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FLUID MOTION SUB-COMMITTEE

AERONAUTICAL RESEARCH COUNCIL

On the Energy Scattered from the Interaction of Turbulence with Sound or Shock Waves

- By -
M. J. Lighthill, B.A.,
(Department of Mathematics, University of Manchester).

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Summary

The energy scattered when a sound wave passes through turbulent fluid flow is studied by means of the author's general theory of sound generated aerodynamically. The energy scattered per unit time from unit volume of turbulence is estimated (§3) as

$$\frac{8\pi^2 L_1}{\Lambda^2} I \frac{\overline{v_1'^2}}{a^2},$$

where I is the intensity and Λ the wavelength of the incident sound, and $\overline{v_1'^2}$ is the mean square velocity and L_1 the macro-scale of the turbulence in the direction of the incident sound. This formula does not assume any particular kind of turbulence, but does assume that Λ/L_1 is less than about 1. For turbulence which is isotropic and homogeneous the energy scattered, and its directional distribution, are obtained for arbitrary values of Λ/L_1. It is predicted that components of the turbulence with wave-number k will scatter sound of wave-number κ at an angle $2\sin^{-1}(k/2\kappa)$. The statistics of multiple successive scatterings is considered (§4), and it is predicted that sound of wavelength less than the micro-scale of the turbulence will become uniform (i.e. quite random) in its directional distribution in a distance approximately $\lambda a^2/\overline{v_1'^2}$.

The theory is extended (§5) to the case of an incident acoustic pulse. However this extended theory cannot be applied directly to the case of a shock wave, for which it would predict infinite scattered energy. This is due to the perfect resonance between successive rays emitted forwards which would occur if the shock wave were propagated at the speed of sound. By taking into account (§6) the true speed of the shock wave (subsonic relative to the fluid behind it), the theory is improved to give a finite value, 0.8s times the kinetic energy of the turbulence traversed by a weak shock of strength s, for the total energy scattered. However the greater part of this energy catches up with the shock wave, and probably is mostly re-absorbed by it, and only the remainder (tabulated as a function of s in Table 1) is freely scattered, behind the shock wave, as sound. The energy thus freely scattered when turbulence is convected through the stationary shock wave pattern in a supersonic jet may form an important part of the sound field of the jet.

Introduction/

1. Introduction

The two main types of oscillatory disturbance in fluids are sound (dilatational) and turbulence (rotational). Sound is propagated; turbulence is not; both the sound pressures and the turbulent vortex lines are convected and diffused. Infinitesimal dilatational and rotational disturbances to a uniform mass of fluid at rest would not interact, as is well known. However between real sound and turbulence there is a non-linear coupling.

One aspect of this coupling is that, when sound waves pass through a given volume of turbulence, some sound is radiated from it at an angle to the incident wave; in other words it is scattered. In the present paper a theory of this scattered sound is deduced from the author's general theory (Lighthill 1952) of sound generated aerodynamically.

The subject has received some attention in Russia, and Blokhintzev (1945, 1946) appears to have given a detailed theory. However the assumptions in this theory are not made clear in the papers available to the present author. In Britain Ellison (1951) gave an account of scattering by a medium with random variations in refractive index, and his theory could probably be modified to deal with scattering by turbulence. It confines its attentions, however, to the distribution of intensity along the wave fronts of plane waves passing through homogeneous turbulence. In the present paper the turbulence is not assumed homogeneous, and the scattered sound emerging in a given direction from a given part of the turbulence is isolated. The theory is therefore particularly suitable for dealing with the effect of a limited region of intense turbulence.

The scattering of sound due to the fluctuating velocities of turbulence bears an obvious analogy to the scattering of light due to fluctuating number density of polarizable molecules. It is shown in §2 that the analogy is even stronger than at first appears, since the quadrupoles, to which on the Lighthill (1952) theory the turbulence is acoustically equivalent, are in fact partly polarized by the incident wave.

The radiation field of those quadrupoles which are polarized is calculated in §3, and its mean square is taken to find the corresponding intensity field, which is shown to be that of the scattered sound. Since the mean has to be taken with respect to the fluctuations in the <u>turbulence</u> (as well as the more rapid fluctuations in the sound waves), a fair time may be needed for the mean to be achieved; in the language of the acoustics, "fading" may be observed. However if the intensity resulting from scattering by a volume of turbulence large compared with the individual eddies is measured, the mean or something near it may well be realised at any instant, because large groups of the eddies traversed may have effectively identical statistical properties.

