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TECHNICAL MEMORANDUM
NO. 31

WIDE-BAND DIRECTIVITY OF
RECEIVING ARRAYS

BY

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MAY 1, 1953

ACOUSTICS RESEARCH LABORATORY
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Office of Naval Research
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Technical Memorandum 31
Wide-Band Directivity of Receiving Arrays
by
James J. Faran, Jr. and Robert Hills, Jr.

May 1, 1953

Summary

The method of maximizing the directional gain of a receiving array (heretofore useful only at a single frequency) is extended to the case of operation at a finite bandwidth. It is also shown how to design for maximum effective gain in the presence of noise which might arise within the individual transducers or their preamplifiers. Some necessary noise-field correlations are computed, and numerical examples are included to show the effects of bandwidth and self-noise on the overall gain for reception which can be achieved. The directional gain of a broadside linear array for operation at a finite bandwidth is always less than that for operation at a single frequency, but is always greater, and often considerably greater, than the gain realized by use of the single-frequency design at the finite bandwidth. Unlike the single-frequency design, whose directional gain falls very rapidly as the operating bandwidth is increased from zero, the wide-band designs operate with good gain over a wide range of bandwidths.

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TM31

PREFACE

The subject of this memorandum is a by-product of a research project under way at this laboratory under the general heading "The Application of Correlation Techniques to Acoustic Receiving Systems." The principal results to this time have been reported in Technical Memoranda Nos. 27 and 28. The authors again wish to express their appreciation to Professor F. V. Hunt, who initiated this project, for his helpful and stimulating direction.

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I.

Introduction

Certain functions which appear in the usual formulation¹ of the problem of maximizing the directional gain of an array can be considered to represent simply special, single-frequency forms of a space correlation function which can, in principle, be evaluated for any spectrum. Analysis of the directivity problem from this point of view yields an extension of present design techniques to the case of maximizing the directional gain of an array for wide-band signal reception. It is becoming well known² that the computations involved in maximizing the gain of an array for operation at a single frequency are very laborious, and lead to specification of element currents, voltages, or sensitivities which are very large compared to those for the uniform array and whose magnitudes must be controlled with fantastic accuracy to achieve the predicted directional gain. By the same token, the spacings and frequency of operation must be accurately controlled to the same degree. The effect of attempting to operate such an array with signals of finite bandwidth (in transmission) or in the presence of a background noise of finite bandwidth (in reception) is no less serious in reducing the directional gain. However, the element sensitivities which maximize the directional gain of an array used in reception can be determined in a straightforward manner for any spectrum by inserting in the original formulation of the problem the proper values of the space correlation function mentioned above. In an isotropic

¹Pritchard, R. L., Directivity of Acoustic Linear Point Arrays, Technical Memorandum No. 21, Acoustics Research Laboratory, Harvard University, Cambridge, Massachusetts. (January 15, 1951).

²Yano, N., "A Note on Super Gain Antenna Arrays", Proc. I. R. E. 39, 1021-1025 (September, 1951).

noise background (in terms of which the directional gain is defined) this space correlation function can be readily computed for at least several spectra, and is found to depend only on the distance between the elements and the relative time delay of the signals after reception, if any. If these background noise correlations can not be computed readily, and if they can be measured, the design for maximum gain can be carried out in terms of the measured values.

Because the reception problem depends almost entirely upon these noise correlations, one may infer that the process of "shading" or selecting these sensitivities for the greatest output signal-to-noise ratio is simply one of cancelling, as much as possible, the coherent parts of the noises received by the various elements. If the spacing is so wide that there is little coherence between the noises received by the various elements, there is little that can be done to increase the directional gain by making the sensitivities other than uniform. Because, in reception, the shading is an operation which is carried out only with regard to the background noise, the design is not dependent upon the spectrum of the signal, as it is in the case of a transmitting array, but only on the spectrum of the background noise at the output of the receiver (assumed to contain no nonlinear circuits) whose pass-band need only be as wide as is necessary to handle the signal.

It has not, perhaps, been generally recognized that the presence of even very small amounts of self-noise in the individual elements or preamplifiers of an array which has been designed for maximum gain can very seriously decrease its effective gain. This is particularly true if any of the sensitivities of the individual elements are considerably larger than the over-all value for the uniform array. It will be demonstrated how such noise, if it exists, can be taken into account in the design of the array, by maximizing a suitably defined effective gain.

II.

Directional Gain and Effective Gain of an Array

Used in Reception

Assume that an array of m receiving transducers is located in a noisy

signal-bearing medium; that the signal and noise from each element is delayed a suitable time τ to compensate for the possibly different travel times from the signal source to the different elements, and adjusted in amplitude by the factor a , τ and a being, in general, different for each element; and that these signals are then added. Such an array is shown in Fig. 1. We assume, for convenience, that the mean-square amplitude of the signal at the output of each transducer is unity, and that the mean-square amplitude of the background noise at the output of each transducer is also unity. After the proper delay compensation for travel time, the signals from all elements will be in phase, and the total signal at the output will be

$$s(t) \sum_{i=1}^m a_i.$$

We apply the constraint

$$R = \sum_{i=1}^m a_i - 1 = 0 \quad (1)$$

to insure that, whatever the set of coefficients a_i , the mean-square signal amplitude at the output of the array will be unity. Now assume that the background noise received by the i^{th} transducer is $n_i(t)$; after compensation and shading, the contribution of the i^{th} transducer to the output noise is $a_i n_i(t - \tau_i)$, and the mean-square output noise from the entire array is

$$\begin{aligned} N^2 &= \left[\sum_{i=1}^m a_i n_i(t - \tau_i) \right]^2 \\ &= \sum_{i=1}^m \sum_{j=1}^m a_i a_j \overline{n_i(t - \tau_i) n_j(t - \tau_j)} \\ &= \sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho_{i,j}(\tau_j - \tau_i) \end{aligned} \quad (2)$$

where $\rho_{i,j}(\tau_j - \tau_i)$ is the cross-correlation function³ of the background noises

³Faran, J. J., Jr., and R. Hills, Jr., Correlators for Signal Reception, Technical Memorandum No. 27, Acoustics Research Laboratory, Harvard University, Cambridge, Massachusetts. September 15, 1952, Chap. 1.

received by the i^{th} and j^{th} elements, and is normalized in the sense that $\rho_{i,i}(0) = \rho_{j,j}(0) = 1$, and also in the sense that

$$\rho_{i,j}(\tau_j - \tau_i) = \frac{n_i(t - \tau_i) n_j(t - \tau_j)}{\sqrt{n_i^2(t) \cdot n_j^2(t)}}.$$

