OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

NR-041-032

Technical Report No. 98

AXIALLY SYMMETRIC PLASTIC STRESS AND VELOCITY FIELDS

by

R. T. Shield

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

September, 1953

A11-98/17

# AXIALLY SYMMETRIC PLASTIC STRESS AND VELOCITY FIELDS

By

R. T. Shield[1]

## 1. Introduction.

This note considers axially symmetric plastic stress and velocity fields in a perfectly plastic material which obeys Tresca's yield criterion of constant maximum shearing stress, k, during plastic deformation. Axial symmetry of the plastic stress field does not require the associated velocity field to be axially symmetric in general. In the following work, however, we shall assume that the velocity field also possesses axial symmetry.

## 2. The Yield Condition.

According to Tresca's yield criterion the maximum shearing stress, which is equal to one-half the difference between the maximum and minimum principal shearing stresses, has the constant value k during plastic deformation. States of stress can be represented by points in a space in which the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ are used as rectangular Cartesian co-ordinates. The states of stress which involve the maximum shearing stress k are represented by the points on the surface of a right prism which has a regular hexagon for its cross section. The axis of the prism is equally inclined to the $\sigma_1$, $\sigma_2$, $\sigma_3$ axes and passes through the origin. The section of the prism by a plane perpendicular to the $\sigma_3$-axis is shown in Fig. 1. The points on the hexagon ABCDEF

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represent states of stress with the maximum shearing stress k and
the centre G of the hexagon has the co-ordinates $(\sigma_1, \sigma_2, \sigma_3)$.

Plastic flow can occur under states of stress re-
presented by points on the surface of the prism, that is by points
on the hexagon in Fig. 1. The material is isotropic so that the
principal axes of the plastic strain rate coincide with the prin-
cipal axes of stress. The principal components of the plastic
strain rate in the $\sigma_1$, $\sigma_2$, $\sigma_3$ directions will be denoted by $\varepsilon_1$, $\varepsilon_2$,
and $\varepsilon_3$ respectively. The plastic strain rate can be represented
in principal stress space by a ray with direction cosines propor-
tional to $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$. The concept of perfect plasticity[1] re-
quires that the ray representing the plastic flow which could occur
under the state of stress represented by a point on the prism is
normal to the side of the prism on which the stress point lies.
If the stress point lies on a corner of the prism, the ray must lie
between the normals to the two sides of the prism which meet at
the corner. Since the axis of the prism is equally inclined to
the $\sigma_1, \sigma_2$, $\sigma_3$ axes, it follows that the incompressibility condi-
tion

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \qquad\qquad (1)$$

must hold. The projection of the ray onto the plane of Fig. 1 is
a ray with direction cosines proportional to $\varepsilon_1$, $\varepsilon_2$ and perpendic-
ular to the side of the hexagon on which the stress point lies.
For a stress point coinciding with the vertex A, for example, the
ray must lie in the angular space shown by the arrows in Fig. 1.

The projection of the ray onto Fig. 1 determines the plastic strain rate to within an arbitrary factor since $\varepsilon_3$ can be found from (1) when $\varepsilon_1$ and $\varepsilon_2$ are known.

We now list the various relations which hold between the stresses and between the strain rates as the stress point moves around the hexagon ABCDEF. In the following, $\lambda$, $\mu$, and $\varepsilon$ denote non-negative arbitrary parameters. We shall assume that $\sigma_1 \geq \sigma_2$ so that we need only consider points to the right of the line O'B.

(i)   Stress point at B.

$$\sigma_1 = \sigma_2 = \sigma_3 + 2k. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \lambda : \varepsilon : -\lambda - \varepsilon.$$

(ii) Stress point on AB.

$$\sigma_1 = \sigma_3 + 2k, \ \sigma_1 > \sigma_2 > \sigma_3. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \lambda : 0 : -\lambda .$$

(iii) Stress point at A.

$$\sigma_1 = \sigma_3 + 2k, \ \sigma_2 = \sigma_3. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \lambda + \mu : -\mu : -\lambda.$$

(iv) Stress point on AF.

$$\sigma_1 = \sigma_2 + 2k, \ \sigma_2 < \sigma_3 < \sigma_1. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \mu : -\mu : 0.$$