It is shown in §2 that the field of the polarized quadrupoles is uncorrelated with the other sound fields which are present, which means that the calculated intensity distribution can be added to theirs to obtain the total intensity field. For some time the author believed that an argument along these lines could be used to prove that the scattered energy was extracted from the turbulence itself; he is grateful to Dr. George Batchelor for refusing to believe this sufficiently pertinaciously for the author to find the flaw in his argument. Some of this work on energy relationships is indicated in the Appendix, which shows how the incident wave is in fact attenuated by an amount comparable with the energy scattered. The author would still guess that some of the energy of turbulence may be reduced by the passage through it of (especially) ultrasonics, but the evidence seems to be that by far the bulk of the energy of the scattered sound is extracted as in other scattering phenomena, from the incident wave.

As/

As the frequency of the incident sound wave increases the total scattered energy increases but the angle at which it is scattered decreases. In consequence the distance in which the direction of a plane wave becomes practically random (§4) varies much less with frequency.

When the theory is extended (§5) from the case of steady incident sound waves to that of an incident acoustic pulse, the scattered sound field is deduced in terms of the spectrum of the total energy of the pulse (instead of the intensity spectrum of the wave). The quantity which is deduced is then the total scattered energy (and its directional distribution), as opposed to the time rate of scattering of energy. But, as with the steady sound wave, this quantity is correct only as a mean, although no longer a mean with respect to time. Rather it is a "stochastic" mean, or mean over a number of experiments. For each eddy reacts to the incident pulse in a manner depending on its state at the instant when the pulse traverses it. However, as before, the mean may be more closely achieved in one experiment if large numbers of similar eddies are traversed.

Once more the scattered energy is found to increase rapidly with the frequency of the pulse, and indeed, in the extreme case when there are discontinuities in pressure in the pulse, the theory gives a non-integrable directional distribution of scattered energy, so that the total scattered energy is infinite. This impossible result is due to an oversimplification in the theory, namely that the speed of a pressure discontinuity (shock wave) has been taken to be exactly that of sound. If this were so then pulses emitted forwards as the incident (perfectly sharp) pulse passes over them would be perfectly superimposed, and this corresponds to the infinity in the directional energy distribution at the direction of the incident pulse.

In §6 it is shown that when the departure of the speed of a real shock wave from that of sound is taken into account, this perfect resonance disappears, and a finite value for the scattered energy emerges. All that is left of the infinity is that the energy which is scattered by the shock is proportional to the first power of the amplitude, for weak shocks, instead of to the square (as it is for continuous pulses). In fact for a weak shock of strength $\Delta p/p = s$, the energy is predicted to be a fraction $0.8s$ of the total kinetic energy of the turbulence traversed by the shock.

However a large part of the scattered energy is probably absorbed by the shock wave itself and goes towards restoring, in part, its strength. This is because the shock wave speed is subsonic relative to the fluid behind it, so that sound emitted forward at a small enough angle (and there is a strong preference for forward emission) must run into the shock. It cannot be transmitted, because the shock is supersonic relative to the fluid ahead of it. Doubtless some is reflected, but there is evidence (Lighthill 1949, 1950) that reflection coefficients of shocks tend very rapidly to zero with their strength.* So it is likely that much is absorbed.

Making the assumption that all is absorbed, we are left (§6) with an expression for the sound energy which is freely scattered, i.e., without hitting the shock again. This expression should be of value in predicting how much sound is produced in a supersonic jet as a result of turbulence passing through the stationary shock waves in the jet.

2./

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* Lighthill (1950) shows this for waves catching up with the shock head-on. It is likely that the same is true for waves oblique to the shock, as the discussion of the stationary case (Lighthill 1949) already indicates.

2. Description and Justification of the Approach Used.

Let v_i be the velocity field (a function of space and time) of a turbulent flow. Similarly let V_i be the velocity field of a sound wave incident upon it. The magnitudes of v_i and V_i are supposed to remain small compared with the speed of sound a.