These correlation coefficients can be calculated easily for nondirectional transducers in an isotropic background noise field having certain spectra; these calculations are presented in the Appendix. For nonisotropic noise fields, if they could not readily be calculated, they might still be measured experimentally with sufficient accuracy to allow shading of an array for optimum operation in that particular nonisotropic background noise field. Because, by (1), the mean-square signal at the output of the array is unity, the mean-square noise output given by (2) is the reciprocal of the output signal-to-noise ratio, and because the signal-to-noise ratio at the input of any transducer was chosen to be unity, (2) is also the reciprocal of the signal-to-noise-improvement power ratio. In the special case where the correlation functions correspond to an isotropic noise background (defined in the Appendix), (2) is the reciprocal of the directivity factor, DF. However, because the analysis applies to cases where the background noise is not isotropic, if the correlation functions can be determined, we shall refer to (2) generally as the reciprocal of the signal-to-noise-improvement factor, IF, whence

$$\frac{1}{\text{IF}} = \sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho_{i,j}(\tau_j - \tau_i). \quad (3)$$

If there is noise generated in the individual transducers (or their preamplifiers), the output signal-to-noise-improvement will be less than that given by (3), and can be considerably less, if the shading coefficients are large. In order to deal with possibility, we define the effective signal-to-noise-improvement factor EF to be the mean-square output signal-to-noise ratio when the mean-square input signal-to-noise ratio is unity. Obviously, if there is no transducer self-noise, $\text{EF} = \text{IF}$. Now let the rms amplitude of the self noise $n_i^s(t)$ at the output of the i^{th} receiver be β_i ; then the mean-square noise at the output of the array (after compensation and

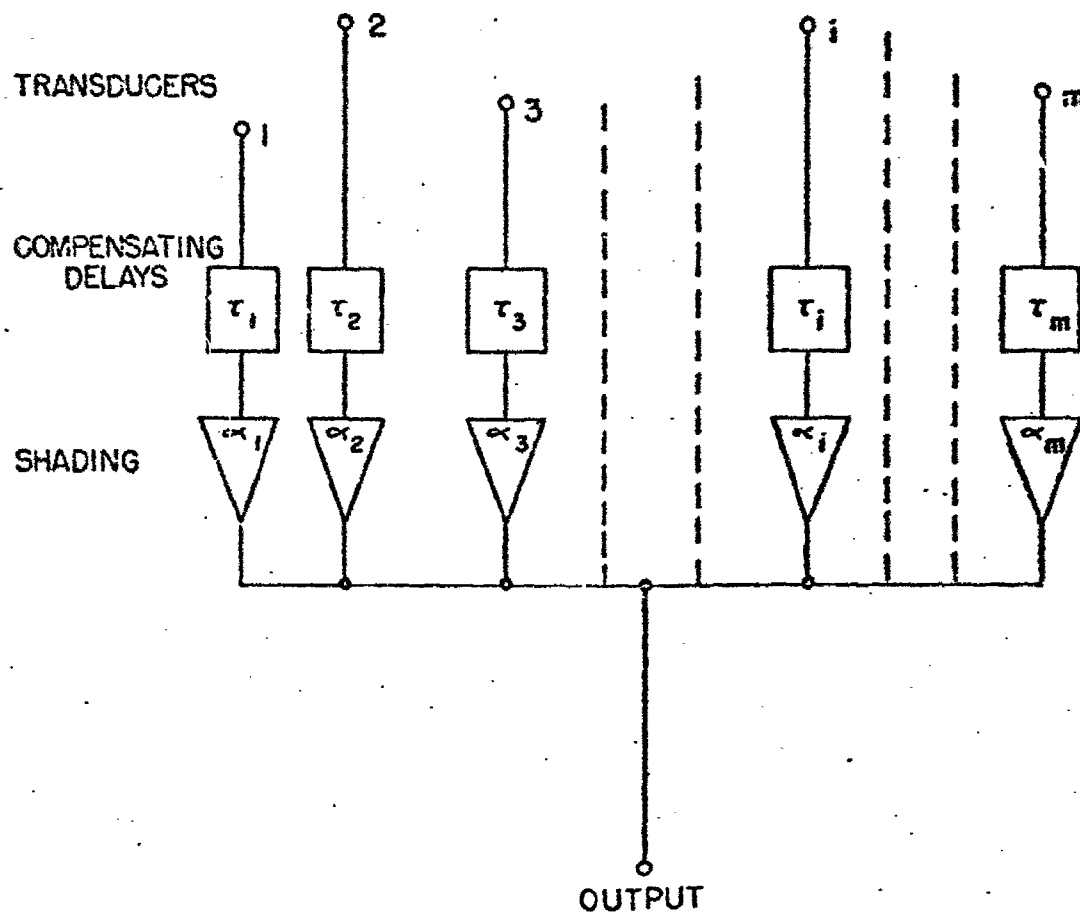


Fig. 1. Array of m receiving transducers with delay networks to compensate for the possibly different travel times from the signal source to the various elements and amplifiers whose gains are equal to the shading coefficients. The sum of the outputs of these amplifiers is the output of the array.

shading) is

$$\begin{aligned}
 N^2 &= \overline{\left(\sum_{i=1}^m a_i [n_i(t-\tau_i) + n_i'(t-\tau_i)] \right)^2} \\
 &= \sum_{i=1}^m \sum_{j=1}^m a_i a_j \overline{n_i(t-\tau_i) n_j(t-\tau_j)} + \sum_{i=1}^m a_i^2 \overline{n_i'^2(t-\tau_i)} \\
 &= \sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho_{i,j}(\tau_j - \tau_i) + \sum_{i=1}^m a_i^2 \beta_i^2,
 \end{aligned}$$

where it has been assumed that the self-noises are not coherent either with each other or with the noises from the medium. The effective signal-to-noise improvement factor is then given by

$$\frac{1}{EF} = \sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho_{i,j}(\tau_j - \tau_i) + \sum_{i=1}^m a_i^2 \beta_i^2. \quad (4)$$

When the correlation coefficients used in the above expressions correspond to an isotropic noise background (and there is no transducer self-noise), IF is equal to the directivity factor, and the directional gain of the array is simply this directivity factor expressed in decibels. In a similar manner (when there is self-noise), we can define an effective gain of the array to be equal to EF, as defined above and expressed in decibels, when the correlation coefficients correspond to an isotropic background.

III.

Maximization of the Directional Gain or of the Effective Gain

The coefficients a_i for maximum DF, IF, or EF are readily determined by finding those values which produce extrema of the expressions (3) or (4) subject to the constraint (1). This is easily accomplished by the method of

the Lagrangian multiplier.⁴ The coefficients are found by solving the following set of $m+1$ simultaneous linear equations in the $m+1$ unknowns $a_1, a_2, \dots, a_m$, and μ for the a 's: For maximum improvement factor,

$$\left. \begin{aligned} \frac{\partial(1/IF)}{\partial a_1} + \mu \frac{\partial R}{\partial a_1} &= 0 \\ \frac{\partial(1/IF)}{\partial a_2} + \mu \frac{\partial R}{\partial a_2} &= 0 \\ \dots\dots\dots \\ \frac{\partial(1/IF)}{\partial a_m} + \mu \frac{\partial R}{\partial a_m} &= 0 \end{aligned} \right\} \quad (5)$$

$$R = 0;$$

for maximum directivity factor DF or maximum effective improvement factor EF, IF in the equations above may be replaced by DF or EF, using the appropriate expression, (3) or (4).