(v)   Stress point at F.

$$\sigma_1 = \sigma_2 + 2k, \ \sigma_1 = \sigma_3. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \mu : -\mu - \varepsilon : \varepsilon.$$

(vi) Stress point on EF.

$$\sigma_2 = \sigma_3 - 2k, \ \sigma_3 > \sigma_1 > \sigma_2. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = 0 : -\varepsilon : \varepsilon.$$

(vii) Stress point at E.

$$\sigma_1 = \sigma_2 = \sigma_3 - 2k. \quad \varepsilon_1 : \varepsilon_2 : \varepsilon_3 = -\lambda : -\varepsilon : \lambda + \varepsilon.$$

## 3. Axial Symmetry.

We take the axis of symmetry to be the z-axis and use cylindrical co-ordinates $(r,\Theta,z)$. Axial symmetry implies that the non-zero stresses are $\sigma_r$, $\sigma_\Theta$, $\sigma_z$ and $\tau_{rz}$, and the equations of equilibrium become

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\Theta}{r} = 0,$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0.$$

$$(2)$$

The $\Theta$-direction is a principal stress direction so that the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$ are given by

$$\sigma_1 = \frac{\sigma_r + \sigma_z}{2} + \left\{ \frac{(\sigma_r - \sigma_z)^2}{4} + \tau_{rz}^2 \right\}^{\frac{1}{2}},$$

$$\sigma_2 = \frac{\sigma_r + \sigma_z}{2} - \left\{ \frac{(\sigma_r - \sigma_z)^2}{4} + \tau_{rz}^2 \right\}^{\frac{1}{2}},$$

$$\sigma_3 = \sigma_\Theta,$$

$$(3)$$

where we have taken the third principal stress direction to be the $\Theta$-direction and have taken $\sigma_1 \geq \sigma_2$.

We denote the velocity components in the $(r,\Theta,z)$ directions by u,v,w respectively. Since the velocity field is axially symmetric, we have

$$u = u(r,z), \quad v = 0, \quad w = w(r,z), \tag{4}$$

and the non-zero plastic strain components are given by

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\Theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}. \tag{5}$$

The incompressibility condition (1) becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{6}$$

The principal components of the plastic strain rate are given by

$$\left.\begin{aligned}
\epsilon_1 &= \frac{\epsilon_r + \epsilon_z}{2} + \frac{1}{2}\left\{ (\epsilon_r - \epsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}}, \\[2mm]
\epsilon_2 &= \frac{\epsilon_r + \epsilon_z}{2} - \frac{1}{2}\left\{ (\epsilon_r - \epsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}}, \\[2mm]
\epsilon_3 &= \epsilon_\Theta.
\end{aligned}\right\} \tag{7}$$

In the following sections we examine the restrictions imposed by axial symmetry upon the plastic stress field and associated velocity field in states of stress represented by points on the hexagon in Fig. 1.

4. <u>Stress Point at B or E.</u>

For the stress points B and E we have

$$\sigma_1 = \sigma_2 = \sigma_3 \pm 2k,$$

where the upper sign refers to the stress point B. The condition $\sigma_1 = \sigma_2$ gives immediately from (3),

$$\sigma_r = \sigma_z, \quad \tau_{rz} = 0.$$

The equations of equilbrium (2) are therefore, since $\sigma_3 = \sigma_\Theta$,

$$\frac{\partial \sigma_r}{\partial r} \pm \frac{2k}{r} = 0 \; ,$$

$$\frac{\partial \sigma_r}{\partial z} = 0.$$

Thus, for this field we have

$$\sigma_r = \sigma_z = \sigma_\theta \pm 2k = \pm 2k \log \frac{A}{r} \; ,$$

where A is a constant and where the upper and lower signs refer to the stress points B and E respectively.