Then as a first approximation (from which the scattered sound will be derived as a second approximation) the combined velocity field will be taken to be $v_i + V_i$; so that the fields are made to combine linearly as they would for infinitesimal disturbances. The value of this assumption as a first approximation is examined critically later in this section.

Now the basic conclusion of Lighthill (1952) was that in any gas flow, with velocity field v_i, the variations of density are governed by the equations of sound with a right-hand side which corresponds physically to a volume distribution of quadrupoles. Further, if temperature variations are small (this point also is reconsidered below), and the magnitude of v_i is small compared with a, then a good approximation to the quadrupole strength is $\rho_o v_i v_j$ per unit volume, where ρ_o is the density on the undisturbed state.

Hence in the present problem a first approximation to the instantaneous quadrupole strength per unit volume is

$$\rho_o(v_i + V_i)(v_j + V_j) = \rho_o v_i v_j + \rho_o V_i V_j + \rho_o(v_i V_j + v_j V_i). \quad \ldots(1)$$

Thus, to obtain a second approximation to the density field, we must add up those due to the incident sound wave and to the quadrupole distribution (1).

The three parts into which this distribution has been divided on the right-hand side of (1) represent, respectively,

(i) the quadrupoles responsible for the sound already generated by the turbulence in the absence of the incident wave;

(ii) those responsible for self-modifications of wave form in the incident wave due to its finite amplitude;

(iii) those responsible for the sound generated by the interaction of the turbulence and the incident wave. These last, of strength $\rho_o(v_i V_j + v_j V_i)$ per unit volume, may be called the <u>scattering quadrupoles</u>.

Note that each scattering quadrupole has <u>one</u> of its axes in the direction of the incident wave V_i. Thus, as foreshadowed in §1, it is in this sense polarized by the incident wave.

From the point of view of the physical conclusion (Lighthill 1952, §2) that aerodynamic generation of sound is due to fluctuations in the flow of momentum across fixed surfaces, the scattered sound is seen to be produced by the momentum of the turbulence being shaken to and fro by the incident sound wave, and vice versa.

Now if the turbulent velocity field can be split up into a mean velocity field $\bar{v}_i$ and a fluctuating field v_i' with zero mean, then the scattering quadrupoles can be correspondingly split. The part

$$\rho_o(\bar{v}_i V_j + \bar{v}_j V_i) \quad . \quad \ldots(2)$$

yields a field which is correlated with the incident wave, and together they constitute the wave <u>as refracted by the mean flow</u>.

On/

On the other hand the turbulent velocities v_i' yield a field of scattering quadrupoles

$$\rho_0(v_i'V_j + v_j'V_i) \qquad \ldots(3)$$

quite uncorrelated with the incident wave. For firstly their mean is zero because that of v_i' is zero and v_i' and V_j are statistically unrelated. Hence the covariance of (3) with V_k, i.e., their mean product of their deviations from the mean, is zero, by a repetition of the same argument. Similarly the quadrupoles (3) are uncorrelated with all the other quadrupole fields which are present. But sound fields which are uncorrelated (or, in the old Rayleigh terminology, unrelated in phase) have the well-known property that their intensities can be added to give the intensity of the combined field.

It follows that the intensity field of the quadrupoles (3), if it can be evaluated, forms a genuine addition to the other sound fields which are present, and may be described briefly as the scattered energy. However one cannot conclude that the incident wave passes through the turbulence unaltered, even if there is no mean flow to refract it, because, as is shown in the Appendix, a correlation which exists between the incident wave and a <u>second</u> approximation to the scattering quadrupoles accounts for a rate of energy loss in the incident wave comparable with the rate at which energy is scattered.

Of course in so far as the incident wave is modified as it passes through the fluid, whether as a result of refraction by the mean flow, attenuation by the turbulence, or ordinary attenuation, it is the modified value of V_i which should be used in the scattering quadrupoles and in expressions which will be deduced for the scattered energy. This is the main limitation on the use of $v_i + V_i$ as a first approximation. On the other hand it is shown in the Appendix that changes in V_i due to the variable phase shifts which result from convection of sound by the turbulent flow can affect the scattered energy only negligibly.

