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THEORETICAL ANALYSIS OF THE LONGITUDINAL STABILITY OF A TANDEM HYDROFOIL SYSTEM IN SMOOTH WATER

by

W.C. Hugli, Jr.
and
Paul Kaplan

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Experimental Towing Tank
Stevens Institute of Technology
Hoboken, New Jersey

THEORETICAL ANALYSIS OF THE LONGITUDINAL STABILITY OF A TANDEM HYDROFOIL SYSTEM IN SMOOTH WATER

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W.C. Hugli, Jr.
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FOREWORD

This report stems from a study performed specifically for Gibbs and Cox, Inc., under an Office of Naval Research sub-contract, in which the stability of a particular configuration previously tested at the Experimental Towing Tank was analyzed (Reference 1). The present development, which was performed under ONR Contract No. Nonr 263, Task Order 01, is more general and detailed, and takes into consideration an additional degree of freedom (surging motion).

SUMMARY

The equations of motion describing the longitudinal stability characteristics of an uncontrolled tandem hydrofoil system in smooth water with three degrees of freedom are developed in the present report. The stability derivatives in the equations are evaluated on the basis of theoretical hydrodynamic formulas, thereby eliminating the necessity of their experimental determination. Comparison of the motions obtained from solution of the equations with those determined from previous experimental stability tests indicates a duplication of the motion experienced in the tank tests. Variations in the location of the longitudinal center-of-gravity position are shown to lead to different types of motion, which is in agreement with the results of standard aircraft stability analyses.

The additional degree of freedom in longitudinal surging is found to indicate a decrease in stability. This mode of motion is of great importance, however, for a study of stability in waves where the waves cause variations in the forward speed.

The equations developed in this report may be extended to studies of motion in waves and controlled motion by incorporating the forces and moments due to wave motion and the dynamics of the control system in the equations. By assuming the resultant system of equations to be linear, the analysis may be performed by using the Laplace transform technique, in the same manner as was done in the present study.

INTRODUCTION

The use of hydrofoils for the support of craft operating on the surface of the water appears to be the most promising way of attaining higher speeds without prodigious amounts of power or without a prohibitive behavior in a seaway. It is for this reason that the hydrofoil principle, while not new, has received considerable attention recently.

A hydrofoil craft in the foil-borne condition is similar to an airplane in flight. Therefore, it is not surprising that the longitudinal stability of hydrofoil boats is one of the problems confronting the designer. A certain amount of work has already been published on the stability of motions of hydrofoil systems in smooth water (see References 2, 3, and 4), with the hydrodynamic derivatives evaluated from experimental data and the downwash of the forward foil of a tandem system obtained from the aerodynamic theory of a foil far removed from the surface. Subsequent studies of the hydrodynamics of foils operating near the free surface have yielded theoretical expressions for the lift and drag forces and the downwash which are close approximations to the actual effects determined experimentally (References 5, 6, 7). The downwash at various distances aft of the forward hydrofoil may even be an upwash, depending on the forward speed and lift coefficient. Alterations in the downwash can, of course, materially affect the longitudinal stability of hydrofoil craft.

The present report deals theoretically with the longitudinal stability of fully submerged tandem hydrofoil systems in smooth water with three degrees of freedom, and utilizes theoretical expressions for the hydrodynamic derivatives. It is a first approach to the problem of developing methods of theoretical treatment for the more general case of longitudinal stability of hydrofoil systems in rough water. The notation used has been adapted from that presented in Reference 8 and affords a simple treatment for this complex stability problem.

In addition, the results of some previous experimental investigations are compared with those obtained from the present theoretical study.

NOTATION

A = aspect ratio

C_{AS} = propeller "drag" coefficient

C_D = $D/\frac{\rho}{2} S V_e^2$, drag force coefficient in equilibrium condition

C_L = $L/\frac{\rho}{2} S V_e^2$, lift force coefficient in equilibrium condition

C_m = $M/\frac{\rho}{2} S V_e^2 c$, pitching moment coefficient

C_T = $T/\frac{\rho}{2} S V_e^2$, thrust coefficient

c = mean hydrofoil chord

D = drag

d = vertical distance of foil to reference axis of hydrocraft

g = acceleration due to gravity

H = vertical displacement of hydrofoil from equilibrium, positive downward

h = submergence of foil below smooth water surface

i_B = K_B^2/ℓ^2, moment of inertia coefficient

K_B = radius of gyration of hydrocraft about the lateral axis

k = coefficient of accession to inertia

L = lift

ℓ = distance from 1/4-chord point of front foil to 1/4-chord point of rear foil, measured along the reference axis

M = pitching moment

m_u, m_w, etc. = stability derivatives

q = angular velocity in pitch (radians/sec.)

$\hat{q}$ = $q\hat{t}$, nondimensional angular velocity

S = total hydrofoil area

S_F = area of forward foil

S_R = area of rear foil

T = propulsive thrust

t = time, sec.

$\hat{t}$ = $W/g\rho S V_e$, unit of aerodynamic time

u = increment of velocity along the x-axis in disturbed flight

$\hat{u}$ = u/V_e , nondimensional increment of velocity along the x-axis in disturbed flight

V = resultant velocity of hydrocraft in disturbed flight

V_e = velocity of hydrocraft in equilibrium condition

W = all-up weight of hydrocraft

w = increment of velocity along the z-axis in disturbed flight

$\hat{w}$ = w/V_e , nondimensional increment of velocity along the z-axis in disturbed flight

x = axis fixed in hydrocraft in disturbed flight in direction of motion in equilibrium condition

x_{ℓ} = distance along reference axis from C.G. to 1/4-chord point of front foil

x_u, x_w, x_q, etc. = stability derivatives

y = lateral axis fixed in hydrocraft

z = axis normal to x-y plane, directed downward

z_u, z_w, z_q, etc. = stability derivatives

α = hydrofoil incidence measured from zero lift

β = $-H/V_e\hat{t}$, nondimensional height increment

Δ = small increment

ϵ = downwash angle at rear foil

θ = angle of rotation of the hydrocraft in pitch from equilibrium condition

λ = root of the stability quartic (or quintic)

μ = $W/g\rho S\ell$, nondimensional mass factor

ρ = water density

τ = $t/\hat{t}$, nondimensional time

Subscripts

e denoting equilibrium value

F denoting front foil

R denoting rear foil

BASIC ASSUMPTIONS UNDERLYING THE THEORY

This particular study is limited to the motions of a noncavitating tandem hydrofoil configuration in the plane of symmetry. The hydrofoils have equal chords, but the spans may differ, allowing different areas and consequently different load distributions on each foil. The water surface is assumed to be smooth except for disturbances due to the motion of the foils that may alter the surface conditions in the immediate neighborhood of the foils. Time lag terms are utilized to express the influence of flow phenomena occurring at the forward foil upon the rear foil at a later time (due to moving through a distance ℓ).

There are five variables: $\hat{u}$, $\hat{w}$, θ , $\hat{q}$, and β . Thus, in addition to the three usual equations of aerodynamic longitudinal stability, there must be two other equations which will be furnished by purely kinematic considerations.

The hydrodynamic force and moment coefficients depend upon only the nondimensionalized vertical and angular displacements and linear and angular velocities. Terms due to buoyancy are neglected since they are not significant in comparison with the dynamic forces.

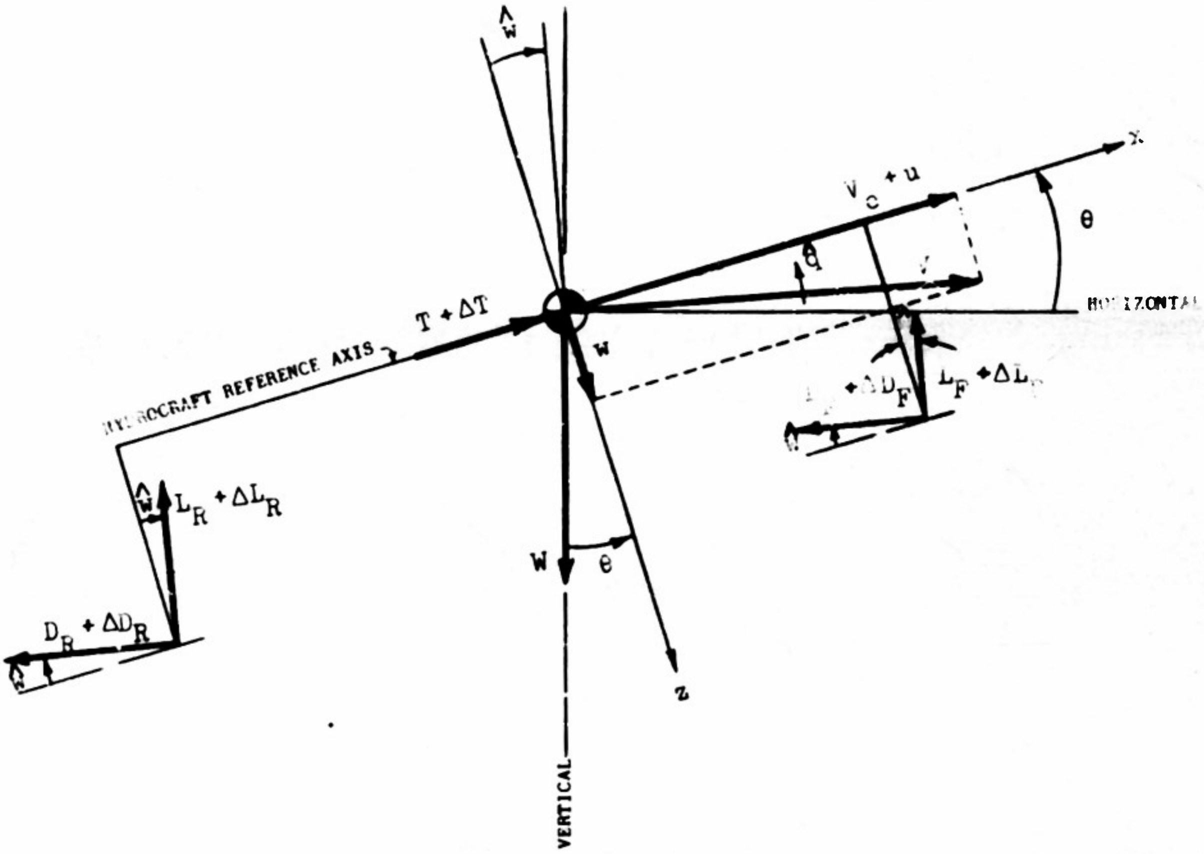
The hydrodynamic lift and drag force coefficients, C_L and C_D, are considered as functions of the incidence angle, α, the speed, V, and also of the depth below the water surface. These coefficients are to be evaluated at the equilibrium position.

The effect of thrust variation with speed and depth can be taken into account, but is considered to be negligible in the present development. The elastic distortion and slipstream effects on the rear foil are to be neglected. The thrust is assumed to act through the center of gravity and along the x-axis.

In addition, all the simplifications applicable to the equations by use of linearization by the method of small disturbances are inherent in the present development.

AXIS AND COORDINATES

The origin is at the center of gravity of the hydrocraft. The x-axis is forward along the water direction in the equilibrium condition and is fixed in the hydrocraft during the disturbed motion. The z-axis is downward in the plane of symmetry of the hydrocraft and perpendicular to the x-axis, while the y-axis is to starboard. The positive direction of the angle of rotation in pitch, θ, is counter-clockwise. The above sign convention, as well as a representation of the forces and velocities associated with the disturbed motion of the hydrocraft, is shown in the following sketch:



SKETCH 1

EQUILIBRIUM CONDITIONS

When the hydrofoil system is in static equilibrium, it is moving forward at a velocity V_e in the x-direction at a definite depth of submergence of the foils and with the reference axis at zero trim. To achieve these conditions, the following requirements must be satisfied:

(a) the sum of the forces in the x- and z-directions must equal zero.

(b) the moment about the center of gravity due to the forces on the foils must equal zero.

These conditions are expressed analytically as

$$\left.\begin{aligned} D_F + D_R &= T \\ L_F + L_R &= W \end{aligned}\right\} \qquad (1)$$

$$M_e = L_F x_{\ell} - L_R(\ell - x_{\ell}) - D_F d_F - D_R d_R = 0 \quad . \qquad (2)$$

EQUATIONS OF MOTION IN DISTURBED FLIGHT

Force Components Along the x-Axis

The forces in the x-direction in disturbed flight are related by the following equation:

$$\frac{W}{g}\left(\frac{du}{dt} + wq\right) = (T + \Delta T) + (L_F + \Delta L_F + L_R + \Delta L_R)\sin\hat{w}$$

$$- (D_F + \Delta D_F + D_R + \Delta D_R)\cos\hat{w} - W\sin\theta \quad . \tag{3}$$

Taking into account the equilibrium conditions (equation (1)) and neglecting small quantities of higher order gives

$$\frac{W}{g}\frac{du}{dt} = \Delta T + (L_F + L_R)\hat{w} - (\Delta D_F + \Delta D_R) - W\theta \quad . \tag{4}$$

Dividing by $\rho S V_e^2$ in order to pass to nondimensional form, and simplifying, yields

$$\frac{d\hat{u}}{d\tau} = \frac{\Delta T}{\rho S V_e^2} + \frac{1}{2} C_L \hat{w} - \frac{\Delta D_F + \Delta D_R}{\rho S V_e^2} - \frac{1}{2} C_L \theta \quad . \tag{5}$$

From consideration of the aerodynamics of propeller thrust (Reference 8), the following equation is obtained:

$$\frac{\Delta T}{\rho S V_e^2} = -C_{AS}\hat{u} \quad \text{with} \quad C_{AS} = -\frac{1}{\rho S V}\frac{\partial T}{\partial V} \quad . \tag{6}$$

However, it will be assumed for the present study that

$$\frac{\Delta T}{\rho S V_e^2} = 0 \quad .$$

The increment of drag, ΔD, is due to the perturbation quantities and is found from the following considerations:

$$\Delta D = \frac{\rho}{2} S(V_e + u + qd)^2\left(C_D + \frac{\partial C_D}{\partial \alpha}\hat{w} + \frac{\partial C_D}{\partial \theta}\theta + \frac{\partial C_D}{\partial H}H + \frac{\partial C_D}{\partial q}q\right) - \frac{\rho}{2} S V_e^2 C_D$$

or

$$\frac{\Delta D}{\rho S V_e^2} = \frac{1}{2}\left(C_D + \frac{\partial C_D}{\partial \alpha}\hat{w} + \frac{\partial C_D}{\partial \theta}\theta + \frac{\partial C_D}{\partial H}H + \frac{\partial C_D}{\partial q}q\right)\left(1 + 2\hat{u} + 2\frac{qd}{V_e}\right) - \frac{1}{2}C_D \ .$$

If it is assumed that the pitching velocity motion causes changes only in the local angle of attack at each foil and does not influence the resultant velocity at the foil, then

$$\frac{\Delta D_F + \Delta D_R}{\rho S V_e^2} = \frac{S_F}{2S}\left[2C_{D_F}\hat{u} + \left(\frac{\partial C_D}{\partial \alpha}\right)_F\hat{w} + \left(\frac{\partial C_D}{\partial \theta}\right)_F\theta + \left(\frac{\partial C_D}{\partial H}\right)_F H + \left(\frac{\partial C_D}{\partial q}\right)_F q\right]$$

$$+ \frac{S_R}{2S}\left[2C_{D_R}\hat{u} + \left(\frac{\partial C_D}{\partial \alpha}\right)_R\hat{w} + \left(\frac{\partial C_D}{\partial \theta}\right)_R\theta + \left(\frac{\partial C_D}{\partial H}\right)_R H + \left(\frac{\partial C_D}{\partial q}\right)_R q\right]$$

$$= C_D\hat{u} + \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R\right]\hat{w} + \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \theta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \theta}\right)_R\right]\theta$$

$$+ \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial H}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial H}\right)_R\right]H + \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial q}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial q}\right)_R\right]q \ . \quad (7)$$

The dependence of drag forces on $\dot{q}$ and $\dot{w}$ are of insignificant magnitude and hence terms expressing any relations of this type will be neglected. The effect of downwash on the rear foil and its influence on the components making up the stability derivatives is discussed below.

