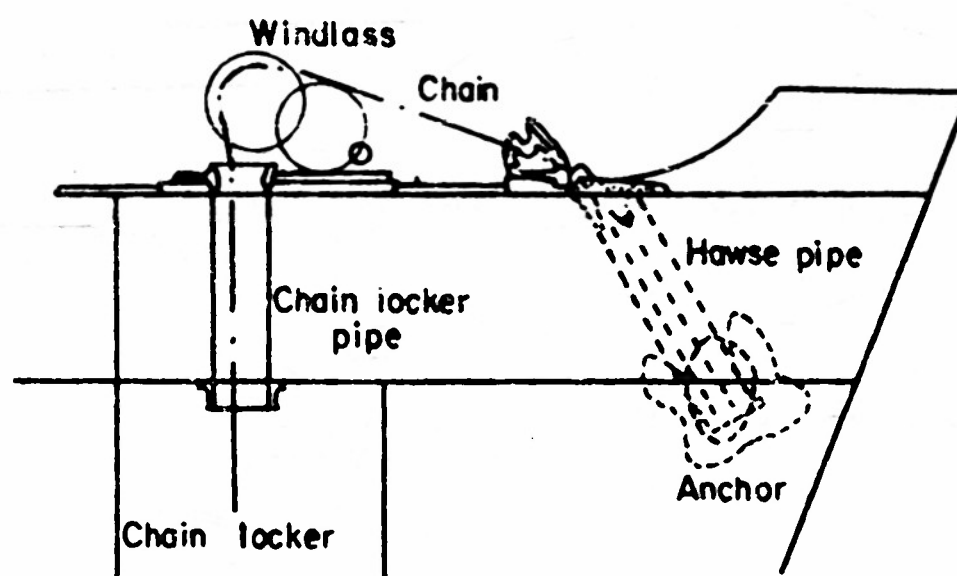


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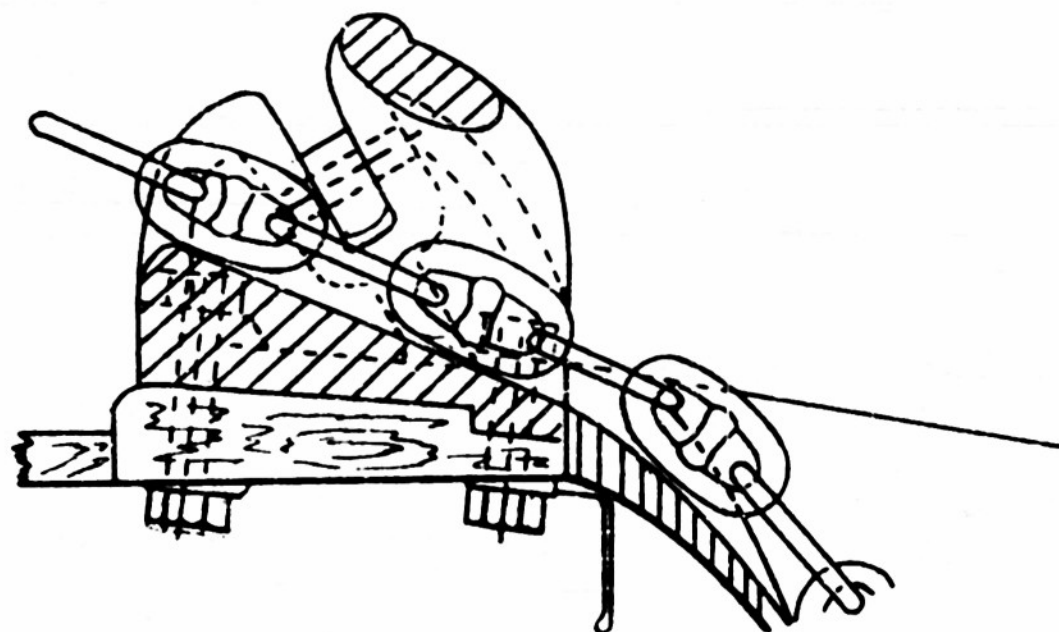