Another source of attenuation is temperature variations, if they are present. To see how they fit into the present approach, note that the quadrupole distribution (Lighthill 1952) resulting from variations in the variables of state is

$$[(p - p_0) - a_0^2(\rho - \rho_0)]\delta_{ij} \doteqdot \frac{p_0}{c_v}(s - s_0)\delta_{ij}, \qquad \ldots(4)$$

where p, ρ, s are pressure, density and specific entropy and suffix zero refers to the undisturbed state. This shows incidentally that those <u>kinetic</u> temperature variations which result from the (nearly adiabatic) pressure fluctuations in the turbulence can be neglected. It is variations of specific entropy, rather than temperature, that produce attenuation by scattering (the two being proportional only if kinetic temperature variations are negligible), or in other words it is temperature variations which originate independently of the turbulence (although their subsequent distribution may have been influenced by turbulent convection).

Now the quadrupoles (4) are correlated with V_i and with other quadrupoles, because approximately (since conditions for a particle of fluid are nearly adiabatic)

$$\frac{\partial s}{\partial t} = -(V_i + \bar{v}_i + v_i')\frac{\partial s}{\partial x_i}, \qquad \ldots(5)$$

where/

where $\partial s/\partial x_i$ changes only slowly with time. However, since $V_i + \overline{v_i} + v_i'$ has zero correlation with the quadrupoles (3), one may infer that the effect of the entropy variations in scattering and attenuating the sound does not substantially modify the scattering due to the turbulent velocities.

Finally it should be noted that the scattered energy may in turn scatter more energy. Although this would be a small effect for moderate volumes of turbulence, it will appear in §4 that for high frequency sounds in large regions of turbulent flow it may be an important factor in producing large changes of direction in the waves.

3. Scattering of a Plane Harmonic Incident Wave

If results for a plane incident wave are worked out, then they can be applied to any incident wave which is approximately plane over distances of the order of those eddy-sizes which bear most of the turbulent energy. For the scattering quadrupole strength (3) at two points at any greater distance apart are uncorrelated, and therefore the corresponding intensity fields can simply be added. Further, if results for a harmonic incident wave are worked out then those for an arbitrary steady incident wave can be deduced if its intensity spectrum is known, as will be seen at the end of this section.

Consider then the plane harmonic incident wave given by

$$V_i = \epsilon a \cos [\kappa(x_1 - at)]\delta_{i1} . \qquad \ldots(6)$$

Its direction of travel is the x_1 direction. Its frequency is $\kappa a/2\pi$, but it is found preferable to work rather with the radian wave-number κ, because the most convenient analysis of the turbulence is with respect to such a wave-number, and because the effectiveness of a component of the turbulence in scattering the sound will be found to depend on the ratio of their wave-numbers. The intensity of the sound wave (6) is

$$I = \tfrac{1}{2}\rho_0 a^3 \epsilon^2 . \qquad \ldots(7)$$

Now the strength per unit volume, (3), of the scattering quadrupoles due to the fluctuating turbulent velocities v_i', becomes

$$\rho_0 \epsilon a \cos [\kappa(x_1 - at)](v_i'\delta_{j1} + v_j'\delta_{i1}) . \qquad \ldots(8)$$

The radiation field (Lighthill 1952) of a quadrupole distribution T_{ij} per unit volume is given by

$$\rho - \rho_0 \sim \frac{1}{4\pi a^4} \frac{x_i x_j}{x^3} \int \frac{\partial^2}{\partial t^2} T_{ij}\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{y}|}{a_0}\right) d\underset{\sim}{y} , \qquad \ldots(9)$$

the integral being taken over the whole field of quadrupoles. (x signifies the magnitude of the vector $\underset{\sim}{x}$.)

Now the form of (9), with T_{ij} given by (8), will be simplified by the assumption that typical frequencies of v_i' at a point are small compared with the frequency $\kappa a/2\pi$ of the incident wave (4), so that the differentiations need only be applied to the cosine term. Notice that if this assumption were false then the wavelength $2\pi/\kappa$ of the sound would greatly exceed the size of the eddies (because this is comparable to their root mean square velocity divided by a typical frequency, which is very small compared with the velocity of sound divided by the frequency of the sound, i.e., $2\pi/\kappa$, if the frequencies are comparable). But one would expect eddies small compared with the wavelength not to affect the sound propagation appreciably, and indeed this expectation is given quantitative support below.