The contribution to the drag force due to downward velocity w has the same effect as that due to changing the angle of incidence at the foils by the additional angle $\hat{w} = w/V_e$. Now, taking into account the change due to w when the angle of incidence at the rear foil is expressed as

$$\alpha_R = \alpha_{e_R} + \frac{w}{V_e} - \frac{d\epsilon}{d\alpha}\left(\alpha_{e_F} + \frac{w}{V_e}\right) \ , \quad (8)$$

with ϵ = downwash angle, and considering changes due to w only gives

$$\alpha_R = \frac{w}{V_e} - \frac{d\epsilon}{d\alpha}\left(\frac{w}{V_e}\right) = \frac{w}{V_e}\left(1 - \frac{d\epsilon}{d\alpha}\right) .$$

Thus the component expressing the dependence of the increment in the drag force on w is given now as

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F + \left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R\right]\hat{w} \quad . \tag{9}$$

The derivative $\partial C_D/\partial \theta$ will not be determined directly as such, since, as mentioned previously, the hydrodynamic derivatives are assumed to be functions of angle of attack, speed, and depth of submergence. This derivative will be transformed so that it will be found in terms of dependence on depth. The relations among the derivatives which allow θ terms to be written in terms of change in depth, H, and subsequently the nondimensional height increment, β, are given below:

$$\frac{\partial C_D}{\partial \theta} = \frac{\partial C_D}{\partial H}\frac{\partial H}{\partial \theta} = \frac{\partial C_D}{\partial \beta}\frac{\partial \beta}{\partial \theta}$$

$$\beta = -\frac{H}{V_e \hat{t}} \quad .$$

From Sketch 1, $-H = x\theta$, and by matching the value of H from the definition of β, it is seen that

$$V_e \hat{t}\beta = x\theta \quad , \quad \frac{\partial \beta}{\partial \theta} = \frac{x}{V_e \hat{t}} \quad , \quad \text{and} \quad \frac{\partial H}{\partial \theta} = -x \quad .$$

In order to maintain the proper orientation of the axes and coordinates, $x = x_\ell$ for the forward foil and $x = -(\ell - x_\ell)$ for the rear foil (θ measured positive for a counter-clockwise rotation). The component expressing dependence of drag force on θ is then given as

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F \frac{x_\ell}{V_e \hat{t}} - \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R \frac{\ell - x_\ell}{V_e \hat{t}}\right]\theta \quad . \tag{10}$$

The dependence of the drag force on the vertical displacement H is converted to a nondimensional dependence on β. Thus the component of drag varying with height is given as

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R\right]\beta \quad . \tag{11}$$

Hydrodynamic forces due to a pitching velocity, q, arise from the change of angle of attack due to the resulting curvature of the streamlines. Derivatives with respect to q can be transformed to derivatives with respect to qx/V_e, or in effect, with respect to an angle-of-attack change due to the pitching velocity of a foil at a distance x from the center of gravity:

$$\frac{\partial C_D}{\partial q} q = \frac{\partial C_D}{\partial\left(\frac{qx}{V_e}\right)}\left(\frac{qx}{V_e}\right) = \frac{\partial C_D}{\partial \alpha}\frac{qx}{V_e} \quad .$$

Since the front foil has an upward velocity equal to qx_ℓ, it has an apparent decrease in angle of attack of qx_ℓ/V_e. The rear foil experiences an apparent increase in angle of attack equal to $q(\ell - x_\ell)/V_e$. Thus the changes of angle of attack due to pitching velocity are $-qx_\ell/V_e$ for the forward foil and $q(\ell - x_\ell)/V_e$ for the rear foil.

Because of the time lag for the downwash from the front foil to reach the rear foil, there is a downwash angle lag at the rear foil due to pitching velocity (Reference 3). The flow at the rear foil at time t is influenced by the downwash which is created at the front foil at time $t - \ell/V_e$. This effect is in addition to the local change in angle of attack at the front due to pitching, $-qx_\ell/V_e$. Thus,

$$\epsilon = \frac{d\epsilon}{d\alpha}\left[\alpha_{e_F} - q\left(t - \frac{\ell}{V_e}\right) - \frac{qx_\ell}{V_e}\right] , \tag{12}$$

and at $t = 0$,

$$\epsilon = \frac{d\epsilon}{d\alpha}\left[\alpha_{e_F} + \frac{q(\ell - x_\ell)}{V_e}\right] \quad .$$

The angle-of-attack change at the rear foil due to pitching velocity is

$$\alpha_R = \frac{q(\ell - x_\ell)}{V_e} - \frac{d\epsilon}{d\alpha}\left[\frac{q(\ell - x_\ell)}{V_e}\right]$$

$$\alpha_R = \frac{q(\ell - x_\ell)}{V_e}\left(1 - \frac{d\epsilon}{d\alpha}\right) \quad . \tag{13}$$

By taking into account the local angle change due to pitching, downwash effects, and modifications due to the time lag, the resultant component of the drag force dependent on q is given as

$$\frac{1}{2}\left\{-\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R\left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}\frac{q\ell}{V_e} \quad . \tag{14}$$

After the nondimensional factor $\mu = W/g\rho S\ell$ is introduced, then $q\ell/V_e = (1/\mu)\hat{q}$. Thus the contribution of pitching velocity to the drag forces may now be written in terms of the nondimensional variables as

$$\frac{1}{2\mu}\left\{-\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R\left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{\ell - x_\ell}{\ell}\right)\right]\right\}\hat{q} \quad . \tag{15}$$

Associated with any accelerated or decelerated motion of a body through a fluid is the virtual mass of the entrained fluid. For a body such as a hydrofoil at small angles of attack, the virtual mass due to acceleration in the x-direction is negligible, and consequently no forces due to virtual mass are included in the equation for the x-direction.

By combining the various components of the forces, including suitable modifications due to downwash effects, the resultant equation for forces in the x-direction becomes

$$\frac{d\hat{u}}{d\tau} - x_u\hat{u} - x_w\hat{w} - x_\theta\theta - x_\beta\beta - \frac{x_q}{\mu}\hat{q} = 0 \quad , \tag{16}$$

where

$$
\left.\begin{aligned}
x_u &= -C_D \\
x_w &= \frac{1}{2}\left\{\frac{S_F}{S} C_{L_F} - \frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F + \left(1 - \frac{d\epsilon}{d\alpha}\right)\left[\frac{S_R}{S} C_{L_R} - \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R\right]\right\} \\
x_\theta &= -\frac{1}{2}\left[C_L + \frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F \frac{x_\ell}{V_e \hat{t}} - \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R \frac{\ell - x_\ell}{V_e \hat{t}}\right] \\
x_\beta &= -\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R\right] \\
x_q &= -\frac{1}{2}\left\{-\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\} \, .
\end{aligned}\right\} \quad (17)
$$

Force Components Along the z-Axis

The equation relating the force components along the z-axis may be written as

$$
\frac{W}{g}\left[\frac{dw}{dt} - (V_e + u)q\right] = -(L_F + \Delta L_F + L_R + \Delta L_R)\cos\hat{w} - (D_F + \Delta D_F + D_R + \Delta D_R)\sin\hat{w} + W\cos\theta \; . \quad (18)
$$

Taking into account the equilibrium conditions (equation (1)) and neglecting small quantities of higher order gives

$$
\frac{W}{g}\left(\frac{dw}{dt} - V_e q\right) = -(\Delta L_F + \Delta L_R) - (D_F + D_R)\hat{w} \quad . \quad (19)
$$

Dividing equation (19) by $\rho S V_e^2$ in order to pass to nondimensional form, and simplifying, yields

$$
\frac{d\hat{w}}{d\tau} - \hat{q} = -\frac{\Delta L_F + \Delta L_R}{\rho S V_e^2} - \frac{1}{2} C_D \hat{w} \quad . \quad (20)
$$

By performing the same type of analysis used in deriving equation (7), the following terms are obtained:

$$\frac{\Delta L}{\rho S V_e^2} = \frac{1}{2}\left(C_L + \frac{\partial C_L}{\partial \alpha}\hat{w} + \frac{\partial C_L}{\partial \theta}\theta + \frac{\partial C_L}{\partial H}H + \frac{\partial C_L}{\partial q}q\right)(1 + 2\hat{u}) - \frac{1}{2}C_L$$

$$\frac{\Delta L_F + \Delta L_R}{\rho S V_e^2} = \frac{S_F}{2S}\left[2C_{L_F}\hat{u} + \left(\frac{\partial C_L}{\partial \alpha}\right)_F\hat{w} + \left(\frac{\partial C_L}{\partial \theta}\right)_F\theta + \left(\frac{\partial C_L}{\partial H}\right)_F H + \left(\frac{\partial C_L}{\partial q}\right)_F q\right]$$

$$+ \frac{S_R}{2S}\left[2C_{L_R}\hat{u} + \left(\frac{\partial C_L}{\partial \alpha}\right)_R\hat{w} + \left(\frac{\partial C_L}{\partial \theta}\right)_R\theta + \left(\frac{\partial C_L}{\partial H}\right)_R H + \left(\frac{\partial C_L}{\partial q}\right)_R q\right]$$

$$= C_L\hat{u} + \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R\right]\hat{w}$$

$$+ \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \theta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \theta}\right)_R\right]\theta + \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial H}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial H}\right)_R\right]H$$

$$+ \frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial q}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial q}\right)_R\right]q \quad . \tag{21}$$

The components of the lift forces given in equation (21) do not contain any quantitative measure of the effects of downwash, and thus have to be corrected to take account of this effect. Also, some of the component terms have to be modified so that they can be expressible in terms of nondimensional variables.

Applying the downwash correction to the component dependent on $\hat{w}$ gives an expression for the lift force variation analagous to equation (9):

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F + \left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R\right]\hat{w} \quad . \tag{22}$$

The component of lift which is dependent upon θ is expressed below in terms of the nondimensional derivatives with respect to β and is equal to

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F \frac{x_\ell}{V_e\hat{t}} - \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R \frac{\ell - x_\ell}{V_e\hat{t}}\right]\theta \quad . \tag{23}$$

The component which is dependent upon the vertical displacement H is converted to a nondimensional dependence on β and is expressed as

$$\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R\right]\beta \quad . \tag{24}$$

Variation of the lift with pitching velocity q is due to the different local angle-of-attack changes at the front and rear foils, the downwash effect, and the time lag. By introducing the effects of these factors, the contribution of the lift due to q is given as

$$\frac{1}{2}\left\{-\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}\frac{q\ell}{V_e} \quad . \tag{25}$$

Using the nondimensional factor μ and converting to the nondimensional variable $\hat{q}$ yields

$$\frac{1}{2\mu}\left\{-\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}\hat{q} \quad . \tag{26}$$

There is a substantial dependence of the lift force on the acceleration variables $\dot{w}$ and $\dot{q}$ that yields terms whose magnitudes are sufficient to be included in the z-force analysis. The two accelerations $\dot{w}$ and $\dot{q}$ produce forces that are due to the virtual mass associated with the accelerations. These virtual mass terms are noncirculatory potential flows that do not produce an induced vortex wake. Thus, there will be no effect on the rear foil due to these forces acting at the forward foil.

In addition to its influence on the virtual mass terms, the acceleration $\dot{w}$ also causes forces on the rear foil due to a time lag factor. The lift at the rear foil at time t is influenced by the vortices which were cast off by the front foil at a time $t - \ell/V_e$. Thus, with the angle of attack at the front foil expressed only in terms of its variation with $\dot{w}$,

$$\left.\begin{aligned}\alpha_F &= \alpha_{e_F} + \frac{\dot{w}t}{V_e} \\ &- \frac{d\epsilon}{d\alpha}\left[\alpha_{e_F} + \frac{\dot{w}}{V_e}\left(t - \frac{\ell}{V_e}\right)\right] \quad .\end{aligned}\right\} \tag{27}$$

Then at time $t = 0$, the effect of $\dot{w}$ on the angle of attack at the rear is

$$\alpha_R = - \frac{d\epsilon}{d\alpha}\left[\frac{\dot{w}}{V_e}\left(- \frac{\ell}{V_e}\right)\right] = \frac{d\epsilon}{d\alpha}\frac{\dot{w}\ell}{{V_e}^2} \quad . \tag{28}$$

The lift force at the rear due to this time lag effect is given as

$$L = \left(\frac{\partial C_L}{\partial \alpha}\right)_R \alpha_R \frac{\rho}{2} S_R {V_e}^2 = \frac{\rho}{2} S_R {V_e}^2 \left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha}\frac{\dot{w}\ell}{{V_e}^2} \quad . \tag{29}$$

L is converted into nondimensional form by dividing equation (29) by $\rho S {V_e}^2$:

$$\frac{L}{\rho S {V_e}^2} = \frac{S_R}{2S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha}\frac{\ell}{{V_e}^2}\dot{w} \quad . \tag{30}$$

Now $\dot{w}$ can be changed into the nondimensional form of $d\hat{w}/d\tau$ by use of the definitions of $\hat{t}$ and μ , yielding the relation

$$\dot{w} = \frac{dw}{dt} = \frac{1}{\mu}\frac{{V_e}^2}{\ell}\frac{d\hat{w}}{d\tau} \quad . \tag{31}$$

Then

$$\frac{L}{\rho S {V_e}^2} = \frac{1}{2\mu}\frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha}\frac{d\hat{w}}{d\tau} \quad . \tag{32}$$

From this expression, a stability derivative can be defined:

$$z_{\dot{w}} = \frac{1}{2}\frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha} \quad . \tag{33}$$

Forces due to virtual mass will now be considered. Because of the similarity between an elliptic disc and the planform of an actual hydrofoil, the theoretical calculations made for an elliptic disc can be used to determine the virtual mass of the hydrofoil. This apparent mass is equal to the mass of an ellipsoid of revolution with the span as axis and the chord as diameter, and of the same density as the fluid. The volume is $(4/3)\pi a^2 b$, where b is the half axis and a the largest radius of the ellipsoid. The inertia factors for the correct aspect ratio are obtained from Reference 9. Thus, the apparent volume K is equal to $k(4/3)\pi a^2 b$, where k is the coefficient of accession to inertia.

It is possible to obtain a foil having the same area and same apparent volume as the elliptic disc, but it will have a somewhat different aspect ratio. The apparent volume of the foil is then $K = k(\pi/4)S_{F,R}c$. Thus, with $c\bar{b}_{F,R} = S_{F,R} = \pi ab$ (where $\bar{b}$ = span of hydrofoil), $c = (16/3\pi)a$. Since the inertia factor k has to be the same in both cases, it depends upon the aspect ratio of the elliptic disc which is equal to $4b/\pi a$. The aspect ratio of the elliptic disc is found to be 1.168 times the aspect ratio of the hydrofoil. Since the inertia factor k does not vary much for aspect ratios greater than 6. and since the factor expressing the ratio of the aspect ratios of the disc and foil is not much larger than 1.0, the values for k given in Reference 9 for the disc aspect ratio may be used without any correction.