FIG. 3

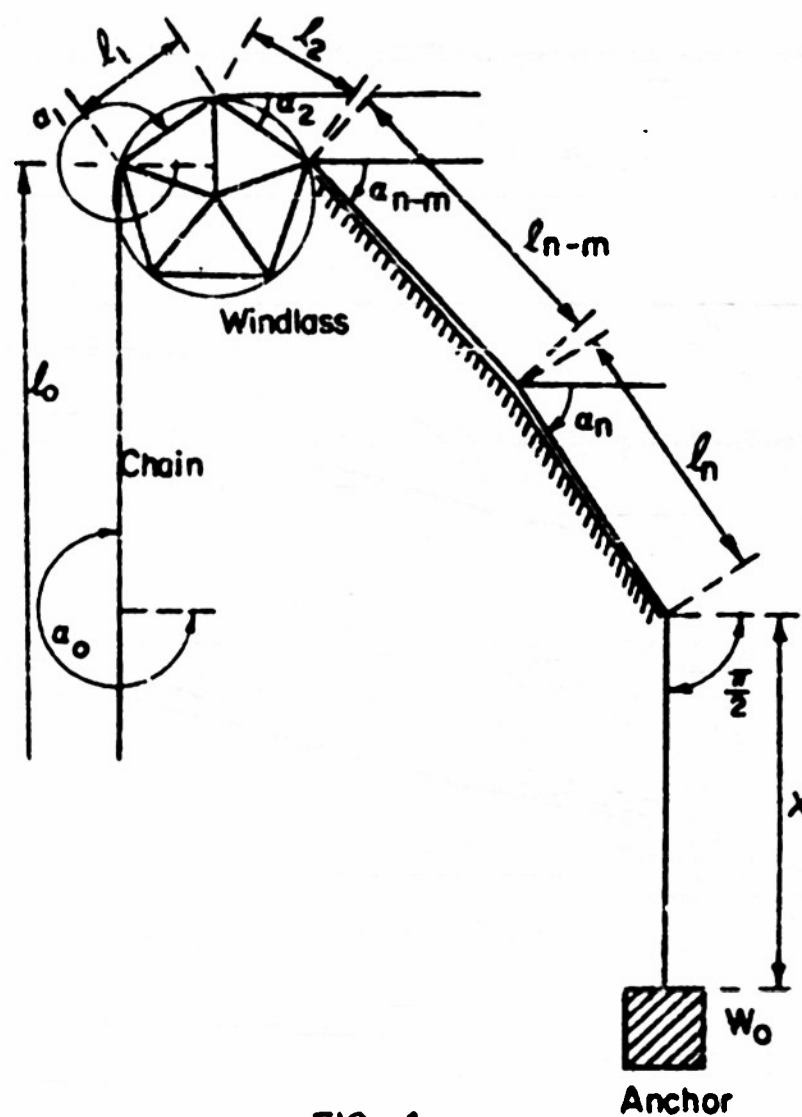


FIG. 4

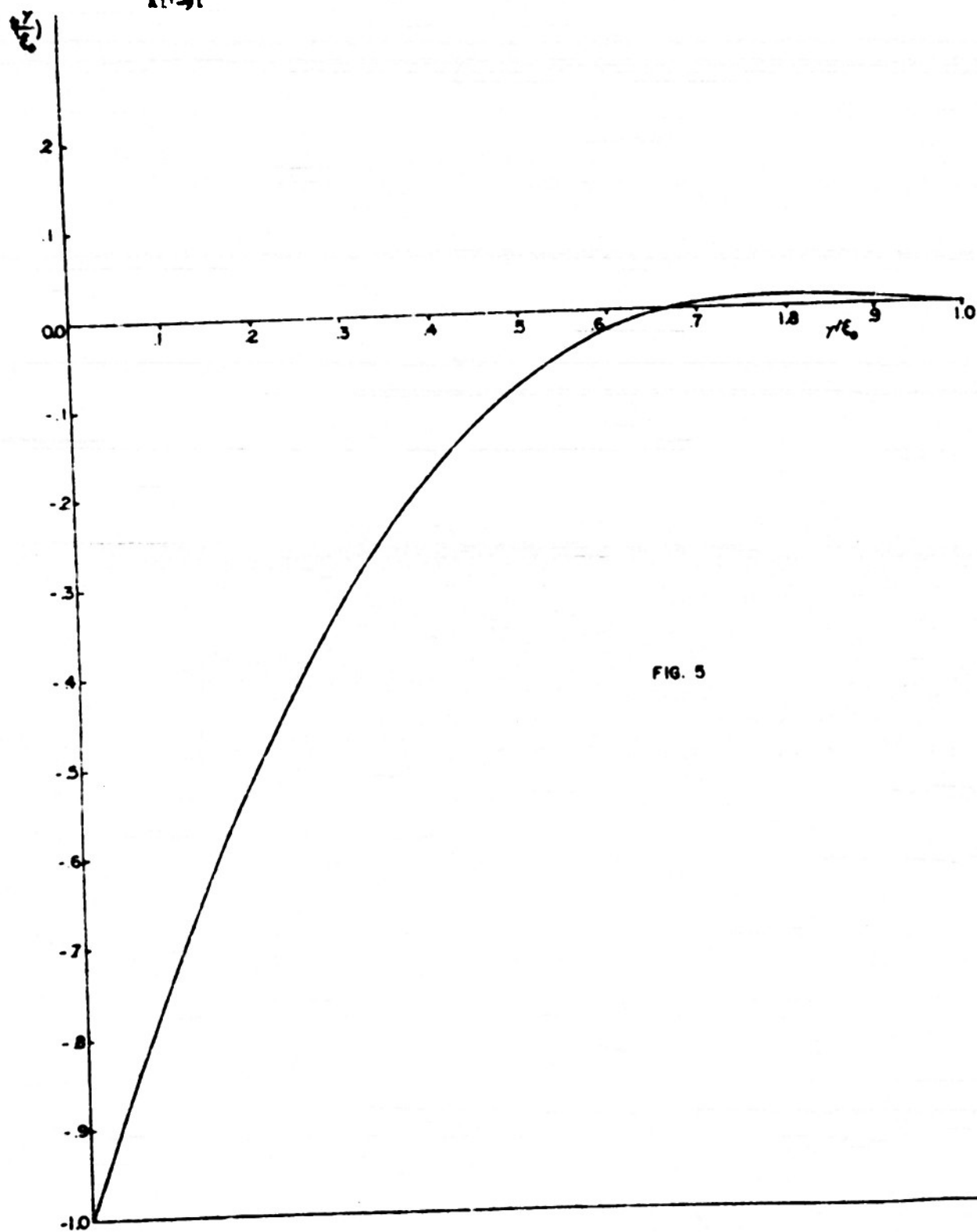