These equations are considerably simplified in the case of a linear array of equally spaced elements arranged for broadside reception in an isotropic background, since no delays are needed and the shading will be symmetrical. We first renumber the elements to take advantage of this symmetry and reduce the number of unknowns. Whereas before, we numbered the m elements

$$1, 2, 3, \dots, m,$$

we now number them

$$-(m-1)/2, \dots, -2, -1, 0, 1, 2, \dots, (m-1)/2,$$

if m is odd, and

$$-m/2, \dots, -2, -1, 1, 2, \dots, m/2,$$

if m is even, where we now understand that $a_{-1} = a_1 = a_m$. For nondirectional transducers in an isotropic background noise field, and with no artificial delays, the background noise crosscorrelations are a function only of the distance between the elements, as is shown in the Appendix. In general, it is convenient to write the crosscorrelation function as $P(d, \tau)$, where d is the distance between the two points and τ is the relative (artificial) delay.

⁴ Sokolnikoff, I. S., and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, 2nd ed., McGraw-Hill, New York (1941), p. 163.

In terms of this function,

$$\rho_{i,j}(0) = P(|i-j|d, 0),$$

where d is the distance between adjacent elements ($d' = |i-j|d$). Then, for m odd,

$$\left. \begin{aligned} \frac{1}{DF} &= \sum_{i=-(m-1)/2}^{(m-1)/2} \sum_{j=-(m-1)/2}^{(m-1)/2} a_{|i|} a_{|j|} P(|i-j|d, 0), \\ \frac{1}{EF} &= \frac{1}{DF} + \sum_{i=-(m-1)/2}^{(m-1)/2} a_{|i|}^2 \beta_{|i|}^2, \\ R &= \sum_{i=-(m-1)/2}^{(m-1)/2} a_{|i|} - 1 = 0. \end{aligned} \right\} \quad (6)$$

and

$$R = \sum_{i=-(m-1)/2}^{(m-1)/2} a_{|i|} - 1 = 0.$$

while, for m even,

$$\left. \begin{aligned} \frac{1}{DF} &= \sum_{i=-m/2}^{m/2} \sum_{j=-m/2}^{m/2} a_{|i|} a_{|j|} P(|i-j|d, 0), \\ \frac{1}{EF} &= \frac{1}{DF} + \sum_{i=-m/2}^{m/2} a_{|i|}^2 \beta_{|i|}^2, \\ R &= \sum_{i=-m/2}^{m/2} a_{|i|} - 1 = 0. \end{aligned} \right\} \quad (7)$$

and

$$R = \sum_{i=-m/2}^{m/2} a_{|i|} - 1 = 0.$$

Since, because of symmetry, differentiation with respect to a_{-j} gives the same equation as does differentiation with respect to a_j , it is only necessary to solve the following set of $(m+1)/2$ equations in the $(m+1)/2$ unknowns $a_0, a_1, \dots, a_{(m-1)/2}$ for the a 's, if m is odd,

$$\left. \begin{aligned} \frac{\partial(\frac{1}{DF})}{\partial a_0} + \mu \frac{\partial R}{\partial a_0} &= 0 \\ \dots\dots\dots \end{aligned} \right\} \quad (8)$$

$$\left. \frac{\partial(\frac{1}{DF})}{\partial a_{(m-1)/2}} + \mu \frac{\partial R}{\partial a_{(m-1)/2}} = 0 \right\}$$

$$R = 0;$$

or to solve the following set of $m/2+1$ equations for the a 's, if m is even,

$$\frac{\partial(\frac{1}{DF})}{\partial a_1} + \mu \frac{\partial R}{\partial a_1} = 0$$

$$\frac{\partial(\frac{1}{DF})}{\partial a_{m/2}} + \mu \frac{\partial R}{\partial a_{m/2}} = 0$$

$$R = 0.$$

(9)

The directivity factor or effective signal-to-noise-ratio improvement factor achieved with a set of coefficients a_i is readily calculated by evaluating the appropriate expression in (6) or (7), whence the directional gain or effective gain is obtained by conversion to decibels.

It is perhaps well to bear in mind that the method of the Lagrangian multiplier only determines sets of coefficients which correspond to extrema of the function under consideration, and that whether this is a minimum (as we desire here, since we are working with the reciprocals of directivity factors) or a maximum must be determined afterwards. We have not, however, encountered anything other than minima in solving these equations.

IV.

Numerical Examples

The excitation coefficients and directivity indices, or directional gains, have been computed by this method for linear arrays of five and nine non-directional transducers. Calculations were made for a wide range of bandwidths of noise whose spectrum is the same as the response spectrum of a single series-tuned-circuit filter. The background noise in a receiving system would have this spectrum if the noise in the medium were essentially "white"

and if there were such a filter in the receiver. The element-to-element spacing of the arrays was $1/8$ wavelength at the center frequency of the spectrum; the over-all lengths of the arrays were then $1/2$ and 1 wavelength, respectively, at the center frequency. The shading coefficients for maximum directional gain at various bandwidths were found first by solving Eqs. (8) using the correlations of Table A-1; then the directional gains which resulted from using each particular set of shading coefficients at a number of different operating bandwidths were computed from the first of Eqs. (6).

The directional gain data for the five-element array are summarized in Table I. The first set of entries is for uniform shading where each a is equal to $1/5$. The directional gain for single-frequency operation ($Q = \infty$) is 1.62 db, and it increases somewhat as the bandwidth is increased. The next group of entries is for the case of shading to produce maximum directional gain for single-frequency operation. If the array could be used with a receiver that would accept only a single frequency, a directional gain of 5.54 db could be realized. If, however, the receiver pass-band was finite (as it must be in a practical case) but narrow, corresponding to a Q of 1000 , for example, the directional gain would fall to -2.84 db, and it would continue to fall, and rapidly so, if the operating bandwidth were further increased. On the other hand, if the array were shaded for maximum directional gain at $Q = 1000$, a directional gain at this bandwidth of 3.91 db would result. As the rest of the data in this table indicate, a significant increase in the directional gain over that of the uniform array can be effected, but the increase is less at the wider bandwidths. Furthermore, the wider the design bandwidth, the wider the range of bandwidths at which the array will operate with a directional gain significantly greater than that for the uniform array. The data of this table are presented graphically in Fig. 2. Schematic diagrams of the shading coefficients which produce maximum directional gain at these different bandwidths are shown in Fig. 3; the wider the design bandwidth, the more nearly the shading coefficients approach the value for the uniform array. The increased weighting given the end-elements seems to be typical of wide-band maximum-gain designs.