The apparent mass is then $\rho k(\pi/4)S_{F,R}c$ and the forces are the mass times the acceleration. The force due to the vertical acceleration $\dot{w}$ is a lift force $-Z'$, where

$$-Z'_{F,R} = \rho k \frac{\pi}{4} S_{F,R} c\dot{w} \quad . \tag{34}$$

The total lift is then

$$-Z' = \rho k \frac{\pi}{4} c(S_F + S_R)\dot{w} = \rho k \frac{\pi}{4} Sc\dot{w} = \rho k \frac{\pi}{4} Sc \frac{1}{\mu}\frac{V_e^2}{\ell}\frac{d\hat{w}}{d\tau} \quad . \tag{35}$$

Dividing equation (35) by $\rho S V_e^2$ to make it nondimensional yields a stability derivative

$$z'_{\dot{w}} = -\frac{k\pi c}{4\ell} \quad . \tag{36}$$

The force due to the virtual mass which is contributed by $\dot{q}$ is considered as a lift force, and is expressed as

$$L'_{F,R} = \rho \frac{k\pi}{4} S_{F,R} c \dot{q} x_{F,R} \quad . \tag{37}$$

This force acts at the center of area of the surface, and since longitudinal reference dimensions are measured from the quarter-chord point, a term must be added to give the proper distances to where the force acts. Therefore,

$$\begin{aligned} L' &= -\rho \frac{k\pi}{4} S_F c\dot{q}\left(x_\ell - \frac{c}{4}\right) + \rho \frac{k\pi}{4} S_R c\dot{q}\left(\ell - x_\ell + \frac{c}{4}\right) \\ &= \rho \frac{k\pi}{4} S c \left[-\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right) + \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)\right]\dot{q} \quad . \end{aligned} \tag{38}$$

The angular acceleration $\dot{q}$ may now be expressed in terms of nondimensional variables by the relation

$$\frac{d\hat{q}}{d\tau} = \frac{\ell^2}{V_e^2}\mu^2 \dot{q} \tag{39}$$

Thus,

$$L' = \rho \frac{k\pi S V_e^2 c}{4\mu^2\ell^2}\left[-\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right) + \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)\right]\frac{d\hat{q}}{d\tau}$$

$$\frac{L'}{\rho S V_e^2} = \frac{1}{\mu^2}\frac{k\pi c}{4\ell}\left(-\frac{S_F}{S}\frac{x_\ell - \frac{c}{4}}{\ell} + \frac{S_R}{S}\frac{\ell - x_\ell + \frac{c}{4}}{\ell}\right)\frac{d\hat{q}}{d\tau} \quad . \tag{40}$$

Another stability derivative is now defined as

$$z'_{\dot{q}} = \frac{k\pi c}{4\ell}\left(\frac{S_F}{S}\,\frac{x_\ell - \frac{c}{4}}{\ell} - \frac{S_R}{S}\,\frac{\ell - x_\ell + \frac{c}{4}}{\ell}\right) \quad . \tag{41}$$

The various components of the z-forces will now be combined, with the virtual mass contributions, time lag factors, and downwash effects included, yielding the final relation

$$\left(1 + \frac{z_{\dot{w}}}{\mu} - \frac{z'_{\dot{w}}}{\mu}\right)\frac{d\hat{w}}{d\tau} - z_u\hat{u} - z_w\hat{w} - z_\theta\theta - z_\beta\beta$$

$$- \left(1 + \frac{z_q}{\mu}\right)\hat{q} - \frac{z'_{\dot{q}}}{\mu^2}\,\frac{d\hat{q}}{d\tau} = 0 \quad , \tag{42}$$

where

$$\left.\begin{aligned}
z_{\dot{w}} &= \frac{1}{2}\,\frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha} \\
z'_{\dot{w}} &= -\frac{k\pi c}{4\ell} \\
z_u &= -C_L \\
z_w &= -\frac{1}{2}\left\{\frac{S_F}{S}\,C_{D_F} + \frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F + \left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_R}{S}\left[C_{D_R} + \left(\frac{\partial C_L}{\partial \alpha}\right)_R\right]\right\} \\
z_\theta &= -\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F \frac{x_\ell}{V_e\hat{t}} - \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R \frac{\ell - x_\ell}{V_e\hat{t}}\right] \\
z_\beta &= -\frac{1}{2}\left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R\right] \\
z_q &= -\frac{1}{2}\left\{-\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \left[\left(\frac{\ell - x_\ell}{\ell}\right)\left(1 - \frac{d\epsilon}{d\alpha}\right)\right]\right\} \\
z'_{\dot{q}} &= \frac{k\pi c}{4\ell}\left(\frac{S_F}{S}\,\frac{x_\ell - \frac{c}{4}}{\ell} - \frac{S_R}{S}\,\frac{\ell - x_\ell + \frac{c}{4}}{\ell}\right) \; .
\end{aligned}\right\} \tag{43}$$

Moments About the y-Axis

The moment equation for pitching about the y-axis is derived below in nondimensional form. The resultant moment is expressed as the sum of equilibrium and hydrodynamic contributions:

$$M = M_e + M_u u + M_w w + M_\theta \theta + M_\beta \beta + M_q q + M_{\dot{w}} \dot{w} \quad . \tag{44}$$

Equating this expression to the product of moment of inertia and angular acceleration, expressed as $(W/g)K_B^{\,2}(dq/dt)$, and subtracting the equilibrium terms which are zero, gives the following equation:

$$\frac{W}{g} K_B^{\,2} \frac{dq}{dt} - M_u u - M_w w - M_\theta \theta - M_\beta \beta - M_q q - M_{\dot{w}} \dot{w} = 0 \quad . \tag{45}$$

This equation is then multiplied by $\mu / \rho S \mathcal{l} V_e^{\,2} i_B$, where $i_B = K_B^{\,2}/\mathcal{l}^2$, K_B being the radius of gyration of the hydrocraft about the y-axis:

$$\frac{\mu W \mathcal{l}^2}{g \rho S \mathcal{l} V_e^{\,2}} \frac{dq}{dt} - \frac{\mu M_u}{\rho S \mathcal{l} V_e^{\,2} i_B} u - \frac{\mu M_w}{\rho S \mathcal{l} V_e^{\,2} i_B} w - \frac{\mu M_\theta}{\rho S \mathcal{l} V_e^{\,2} i_B} \theta$$

$$- \frac{\mu M_\beta}{\rho S \mathcal{l} V_e^{\,2} i_B} \beta - \frac{\mu M_q}{\rho S \mathcal{l} V_e^{\,2} i_B} q - \frac{\mu M_{\dot{w}}}{\rho S \mathcal{l} V_e^{\,2} i_B} \dot{w} = 0 \quad . \tag{46}$$

Now with $\hat{q} = q\hat{t}$ and $t = \hat{t}\tau$, then

$$\frac{dq}{dt} = \frac{\dfrac{d\hat{q}}{\hat{t}}}{\hat{t}\, d\tau} = \frac{1}{\hat{t}^2} \frac{d\hat{q}}{d\tau} \quad ,$$

and with $\hat{t}$ and μ defined as

$$\hat{t} = \frac{W}{g \rho S V_e}$$

$$\mu = \frac{W}{g \rho S \mathcal{l}} \quad ,$$

the following rel[illegible]ion results:

$$\frac{d\hat{q}}{d\tau} - \frac{\mu}{i_B}\frac{M_u}{\rho S \ell V_e}\hat{u} - \frac{\mu}{i_B}\frac{M_w}{\rho S \ell V_e}\hat{w} - \frac{\mu}{i_B}\frac{M_\theta}{\rho S \ell V_e^2}\theta$$

$$- \frac{\mu}{i_B}\frac{M_\beta}{\rho S \ell V_e^2}\beta - \frac{M_q}{\rho S \ell^2 V_e i_B}\hat{q} - [illegible] = 0 \quad . \tag{47}$$

Equation (47) may now be written as

$$\frac{d\hat{q}}{d\tau} - \frac{\mu m_u}{i_B}\hat{u} - \frac{\mu m_w}{i_B}\hat{w} - \frac{\mu m_\theta}{i_B}\theta - \frac{\mu m_\beta}{i_B}\beta - \frac{m_q}{i_B}\hat{q} - \frac{\mu m_{\dot{w}}}{i_B}\frac{d\hat{w}}{d\tau} = 0 \quad , \tag{48}$$

where m_u , m_w , m_θ , m_β , m_q , and $m_{\dot{w}}$ are nondimensional partial derivatives (stability derivatives).

The terms m_w , m_θ , m_β , and m_q may be easily found in terms of the moment coefficient C_m . The moment is put into nondimensional form by dividing by $\rho S V_e^2 \ell$, giving $M/\rho S V_e^2 \ell$. However, $C_m = M/(\rho/2) S V_e^2 c$ and the two forms are related by $M/\rho S V_e^2 \ell = (c/2\ell) C_m$. Thus,

$$m_w = \frac{c}{2\ell}\frac{\partial C_m}{\partial \alpha}$$

$$m_\theta = \frac{c}{2\ell}\frac{\partial C_m}{\partial \theta}$$

$$m_\beta = \frac{c}{2\ell}\frac{\partial C_m}{\partial \beta}$$

$$m_q = \frac{c}{2\ell}\frac{\partial C_m}{\partial\left(\frac{q\ell}{V_e}\right)}$$

In later sections of the report, the term m_u will be found and shown to be zero due to the equilibrium conditions, and the term $m_{\dot{w}}$ due to time lag will also be determined. In addition, moments due to the lift forces generated by virtual masses will be found and included in the final moment equation.

KINEMATIC CONDITI[illegible]

The two kinematic [illegible] necess[illegible] in order to have five equations [illegible]. The first condition expresses the equality [illegible] the time rate of change of pitch angle:

$$\hat{q} - \frac{d\theta}{d\tau} = 0 \quad . \tag{49}$$

It is now necessary to express the relation between the vertical velocity component dH/dt and its x and z components. It is seen from Sketch 1, page 7, that

$$-\frac{dH}{dt} = (V_e + u)\sin\theta - w\cos\theta \quad , \tag{50}$$

or, if only small quantities of the first order are retained, and non-dimensional variables are introduced, then

$$\frac{d\beta}{d\tau} + \hat{w} - \theta = 0 \quad . \tag{51}$$

LONGITUDINAL STABILITY DERIVATIVES

In this section, the longitudinal stability derivatives are discussed in detail. The various hydrodynamic derivatives appearing in the expressions for the stability derivatives will be evaluated in a later section of the report. Experimental values of these hydrodynamic derivatives are obtainable from towing basin tests.

m_u , Change in Pitching Moment with Perturbation Velocity

The moments will be broken down into drag and lift forces multiplied by their respective moment arms.

The drag and lift forces due to the longitudinal perturbation velocity u are obtained from equations (7) and (21), respectively:

$$\left.\begin{aligned} &\text{Drag Forces:} \quad \rho S V_e^{2} C_D \hat{u} \\ &\text{Lift Forces:} \quad \rho S V_e^{2} C_L \hat{u} \end{aligned}\right\} \qquad (52)$$

These forces may be written in a slightly different form in order to separate the actions on the forward and rear foils:

$$\left.\begin{aligned} &\text{Drag Forces:} \quad \rho S V_e^{2} \left(\frac{S_F}{S} C_{D_F} + \frac{S_R}{S} C_{D_R} \right) \hat{u} \\ &\text{Lift Forces:} \quad \rho S V_e^{2} \left(\frac{S_F}{S} C_{L_F} + \frac{S_R}{S} C_{L_R} \right) \hat{u} \end{aligned}\right\} \qquad (53)$$

The moments about the center of gravity are then given as

$$M = \rho S V_e^{2} \left[\frac{S_F}{S} C_{L_F} x_{\ell} - \frac{S_R}{S} C_{L_R} (\ell - x_{\ell}) - \frac{S_F}{S} C_{D_F} d_F - \frac{S_R}{S} C_{D_R} d_R \right] \hat{u}$$

$$= 2 \left[L_F x_{\ell} - L_R (\ell - x_{\ell}) - D_F d_F - D_R d_R \right] \hat{u} \qquad . \qquad (54)$$

From the equilibrium conditions (equation (2)), $M = 0$; hence, $M_u = 0$ and $m_u = 0$.

m_w . Change in Pitching Moment with Vertical Velocity

The drag and lift forces due to vertical velocity are found from equations (9), (17), (22), and (43):

$$\left.\begin{aligned}
&\text{Drag Forces: } -\frac{\rho}{2} SV_e^2 \left\{ \left[\frac{S_F}{S} C_{L_F} - \frac{S_F}{S} \left(\frac{\partial C_D}{\partial \alpha} \right)_F \right] \right. \\
&\qquad \left. + \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[\frac{S_R}{S} C_{L_R} - \frac{S_R}{S} \left(\frac{\partial C_D}{\partial \alpha} \right)_R \right] \right\} \frac{w}{V_e} \\
&\text{Lift Forces: } \frac{\rho}{2} SV_e^2 \left\{ \left[\frac{S_F}{S} C_{D_F} + \frac{S_F}{S} \left(\frac{\partial C_L}{\partial \alpha} \right)_F \right] \right. \\
&\qquad \left. + \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[\frac{S_R}{S} C_{D_R} + \frac{S_R}{S} \left(\frac{\partial C_L}{\partial \alpha} \right)_R \right] \right\} \frac{w}{V_e} \quad .
\end{aligned}\right\} \qquad (55)$$

The moments about the center of gravity are then given by

$$\begin{aligned}
M = \frac{\rho}{2} SV_e^2 &\left\{ \frac{S_F}{S} \left[C_{D_F} + \left(\frac{\partial C_L}{\partial \alpha} \right)_F \right] x_\ell - \frac{S_R}{S} \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[C_{D_R} + \left(\frac{\partial C_L}{\partial \alpha} \right)_R \right] (\ell - x_\ell) \right\} \frac{w}{V_e} \\
+ \frac{\rho}{2} SV_e^2 &\left\{ \frac{S_F}{S} \left[C_{L_F} - \left(\frac{\partial C_D}{\partial \alpha} \right)_F \right] d_F + \frac{S_R}{S} \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[C_{L_R} - \left(\frac{\partial C_D}{\partial \alpha} \right)_R \right] d_R \right\} \frac{w}{V_e} \quad , \qquad (56)
\end{aligned}$$

and subsequently,

$$\begin{aligned}
\frac{\partial C_m}{\partial \alpha} = \frac{1}{c} &\left\{ \overbrace{\frac{S_F}{S} \left[C_{D_F} + \left(\frac{\partial C_L}{\partial \alpha} \right)_F \right] x_\ell - \frac{S_R}{S} \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[C_{D_R} + \left(\frac{\partial C_L}{\partial \alpha} \right)_R \right] (\ell - x_\ell)}^{1} \right\} \\
+ \frac{1}{c} &\left\{ \overbrace{\frac{S_F}{S} \left[C_{L_F} - \left(\frac{\partial C_D}{\partial \alpha} \right)_F \right] d_F + \frac{S_R}{S} \left(1 - \frac{d\epsilon}{d\alpha} \right) \left[C_{L_R} - \left(\frac{\partial C_D}{\partial \alpha} \right)_R \right] d_R}^{2} \right\} \quad . \qquad (57)
\end{aligned}$$

But

$$M_w = \frac{\rho}{2} SV_e \left\{ (1) \right\} + \frac{\rho}{2} SV_e \left\{ (2) \right\}$$

and

$$m_w = \frac{M_w}{\rho S \ell V_e}$$

by definition. Hence,

$$m_w = \frac{1}{2\ell} \left\{ (1) \right\} + \frac{1}{2\ell} \left\{ (2) \right\} \tag{58}$$

and

$$m_w = \frac{c}{2\ell} \frac{\partial C_m}{\partial \alpha} \quad .$$

m_θ , Change in Pitching Moment with Pitch Attitude

A rotation of the reference axis in the plane of symmetry causes changes in the depths of submergence of the hydrofoils, thus altering the drag and lift forces on each foil. These variations can then be converted into a variation in the pitching moment about the center of gravity.

The drag and lift forces due to submersion changes caused by rotation can be obtained from equations (10), (17), and (23):

$$\left.\begin{aligned} &\text{Drag Forces: } \frac{\rho}{2} SV_e^2 \left[C_L + \frac{S_F}{S} \left(\frac{\partial C_D}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} - \frac{S_R}{S} \left(\frac{\partial C_D}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] \theta \\ &\text{Lift Forces: } \frac{\rho}{2} SV_e^2 \left[\frac{S_F}{S} \left(\frac{\partial C_L}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} - \frac{S_R}{S} \left(\frac{\partial C_L}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] \theta \quad . \end{aligned}\right\} \tag{59}$$

The C_L term in the drag force component acts through the center of gravity and thus does not affect the moments. The moments about the center of gravity are then

$$M = \frac{\rho}{2} S V_e^{\,2} \left\{ \frac{S_F}{S} \left[\left(\frac{\partial C_L}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} \right] x_\ell + \frac{S_R}{S} \left[\left(\frac{\partial C_L}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] (\ell - x_\ell) \right\} \theta$$

$$- \frac{\rho}{2} S V_e^{\,2} \left\{ \frac{S_F}{S} \left[\left(\frac{\partial C_D}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} \right] d_F - \frac{S_R}{S} \left[\left(\frac{\partial C_D}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] d_R \right\} \theta \qquad (60)$$

$$C_m = \frac{1}{c} \overbrace{\left\{ \frac{S_F}{S} \left[\left(\frac{\partial C_L}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} \right] x_\ell + \frac{S_R}{S} \left[\left(\frac{\partial C_L}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] (\ell - x_\ell) \right\}}^{③} \theta$$

$$- \frac{1}{c} \overbrace{\left\{ \frac{S_F}{S} \left[\left(\frac{\partial C_D}{\partial \beta} \right)_F \frac{x_\ell}{V_e \hat{t}} \right] d_F - \frac{S_R}{S} \left[\left(\frac{\partial C_D}{\partial \beta} \right)_R \frac{\ell - x_\ell}{V_e \hat{t}} \right] d_R \right\}}^{④} \theta \qquad (61)$$

$$\frac{\partial C_m}{\partial \theta} = \frac{1}{c}\{③\} - \frac{1}{c}\{④\} \; .$$

But

$$M_\theta = \frac{\rho}{2} S V_e^{\,2}\{③\} - \frac{\rho}{2} S V_e^{\,2}\{④\}$$

and

$$m_\theta = \frac{M_\theta}{\rho S \ell V_e^{\,2}} \; .$$

Thus,

$$m_\theta = \frac{1}{2\ell}\{③\} - \frac{1}{2\ell}\{④\} \qquad (62)$$

and

$$m_\theta = \frac{c}{2\ell} \frac{\partial C_m}{\partial \theta} \; .$$

m_β , Change in Pitching Moment with Vertical Displacement of the Center of Gravity

Changes in the drag and lift forces with vertical displacement produce moment changes about the center of gravity.