FIG. 5

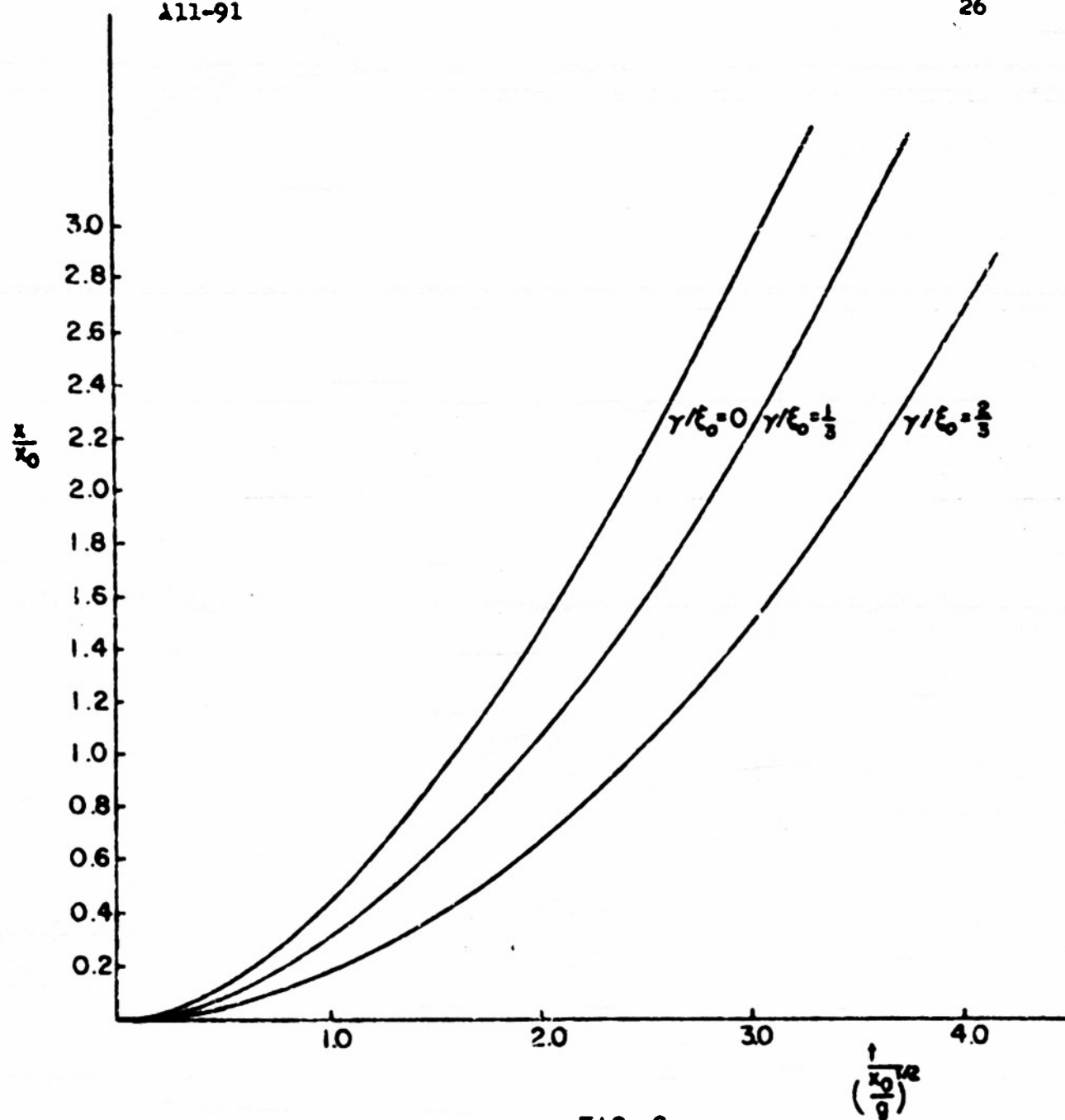


FIG. 6