The plastic strain rates must be such that

$$\epsilon_1 : \epsilon_2 : \epsilon_\theta = \pm \lambda : \pm \epsilon : \mp \lambda \mp \epsilon,$$

using the same convention for the signs. Since $\epsilon_\theta = u/r$, it follows that

$$u \leq 0 \text{ for B and } u \geq 0 \text{ for E.}$$

The components u and w must be such that

$$\epsilon_r + \epsilon_z \geq \left\{ (\epsilon_r - \epsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}} \text{ for B,}$$

$$|\epsilon_r + \epsilon_z| \geq \left\{ (\epsilon_r - \epsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}} \text{ for E,}$$

since $\epsilon_1$, $\epsilon_2$ are both positive for B and both negative for E.

A discontinuity in the velocity field can only occur across a surface on which the shearing stress has the maximum value k. The shearing stress in the r,z plane is everywhere zero so that the components u,w must be continuous functions.

## 5. Stress Point on AB or EF.

For states of stress represented by points on AB, we have

$$\epsilon_1 : \epsilon_2 : \epsilon_\Theta = \lambda : 0 : -\lambda .$$

It follows from (5) and (7) that

$$u \leq 0, \quad \frac{\epsilon_r + \epsilon_z}{2} - \frac{1}{2} \left\{ (\epsilon_r - \epsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}} = 0.$$

Thus, u and w are to be determined from the equations

$$\left. \begin{array}{c} \dfrac{\partial u}{\partial r} + \dfrac{u}{r} + \dfrac{\partial w}{\partial z} = 0, \\[2mm] \dfrac{u^2}{r^2} = (\dfrac{\partial u}{\partial r} + \dfrac{\partial w}{\partial z})^2 = (\dfrac{\partial u}{\partial r} - \dfrac{\partial w}{\partial z})^2 + (\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial r})^2 \end{array} \right\} \quad (8)$$

and the condition $u \leq 0$. The field is therefore kinematically determinate in the sense that there are two differential equations for the two components of velocity. When the velocity field is known, the stress field is obtained from the two equations of equilibrium, the yield condition

$$\sigma_1 = \sigma_3 + 2k, \quad \sigma_1 > \sigma_2 > \sigma_3,$$

and the condition that the principal axes of stress must coincide with the principal axes of the strain rate.

Analogous remarks apply when the stress point lies on EF. u and w are determined from equations (8) and the condition $u \geq 0$.

The velocity components u, w must be continuous since the shearing stress in the r,z plane is less than k.

## 6. Stress Point on AF.

In this case,

$$\varepsilon_1 : \varepsilon_2 : \varepsilon_\Theta = \mu : -\mu : 0.$$

Since $\varepsilon_\Theta = u/r$, this gives immediately $u = 0$. The incompressibility condition then shows that $\partial w/\partial z = 0$ so that $w = w(r)$. The strain components are therefore

$$\varepsilon_r = \varepsilon_z = \varepsilon_\Theta = 0, \quad \gamma_{rz} = \frac{dw}{dr} .$$

Since the principal axes of stress and strain coincide we must have

$$\sigma_r = \sigma_z .$$

Also $\sigma_1 = \sigma_2 + 2k$, so that $\tau_{rz} = \pm k$ using (3). The equations of equilibrium (2) become

$$\left. \begin{array}{c} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{\sigma_r - \sigma_\Theta}{r} = 0, \\[2mm] \dfrac{\partial \sigma_r}{\partial z} \pm \dfrac{k}{r} = 0, \end{array} \right\}$$

and have the solution

$$\left. \begin{array}{l} \sigma_r = \mp \dfrac{kz}{r} + f(r), \\[3mm] \sigma_\Theta = \mp \dfrac{kz}{r} + f(r) \pm \dfrac{kz}{r} + r\, f'(r) = \dfrac{1}{r} \dfrac{\partial}{\partial r} \left\{ r\, f(r) \right\}. \end{array} \right\}$$

The function $f(r)$ is restricted by the condition

$$\sigma_r - k < \sigma_\Theta < \sigma_r + k$$

## 7. Stress Point at A or F.

The hypothesis of Haar and von Kármán [2] postulates that during axially symmetric plastic flow the circumferential stress $\sigma_\Theta$ is equal to one of the principal stresses $\sigma_1$, $\sigma_2$ in the r,z plane. In certain problems, the stress field is statically determinate under this hyopthesis. Ishlinsky [3] has used the hypothesis to calculate the indentation pressure in the indentation of a plane surface by a flat circular punch. The author obtained a plastic stress field around the punch but it was not shown that the field could be extended throughout the body in a satisfactory manner. Also it remains to be shown whether a velocity field can be associated with the stress field or not.