The drag and lift forces due to vertical displacement can be obtained from equations (11) and (24):

$$\left.\begin{aligned} &\text{Drag Forces: } \frac{\rho}{2} S V_e^2 \left[\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R\right]\beta \\ &\text{Lift Forces: } \frac{\rho}{2} S V_e^2 \left[\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R\right]\beta \quad . \end{aligned}\right\} \tag{63}$$

The moments about the center of gravity are then

$$M = \frac{\rho}{2} S V_e^2 \left\{\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F x_\ell - \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R (\ell - x_\ell)\right\}\beta$$

$$- \frac{\rho}{2} S V_e^2 \left\{\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F d_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R d_R\right\}\beta \tag{64}$$

$$C_m = \frac{1}{c}\overbrace{\left\{\frac{S_F}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_F x_\ell - \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \beta}\right)_R (\ell - x_\ell)\right\}}^{5}\beta$$

$$- \frac{1}{c}\overbrace{\left\{\frac{S_F}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_F d_F + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \beta}\right)_R d_R\right\}}^{6}\beta \tag{65}$$

$$\frac{\partial C_m}{\partial \beta} = \frac{1}{c}\{5\} - \frac{1}{c}\{6\} .$$

But

$$M_\beta = \frac{\rho}{2} S V_e^2 \{5\} - \frac{\rho}{2} S V_e^2 \{6\}$$

and

$$m_\beta = \frac{M_\beta}{\rho S \ell V_e^2}$$

by definition. Thus,

$$m_\beta = \frac{1}{2\ell}\left\{⑤\right\} - \frac{1}{2\ell}\left\{⑥\right\} \qquad (66)$$

and

$$m_\beta = \frac{c}{2\ell}\frac{\partial C_m}{\partial \beta} \quad .$$

m_q , Change in Pitching Moment with Pitching Velocity

The pitching velocity causes changes in the angle of attack at each hydrofoil which in turn bring about changes in the drag and lift forces. These force changes then contribute to the pitching moment about the center of gravity.

From equations (14) and (25), the drag and lift forces due to pitching velocity are obtained:

$$\left.\begin{aligned} &\text{Drag Forces} = \frac{\rho}{2} S V_e^2 \left\{ - \frac{S_F}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_D}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}\frac{q\ell}{V_e} \\ &\text{Lift Forces} = \frac{\rho}{2} S V_e^2 \left\{ - \frac{S_F}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell} + \frac{S_R}{S}\left(\frac{\partial C_L}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}\frac{q\ell}{V_e} \,. \end{aligned}\right\} \qquad (67)$$

Moments about the center of gravity are then

$$\begin{aligned} M = {} & \frac{\rho}{2} S V_e^2 \left\{ - \frac{S_F}{S}\left[\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell}\right] x_\ell - \frac{S_R}{S}\left\{\left(\frac{\partial C_L}{\partial \alpha}\right)_R \left[\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{\ell - x_\ell}{\ell}\right]\right\}(\ell - x_\ell)\right\}\frac{q\ell}{V_e} \\ & - \frac{\rho}{2} S V_e^2 \left\{ - \frac{S_F}{S}\left[\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell}\right] d_F + \frac{S_R}{S}\left\{\left(\frac{\partial C_D}{\partial \alpha}\right)_R \left[\frac{\ell - x_\ell}{\ell}\left(1 - \frac{d\epsilon}{d\alpha}\right)\right]\right\} d_R\right\}\frac{q\ell}{V_e} \qquad (68) \end{aligned}$$

$$C_m = \frac{1}{c}\left\{\overbrace{-\frac{S_F}{S}\left[\left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{x_\ell}{\ell}\right]x_\ell - \frac{S_R}{S}\left\{\left(\frac{\partial C_L}{\partial \alpha}\right)_R\left[\frac{\ell - x_\ell}{\ell}\left(1 - \frac{d\epsilon}{d\alpha}\right)\right]\right\}(\ell - x_\ell)}^{⑦}\right\}\frac{q\ell}{V_e}$$

$$- \frac{1}{c}\left\{\overbrace{-\frac{S_F}{S}\left[\left(\frac{\partial C_D}{\partial \alpha}\right)_F \frac{x_\ell}{\ell}\right]d_F + \frac{S_R}{S}\left\{\left(\frac{\partial C_D}{\partial \alpha}\right)_R\left[\frac{\ell - x_\ell}{\ell}\left(1 - \frac{d\epsilon}{d\alpha}\right)\right]\right\}d_R}^{⑧}\right\}\frac{q\ell}{V_e} \qquad (69)$$

$$\frac{\partial C_m}{\partial\left(\frac{q\ell}{V_e}\right)} = \frac{1}{c}\left\{⑦\right\} - \frac{1}{c}\left\{⑧\right\} .$$

But

$$M_q = \frac{\rho}{2} S V_e \ell \left\{⑦\right\} - \frac{\rho}{2} S V_e \ell \left\{⑧\right\}$$

and

$$m_q = \frac{M_q}{\rho S \ell^2 V_e}$$

by definition. Thus,

$$m_q = \frac{1}{2\ell}\left\{⑦\right\} - \frac{1}{2\ell}\left\{⑧\right\} \qquad (70)$$

and

$$m_q = \frac{c}{2\ell}\,\frac{\partial C_m}{\partial\left(\frac{q\ell}{V_e}\right)} .$$

$m_{\dot{w}}$, Change in Pitching Moment with Vertical Acceleration

The importance of the term $m_{\dot{w}}$ arises from the fact that the rear hydrofoil at time t is influenced by the vortices which were cast aft by the first hydrofoil at time $t - \ell/V_e$. Thus, the expression for lift on the rear foil given in equation (29) is obtained:

$$L = \frac{\rho}{2} S_R V_e^2 \left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha} \frac{\dot{w}\ell}{V_e^2} \quad .$$

The moment produced by the rear foil is given as

$$\left.\begin{aligned} M &= -\frac{\rho}{2} S_R V_e^2 \left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha} \frac{\dot{w}\ell}{V_e^2} (\ell - x_\ell) \\ M_{\dot{w}} &= -\frac{\rho}{2} S_R \ell \left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha} (\ell - x_\ell) \quad . \end{aligned}\right\} \quad (71)$$

By definition, $\mu m_{\dot{w}} = M_{\dot{w}}/\rho S \ell^2$; thus,

$$m_{\dot{w}} = -\frac{1}{2\mu} \left[\frac{S_R}{S} \left(\frac{\partial C_L}{\partial \alpha}\right)_R \frac{d\epsilon}{d\alpha}\right] \frac{\ell - x_\ell}{\ell} \quad . \quad (72)$$

<u>Pitching Moments Due to Virtual Mass</u>

The accelerations $\dot{w}$ and $\dot{q}$ produce forces that are due to the virtual masses associated with the accelerations. Since these forces are of the nature of lift forces, the moments due to such forces may be determined.

The lift force due to the virtual mass associated with the acceleration $\dot{w}$ is given by equation (35):

$$\text{Lift} = -Z' = \rho \frac{k\pi c}{4} (S_F + S_R)\dot{w} = \rho \frac{k\pi c}{4} S\dot{w} \quad .$$

The moment derived from this force is found by use of a moment arm to the center of area of the foil surface and hence is corrected by referring distances to the quarter-chord point as in the development of equation (38). Moments about the center of gravity are then

$$M' = \rho \frac{k\pi c}{4} \left[S_F \left(x_\ell - \frac{c}{4}\right) - S_R \left(\ell - x_\ell + \frac{c}{4}\right)\right] \dot{w} \quad (73)$$

$$M'_{\dot{w}} = \rho \frac{k\pi c}{4} S \left[\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right) - \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)\right]$$

$$\mu m'_{\dot{w}} = \frac{M'_{\dot{w}}}{\rho S \ell^2} \quad .$$

Thus,

$$m'_{\dot{w}} = \frac{k\pi c}{4\mu\ell}\left(\frac{S_F}{S}\,\frac{x_\ell - \frac{c}{4}}{\ell} - \frac{S_R}{S}\,\frac{\ell - x_\ell + \frac{c}{4}}{\ell}\right) \quad . \tag{74}$$

The lift due to the virtual mass associated with the angular acceleration $\dot{q}$ is given by equation (38):

$$\text{Lift} = L' = \rho \frac{k\pi c}{4} S \left[-\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right) + \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)\right]\dot{q} \quad .$$

Moments about the center of gravity are then

$$\left.\begin{aligned} M' &= \rho \frac{k\pi c}{4} S \left[-\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right)^2 - \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)^2\right]\dot{q} \\ M'_{\dot{q}} &= -\rho \frac{k\pi c}{4} S \left[\frac{S_F}{S}\left(x_\ell - \frac{c}{4}\right)^2 + \frac{S_R}{S}\left(\ell - x_\ell + \frac{c}{4}\right)^2\right] \quad . \end{aligned}\right\} \tag{75}$$

It can be shown that $m'_{\dot{q}} = M'_{\dot{q}}/\rho S \ell^3$, which then goes into the moment equation (48) as $-(m'_{\dot{q}}/\mu i_B)(d\hat{q}/d\tau)$. Thus,

$$m'_{\dot{q}} = -\frac{k\pi c}{4\ell}\left[\frac{S_F}{S}\,\frac{\left(x_\ell - \frac{c}{4}\right)^2}{\ell^2} + \frac{S_R}{S}\,\frac{\left(\ell - x_\ell + \frac{c}{4}\right)^2}{\ell^2}\right] \quad . \tag{76}$$

The final moment equation including the effects of time lag and virtual mass contributions is then

$$\left(1 - \frac{m'_{\dot{q}}}{\mu i_B}\right)\frac{d\hat{q}}{d\tau} - \frac{\mu m_w}{i_B}\hat{w} - \frac{\mu m_\theta}{i_B}\theta - \frac{\mu m_\beta}{i_B}\beta - \frac{m_q}{i_B}\hat{q} - \frac{\mu(m_{\dot{w}} + m'_{\dot{w}})}{i_o}\frac{d\hat{w}}{d\tau} = 0 \ , \tag{77}$$

where the coefficients are obtained from equations (58), (62), (66), (70), (72), (74), and (76).

HYDRODYNAMIC DERIVATIVES

The theoretical values of the hydrodynamic lift and drag force coefficients C_L and C_D, as well as their derivatives, are given in this section. These coefficients are functions of incidence angle α, speed V, and depth h below the smooth water surface.

A hydrofoil moving near a free surface will suffer a decrease in lift as compared to one at infinite depth, due to the reduced virtual mass associated with the foil and the flow modification due to wave formation. The theoretical expression for the lift coefficient of a hydrofoil has been determined by different investigators (e.g., References 5 and 6). Analysis and comparison of the lift theories indicate that the one due to Keldysch and Lavrentiev (Reference 5) leads to results which are more consistent with experimental data.

The Keldysch-Lavrentiev theory gives the following expression for the lift coefficient of a two-dimensional flat plate which approximates the actual thin airfoil:

$$C_{\ell} = 2\pi\alpha_g(P_1 - \alpha_g P_2) \quad , \tag{78}$$

where α_g is the geometric setting relative to the smooth water surface (two-dimensional angle of attack), and P_1 and P_2 are dimensionless expressions dependent upon depth and Froude number, V_0^2/gc. For the small angles usually experienced in the motions considered in this report, the two-dimensional lift coefficient may be approximated by

$$C_{\ell} = 2\pi\alpha_g P_1 \quad . \tag{79}$$

For an actual three-dimensional foil, it has been determined in Reference 7 that

$$\frac{\partial C_L}{\partial \alpha} = 2\pi P_1 \frac{\pi A + 8h/c}{\pi A + (8h/c) + 4\pi P_1} \quad , \tag{80}$$

and hence,

$$C_L = 2\pi\alpha P_1 \frac{\pi A + 8h/c}{\pi A + (8h/c) + 4\pi P_1} \quad , \tag{81}$$

where α in this case is the actual angle of attack of the foil relative to the zero lift angle. This expression for lift coefficient can be used to determine the derivative of lift coefficient with respect to depth, $\partial C_L/\partial h$, and thus can be related to the components making up the longitudinal stability derivatives.

The value of $\partial C_L/\partial h$ is given as

$$\frac{\partial C_L}{\partial h} = 2\pi\alpha \left\{ \frac{\pi A + 8h/c}{\pi A + (8h/c) + 4\pi P_1} \frac{\partial P_1}{\partial h} + P_1 \frac{(32\pi P_1/c) - 4\pi \frac{\partial P_1}{\partial h} (\pi A + 8h/c)}{\left[\pi A + (8h/c) + 4\pi P_1\right]^2} \right\} . \tag{82}$$

Since the coefficients are to be evaluated at the equilibrium position and since H is defined as the deviation from equilibrium depth, the following relation exists:

$$\left(\frac{\partial C_L}{\partial h}\right)_e = \left(\frac{\partial C_L}{\partial H}\right)_{H=0} = -\frac{1}{V_e \hat{t}} \left(\frac{\partial C_L}{\partial \beta}\right)_{H=0} \quad , \tag{83}$$

where the subscript "e" denotes equilibrium condition.

Motion of the hydrofoil below the free surface results in wavemaking with which there is associated a wave drag. This effect is due to the proximity of the bound vortices to the surface. In a finite-depth channel, there is a critical speed above which there is no wave drag, which is defined as

$$V_c = \sqrt{gh_B} \quad , \tag{84}$$

where h_B is the tank water depth. Since the Keldysch-Lavrentiev theory is derived on the basis of infinite depth, there is no mathematical mechanism that will make this wave drag component disappear; therefore, it is necessary to determine whether the wave drag is to be included in an analysis of stability tests in a finite-depth channel.

The theoretical expression for the wave drag coefficient is given in Reference 5 as

$$C_{D_W} = \frac{\pi^2 \alpha_g^2}{\kappa} (Q_1 - \alpha_g Q_2) \quad , \tag{85}$$

where κ is a form of Froude number, i.e.,

$$\kappa = \frac{V_e^2}{2gc} \quad , \tag{86}$$

and Q_1 and Q_2 are functions similar to the previous P_1 and P_2. The same type of approximation leading to equation (79) is used for the wave drag coefficient, resulting in

$$C_{D_W} = \frac{\pi^2 \alpha_g^2}{\kappa} Q_1 \quad . \tag{87}$$

In addition to the wave drag, there is the induced drag due to the trailing vortices of the finite span and the profile drag due to frictional and form drags. The expression for the induced drag coefficient is given in Reference 7 as

$$C_{D_i} = \frac{2C_L^2}{\pi A + 8h/c} \quad . \tag{88}$$

It is assumed that the profile drag does not vary much with angle of attack or depth; hence,

$$\frac{\partial C_{D_p}}{\partial \alpha} = \frac{\partial C_{D_p}}{\partial h} = 0 \quad . \tag{89}$$

The values of profile drag to be used in evaluating the total drag of a hydrofoil may be determined from wind-tunnel tests at the appropriate Reynolds number.

The final expression for the drag coefficient is then the sum of the contributions of profile drag, induced drag, and wave drag, i.e.,

$$\left.\begin{aligned} C_D &= C_{D_P} + C_{D_i} + C_{D_W} \\ C_D &= C_{D_P} + \frac{2C_L^2}{\pi A + 8h/c} + \frac{\pi^2\alpha_g^2}{\kappa} Q_1 \quad . \end{aligned}\right\} \tag{90}$$

With this expression for C_D, it is a simple matter to determine the required derivatives. Thus,

$$\frac{\partial C_D}{\partial h} = 4\left[\frac{(\pi A + 8h/c)C_L \dfrac{\partial C_L}{\partial h} - 4C_L^2/c}{(\pi A + 8h/c)^2}\right] + \frac{\pi^2\alpha_g^2}{\kappa}\frac{\partial Q_1}{\partial h} \tag{91}$$

and

$$\frac{\partial C_D}{\partial \alpha} = \frac{4C_L \dfrac{\partial C_L}{\partial \alpha}}{\pi A + 8h/c} + \frac{2\pi^2\alpha_g}{\kappa} Q_1 \frac{\partial \alpha_g}{\partial \alpha} \quad . \tag{92}$$

Since $\alpha = \alpha_g - \alpha_o$, where α_o is the angle of zero lift which is practically constant for the range of speeds considered, $\partial\alpha_g/\partial\alpha = 1$ and hence

$$\frac{\partial C_D}{\partial \alpha} = \frac{4C_L \dfrac{\partial C_L}{\partial \alpha}}{\pi A + 8h/c} + \frac{2\pi^2\alpha_g}{\kappa} Q_1 \quad . \tag{93}$$

The variations of lift and drag coefficients with angle of attack and with depth have been theoretically expressed and may be evaluated for application within the expressions comprising the stability derivatives. The values of these coefficients and their derivatives for the rear foil are found in the same manner as for the forward foil. Any modification of these factors due to downwash is accounted for by the presence of downwash terms involving $d\epsilon/d\alpha$.