The same data for the nine-element array are presented in Table II and

in graphical form in Fig. 4. Even starting with 13 significant figures, and carrying 12 through the computation, it was impossible to evaluate the coefficients with sufficient accuracy (8 significant figures) to compute the directional gain for this array for operation at a single frequency. In order to make this example as complete as possible, a probable value for the directional gain for single-frequency operation, 7.6 db, has been estimated from Fig. 4-7 of Technical Memorandum No. 21.1. It was not possible to compute the directional gain for this shading at any other bandwidth, either, for the same reason, but the gain must fall very, very rapidly as the bandwidth is increased, if we are to judge by the performance of the five-element array. No difficulty was encountered in computing on a desk calculator the cases for the nine-element array where $Q = 100, 8, \text{ or } 2$, and these data show much the same behaviour as those for the smaller array. The shading coefficients are presented schematically for this case in Fig. 5. Here we see that by increasing the design bandwidth from zero to correspond to $Q = 100$, the coefficients are made to fall from the order of 30,000 to the order of 11. The directional gain achieved at $Q = 100$ is 1.42 db greater than that for the uniform array, and is unquestionably far greater than that for the single-frequency design, if it were operated at this bandwidth.

The effect of self-noise in the individual transducers (or their preamplifiers) on the effective gain of an array can be calculated from the second of Eqs. (6). For example, for the five-element linear array discussed above, designed for maximum directional gain for single-frequency operation, although the directional gain is 5.54 db, transducer self-noise 40 db below the noise from the medium at the output of each transducer will decrease the effective gain of the array to 0.15 db. If the self-noise is only 20 db below the noise from the medium, the effective gain of the array is reduced to -18.40 db! If, on the other hand, the array is designed for maximum effective gain in the presence of self-noise which is 20 db down, the effective gain can be made 3.63 db. These data are included with others for wide-band operation in Table III. Because

* It has been suggested that, once these coefficients are known approximately, by changing to a new set of variables scaled so that each of the new variables is of the order of unity, it might be possible to make the necessary improvement in the accuracy of the determined values. Unfortunately, time did not permit investigation of this technique.

Table I

DIRECTIONAL GAIN IN RECEPTION

for five-element linear array with spacing equal to $1/8$ wavelength at center frequency of tuned-circuit spectrum

Shading	Q of Operating Spectrum	Directional Gain
Uniform	∞	1.62 db
"	8	1.84
"	2	2.47
Max. Gain at Q = ∞	∞	5.54
"	1000	-2.84
"	100	-12.22
"	8	-22.94
"	2	-29.12
Max. Gain at Q = 1000	∞	4.04
"	1000	3.91
"	100	2.99
"	8	-2.32
"	2	-7.50
Max. Gain at Q = 100	∞	3.83
"	1000	3.82
"	100	3.74
"	8	2.79
"	2	0.92
Max. Gain at Q = 8	∞	3.51
"	100	3.48
"	8	3.18
"	2	2.47
Max. Gain at Q = 2	∞	2.70
"	100	2.71
"	8	2.81
"	2	3.08

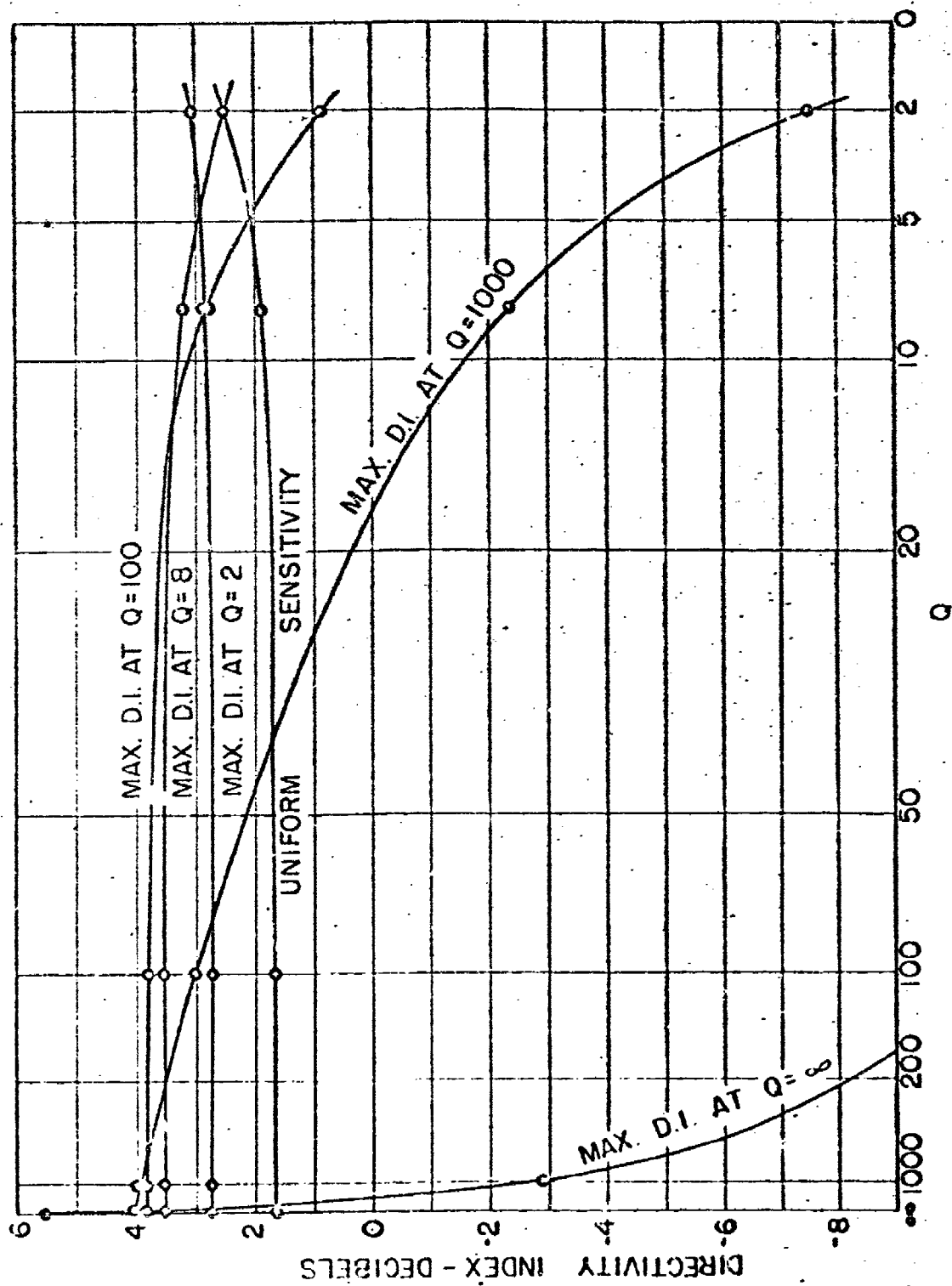


Fig. 2. Directivity Index (directional gain) of a five-element linear array with spacing equal to $1/8$ wavelength at the center frequency of a tuned-circuit spectrum as a function of the Q of the spectrum at which it is operated; for the uniformly sensitive array, and for arrays designed for maximum directivity index at the indicated values of the spectrum Q .