Making/

Making the assumption described, (8) and (9) give

$$\rho - \rho_0 \sim - \frac{\rho_0 \epsilon \kappa^2}{2\pi a} \frac{x_i x_1}{x^3} \int \cos\left[\kappa(y_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - at)\right] v_i'\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{y}|}{a}\right) d\underset{\sim}{y}, \quad \ldots(10)$$

The intensity of the scattered energy, I_s, taken as a mean over times characteristic of the turbulent fluctuations as described in §1, is deduced as a^3/ρ_0 times the mean square of (10), namely as

$$I_s \sim \frac{\rho_0 \epsilon^2 \kappa^4 a}{4\pi^2} \frac{x_i x_j x_1^2}{x^6} \iint \overline{\cos\left[\kappa(y_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - at)\right] \cos\left[\kappa(z_1 + |\underset{\sim}{x} - \underset{\sim}{z}| - at)\right]}$$

$$\overline{v_i'\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{y}|}{a}\right) v_j'\left(\underset{\sim}{z}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{z}|}{a}\right)} d\underset{\sim}{y} d\underset{\sim}{z}. \quad \ldots(11)$$

In the integrand of (11) the mean values of the products of cosines and of turbulent velocities can be taken separately because the fluctuations of the two are statistically independent.

The mean product of cosines in (11) is equal to

$$\tfrac{1}{2} \cos\left[\kappa(y_1 - z_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - |\underset{\sim}{x} - \underset{\sim}{z}|)\right], \quad \ldots(12)$$

and this can be simplified when x is large compared with $|\underset{\sim}{y} - \underset{\sim}{z}|$, as it is in the radiation field, to give

$$\tfrac{1}{2} \cos\left[\underset{\sim}{k} \cdot (\underset{\sim}{y} - \underset{\sim}{z})\right], \text{ where } k_i = \kappa\left(\frac{x_i}{x} - \delta_{i1}\right). \quad \ldots(13)$$

Here $\underset{\sim}{k}$ is the so-called scattering vector, whose direction bisects that of the incident wave reversed and that of the point x where the intensity is to be determined.

The mean product of velocity deviations in (11), or "covariance" (of which a correlation coefficient is a non-dimensional form) is negligible if $\underset{\sim}{y}$ and $\underset{\sim}{z}$ are points which have no eddy in common containing a significant part of the turbulent energy. But when they have such an eddy in common, the difference between the times at which those two velocities in (11) are taken must be negligible, because the ratio of a typical eddy diameter to a time significant in the turbulent fluctuations is much smaller than the velocity of sound if the Mach number of the turbulence is low. As a result one can rewrite the covariance as a simultaneous covariance, and cease to display the time explicitly, remembering always that the time at which it is supposed to be evaluated is the time of emission of the scattered energy.

Equation (11), with the simplifications which have been discussed, and substituting also for ϵ in terms of the intensity (7) of the incident wave, becomes

$$I_s \sim \frac{2\pi I \kappa^4}{a^2} \frac{x_i x_j x_1^2}{x^6} F_{ij}(\underset{\sim}{k}), \quad \ldots(14)$$

where

$$F_{ij}(\underset{\sim}{k}) = \frac{1}{8\pi^3} \iint \overline{v_i'(\underset{\sim}{y}) v_j'(\underset{\sim}{z})} \cos\left[\underset{\sim}{k} \cdot (\underset{\sim}{y} - \underset{\sim}{z})\right] d\underset{\sim}{y} d\underset{\sim}{z}. \quad \ldots(15)$$

Now/

Now the quantity $F_{ij}(\underset{\sim}{k})$, in terms of which the intensity of scattered sound has been expressed, can be interpreted as the <u>spectrum</u>, with respect to the vector wave-number $\underset{\sim}{k}$, of the single volume integral

$$\int v_i'(\underset{\sim}{y})v_j'(\underset{\sim}{y})d\underset{\sim}{y}\ , \qquad \ldots(16)$$

taken over the whole turbulent flow. (In particular, $\frac{1}{2}\rho_0 F_{ii}(\underset{\sim}{k})$ is the spectrum of the total kinetic energy of the turbulent field.) To see this, note that by Fourier's integral theorem the integral of $F_{ij}(\underset{\sim}{k})$ over the whole of $\underset{\sim}{k}$-space gives (16). Also $F_{ij}(\underset{\sim}{k})$ is evidently small when k is either large or small compared with the wave-numbers of the energy-bearing eddies, respectively by the Riemann-Lebesgue theorem and because for small k it becomes more and more nearly proportional to the variance of the total momentum of the turbulence, which is zero.