At the stress points A and F the circumferential stress is equal to the minimum or maximum principal stress in the r,z plane respectively, as in the hypothesis. We consider first the stress point F.

For states of stress represented by the point F,

$$\sigma_1 = \sigma_2 + 2k \ , \ \sigma_1 = \sigma_3$$

so that, from (3),

$$\frac{1}{4} (\sigma_r - \sigma_z)^2 + \tau_{rz}^2 = k^2 \ , \ \sigma_\Theta = \frac{\sigma_r + \sigma_z}{2} + k. \tag{9}$$

We put

$$p = - \frac{1}{2} (\sigma_r + \sigma_z)$$

and denote by $\varphi$ the angle of inclination of the first shear line

in the r,z plane to the r-axis, as shown in Fig. 2. From the yield condition (9) it follows that

$$
\left.\begin{array}{l}
\sigma_r = - p - k \sin 2\varphi \ , \\[4pt]
\sigma_z = - p + k \sin 2\varphi \ , \\[4pt]
\tau_{rz} = k \cos 2\varphi \ , \\[4pt]
\sigma_\Theta = - p + k \ .
\end{array}\right\}
\qquad (10)
$$

It can be shown that the shear lines are the characteristic lines of the hyperbolic system of differential equations (3) and (9), and the following relations hold

$$
\left.\begin{array}{l}
dp + 2k \, d\varphi + k \, (1 + \tan \varphi) \dfrac{dr}{r} = 0 \text{ on an } \alpha\text{-line,} \\[8pt]
dp - 2k \, d\varphi + k \, (1 + \cos \varphi) \dfrac{dr}{r} = 0 \text{ on an } \beta\text{-line.}
\end{array}\right\}
\qquad (11)
$$

In the relations (10) the first and second shear lines are called $\alpha$- and $\beta$-lines respectively.

The plastic strain rates satisfy

$$
\varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \mu : - \mu - \varepsilon : \varepsilon
$$

so that we have, with (5) and (7),

$$
\varepsilon_r + \varepsilon_z = - \varepsilon, \ \frac{u}{r} = \varepsilon_\Theta = \varepsilon \ ,
$$

$$
\left\{ (\varepsilon_r - \varepsilon_z)^2 + \gamma_{rz}^2 \right\}^{\frac{1}{2}} = \varepsilon + 2\mu \ ,
$$

where $\mu$ and $\varepsilon$ are positive. Thus the velocity field must be such that

$$u \geq 0, \quad \left(\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 \geq \frac{u^2}{r^2} . \tag{12}$$

The velocity components are determined from the incompressibility condition (6) and the condition that the principal axes of stress and strain coincide, i.e.

$$\frac{\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}}{\dfrac{\partial u}{\partial r} - \dfrac{\partial w}{\partial z}} = - \cot 2\varphi .$$

It is found that the characteristics of the velocity field coincide with the characteristics of the stress field. The relations along the characteristics are

$$\left. \begin{aligned} du + \tan \varphi \, dw + \frac{u}{2\cos^2\varphi} \frac{dr}{r} &= 0 \text{ on an } \alpha\text{-line,} \\[2mm] du - \cot \varphi \, dw + \frac{u}{2\sin^2\varphi} \frac{dr}{r} &= 0 \text{ on a } \beta\text{-line.} \end{aligned} \right\} \tag{13}$$

If $u_\alpha$, $u_\beta$ denote the velocity components along the $\alpha$, $\beta$ lines so that

$$u_\alpha = u \cos \varphi + w \sin \varphi,$$
$$u_\beta = -u \sin \varphi + w \cos \varphi,$$

the relations (11) can be written

$$\left. \begin{aligned} du_\alpha - u_\beta \, d\varphi &= \frac{-u}{2\cos\varphi} \frac{dr}{r} = \frac{-u}{2r} \, ds_\alpha \text{ on an } \alpha\text{-line,} \\[2mm] du_\beta + u_\alpha \, d\varphi &= \frac{u}{2\sin\varphi} \frac{dr}{r} = \frac{-u}{2r} \, ds_\beta \text{ on a } \beta\text{-line,} \end{aligned} \right\} \tag{14}$$

where $ds_\alpha$, $ds_\beta$ are the elements of length along the $\alpha$, $\beta$ lines.