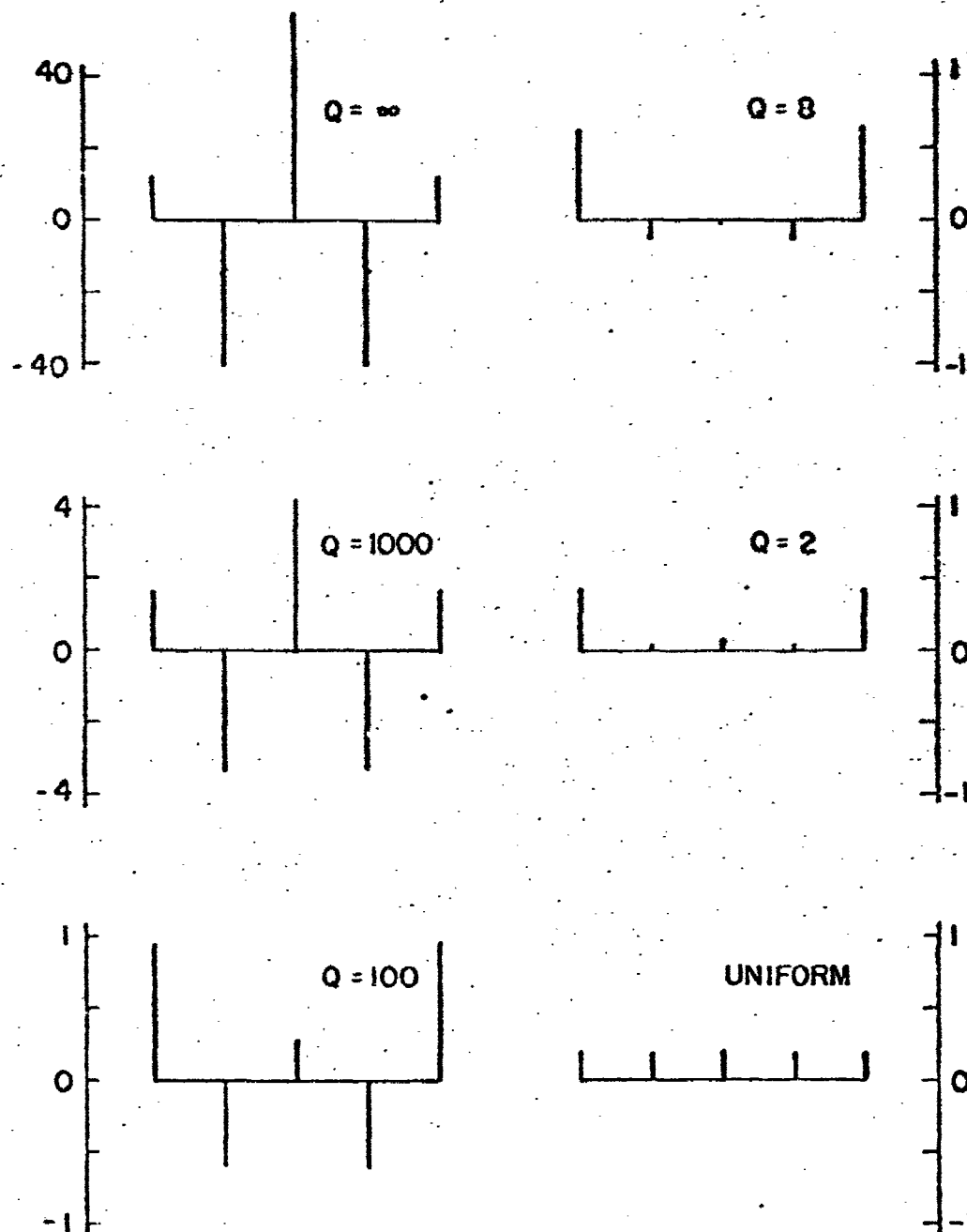


Fig. 3. Schematic diagrams of the shading coefficients for a five-element linear array with spacing equal to $1/8$ wavelength at the center frequency of a tuned-circuit spectrum; for the uniformly sensitive array, and for arrays designed for maximum directivity index (directional gain) at the indicated value of the spectrum Q .

Table II
DIRECTIONAL GAIN IN RECEPTION

for nine-element linear array with spacing equal to $1/8$ wavelength at center frequency of tuned-circuit spectrum

Shading	Q of Operating Spectrum	Directional Gain
Uniform	∞	3.94 db
"	100	3.96
"	8	4.12
"	2	4.66
Max. Gain at $Q = \infty$	∞	7.6 (estimated)
Max. Gain at $Q = 100$	∞	5.68
"	1000	5.65
"	100	5.38
"	8	2.93
"	2	-0.56
Max. Gain at $Q = 8$	∞	4.88
"	100	4.86
"	8	4.82
"	2	4.71
Max. Gain at $Q = 2$	∞	4.61
"	1000	4.61
"	100	4.62
"	8	4.71
"	2	5.07