It follows therefore from (14) that the scattered intensity is large only at points where the "scattering vector" $\underset{\sim}{k}$ is comparable in magnitude with the size of the energy-bearing eddies. Since by (13) its magnitude is

$$k = 2\kappa \sin \tfrac{1}{2}\theta\ , \qquad \ldots(17)$$

where θ is the direction of the point $\underset{\sim}{x}$ relative to the direction of the incident wave, we can conclude

(i) that if 2κ is less than the wave-numbers of the main energy-bearing eddies there is relatively little scattered sound; this arises from the biggest eddies only and its directional maximum is at $\theta = \pi$ (opposite to the direction of the incident wave);

(ii) that if κ is of the same order of magnitude as the wave-numbers of the main energy-bearing eddies, there is a fairly even directional distribution of the scattered sound, except for a marked falling off in a cone near $\theta = 0$, wherein the k given by (17) falls below the wave-numbers of the energy-bearing eddies*;

(iii) that if κ is greater than those wave-numbers, then the bulk of the scattered sound is thrown forward in a cone with θ small (although with an interior cone, in which θ is much smaller, still excluded); the restriction on the direction to a solid angle of order κ^{-2} for large κ then reduces the rate at which the <u>total</u> scattered energy increases with κ from κ^4 as in (14) to κ^2.

One can obtain quite a simple expression for the total scattered energy in this last case of very high frequency sound, for which the energy scattered is greatest. For the total energy scattered per unit time is the integral of (14) over a large sphere with centre $\underset{\sim}{x} = 0$. Now by (13) this is the same as x^2/κ^2 times the integral over a large sphere in $\underset{\sim}{k}$-space with centre $(-\kappa, 0, 0)$ and radius κ. Thus the scattered power is

$$P_s = \frac{2\pi I \kappa^2}{a^2}\int\left(\delta_{i1} + \frac{k_i}{\kappa}\right)\left(\delta_{j1} + \frac{k_j}{\kappa}\right)\left(1 + \frac{k_1}{\kappa}\right)^2 F_{ij}(\underset{\sim}{k})dS\ , \qquad \ldots\ldots(18)$$

over such a sphere. But, in cases where $F_{ij}(\underset{\sim}{k})$ is significant only where k/κ is small, the integral need be taken only over the part of the sphere

where/

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* Note also that in all cases a minimum in scattered energy at the direction 90° would be expected, since on the approximations which have led to (14) $I_s = 0$ when $x_1 = 0$, i.e., when $\theta = 90°$.

where this is so, and this part (see Fig.1) is approximately the part of the plane $k_1 = 0$ in which $F_{ij}(\underset{\sim}{k})$ is significant. From this, approximating also the coefficient of $F_{ij}(\underset{\sim}{k})$ in the integral, we obtain

$$P_s \doteqdot \frac{2\pi I\kappa^2}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{11}(0,\ k_2,\ k_3)dk_2dk_3$$

$$= \frac{I\kappa^2}{a^2}\int d\underset{\sim}{y}\int_{-\infty}^{\infty} \overline{v_1'(y_1,\ y_2,\ y_3)v_1'(y_1 + w,\ y_2,\ y_3)}dw\ , \quad \ldots(19)$$

the integrals being equal by (15), and by Fourier's integral theorem.

The inner integral in (19) is often written where L_1 (Taylor's "Macro-scale of turbulence", or correlation radius) is a measure of the width in the x_1- direction of the energy-bearing eddies at the point y. In terms of this length, the rate at which energy is scattered from <u>unit volume of turbulence</u>, by an incident wave in the x_1- direction with intensity I, and with wave-number κ large compared with those of the energy-bearing eddies, is*

$$p_s \doteqdot 2I\kappa^2 L_1 \frac{\overline{v_1'^2}}{a^2}\ . \quad \ldots (20)$$

The quantitative significance of this result, as of the others in this section, will be discussed in §4.