The velocity field is determined from equations (13) or (14) and conditions (12).

The situation at the stress point A is very similar. The first of equations (9) and the first three of equations (10) hold while

$$\sigma_\Theta = \frac{\sigma_r + \sigma_z}{2} - k = - p - k.$$

The relations along the characteristic lines (the shear lines) are now

$$\left. \begin{array}{l} dp + 2k \, d\varphi - k \, (1 - \tan \varphi) \, \dfrac{dr}{r} = 0 \text{ on an } \alpha\text{-line,} \\[2ex] dp - 2k \, d\varphi - k \, (1 - \cot \varphi) \, \dfrac{dr}{r} = 0 \text{ on a } \beta\text{-line.} \end{array} \right\} \quad (15)$$

The velocity components must be such that

$$u \leq 0, \quad \left(\frac{\partial u}{\partial r} - \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 \geq \frac{u^2}{r^2}, \qquad (16)$$

and along the characteristics the relations (13) and (14) hold.

It can be seen from above that the equations for the stress and velocity components are similar, although more complicated, to the corresponding equations for plastic flow under conditions of plane strain. The conditions (12) or (16) are analytical statements of the condition that the rate of doing plastic work during the deformation must be positive. The corresponding condition for plane-strain plastic flow has been occasionally overlooked in the literature.

## 8.  A Simple Example.

In this section a simple example will be given of an axially symmetric stress field with an associated velocity field.

In Fig. 3, the circular cylinder OAD is compressed by a flat smooth punch on the end OA. The curved surface of the cylinder is stress free and the plastic stress field is simply a longitudinal compression of amount 2k and zero radial and circumferential stresses. In the terminology of this note, stress field is everywhere at the stress point F in Fig. 1. The shear lines are straight lines inclined at 45 degrees to the axis of the cylinder; AB is an α-line, BD is a β-line.

The velocity boundary conditions are that the end of the cylinder OA moves as a plane surface so that w has a constant value on OA. The shear rate $\gamma_{rz}$ must be zero, since $\tau_{rz}$ is zero, so that in view of the incompressibility condition (6) the conditions (12) will be satisfied if $u \geq 0$ and if $\partial u/\partial r$, $\partial w/\partial z$ are of different sign. If the region of plastic flow is restricted to the region OAC, a velocity field satisfying these conditions and the velocity boundary conditions cannot be found. We outline a velocity field which involves plastic flow in the region OADB.

In OAB the velocity components are taken to be given by

$$u = 1/2r, \quad w = 1 - z, \tag{17}$$

where the length of OB is the unit of length and the upward velocity of the punch is taken to be unity. The velocity field (17) satisfies

incompressibility, the boundary conditions on OA, the condition
$Y_{rz} = 0$, and the conditions (12) for positive plastic work. The
velocity field in the region ABC is determined from the equations
(13) and the velocity conditions across BD, AB. On the character-
istic BD, which separates the material at rest from the material in
motion, we must have u = w, and since the velocity is zero at B,
the velocity components are zero along BD, from the first of equa-
tions (13). The normal velocity across AB is known from the velocity
field in OAB, and it is found that the velocity field is continuous
across AB. The field determined in this way in region ABD is found
to satisfy the conditions (12). Thus the velocity field is com-
patible with the stress field. Fig. 4 shows the deformation of a
square grid in the r,z plane which would occur if the incipient
velocity field was maintained for a short period of time.

Reversing the sign of the stress $\sigma_z$ and the signs of
the velocity component gives a solution to the case when the end
OA is subject to a uniform tension 2k. The stress field is then
represented by the stress point A.