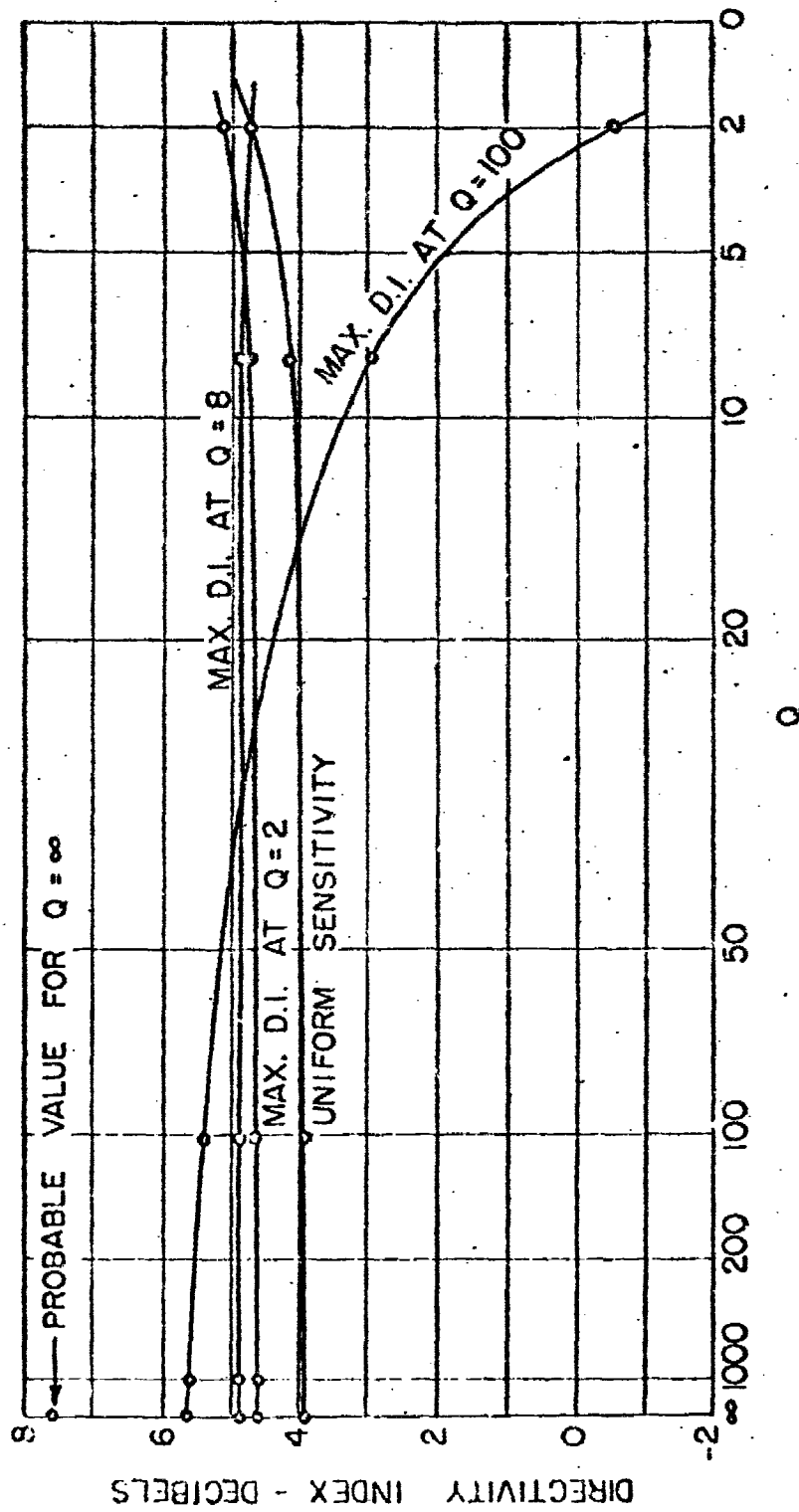


Fig. 4. Directivity index (directional gain) of a nine-element linear array with spacing equal to $1/8$ wavelength at the center frequency of a tuned-circuit spectrum as a function of the Q of the spectrum at which it is operated; for the uniformly sensitive array, and for arrays designed for maximum directivity index at the indicated values of the spectrum Q .

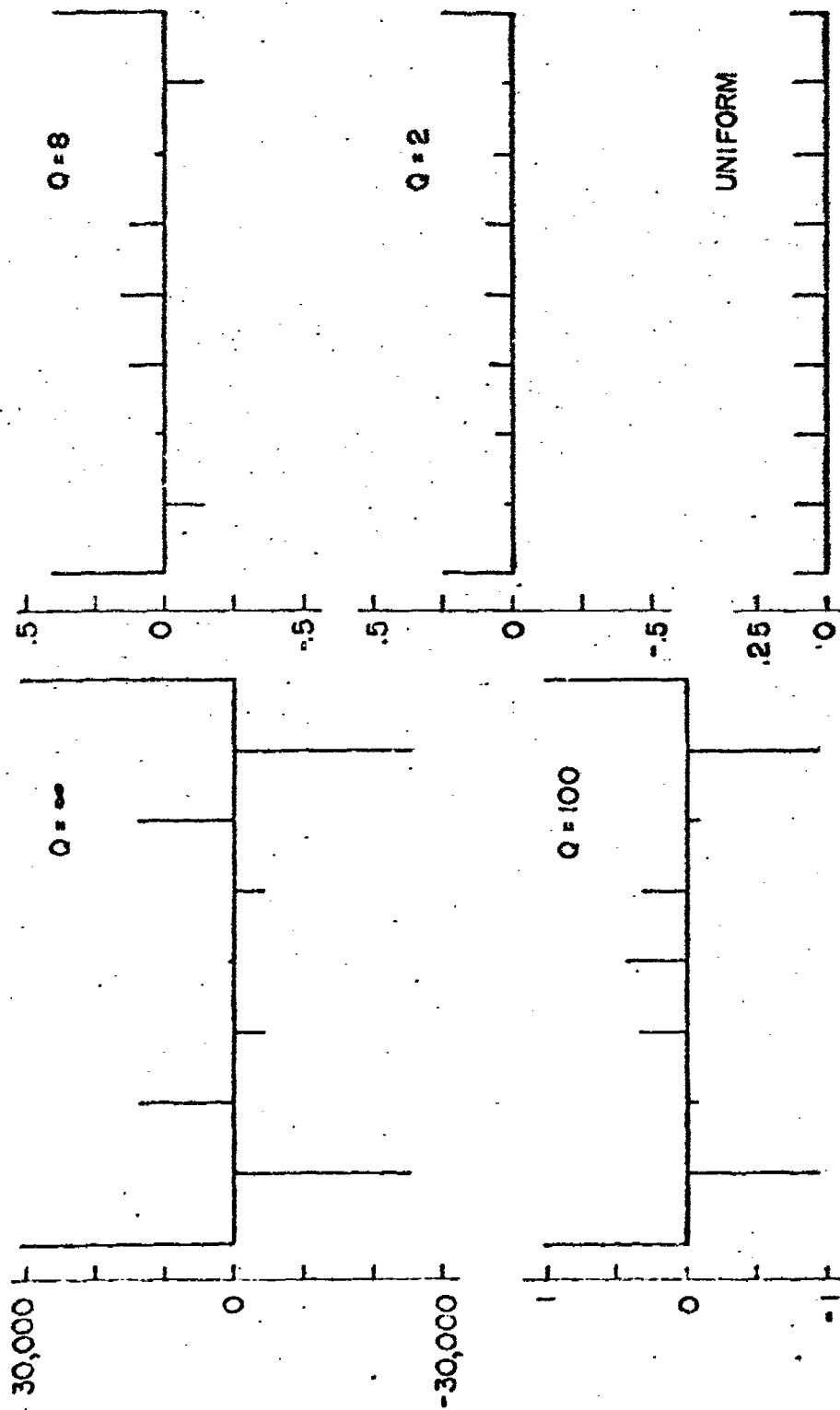


Fig. 5. Schematic diagrams of the shading coefficients for a nine-element linear array with spacing equal to $1/8$ wavelength at the center frequency of a tuned-circuit spectrum; for the uniformly sensitive array, and for arrays designed for maximum directivity index (directional gain) at the indicated value of the spectrum Q .

self-noise reduces the effective gain drastically only when the shading coefficients are very large, its effect is slight on arrays designed for wide-band operation, where, as we see from Fig. 3, the shading coefficients are relatively small. Included in Table III are the directional gains which these self-noise designs would have if the self-noise were not present. These figures indicate that it is possible to design so as to protect against the possible presence of self-noise, while achieving a design which would still produce a fairly large directional gain.