DETERMINATION OF DOWNWASH ANGLE ϵ AND OF $d\epsilon/d\alpha$

The angle of attack at the rear foil of a tandem hydrofoil system is the difference between the angular setting of the rear foil (relative to zero lift angle) and the mean downwash angle ϵ. Thus,

$$\alpha_R = \alpha - \epsilon \quad , \tag{94}$$

where α is the difference between the geometric setting of the rear foil relative to the surface of the undisturbed fluid and the angle of zero lift.

The sense of ϵ is positive when it tends to reduce the angle of attack at the rear foil. It is determined from the slope of the sinusoidal surface wave generated by the bound vortex of the forward foil (based on two-dimensional theory). Beyond the first 1/4 wave length downstream, the surface is represented by a sine wave whose equation (Reference 10) is given as

$$\zeta = - C_{L_F} c e^{-\frac{gh}{V_e^2}} \sin \frac{gs}{V_e^2} \quad , \tag{95}$$

where ζ is the displacement of the surface, s is the distance between the quarter-chord points of the forward and rear foils, and h is the depth of submersion of the forward foil. For the small slopes considered in the linearized wave theory, the downwash angle ϵ is determined as the negative of the slope of the surface at the particular point in consideration. The surface wave slope is then

$$\frac{d\zeta}{ds} = - C_{L_F} \frac{gc}{V_e^2} e^{-\frac{gh}{V_e^2}} \cos \frac{gs}{V_e^2} \quad . \tag{96}$$

Hence, the angle of downwash at a distance s behind the forward foil and at a depth h' below the water surface is expressed as

$$\epsilon = - \frac{d\zeta}{ds} e^{-\frac{gh'}{V_e^2}} = C_{L_F} \frac{gc}{V_e^2} e^{-\frac{g(h+h')}{V_e^2}} \cos \frac{gs}{V_e^2} \quad . \tag{97}$$

With the aid of this expression for ϵ , the value of $d\epsilon/d\alpha$ is found to be

$$\frac{d\epsilon}{d\alpha} = \left(\frac{\partial C_L}{\partial \alpha}\right)_F \frac{gc}{V_e^2} e^{-\frac{g(h+h')}{V_e^2}} \cos \frac{gs}{V_e^2} \quad . \qquad (98)$$

SUMMARY OF EQUATIONS

The differential equations of a disturbed longitudinal motion of the tandem hydrofoil system are summarized below. They were derived as equations (16), (42), (49), (51), and (77) and are collected here for convenience.

$$
\left.
\begin{aligned}
&\frac{d\hat{u}}{d\tau} - x_u \hat{u} - x_w \hat{w} - x_\theta \theta - x_\beta \beta - \frac{x_q}{\mu} \hat{q} = 0 \\
&- z_u \hat{u} + \left(1 + \frac{z_{\dot{w}}}{\mu} - \frac{z'_{\dot{w}}}{\mu}\right) \frac{d\hat{w}}{d\tau} - z_w \hat{w} - z_\theta \theta - z_\beta \beta - \frac{z'_{\dot{q}}}{\mu^2} \frac{d\hat{q}}{d\tau} - \left(1 + \frac{z_q}{\mu}\right) \hat{q} = 0 \\
&- \frac{\mu (m_{\dot{w}} + m'_{\dot{w}})}{i_B} \frac{d\hat{w}}{d\tau} - \frac{\mu m_w}{i_E} \hat{w} - \frac{\mu m_\theta}{i_B} \theta - \frac{\mu m_\beta}{i_B} \beta + \left(1 - \frac{m'_{\dot{q}}}{\mu i_B}\right) \frac{d\hat{q}}{d\tau} - \frac{m_q}{i_B} \hat{q} = 0 \\
&- \frac{d\theta}{d\tau} + \hat{q} = 0 \\
&\hat{w} - \theta + \frac{d\beta}{d\tau} = 0
\end{aligned}
\right\} \quad (99)
$$

STABILITY DETERMINANTS

The complete system of differential equations may be solved by expressing each of the five variables as a sum of exponentials of the power $\lambda_i \tau$, i.e.,

$$\left.\begin{aligned} \hat{u} &= \sum \hat{u}_i e^{\lambda_i \tau} \quad , \quad \hat{w} = \sum \hat{w}_i e^{\lambda_i \tau} \quad , \quad \theta = \sum \theta_i e^{\lambda_i \tau} \quad , \\ \beta &= \sum \beta_i e^{\lambda_i \tau} \quad , \quad \hat{q} = \sum \hat{q}_i e^{\lambda_i \tau} \quad . \end{aligned}\right\} \tag{100}$$

The differential equations are reduced to a system of five algebraic equations for λ , the values of which determine the stability of the motion. In order to have a consistent set of equations, the determinant of these equations must vanish. Now, by defining

$$a = 1 + \frac{z_{\dot{w}}}{\mu} - \frac{z'_{\dot{w}}}{\mu} \quad , \quad b = \frac{-\mu(m_{\dot{w}} + m'_{\dot{w}})}{i_B} \quad , \quad c = 1 + \frac{z_q}{\mu} \quad , \quad d = 1 - \frac{m'_{\dot{q}}}{\mu i_B} \; ,$$

then

$$\begin{vmatrix} \lambda - x_u & -x_w & -x_\theta & -x_\beta & -\frac{x_q}{\mu} \\ -z_u & a\lambda - z_w & -z_\theta & -z_\beta & -\frac{z'_{\dot{q}}}{\mu^2}\lambda - c \\ 0 & b\lambda - \frac{\mu m_w}{i_B} & -\frac{\mu m_\theta}{i_B} & -\frac{\mu m_\beta}{i_B} & d\lambda - \frac{m_q}{i_B} \\ 0 & 0 & -\lambda & 0 & 1 \\ 0 & 1 & -1 & \lambda & 0 \end{vmatrix} = 0 \tag{101}$$

The system is of the fifth order. The stability quintic will now be obtained in the usual way by equating to zero the determinant of the system (101). The resultant quintic expression may be written as

$$\lambda^5 + B\lambda^4 + C\lambda^3 + D\lambda^2 + E\lambda + F = 0 \quad , \tag{102}$$

where

$$B = \frac{bc - \dfrac{am_q}{i_B} - z_w d - \dfrac{m_w z'_{\dot q}}{\mu i_B} - x_u\left(ad + \dfrac{bz'_{\dot q}}{\mu^2}\right)}{ad + bz'_{\dot q}/\mu^2}$$

$$C = \frac{d(z_\beta + x_u z_w - z_u x_w) + b\left(z_\theta - cx_u + \dfrac{x_q z_u}{\mu}\right)}{ad + bz'_{\dot q}/\mu^2}$$

$$- \frac{\dfrac{\mu}{i_B}\left(m_w c + am_\theta - \dfrac{m_\beta z'_{\dot q}}{\mu^2} - \dfrac{m_w x_u z'_{\dot q}}{\mu^2}\right) - \dfrac{m_q}{i_B}(z_w + ax_u)}{ad + bz'_{\dot q}/\mu^2}$$

$$D = \frac{\dfrac{\mu}{i_B}\left[m_\beta(c - a) + z_w m_\theta - m_w z_\theta + am_\theta x_u + m_w cx_u - \dfrac{m_\beta x_u z'_{\dot q}}{\mu^2}\right] + d(z_u x_\beta - x_u z_\beta)}{ad + bz'_{\dot q}/\mu^2}$$

$$+ \frac{\dfrac{1}{i_B}(m_q z_u x_w - m_q z_\beta - m_w z_u x_q - m_q x_u z_w) + b(z_\beta - z_\theta x_u + z_u x_\theta)}{ad + bz'_{\dot q}/\mu^2}$$

$$\left.\right\} \quad (103)$$

$$E = \frac{\dfrac{\mu}{i_B}\left[m_\beta z_\theta - m_\theta z_\beta + m_\beta z_w - m_w z_\beta + m_w z_\theta x_u - m_\theta z_w x_u + m_\beta x_u(a - c)\right]}{ad + bz'_{\dot q}/\mu^2}$$

$$+ \frac{\dfrac{1}{i_B}(m_\beta x_q z_u + m_q z_\beta x_u - m_q z_u x_\beta) + b(z_u x_\beta - z_\beta x_u)}{ad + bz'_{\dot q}/\mu^2}$$

$$F = \frac{\dfrac{\mu x_u}{i_B}(m_\theta z_\beta - m_\beta z_\theta + m_w z_\beta - m_\beta z_w) + \dfrac{\mu z_u}{i_B}(m_\beta x_w - m_w x_\beta + m_\beta x_\theta - m_\theta x_\beta)}{ad + bz'_{\dot q}/\mu^2} .$$

The method of analysis utilized to determine the dynamic stability without actually solving the equations of motion is by means of the Hurwitz criteria or Routh's discriminant (Reference 11). For the particular case of a quintic expression in λ, the following conditions must hold in order to have stable solutions (real part of λ less than zero). With an expression of the form

$$a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \quad ,$$

the criteria of stability requires that

1) all the $a_i > 0$

2) $\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \; ; \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0 \; ; \quad \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix} > 0 \; .$

In the longitudinal stability tests of Reference 12, constant speed was maintained and no longitudinal surging motion was encountered; hence, the entire x-equation can be neglected for this case. The remaining four equations may then be analyzed in the same manner as for the three degrees of freedom system. The stability determinant will be a quartic in this case, derived from the determinantal equation

$$\begin{vmatrix} a\lambda - z_w & -z_\theta & -z_\beta & -\dfrac{z'_q}{\mu^2}\lambda - c \\ b\lambda - \dfrac{\mu m_w}{i_B} & -\dfrac{\mu m_\theta}{i_B} & -\dfrac{\mu m_\beta}{i_B} & d\lambda - \dfrac{m_q}{i_B} \\ 0 & -\lambda & 0 & 1 \\ 1 & -1 & \lambda & 0 \end{vmatrix} = 0 \quad . \qquad (104)$$

The quartic may be written as

$$\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0 \quad , \tag{105}$$

where

$$\left.\begin{aligned}
B_1 &= \frac{bc - \dfrac{am_q}{i_B} - z_w d - \dfrac{m_w z'_{\dot{q}}}{\mu i_B}}{ad + bz'_{\dot{q}}/\mu^2} \\
C_1 &= \frac{z_\beta d + z_\theta b + \dfrac{z_w m_q}{i_B} - \dfrac{\mu}{i_B}\left(m_w c + m_\theta a - \dfrac{m_\beta z'_{\dot{q}}}{\mu^2}\right)}{ad + bz'_{\dot{q}}/\mu^2} \\
D_1 &= \frac{z_\beta\left(b - \dfrac{m_q}{i_B}\right) + \dfrac{\mu}{i_B}\left[z_w m_\theta - z_\theta m_w + m_\beta(c - a)\right]}{ad + bz'_{\dot{q}}/\mu^2} \\
E_1 &= \frac{\dfrac{\mu}{i_B}\left(m_\beta z_\theta - m_\theta z_\beta + m_\beta z_w - m_w z_\beta\right)}{ad + bz'_{\dot{q}}/\mu^2} \quad .
\end{aligned}\right\} \tag{106}$$

The previously outlined stability analysis based upon the Hurwitz criteria is applicable to this case as well. When the expanded determinants are greater than zero, the values of λ will be such that the motion is stable and the disturbances are damped out.

SOLUTION OF EQUATIONS OF MOTION

An exact solution of the equations of motion that will yield the trajectories of the system will be obtained through use of the Laplace transform. The five equations previously found can be reduced to a system of three equations through elimination of the variables w and q, which are related by purely kinematic conditions to the other variables. Thus, the following equations are obtained:

$$\left.\begin{aligned}
&\frac{d\hat{u}}{d\tau} - x_u\hat{u} - (x_w + x_\theta)\theta - \frac{x_q}{\mu}\frac{d\theta}{d\tau} - x_\beta\beta + x_w\frac{d\beta}{d\tau} = 0 \\
&- z_u\hat{u} - (z_w + z_\theta)\theta + (a - c)\frac{d\theta}{d\tau} - \frac{z'_q}{\mu^2}\frac{d^2\theta}{d\tau^2} - z_\beta\beta + z_w\frac{d\beta}{d\tau} - a\frac{d^2\beta}{d\tau^2} = 0 \\
&- \frac{\mu}{i_B}(m_w + m_\theta)\theta + \left(b - \frac{m_q}{i_B}\right)\frac{d\theta}{d\tau} + d\frac{d^2\theta}{d\tau^2} - \frac{\mu m_\beta}{i_B}\beta + \frac{\mu m_w}{i_B}\frac{d\beta}{d\tau} - b\frac{d^2\beta}{d\tau^2} = 0 .
\end{aligned}\right\} \quad (107)$$

The solution for the case of constant speed with no longitudinal surging will be considered first. This case leads to two equations with β and θ as variables, the entire x-equation and terms such as $z_u\hat{u}$ not being considered. The two equations are then

$$\left.\begin{aligned}
&-(z_w + z_\theta)\theta + (a - c)\frac{d\theta}{d\tau} - \frac{z'_q}{\mu^2}\frac{d^2\theta}{d\tau^2} - z_\beta\beta + z_w\frac{d\beta}{d\tau} - a\frac{d^2\beta}{d\tau^2} = 0 \\
&- \frac{\mu}{i_B}(m_w + m_\theta)\theta + \left(b - \frac{m_q}{i_B}\right)\frac{d\theta}{d\tau} + d\frac{d^2\theta}{d\tau^2} - \frac{\mu m_\beta}{i_B}\beta + \frac{\mu m_w}{i_B}\frac{d\beta}{d\tau} - b\frac{d^2\beta}{d\tau^2} = 0 .
\end{aligned}\right\} \quad (108)$$

The Laplace transform will now be applied to these equations. The following relations define the operation of the transform on the variables θ and β and on their derivatives (Reference 13):

$$L\left[\theta(\tau)\right] = \int_0^\infty e^{-p\tau}\theta(\tau)d\tau = \theta(p)$$

$$L\left[\beta(\tau)\right] = \int_0^\infty e^{-p\tau}\beta(\tau)d\tau = \beta(p)$$

$$L\left[\frac{d\theta}{d\tau}\right] = p\theta(p) - \theta(0)$$

$$L\left[\frac{d^2\theta}{d\tau^2}\right] = p^2\theta(p) - p\theta(0) - \theta'(0)$$

$$L\left[\frac{d\beta}{d\tau}\right] = p\beta(p) - \beta(0)$$

$$L\left[\frac{d^2\beta}{d\tau^2}\right] = p^2\beta(p) - p\beta(0) - \beta'(0) \quad ,$$

where $\theta(0)$, $\theta'(0)$, $\beta(0)$, and $\beta'(0)$ are the initial values $(\tau = 0)$ of θ , $d\theta/d\tau$, β , and $d\beta/d\tau$, respectively.