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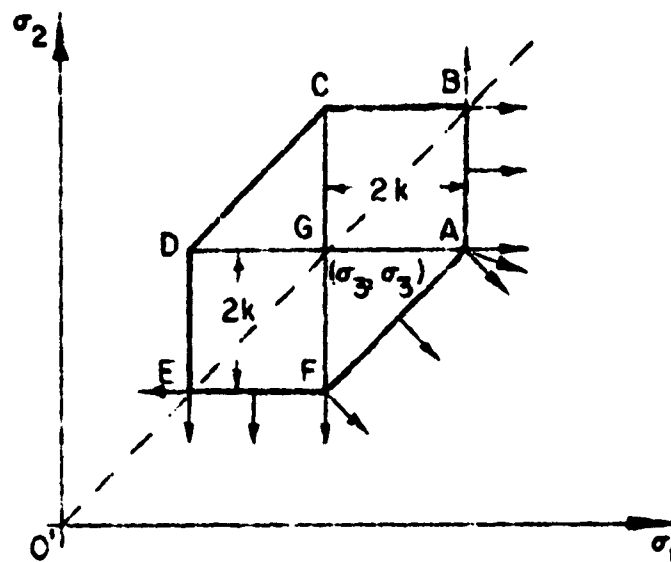        Applied Mathematics Group, Brown University, 1946.
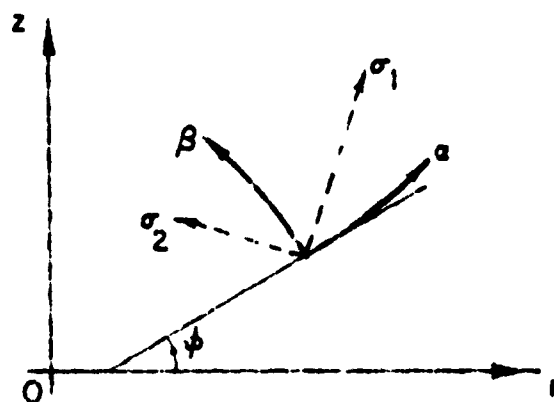
FIG. 1. TRESCA YIELD CONDITION
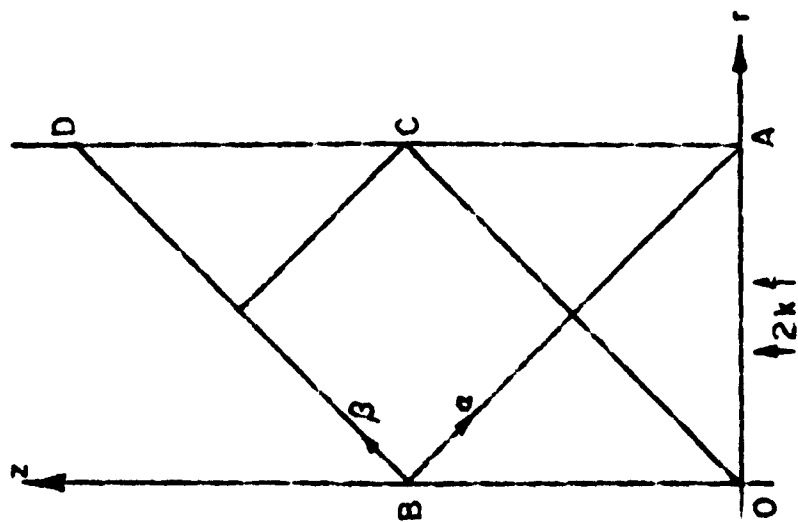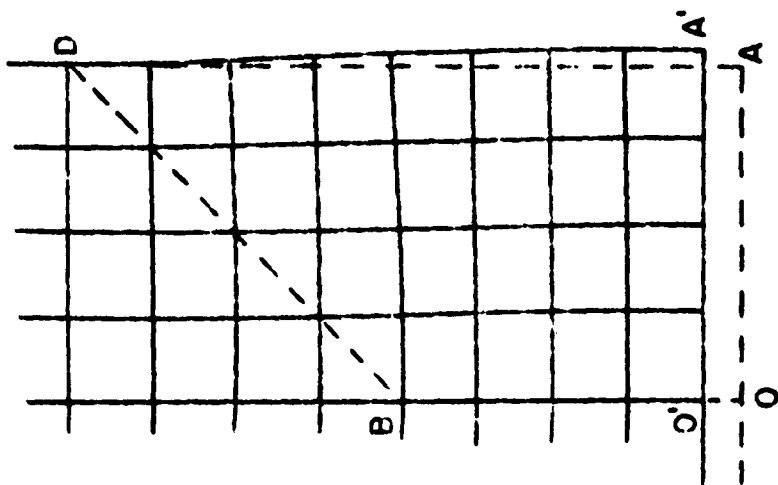