V.

Summary

A method has been demonstrated for determining the shading coefficients of a receiving array which produces the greatest possible mean-square signal-to-noise ratio at the output, in the presence of noise arising either in the signal-bearing medium or in the transducers themselves or both. From the numerical examples presented, it appears that the maximum directional gain of an equally-spaced broadside array for wide-band operation is less than that for single-frequency operation, but usually much more than that obtained by use of the single-frequency design at the wide bandwidth, and, except for very wide-band cases, is significantly greater than that obtained by using uniform sensitivity. The effect of small amounts of self-noise arising in the transducers is small except in arrays designed for maximum directional gain at very narrow bandwidths; in these cases, by taking this self-noise into account, the effective gain of the array can be made near to what the directional gain would be in the absence of such self-noise.

Although it is now possible in principle to insert the functional form of the noise crosscorrelation function into (3) or (4) and maximize the gain of the array not only with respect to the shading coefficients but also with respect to the spacings of the elements (along a line), this procedure has not been attempted here, for fairly obvious reasons. By further accounting for the effects of various time delays applied to the signals from the different elements, it might be possible to maximize the gain of the array in terms of its spatial configuration. Certainly only slight changes in the functions of Table A-1 result in vastly different sets of coefficients for the equally spaced linear array; if some of these slight changes can be compensated by changes in spacing along a line or

in spatial configuration of the array, increased directional gains at all bandwidths might be found.

Appendix

Computation of Background Noise Crosscorrelation in an Isotropic Noise Field*

An isotropic noise field may be defined as one in which the total noise power received by a directional receiver is independent of both its location and its angular orientation. For example, imagine a uniform distribution of infinitesimal statistically-independent random noise sources on the inner surface of a very large sphere. If all these sources emit noise having the same spectrum, the sound field in the vicinity of the center of the sphere may be considered isotropic.** The crosscorrelation between the total pressures at two different points in such an isotropic noise field can be calculated by integrating the crosscorrelation between the pressures produced at those two points by a single point source on the large sphere over all possible positions of the point source on the sphere. This crosscorrelation has been derived previously and numerically evaluated for the case of a rectangular spectrum one octave in width.⁵ We here again derive these results in somewhat different notation, and also evaluate the crosscorrelation for tuned-circuit spectra.

Assume that a single point source is located on a large sphere concentric with the origin at the spherical coordinate angles θ and ϕ . Assume that this source generates random noise and that the pressure at the origin generated

*The context of this appendix was previously presented as Chapter III of Technical Memorandum No. 28, where use was made of it in a different connection. Because of its fundamental importance to this work, it is included here in substantially the same form, for the convenience of the reader.

**The above definition is a broad-band equivalent of the background distribution specified in the standard definition of directivity factor in American Standard Acoustical Terminology, American Standards Association, Inc., New York (July 31, 1951).

⁵Marsh, H. W., Jr., "Correlation in Wave Fields", A declassified portion of the Quarterly Report for the period ending March 31, 1950, U. S. Navy Underwater Sound Laboratory, New London, Connecticut.

Table III

EFFECTIVE GAIN IN RECEPTION WITH NOISY TRANSDUCERS

for five-element linear array with spacing equal to $1/8$ wavelength at center frequency of tuned-circuit spectrum

Q of Spectrum	Self-Noise*	Designed for Maximum Directional Gain		Designed for Maximum Effective Gain	
		Directional Gain	Effective Gain	Directional Gain	Effective Gain
∞	-40 db	5.54 db	0.15 db		
"	-30	"	-8.56		
"	-20	"	-18.40	3.80 db	3.63 db
1000	-20	3.91	0.62	3.79	3.64
100	-20	3.74	3.47	3.71	3.57
8	-20	3.18	3.10	3.18	3.11
2	-20	3.08	3.05	3.08	3.05

* at output of each transducer in decibels relative to noise from medium at output of that transducer

thereby may be represented by the random function $f(t)$. The pressure at a distance d from the origin in the positive direction on the axis of ϕ (z -axis) is then $f[t + (d/c)\cos\theta]$, where c is the sound velocity in the medium. If $\rho(\tau)$ is the normalized autocorrelation function of $f(t)$, the normalized cross-correlation of the pressures at these two points is $\rho[\tau + (d/c)\cos\theta]$.

If the isotropic noise field is assumed to have a mean-square pressure of unity in the vicinity of the origin, the contribution to this mean-square pressure from an element of solid angle $\sin\theta d\theta d\phi$ will be $(1/4\pi) \sin\theta d\theta d\phi$. The expression for the crosscorrelation in the isotropic noise field is then

$$\begin{aligned} P(d, \tau) &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \rho[\tau + (d/c)\cos\theta] \sin\theta d\theta d\phi \\ &= \frac{1}{2} \int_0^\pi \rho[\tau + (d/c)\cos\theta] \sin\theta d\theta. \end{aligned} \quad (A1)$$

The symbol capital rho (P) is used here to indicate that this crosscorrelation function is normalized (in the sense that $P(0, 0) = 1$). There are two possible methods of evaluating this integral:

- (1). We can introduce the new variable $\tau' = \tau + (d/c)\cos\theta$ so that

$$P(d, \tau) = \frac{c}{2d} \int_{-d/c+\tau}^{d/c+\tau} \rho(\tau') d\tau'.$$

This formula must be used with care whenever $\rho(\tau)$ is explicitly a function of $|\tau|$; otherwise its use is convenient whenever the indefinite integral of $\rho(\tau)$ is easily found.

- (2). The autocorrelation function $\rho(\tau)$ may be written as the Fourier transform of the intensity spectrum of $f(t)$:

* See reference 3. For convenience in carrying out integration in the complex plane we assume here that $W(\omega)$ is an even function of ω on the real axis, which is possible because $\rho(\tau)$ is real. A difference of a factor 2 is implicit in the definitions of the spectra for positive, real ω .

$$\rho(\tau) = \int_{-\infty}^{\infty} W(\omega) e^{j\omega\tau} d\omega.$$

Since $\rho(\tau)$ is a normalized autocorrelation function, the intensity spectrum $W(\omega)$ above is also assumed to be normalized so that

$$\int_{-\infty}^{\infty} W(\omega) d\omega = 1. \quad (A2)$$

Equation (A1) may then be written

$$\begin{aligned} P(d, \tau) &= \frac{1}{2} \int_0^\pi \int_{-\infty}^{\infty} W(\omega) e^{j\omega[(d/c)\cos\theta + \tau]} \sin\theta d\omega d\theta \\ &= \int_{-\infty}^{\infty} W(\omega) \frac{\sin(\omega d/c)}{(\omega d/c)} e^{j\omega\tau} d\omega. \end{aligned} \quad (A3)$$

This formula is useful when $W(\omega)$ can be written explicitly as an analytic function of ω , in which case it may be readily evaluated by contour integration.