The transformed equations are

$$\left[\frac{z'_{\dot{q}}}{\mu^2}p^2 - (a - c)p + (z_w + z_\theta)\right]\theta(p) + \left[ap^2 - z_w p + z_\beta\right]\beta(p) =$$
$$\left[a\beta(0) + \frac{z'_{\dot{q}}}{\mu^2}\theta(0)\right]p + \left[\frac{z'_{\dot{q}}}{\mu^2}\theta'(0) - (a - c)\theta(0) - z_w\beta(0) + a\beta'(0)\right] \tag{109}$$

$$\left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right]\theta(p) + \left[-bp^2 + \frac{\mu m_w}{i_B}p - \frac{\mu m_\beta}{i_B}\right]\beta(p) =$$
$$\left[d\theta(0) - b\beta(0)\right]p + \left[\left(b - \frac{m_q}{i_B}\right)\theta(0) + d\theta'(0) + \frac{\mu m_w}{i_B}\beta(0) - b\beta'(0)\right] . \tag{110}$$

Solving these two simultaneous linear equations for $\theta(p)$ with the aid of the definitions

$$X = \left[a\beta(0) + \frac{z'_{\dot{q}}}{\mu^2}\,\theta(0)\right]p + \left[\frac{z'_{\dot{q}}}{\mu^2}\,\theta'(0) - (a - c)\theta(0) - z_w\beta(0) + a\beta'(0)\right] \tag{111}$$

$$Y = \left[d\theta(0) - b\beta(0)\; p\right] + \left[\left(b - \frac{m_q}{i_B}\right)\theta(0) + d\theta'(0) + \frac{\mu m_w}{i_B}\,\beta(0) - b\beta'(0)\right] \tag{112}$$

yields

$$\theta(p) = \frac{\begin{vmatrix} X & (ap^2 - z_w p + z_\beta) \\ Y & \left(-bp^2 + \frac{\mu m_w}{i_B}\,p - \frac{\mu m_\beta}{i_B}\right) \end{vmatrix}}{\begin{vmatrix} \left[\frac{z'_{\dot{q}}}{\mu^2}\,p^2 - (a - c)p + (z_w + z_\theta)\right] & (ap^2 - z_w p + z_\beta) \\ \left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right] & \left(-bp^2 + \frac{\mu m_w}{i_B}\,p - \frac{\mu m_\beta}{i_B}\right) \end{vmatrix}} \tag{113}$$

$$\theta(p) = \frac{\begin{vmatrix} X & (ap^2 - z_w p + z_\beta) \\ Y & \left(-bp^2 + \frac{\mu m_w}{i_B}\,p - \frac{\mu m_\beta}{i_B}\right) \end{vmatrix}}{-(ad + bz'_{\dot{q}}/\mu^2)(p^4 + B_1p^3 + C_1p^2 + D_1p + E_1)} \tag{114}$$

where the denominator contains the stability quartic previously determined (equation (105)). Evaluation of the determinant in the numerator then yields the final value of $\theta(p)$:

$$\theta(p) = \frac{\theta(0)p^3 + \left[B_1\theta(0) + \theta'(0)\right]p^2 + \phi p + \psi}{p^4 + B_1p^3 + C_1p^2 + D_1p + E_1} \tag{115}$$

where

$$\left.\begin{aligned}
\phi &= \frac{\left\{\frac{\mu}{i_B}\left[(a-c)m_w + \frac{m_\beta z'_{\dot q}}{\mu^2}\right] + dz_\beta - z_w\left(b - \frac{m_q}{i_B}\right)\right\}\theta(0)}{ad + bz'_{\dot q}/\mu^2} \\
&- \frac{\left(dz_w + \frac{m_w z'_{\dot q}}{\mu i_B}\right)\theta'(0) + \left(bz_\beta - \frac{\mu m_\beta a}{i_B}\right)\beta(0) + \left(\frac{\mu m_w a}{i_B} - bz_w\right)\beta'(0)}{ad + bz'_{\dot q}/\mu^2} \\
\psi &= \frac{\left[z_\beta\left(b - \frac{m_q}{i_B}\right) - \frac{\mu m_\beta}{i_B}(a-c)\right]\theta(0) + \left(dz_\beta + \frac{m_\beta z'_{\dot q}}{\mu i_B}\right)\theta'(0)}{ad + bz'_{\dot q}/\mu^2} \\
&- \frac{\frac{\mu}{i_B}(m_\beta z_w - m_w z_\beta)\beta(0) + \left(bz_\beta - \frac{\mu m_\beta a}{i_B}\right)\beta'(0)}{ad + bz'_{\dot q}/\mu^2}
\end{aligned}\right\} \quad (116)$$

Similarly, the value of $\beta(p)$ is given as

$$\beta(p) = \frac{\begin{vmatrix} \left[\frac{z'_{\dot q}}{\mu^2}p^2 - (a-c)p + (z_w + z_\theta)\right] & X \\ \left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right] & Y \end{vmatrix}}{-(ad + bz'_{\dot q}/\mu^2)(p^4 + B_1p^3 + C_1p^2 + D_1p + E_1)} \quad (117)$$

$$\beta(p) = \frac{\beta(0)p^3 + \left[B_1\beta(0) + \beta'(0)\right]p^2 + \xi p + \eta}{p^4 + B_1p^3 + C_1p^2 + D_1p + E_1} \quad (118)$$

where

$$
\left.
\begin{aligned}
\xi = & -\frac{\left[d(z_w + z_\theta) + \frac{z'_{\dot q}}{\mu^2}(m_w + m_\theta)\right]\theta(0) + \left[d(a - c) + \frac{z'_{\dot q}}{\mu^2}\left(b - \frac{m_q}{i_B}\right)\right]\theta'(0)}{ad + bz'_{\dot q}/\mu^2} \\
& - \frac{\left[\frac{\mu a}{i_B}(m_w + m_\theta - m_\beta) + \frac{\mu m_\beta c}{i_B} - bz_\theta - \frac{m_q z_w}{i_B}\right]\beta(0) + \left(\frac{a m_q}{i_B} - bc\right)\beta'(0)}{ad + bz'_{\dot q}/\mu^2} \\
\eta = & \frac{\left[\frac{\mu}{i_B}(m_w + m_\theta)(a - c) - \left(b - \frac{m_q}{i_B}\right)(z_w + z_\theta)\right]\theta(0)}{ad + bz'_{\dot q}/\mu^2} \\
& - \frac{\left[d(z_w + z_\theta) + \frac{z'_{\dot q}(m_w + m_\theta)}{\mu i_B}\right]\theta'(0) - \frac{\mu}{i_B}\left[z_w(m_w + m_\theta) - m_\beta(z_w + z_\theta)\right]\beta(0)}{ad + bz'_{\dot q}/\mu^2} \\
& + \frac{\left[b(z_w + z_\theta) - \frac{\mu a}{i_B}(m_w + m_\theta)\right]\beta'(0)}{ad + bz'_{\dot q}/\mu^2} \quad .
\end{aligned}
\right\} \quad (119)
$$

The functions $\theta(\tau)$ and $\beta(\tau)$ can now be evaluated by means of the Mellin inversion theorem, using an integration in the complex plane along the so-called Bromwich path, with $p = u + iv$ being the complex variable:

$$
\left.
\begin{aligned}
\theta(\tau) &= \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} e^{p\tau}\theta(p)dp \\
\beta(\tau) &= \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} e^{p\tau}\beta(p)dp \quad .
\end{aligned}
\right\} \quad (120)
$$

The singular points of the integrand are the zeros of the denominator, which in this case is the stability quartic. After determining the four roots of this quartic polynomial, it is expressed as

$$(p - \lambda_1)(p - \lambda_2)(p - \lambda_3)(p - \lambda_4) \quad ,$$

where λ_i are the roots. The solution is then given as

$$\theta(\tau) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} e^{p\tau} \frac{\theta(0)p^3 + \left[B_1\theta(0) + \theta'(0)\right]p^2 + \phi p + \psi}{(p - \lambda_1)(p - \lambda_2)(p - \lambda_3)(p - \lambda_4)} dp \tag{121}$$

$$\theta(\tau) = \sum_{i=1}^{4} \text{Residues of } \left[e^{p\tau}\theta(p)\right] \text{ at } \lambda_i \quad . \tag{122}$$

Similarly, for $\beta(\tau)$,

$$\beta(\tau) = \sum_{i=1}^{4} \text{Residues of } \left[e^{p\tau}\beta(p)\right] \text{ at } \lambda_i \quad . \tag{123}$$

Thus the determination of the trajectories of the motion is reduced to a solution of an algebraic equation and an evaluation of the stability derivatives based upon equilibrium parameters.

For the case in which there is an additional degree of freedom, i.e., longitudinal surging, there are three equations for the three unknowns $\hat{u}$, β , and θ given by equation (107). The transformed equations are then

$$(p - x_u)\hat{u}(p) - \left[\frac{x_q}{\mu} p + (x_w + x_\theta)\right] \theta(p) + (x_w p - x_\beta)\beta(p) =$$

$$\left[\hat{u}(0) - \frac{x_q}{\mu} \theta(0) + x_w \beta(0)\right] \tag{124}$$

$$z_u \hat{u}(p) + \left[\frac{z'_q}{\mu^2} p^2 - (a - c)p + (z_w + z_\theta)\right] \theta(p) + \left[ap^2 - z_w p + z_\beta\right] \beta(p) =$$

$$\left[a\beta(0) + \frac{z'_q}{\mu^2} \theta(0)\right] p + \left[\frac{z'_q}{\mu^2} \theta'(0) - (a - c)\theta(0) - z_w \beta(0) + a\beta'(0)\right] \tag{125}$$

$$\left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right]\theta(p) + \left[-bp^2 + \frac{\mu m_w}{i_B}p - \frac{\mu m_\beta}{i_B}\right]\beta(p) =$$

$$\left[d\theta(0) - b\beta(0)\right]p + \left[\left(b - \frac{m_q}{i_B}\right)\theta(0) + d\theta'(0) + \frac{\mu m_w}{i_B}\beta(0) - b\beta'(0)\right] . \tag{126}$$

Since, initially, the surge velocity $\hat{u}$ is zero, $\hat{u}(0) = 0$ and equation (124) can be re-written as

$$(p - x_u)\hat{u}(p) - \left[\frac{x_q}{\mu}p + (x_w + x_\theta)\right]\theta(p) + (x_w p - x_\beta)\beta(p) = \left[-\frac{x_q}{\mu}\theta(0) + x_w\beta(0)\right] . \tag{127}$$

Solving for $\hat{u}(p)$ gives

$$\hat{u}(p) = \frac{\begin{vmatrix} \left[-\frac{x_q}{\mu}\theta(0) + x_w\beta(0)\right] & -\left[\frac{x_q}{\mu}p + (x_w + x_\theta)\right] & (x_w p - x_\beta) \\ X & \left[\frac{z'_{\dot{q}}}{\mu^2}p^2 - (a - c)p + (z_w + z_\theta)\right] & (ap^2 - z_w p + z_\beta) \\ Y & \left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right] & \left(-bp^2 + \frac{\mu m_w}{i_B}p - \frac{\mu m_\beta}{i_B}\right) \end{vmatrix}}{\begin{vmatrix} (p - x_u) & -\left[\frac{x_q}{\mu}p + (x_w + x_\theta)\right] & (x_w p - x_\beta) \\ z_u & \left[\frac{z'_{\dot{q}}}{\mu^2}p^2 - (a - c)p + (z_w + z_\theta)\right] & (ap^2 - z_w p + z_\beta) \\ 0 & \left[dp^2 + \left(b - \frac{m_q}{i_B}\right)p - \frac{\mu}{i_B}(m_w + m_\theta)\right] & \left(-bp^2 + \frac{\mu m_w}{i_B}p - \frac{\mu m_\beta}{i_B}\right) \end{vmatrix}} \tag{128}$$

where X and Y are given by equations (111) and (112).

After expanding the denominator, the stability quintic for this case (equation (102)) multiplied by the term $-(ad + bz'_{\dot{q}}/\mu^2)$ is obtained. Evaluation of the numerator then gives

$$\hat{u}(p) = \frac{\gamma p^3 + \delta p^2 + \nu p + \eta}{p^5 + Bp^4 + Cp^3 + Dp^2 + Ep + F} \tag{129}$$

where

$$\left.\begin{aligned}
\gamma &= (x_w + x_\theta)\theta(0) + \frac{x_q}{\mu}\theta'(0) + x_\beta\beta(0) - x_w\theta'(0) \\[2ex]
\delta &= \left[\frac{\frac{z'_{\dot q}}{\mu i_B}(m_\theta x_w - m_w x_\theta) - \frac{bx_q}{\mu}(z_\theta + z_w) - \frac{a}{i_B}(m_q x_w - m_w x_q)}{ad + bz'_{\dot q}/\mu^2} \right. \\
&\qquad \left. + \frac{bc(x_\theta + x_w) - d(z_w x_\theta - z_\theta x_w)}{ad + bz'_{\dot q}/\mu^2}\right]\theta(0) \\
&\quad + \left[x_\theta + \frac{\frac{z'_{\dot q}}{\mu^2 i_B}(m_q x_w - m_w x_q) + d\left(cx_w - \frac{x_q z_w}{\mu}\right)}{ad + bz'_{\dot q}/\mu^2}\right]\theta'(0) \\
&\quad + \frac{\left[d(x_w z_\beta - x_\beta z_w) + \frac{z'_{\dot q}}{\mu i_B}(m_\beta x_w - m_w x_\beta) - \frac{a m_q x_\beta}{i_B} + b\left(cx_\beta - \frac{x_q z_\beta}{\mu}\right)\right]\beta(0)}{ad + bz'_{\dot q}/\mu^2} \\
&\quad + \left[x_\beta + \frac{\frac{a}{i_B}(m_q x_w - m_w x_q) + b\left(\frac{x_q z_w}{\mu} - cx_w\right)}{ad + bz'_{\dot q}/\mu^2}\right]\beta'(0)
\end{aligned}\right\} \tag{130}$$

$$
\begin{aligned}
\nu = {} & \frac{\left\{\dfrac{z'_{\dot{q}}}{\mu i_B}\left(m_\beta x_w - m_w x_\beta + m_\beta x_\theta - m_\theta x_\beta\right) - \dfrac{x_q}{i_B}\left(m_\theta z_w - m_w z_\theta\right) - \dfrac{\mu(a-c)}{i_B}\left(m_\theta x_w - m_w x_\theta\right)\right.}{ad + bz'_{\dot{q}}/\mu^2} \\
& - \frac{\left.\left(b - \dfrac{m_q}{i_B}\right)\left(z_w x_\theta - z_\theta x_w\right) + d\left[x_\beta\left(z_w + z_\theta\right) - z_\beta\left(x_w + x_\theta\right)\right]\right\}\theta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
& + \left\{x_\beta + \frac{\dfrac{z'_{\dot{q}}}{\mu^2 i_B}\left[m_\beta x_q - m_q x_\beta + \mu\left(m_\theta x_w - m_w x_\theta\right)\right] - d\left(z_w x_\theta - z_\theta x_w - \dfrac{x_q z_\beta}{\mu} + cx_\beta\right)}{ad + bz'_{\dot{q}}/\mu^2}\right\}\theta'(0) \\
& + \frac{\left[\dfrac{\mu}{i_B}\left(m_\beta x_w c - m_w x_\beta c + m_\beta x_\theta a - m_\theta x_\beta a\right) + b\left(x_\beta z_\theta - x_\theta z_\beta\right)\right.}{ad + bz'_{\dot{q}}/\mu^2} \\
& - \frac{\left.\dfrac{z_w}{i_B}\left(m_\beta x_q - m_q x_\beta\right) + \dfrac{z_\beta}{i_B}\left(m_q x_w - m_w x_q\right)\right]\beta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
& + \frac{\left\{a\left[\dfrac{m_\beta x_q}{i_B} - \dfrac{m_q x_\beta}{i_B} + \dfrac{\mu}{i_B}\left(m_\theta x_w - m_w x_\theta\right)\right] + b\left(z_w x_\theta - z_\theta x_w + cx_\beta - \dfrac{x_q z_\beta}{\mu}\right)\right\}\beta'(0)}{ad + bz'_{\dot{q}}/\mu^2}
\end{aligned}
\tag{130 cont.}
$$

$$
\left.
\begin{aligned}
n = &-\frac{\left\{\frac{\mu}{i_B}\left(\frac{x_q}{\mu} + a - c\right)(m_\beta z_w - m_w z_\beta + m_\beta z_\theta - m_\theta z_\beta)\right.}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left.\left(b - \frac{m_q}{i_B}\right)\left[x_\beta(z_w + z_\theta) - z_\beta(x_w + x_\theta)\right]\right\}\theta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left\{\frac{z'_{\dot{q}}}{\mu i_B}(m_\beta x_w - m_w x_\beta + m_\beta x_\theta - m_\theta x_\beta) - d\left[x_\beta(z_w + z_\theta) - z_\beta(x_w + x_\theta)\right]\right\}\theta'(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&- \frac{\frac{\mu}{i_B}\left\{(z_w - x_w)(m_\beta z_w - m_w z_\beta + m_\beta z_\theta - m_\theta z_\beta) + m_w\left[x_\beta(z_w + z_\theta) - z_\beta(x_w + x_\theta)\right]\right\}\beta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left\{\frac{\mu a}{i_B}(m_\beta x_w - m_w x_\beta + m_\beta x_\theta - m_\theta x_\beta) + b\left[x_\beta(z_w + z_\theta) - z_\beta(x_w + x_\theta)\right]\right\}\beta'(0)}{ad + bz'_{\dot{q}}/\mu^2} .
\end{aligned}
\right\} \quad \begin{array}{l}(130)\\ \text{cont.}\end{array}
$$