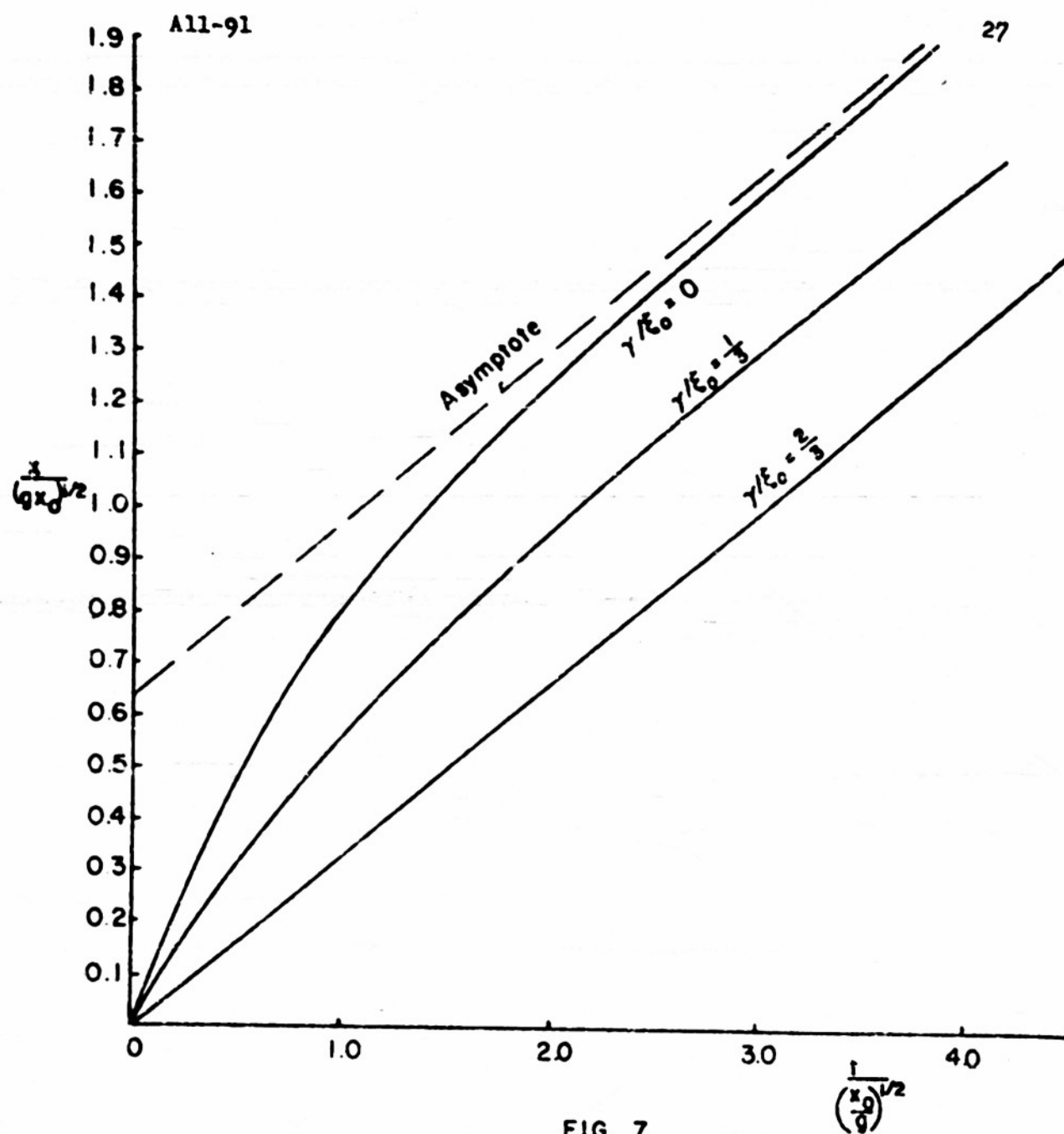


FIG. 7

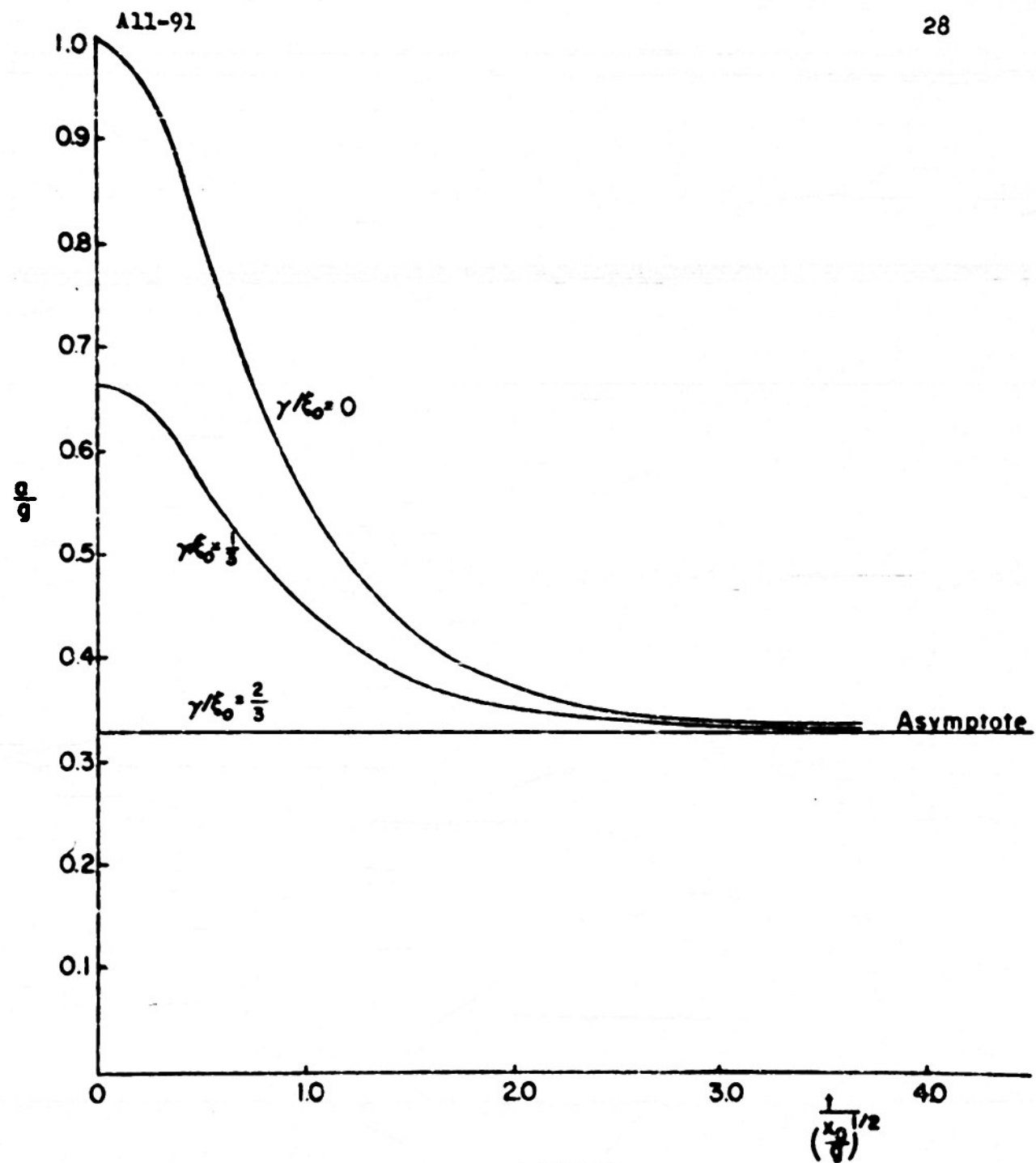