We first find $P(d, 0)$ for a rectangular spectrum. The intensity spectrum of the noise field is assumed to have the constant value $1/2\Delta\omega$ between $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$, and to be zero elsewhere. From (A3) we then have

$$\begin{aligned} P(d, 0) &= \frac{1}{2\Delta\omega} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} \frac{\sin(\omega d/c)}{(\omega d/c)} d\omega \\ &= \frac{c}{2d\Delta\omega} \int_0^{\frac{d}{c}(\omega_0 + \Delta\omega)} \frac{\sin x}{x} dx - \frac{c}{2d\Delta\omega} \int_0^{\frac{d}{c}(\omega_0 - \Delta\omega)} \frac{\sin x}{x} dx \\ &= \frac{c}{2d\Delta\omega} \left[\text{Si}\left(\frac{\omega_0 d}{c} + \frac{\Delta\omega d}{c}\right) - \text{Si}\left(\frac{\omega_0 d}{c} - \frac{\Delta\omega d}{c}\right) \right] \\ &= \frac{c}{4\pi d\Delta f} \left[\text{Si}\left[\frac{2\pi d}{c}(f_0 + \Delta f)\right] - \text{Si}\left[\frac{2\pi d}{c}(f_0 - \Delta f)\right] \right], \end{aligned}$$

where $\text{Si}(\)$ is the sine-integral function.

VALUES OF THE FUNCTION $P(d, 0)$ FOR THE TUNED-CIRCUIT SPECTRUM

as a function of d/λ_0 , where λ_0 is the wavelength at the center frequency of the tuned-circuit spectrum, it having been assumed that $\omega' = \omega_0$; for various values of Q .

d/λ_0	$P(d, 0)$	d/λ_0	$P(d, 0)$
$Q = \infty$		$Q = 8$	
0.0	1.0000 0000	0.0	1.0000 0000
0.125	0.9003 1632	0.125	0.8571 8930
0.25	0.6366 1977	0.25	0.5770 8975
0.375	0.3001 0544	0.375	0.2590 1131
0.5	0.0000 0000	0.5	0.0000 0000
0.625	-0.1800 6326	0.625	-0.1408 7477
0.75	-0.2122 0659	0.75	-0.1580 6968
0.875	-0.1286 1662	0.875	-0.0912 1545
1.0	0.0000 0000	1.0	0.0000 0000
$Q = 1000$		$Q = 2$	
0.0	1.0000 0000	0.0	1.0000 0000
0.125	0.8999 6283	0.125	0.7398 1238
0.25	0.6361 1997	0.25	0.4298 6598
0.375	0.2997 5209	0.375	0.1665 1497
0.5	0.0000 0000	0.5	0.0000 0000
0.625	-0.1797 1005	0.625	-0.0674 6173
0.75	-0.2117 0718	0.75	-0.0653 3076
0.875	-0.1282 6355	0.875	-0.0325 3737
1.0	0.0000 0000	1.0	0.0000 0000
$Q = 100$			
0.0	1.0000 0000		
0.125	0.8967 8772		
0.25	0.6316 3935		
0.375	0.2965 9065		
0.5	0.0000 0000		
0.625	-0.1765 6221		
0.75	-0.2072 6503		
0.875	-0.1251 2924		
1.0	0.0000 0000		

We now find the complete function $P(d, \tau)$ for the tuned-circuit spectrum. The intensity spectrum of "white" noise which has been passed through a series-tuned circuit is (normalized to satisfy (A2))

$$W(\omega) = \frac{(2\omega_F/\pi)\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\omega^2\omega_F^2}$$

where $\omega_0/2\pi$ is the center frequency, $\omega_F/2\pi$ is the half-band-width, and $Q = \omega_0/2\omega_F$. The function $W(\omega)$ is an analytic function of the complex variable ω having four simple poles, two each in the upper and lower half-planes. If the sine function in the integrand of (A3) is written as the sum of two exponentials, the integral may be evaluated by contour integration. On the real axis, $W(\omega)$ is a real, even function of ω , and $[\sin(\omega d/c)]/(\omega d/c)$ is also real and even. Because the Fourier transform of a real, even function is also a real, even function, it is only necessary to evaluate $P(d, \tau)$ for positive τ , the values for negative τ being obtained from

$$P(d, -\tau) = P(d, \tau).$$

We thus evaluate

$$P(d, \tau) = \frac{c}{2jd} \left[\int_{-\infty}^{\infty} \frac{W(\omega)}{\omega} e^{j\omega(d/c + \tau)} d\omega - \int_{-\infty}^{\infty} \frac{W(\omega)}{\omega} e^{-j\omega(d/c - \tau)} d\omega \right].$$

The contour integration is carried out in straightforward manner, assuming that $\tau \geq 0$, it being noted that the second integral is carried around the lower or upper half-plane depending respectively, on whether τ is less than, or greater than, d/c . The result is, for all τ ,

$$P(d, \tau) = \frac{ce^{-\omega_F d/c}}{2d\omega'} \left[e^{-\omega_F |\tau|} \sin \omega' \left(\frac{d}{c} + |\tau| \right) + e^{\omega_F |\tau|} \sin \omega' \left(\frac{d}{c} - |\tau| \right) \right], \quad |\tau| \leq d/c,$$

$$= \frac{ce^{-\omega_F |\tau|}}{2d\omega'} \left[e^{-\omega_F d/c} \sin \omega' \left(\frac{d}{c} + |\tau| \right) + e^{\omega_F d/c} \sin \omega' \left(\frac{d}{c} - |\tau| \right) \right], \quad |\tau| \geq d/c,$$

where

$$\omega' = \sqrt{\omega_0^2 - \omega_F^2} = \omega_0 \sqrt{1 - 1/4Q^2}.$$

When $\tau = 0$ this crosscorrelation function reduces to

$$P(d, 0) = e^{-\omega_F d/c} \left[\frac{\sin(\omega' d/c)}{(\omega' d/c)} \right]. \quad (A4)$$

Values of the function $P(d, 0)$ computed from (A4) for various values of Q are tabulated in Table A-I. In computing these values, the simplifying assumption that $\omega' = \omega_0$ was made. The numerical examples cited in this memorandum were based on the values given in this table; the result of having made the assumption is that the correct spacings of the arrays for which values were computed are larger than the value quoted by a factor

$$\frac{1}{\sqrt{1 - 1/4Q^2}}$$

are the true values of Q are smaller by a factor

$$\sqrt{1 - 1/4Q^2}$$

The effect on the computed directional gains of having made this assumption is almost certainly negligible.