Similarly, for $\theta(p)$ and $\beta(p)$,

$$
\theta(p) = \frac{\theta(0)p^4 + \left[B\theta(0) + \theta'(0)\right]p^3 + \sigma p^2 + \chi p + \omega}{p^5 + Bp^4 + Cp^3 + Dp^2 + Ep + F} \tag{131}
$$

$$
\beta(p) = \frac{\beta(0)p^4 + \left[B\beta(0) + \beta'(0)\right]p^3 + \Gamma p^2 + \Delta p + \Omega}{p^5 + Bp^4 + Cp^3 + Dp^2 + Ep + F} \tag{132}
$$

where

$$
\left.
\begin{aligned}
\sigma = {} & \frac{\left[dz_\beta + \frac{bx_q z_u}{\mu} - a(x_w z_u - x_u z_w) - bcx_u + \frac{ax_u m_q}{i_B} \right.}{ad + bz'_{\dot q}/\mu^2} \\
& + \frac{\left. \frac{x_u m_w z'_{\dot q}}{\mu i_B} + \frac{m_\beta z'_{\dot q}}{\mu i_B} - z_w \left(b - \frac{m_q}{i_B} \right) + \frac{\mu(a - c)}{i_B} m_w \right] \theta(0)}{ad + bz'_{\dot q}/\mu^2} \\
& - \left[x_u + \frac{dz_w + \frac{m_w z'_{\dot q}}{\mu i_B}}{ad + bz'_{\dot q}/\mu^2} \right] \theta'(0) + \frac{\frac{\mu a m_\beta}{i_B} - bz_\beta}{ad + bz'_{\dot q}/\mu^2} \beta(0) + \frac{bz_w - \frac{\mu a m_w}{i_B}}{ad + bz'_{\dot q}/\mu^2} \beta'(0) \\
\chi = {} & \frac{\left[d(x_\beta z_u - x_u z_\beta) - \left(b - \frac{m_q}{i_B} \right) (x_w z_u - x_u z_w + z_\beta) \right.}{ad + bz'_{\dot q}/\mu^2} \\
& - \frac{\left. \frac{x_q m_w z_u}{i_B} + \frac{x_u m_\beta z'_{\dot q}}{\mu i_B} + \frac{\mu(a - c)}{i_B} (m_w x_u + m_\beta) \right] \theta(0)}{ad + bz'_{\dot q}/\mu^2} \\
& + \frac{\left[dz_\beta + \frac{m_\beta z'_{\dot q}}{\mu i_B} + \frac{x_u m_w z'_{\dot q}}{\mu i_B} - d(x_w z_u - x_u z_w) \right] \theta'(0)}{ad + bz'_{\dot q}/\mu^2} \\
& - \frac{\left[b(x_\beta z_u - x_u z_\beta) + \frac{\mu}{i_B} (m_\beta z_w - m_w z_\beta + ax_u m_\beta) \right] \beta(0)}{ad + bz'_{\dot q}/\mu^2} \\
& + \frac{\left[b(x_w z_u - x_u z_w - z_\beta) + \frac{\mu a}{i_B} (m_w x_u + m_\beta) \right] \beta'(0)}{ad + bz'_{\dot q}/\mu^2}
\end{aligned}
\right\} \quad (133)
$$

$$\omega = \frac{\left\{ b(x_\beta z_u - x_u z_\beta) + \frac{x_u}{i_B}\left[\mu(a-c)m_\beta + m_q z_\beta\right] - \frac{z_u}{i_B}(m_q x_\beta - m_\beta x_q) \right\}\theta(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \frac{\left[d(x_\beta z_u - x_u z_\beta) - \frac{x_u m_\beta z'_{\dot{q}}}{\mu i_B}\right]\theta'(0)}{ad + bz'_{\dot{q}}/\mu^2} + \frac{\left[\frac{\mu z_u}{i_B}(m_w x_\beta - m_\beta x_w) - \frac{\mu x_u}{i_B}(m_\beta z_w - m_w z_\beta)\right]\beta(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$- \frac{\left[b(x_\beta z_u - x_u z_\beta) + \frac{\mu a x_u m_\beta}{i_B}\right]\beta'(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$\Gamma = \frac{-\left[d(z_w + z_\theta) + \frac{z'_{\dot{q}}}{\mu^2}(m_w + m_\theta)\right]\theta(0)}{ad + bz'_{\dot{q}}/\mu^2} + \frac{\left[d(a-c) + \frac{z'_{\dot{q}}}{\mu^2}\left(b - \frac{m_q}{i_B}\right)\right]\theta'(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \left\{ \frac{\left[bz_\theta + \frac{z_w m_q}{i_B} + \frac{b z_u x_q}{\mu} - d x_w z_u - \frac{\mu c m_\beta}{i_B} - \frac{\mu a}{i_B}(m_w + m_\theta - m_\beta)\right]}{ad + bz'_{\dot{q}}/\mu^2} - B_1 x_u \right\}\beta(0)$$

$$+ \left(\frac{bc - \frac{a m_q}{i_B}}{ad + bz'_{\dot{q}}/\mu^2} - x_u \right)\beta'(0)$$

(133) cont.

$$\Delta = \frac{\left\{ d\left[x_u(z_w + z_\theta) - z_u(x_w + x_\theta)\right] + \dfrac{x_u z'_{\dot{q}}}{\mu^2}(m_w + m_\theta)\right.}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \frac{\left.\dfrac{\mu}{i_B}(a - c)(m_w + m_\theta) - \left(b - \dfrac{m_q}{i_B}\right)(z_w + z_\theta)\right\}\theta(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$- \frac{\left[\dfrac{dx_q z_u}{\mu} + d(a - c)x_u + \dfrac{x_u z'_{\dot{q}}}{\mu^2}\left(b - \dfrac{m_q}{i_B}\right) + d(z_w + z_\theta) + \dfrac{z'_{\dot{q}}}{\mu i_B}(m_w + m_\theta)\right]\theta'(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \frac{\left\{ b\left[z_u(x_w + x_\theta) - x_u z_\theta\right] - \dfrac{x_q m_w z_u}{i_B} - \dfrac{x_u m_q z_w}{i_B} - x_w z_u\left(b - \dfrac{m_q}{i_B}\right)\right.}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \frac{\left.\dfrac{\mu}{i_B}\left[a x_u(m_w + m_\theta) - (a - c)x_u m_\beta + z_w(m_w + m_\theta) - m_\beta(z_w + z_\theta)\right]\right\}\beta(0)}{ad + bz'_{\dot{q}}/\mu^2}$$

$$+ \frac{\left[\dfrac{bx_q z_u}{\mu} + x_u\left(\dfrac{am_q}{i_B} - bc\right) + b(z_w + z_\theta) - \dfrac{\mu a}{i_B}(m_w + m_\theta)\right]\beta'(0)}{ad + bz'_{\dot{q}}/\mu^2} \qquad (133) \text{ cont.}$$

$$
\begin{aligned}
\Omega = &- \frac{\left\{\frac{x_q z_u}{i_B}(m_w + m_\theta) + \left[z_u(x_w + x_\theta) - x_u(z_w + z_\theta)\right]\left(b - \frac{m_q}{i_B}\right)\right.}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left.\frac{\mu x_u}{i_B}(a - c)(m_w + m_\theta)\right\}\theta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left\{d\left[x_u(z_w + z_\theta) - z_u(x_w + x_\theta)\right] + \frac{x_u z'_{\dot{q}}}{\mu i_B}(m_w + m_\theta)\right\}\theta'(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\frac{\mu}{i_B}\left[z_u(x_w m_\theta - m_w x_\theta) + x_u m_\beta(z_w + z_\theta) - x_u z_w(m_w + m_\theta)\right]\beta(0)}{ad + bz'_{\dot{q}}/\mu^2} \\
&+ \frac{\left\{b\left[z_u(x_w + x_\theta) - x_u(z_w + z_\theta)\right] + \frac{\mu a x_u}{i_B}(m_w + m_\theta)\right\}\beta'(0)}{ad + bz'_{\dot{q}}/\mu^2}
\end{aligned}
\qquad (133) \text{ cont.}
$$

The solutions for $\hat{u}(\tau)$, $\theta(\tau)$, and $\beta(\tau)$ are found in exactly the same manner as previously for the case of two degrees of freedom. These solutions are given by

$$\hat{u}(\tau) = \sum_{i=1}^{5} \text{Residues of } \left[e^{p\tau}\hat{u}(p)\right] \text{ at } \lambda_i \qquad (134)$$

$$\theta(\tau) = \sum_{i=1}^{5} \text{Residues of } \left[e^{p\tau}\theta(p)\right] \text{ at } \lambda_i \qquad (135)$$

$$\beta(\tau) = \sum_{i=1}^{5} \text{Residues of } \left[e^{p\tau}\beta(p)\right] \text{ at } \lambda_i \qquad (136)$$

where λ_i are the solutions of the stability quintic for this case (equation (102)).

APPLICATION OF EQUATIONS OF MOTION TO LONGITUDINAL STABILITY TESTS

The previously derived equations of motion (99) will now be applied to actual experimental studies. The longitudinal stability tests of Reference 12 will be used. Since, in these tests, constant speed was maintained, the whole x-equation can be left out of the calculations. The resulting equations for this case are given by equation (108) and the stability determinant is given by equations (104), (105), and (106).

The tests were made on a tandem system made up of two aspect ratio 20 hydrofoils of rectangular planform with 0° dihedral. The physical dimensions and conditions that remained constant in the tests are listed below:

$\ell = \frac{50}{12} = 4.167$ ft. $\qquad W = 63.3$ lb.

$c = \frac{2.5}{12} = 0.2083$ ft. $\qquad d_{F,R} = \frac{13.54}{12} = 1.128$ ft.

$S = 1.736$ ft.^{2} $\qquad \mu = \frac{W}{g\rho S\ell} = 0.1404$

$S_F = S_R = 0.868$ ft.^{2}

The equilibrium depth was one chord, $h_e = 0.2083$ ft., and equilibrium trim was 0°. Equilibrium speed was in the neighborhood of 8.50 ft./sec., but was slightly different in each of the cases tested. The moment of inertia of the system, the forward and rear pre-set angles of attack, and the location of the longitudinal center of gravity were also varied for each case. The center-of-gravity location was determined by the equilibrium conditions necessary for a static balance, equations (1) and (2).

The first case to be treated is that of the far-forward center-of-gravity position, $x_\ell/\ell = 0.25$. Equilibrium speed V_e was 8.40 ft./sec., the moment of inertia was such that $i_B = K_B^2/\ell^2 = 0.3601$, and the angles of attack (relative to zero lift) were $\alpha_{e_F} = 0.1745$ radian and $\alpha_{e_R} = 0$ radian. The hydrodynamic coefficients and derivatives are to be determined on the basis of the above conditions and will then be used to determine the stability derivatives.

For this case, the stability quartic is determined and the coefficients of the powers of λ are evaluated with the following results:

$a_0 = 1$ $\qquad a_3 = D_1 = 0.3464$

$a_1 = B_1 = 5.5093$ $\qquad a_4 = E_1 = 0.0505$.

$a_2 = C_1 - 4.6345$

Application of the Hurwitz criteria shows that all the $a_i > 0$:

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_0a_3 = 25.1865 > 0$$

$$\begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} = a_1(a_2a_3 - a_1a_4) - a_0a_3^2 = 7.1919 > 0 \ .$$

Since the determinants are greater than zero, the values of λ are such that the motion is stable and the disturbances are damped out.

The roots of the quartic are found to be

$\lambda_1 = -0.948$ $\qquad \lambda_3 = -0.0335 + 0.1043\ i$

$\lambda_2 = -4.494$ $\qquad \lambda_4 = -0.0335 - 0.1043\ i$.

The existence of the conjugate complex roots indicates oscillatory stability, which was discovered to be the condition in the stability experiments for this configuration.

For this case, the initial values of the variables were a positive trim of 2° and a submergence of two chords, so that the hydrofoil system was initially displaced from equilibrium. The initial values of the variables and their derivatives were:

$$\theta(0) = 2^o = 0.0349 \text{ radian} \; ; \; \left.\frac{d\theta}{d\tau}\right|_{\tau=0} = \theta'(0) = 0$$

$$\beta(0) = -0.3561 \; ; \; \left.\frac{d\beta}{d\tau}\right|_{\tau=0} = \beta'(0) = 0.0349 \; .$$

By solving the equations of motion, the trajectories are found to be

$$\theta(\tau) = 0.0037\, e^{-.948\tau} - 0.00042\, e^{-4.494\tau} + e^{-.0335\tau}(0.03393 \cos 0.1043\tau + 0.00914 \sin 0.1043\tau) \quad (137)$$

$$\beta(\tau) = 0.05261\, e^{-.948\tau} - 0.01815\, e^{-4.494\tau} - e^{-.0335\tau}(0.45994 \cos 0.1043\tau - 0.39662 \sin 0.1043\tau). \quad (138)$$

These solutions are plotted in Figures 1 and 2 and indicate a low amplitude, low frequency, damped oscillation, which is in agreement with the experimental findings. They are close to the trajectories which would result from an exact solution of the equations but differ slightly because of the small numbers dealt with and the manipulation of the complex numbers to determine the coefficients of the exponential terms. This can be seen in Figures 1 and 2 at $\tau = 0$ where the initial conditions are not exactly satisfied.

The next case to be investigated here is the one with the center of gravity located at $x_\ell/\ell = 0.569$. The value of V_e was 8.45 ft./sec., the value of i_B was 0.2148, and the angles of incidence were $\alpha_{e_F} = \alpha_{e_R} = 0.1047$ for the forward and rear foils. Evaluation of the stability determinant for this condition gives the following values for the coefficients:

$$a_0 = 1 \qquad a_3 = D_1 = 0.7287$$
$$a_1 = B_1 = 5.4545 \qquad a_4 = E_1 = 0.0201 .$$
$$a_2 = C_1 = 6.9506$$

These values then satisfy the first requisite for stability under the Hurwitz criteria, i.e., that all $a_i > 0$:

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 = 37.1833 > 0$$

$$\begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} = a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^{\,2} = 26.4977 > 0 .$$

Since the determinants are greater than zero, the motion in this case is stable.

By assuming the same initial conditions for this configuration as for the previous case of the far-forward center of gravity, and solving the equations of motion, the trajectories are found to be

$$\theta(\tau) = 0.01602\, e^{-.044\tau} + 0.02005\, e^{-.068\tau} - 0.00115\, e^{-1.785\tau} - 0.00001\, e^{-3.557\tau} \tag{139}$$

$$\delta(\tau) = 1.9707\, e^{-.044\tau} - 2.35535\, e^{-.068\tau} + 0.03481\, e^{-1.785\tau} - 0.00658\, e^{-3.557\tau} \tag{140}$$

plots of which are given in Figures 3 and 4.

For the case of the center of gravity located at $x_\ell/\ell = 0.604$, V_e was 8.37 ft./sec., i_B was 0.2186, and the angles of incidence were $\alpha_{e_F} = 0.0873$ radian and $\alpha_{e_R} = 0.1222$ radian. The stability quartic is determined and the coefficients are

$a_0 = 1$ $\qquad$ $a_3 = D_1 = 0.6107$

$a_1 = B_1 = 5.5020$ $\qquad$ $a_4 = E_1 = 0.0029$.

$a_2 = C_1 = 8.0804$

These values of $a_i > 0$ satisfy the first requisite for stability under the Hurwitz criteria. The required determinants will also be greater than zero, indicating stability for this case. However, the value of the coefficient E_1 is of such small magnitude that in the region under consideration, there cannot be any definite statement regarding the stability of the motion, because of the definition of E_1 as the sum of products of small numbers. The actual motion of this configuration in the stability tests was a slow, unstable, diving divergence and was very close to the divergence boundary, the actual location of which was approximately at $x_{\ell}/\ell = 0.60$.

In order to show the trend of the solutions of the equations in agreeing with the experimental results, a static balance was found by using the data of Reference 14 at a center-of-gravity location given by $x_{\ell}/\ell = 0.735$. The equilibrium speed was 8.50 ft./sec. and the moment of inertia was chosen such that $i_B = 0.4094$. The angles of incidence on the forward and rear foils were $\alpha_{e_F} = 0.0698$ radian and $\alpha_{e_R} = 0.1571$ radian. The stability quartic is evaluated for this case and the coefficients are found to be

$a_0 = 1$ $\qquad$ $a_3 = D_1 = 0.4169$

$a_1 = B_1 = 4.2538$ $\qquad$ $a_4 = E_1 = -0.0133$.

$a_2 = C_1 = 3.7112$

Since a_4 is a negative number, the first requirement of the criteria for stability is not satisfied and hence the motion is unstable.

The effect of an additional degree of freedom (surge motion) is now investigated for the stable case at $x_{\ell}/\ell = 0.569$. The resulting stability quintic has the following coefficients:

$a_0 = 1$ $\qquad a_3 = D = 1.0667$

$a_1 = B = 5.4763$ $\qquad a_4 = E = 0.0336$

$a_2 = C = 7.1749$ $\qquad a_5 = F = 0.0003$.

Valid conclusions as to the stability of this configuration are not possible because of the small order of magnitude of the coefficient F . However, the fact that F is very small is somewhat indicative of a lessening of stability due to this additional degree of freedom.

CONCLUSIONS

The equations derived in this report, which are based upon the assumptions outlined on page 6 , adequately describe the motion of a tandem hydrofoil system in smooth water. Analysis of the resultant equations, with the coefficients evaluated on the basis of theoretical hydrodynamic formulas, indicates a duplication of the type of motion experienced in tank tests. The trend in the motion with oscillatory stability for a far-forward center-of-gravity location, damped stable motion at a center-of-gravity position near the middle of the configuration, and divergent instability at points further back indicates the validity of the theory developed herein. Therefore, for constant speed, the derived equations are sufficient for investigating the longitudinal dynamic stability.