FIG. 2. LINES OF MAXIMUM SHEAR

FIG. 4. RESULTING DEFORMATION OF SQUARE
GRID IN $r, z$ PLANE IF INCIPIENT
VELOCITY FIELD IS MAINTAINED FOR
SHORT TIME



FIG. 3. FLAT PUNCH ON END
OF CIRCULAR CYLINDER

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Gainesville, Florida

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M-1             Dr. George F. Carrier, 309 Pierce Hall

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Chicago, Illinois
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Urbana, Illinois
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                tics, Attn: Prof. M. C. Steele

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Bloomington, Indiana
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Johns Hopkins University
Baltimore, Maryland
M-1      Attn:  Dr. W. H. Hoppman, II

M-1   Director, Applied Physics Laboratory
Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland

Lehigh University
Bethlehem, Pennsylvania
R-1      Attn:  Mr. Lynn S. Beedle

Massachusetts Institute of Technology
Cambridge 39, Massachusetts
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R-1           Dr. W. Kaplan

New York University
Institute for Mathematics & Mechanics
45 Fourth Avenue
New York 3, New York
R-1     Attn: Professor R. Courant
R-1           Dr. G. Hudson

New York University
New York 53, New York
R-1     Attn: Dr. C. T. Wang, Dept. of Aeronautics

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Stanford, California
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Hoboken, New Jersey
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Salt Lake City, Utah
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Norton, Massachusetts
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New Kensington, Pennsylvania
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M-1          H. N. Hill, Aluminum Research Laboratory

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Murray Hill, New Jersey
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Detroit, Michigan
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Department 72-25, Factory A-1, Building 66
Burbank, California
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Midwest Research Institute
Kansas City, Missouri
R-1      Attn: M. Goland

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East Hartford, Connecticut
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U. S. Rubber Company
Passaic, New Jersey
R-1      Attn: H. Smallwood

Welding Research Council
Engineering Foundation
29 West 39 Street
New York 18, New York
M-1      Attn: W. Spraragen, Director

Westinghouse Research Laboratories
East Pittsburgh, Pennsylvania
R-1      Attn: Dr. E. A. Davis

M-1      Dr. A. Nadai
136 Cherry Valley Road
Pittsburgh 21, Pennsylvania

Westinghouse Electric Corporation
Lester Branch P.O.
Philadelphia, Pennsylvania
R-1      Attn: R. P. Kroon, Manager of Engineering, AGT Division

University of Pennsylvania
Towne Scientific School
Philadelphia 4, Pennsylvania
R-1      Attn: Dr. C. W. MacGregor, Vice President in Charge of
             Scientific and Engineering Studies

R-1  Professor G. Wästlund
    Cement & Concrete Research Institute
    Royal Institute of Technology
    Stockholm 70, SWEDEN

M-1  Professor John E. Goldberg
    Department of Structural Engineering
    Purdue University
    Lafayette, Indiana

R-1  Milad F. Hanna
    % F.S.S.B.
    Massachusetts Institute of Technology
    Cambridge 39, Massachusetts

R-1  Dr. W. Freiberger
    Department of Supply
    Aeronautical Research Laboratories
    Box 4331 GPO
    Melbourne, AUSTRALIA

M-1  Professor B. W. Shaffer
    Department of Mechanical Engineering
    New York University
    New York 53, New York

M-1  Professor G. Sachs
    Division of Metallurgical Research
    Engineering & Science Campus
    East Syracuse 4, N. Y.

R-1  Professor Aziz Ghali
    Head, Structural Analysis Department
    Fonad University
    Giza, EGYPT

M-1  Professor Hugh Ford
    Mechanical Engineering Department
    Imperial College of Science & Technology
    London, S.W.7
    ENGLAND

R-1  Dr. R. H. Wood
    D.S.I.R. Building Research Station
    Garston, Watford, Herts
    ENGLAND