FIG. 8

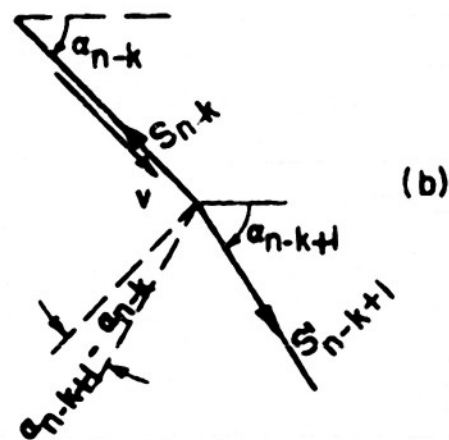
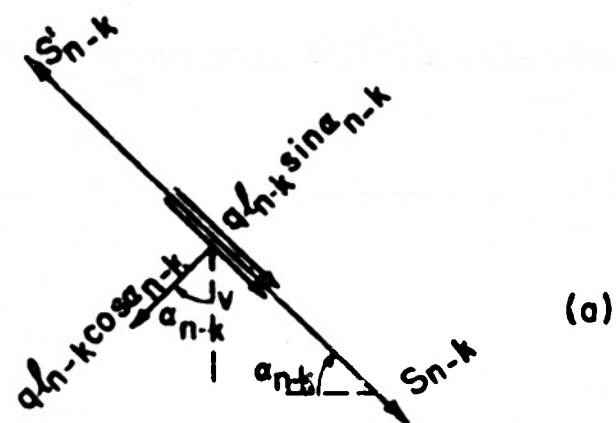


FIG. 9

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