A preliminary study on the effect of including the x-equation in the analysis for the case of varying speed indicates a lessening of stability. This additional degree of freedom in longitudinal surging is, however, not too important in the present case, i.e., motion under an undisturbed free surface. It is of great importance though for the case of motion in waves where the waves cause variations in the forward speed.

The equations developed in this report can be extended to a study of motion in waves and controlled motion, by employing the proper types of forcing functions on the right-hand side. Solution of the resulting equations in this case by use of the Laplace transform method will involve only additional algebraic operations. Further investigation of the present system for varied equilibrium conditions will enable an investigator to determine the proper stable range of operation for a particular design having fully submerged hydrofoils in tandem.

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TRAJECTORY OF TANDEM HYDROFOIL SYSTEM
PITCH ANGLE CHANGE VS. TIME (NONDIMENSIONAL)
CENTER OF GRAVITY AT $\frac{x_o}{l}$ = 0.25

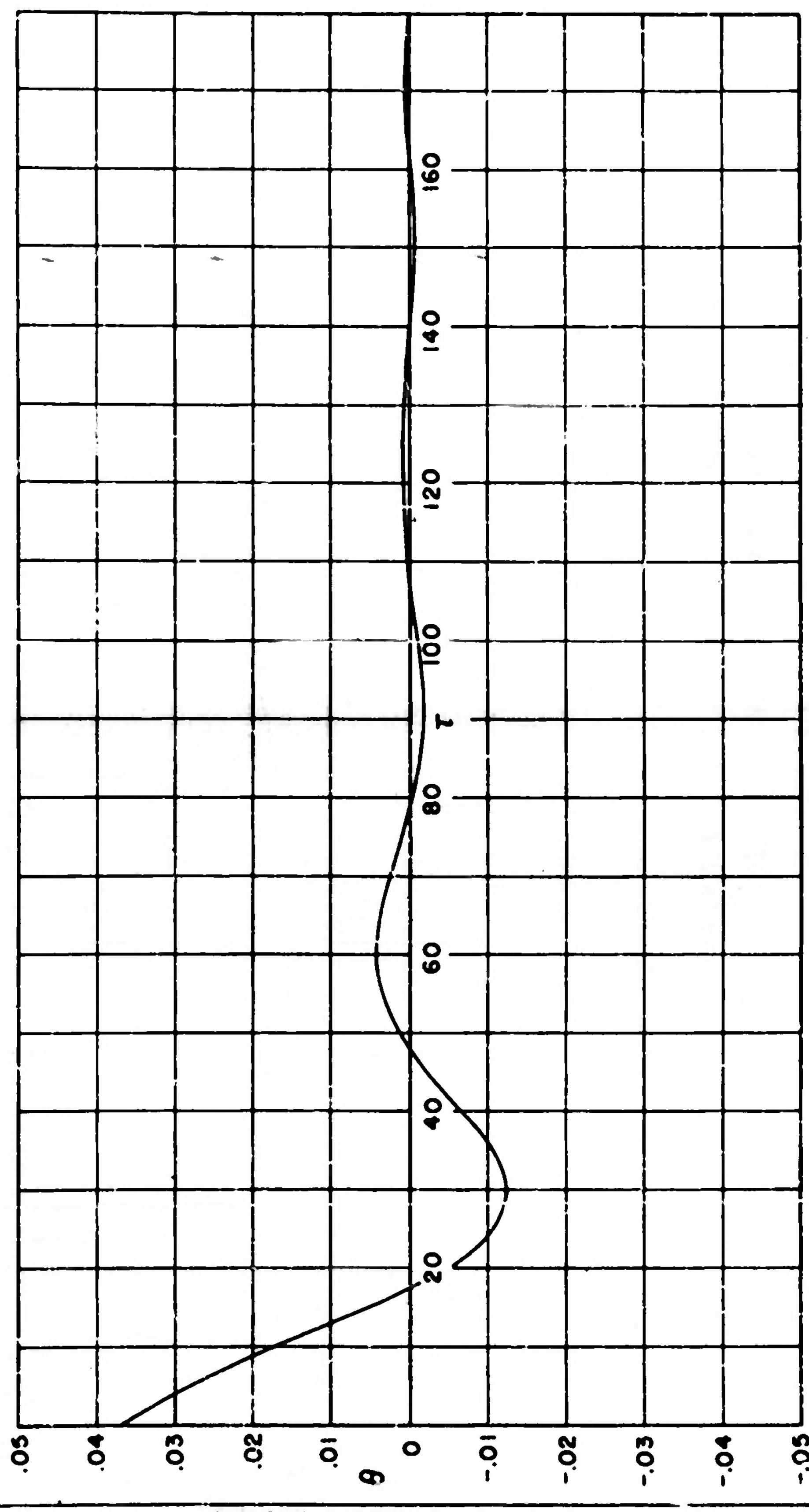

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FIGURE 2

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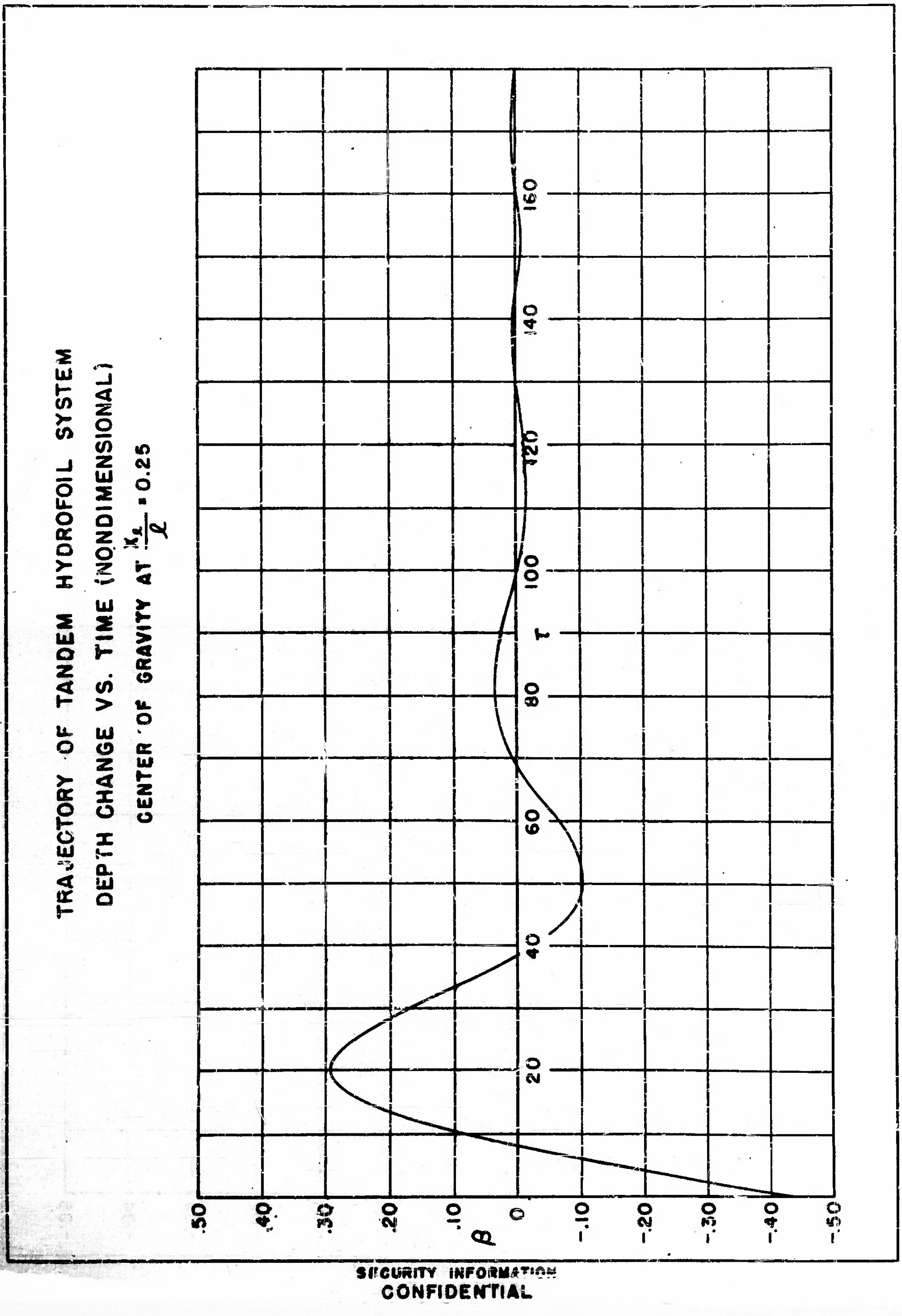

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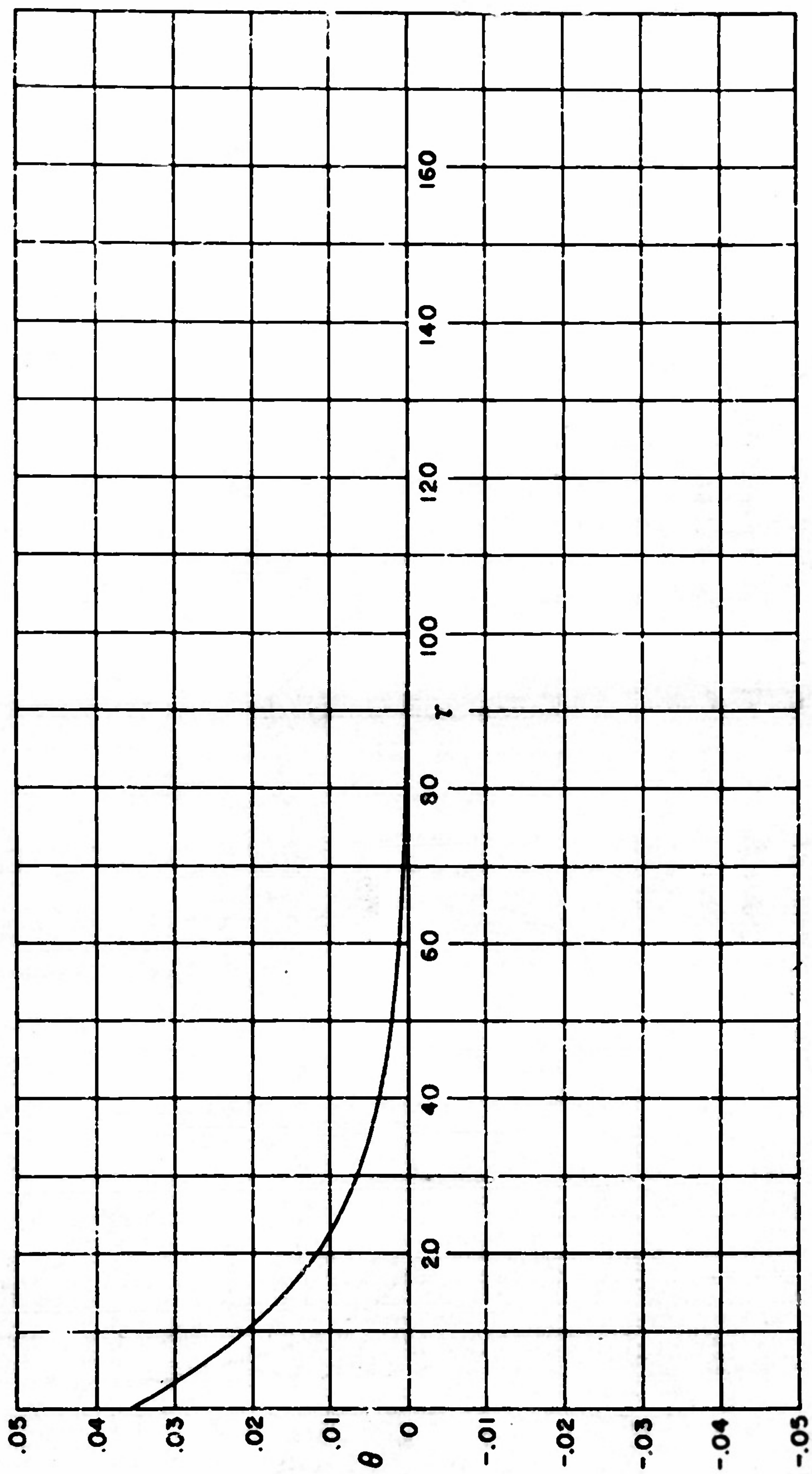
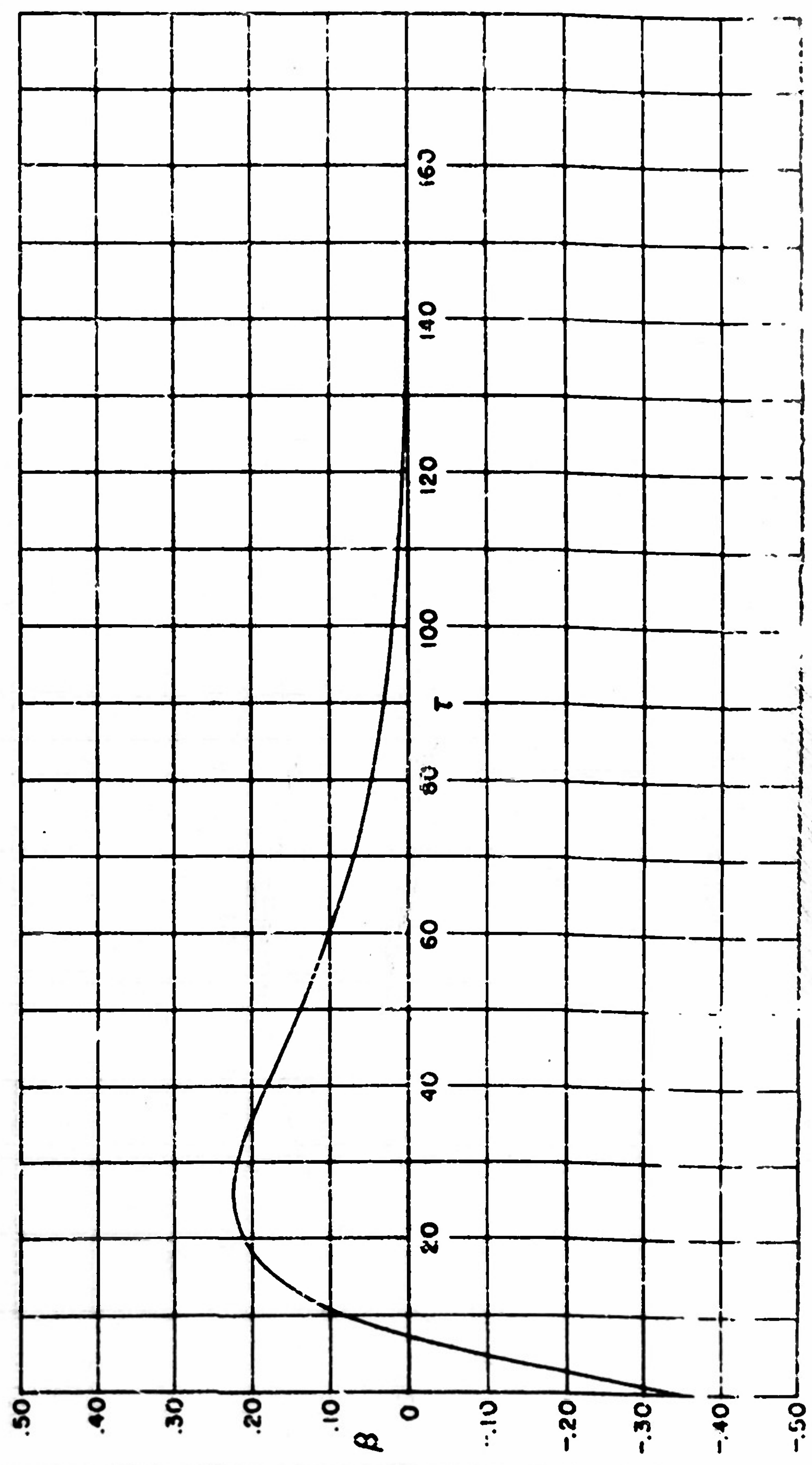
TRAJECTORY OF TANDEM HYDROFOIL SYSTEM

PITCH ANGLE CHANGE VS. TIME (NONDIMENSIONAL)

CENTER OF GRAVITY AT $\frac{x_g}{\ell} = 0.589$



K-479
FIGURE 4



TRAJECTORY OF TANDEM HYDROFOIL SYSTEM
DEPTH CHANGE VS. TIME (NONDIMENSIONAL)
CENTER OF GRAVITY AT $\frac{X_2}{\ell} = 0.569$