A similar result <u>without</u> the restriction on the size of κ can be obtained in a useful form only in the special case of homogeneous isotropic turbulence. For homogeneous turbulence the covariance in (15) depends only on the <u>relative</u> position of $\underset{\sim}{y}$ and $\underset{\sim}{z}$, and it is usually written

$$\overline{v_i'(\underset{\sim}{y})v_j'(\underset{\sim}{z})} = R_{ij}(\underset{\sim}{z} - \underset{\sim}{y})\ . \quad \ldots(21)$$

The spectral tensor is then defined (Batchelor 1949) as

$$\Gamma_{ij}(\underset{\sim}{k}) = \frac{1}{8\pi^3}\int R_{ij}(\underset{\sim}{y})e^{-i\underset{\sim}{k}\cdot\underset{\sim}{y}}d\underset{\sim}{y}\ . \quad \ldots(22)$$

Evidently the spectral tensor used above in the general case, namely $F_{ij}(\underset{\sim}{k})$, has the value $\mathcal{R}\Gamma_{ij}(\underset{\sim}{k})$ per unit volume of homogeneous turbulence, where $\mathcal{R}$ signifies "real part of".

Now in isotropic turbulence $\Gamma_{ij}(\underset{\sim}{k})$ is purely real, and is expressible uniquely in terms of the spectrum $E(k)$ of the turbulent energy (per unit mass) with respect to the scalar wave-number k. In fact

$$\Gamma_{ij}(\underset{\sim}{k}) = \frac{E(k)}{4\pi k^4}(k^2\delta_{ij} - k_ik_j)\ . \quad \ldots(23)$$

Thus the intensity of scattered sound per unit volume of turbulence, by (14) with $F_{ij}(\underset{\sim}{k})$ replaced by (23), is

$$i_s \sim \frac{I\kappa^4}{2a^2}\frac{x_1^2}{x^4}\frac{E(k)}{k^4}\left(k^2 - \left(\underset{\sim}{k}\cdot\frac{\underset{\sim}{x}}{x}\right)^2\right). \quad \ldots(24)$$

Again/

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* Small letters (p_s and i_s) will be used for the values, per unit volume of turbulence, of the scattered power and intensity P_s and I_s.

Again writing θ for the angle between the directions of incidence and emission, so that $x_1 = x\cos\theta$ and k is given by (17), and by (13) $\mathbf{k}\cdot\mathbf{x}/x = \kappa(1-\cos\theta)$, one deduces from (24) that

$$i_s \sim \frac{I\kappa^2}{8a^2}\,\frac{\cos^2\theta\,\cot^2(\tfrac{1}{2}\theta)}{x^2}\,E(2\kappa\sin\tfrac{1}{2}\theta)\;. \qquad \ldots(25)$$

The form of the term $E(2\kappa\sin\frac{1}{2}\theta)$ in (25) makes clear once more points (i), (ii) and (iii) above concerning the direction of scattering. The coefficient $\cos^2\theta$ exemplifies the general result concerning a minimum at 90°. The coefficient $\cot^2(\frac{1}{2}\theta)$ indicates, as a consequence of the special assumption of isotropy, a further minimum at 180°. It does not however invalidate the conclusion that $i_s \to 0$ as $\theta \to 0$ because as is well known $E(k) = O(k^4)$ as $k \to 0$.

Integrating (25) over a large sphere with centre $x = 0$ we deduce the rate at which energy is scattered from unit volume of isotropic turbulence as

$$P_s = \frac{\pi I\kappa^2}{4a^2}\int_0^\pi \cos^2\theta\,\cot^2(\tfrac{1}{2}\theta)\,E(2\kappa\sin\tfrac{1}{2}\theta)\,\sin\theta\,d\theta$$

$$= \frac{\pi I\kappa^2}{a^2}\int_0^{2\kappa}\frac{E(k)}{k}\,M\!\left(\frac{k}{\kappa}\right)dk\;, \qquad \ldots(26)$$

where

$$M(x) = (1-\tfrac{1}{2}x^2)^2(1-\tfrac{1}{4}x^2)\;. \qquad \ldots(27)$$

It should be noticed that when κ is much larger than the values of k for which $E(k)$ is significant formula (26) becomes

$$P_s = \frac{\pi I\kappa^2}{a^2}\int_0^\infty \frac{E(k)}{k}\,dk\;, \qquad \ldots(28)$$

which agrees with (20) in view of the known formula for isotropic turbulence

$$L_1 = \frac{\pi}{2v_1'^2}\int_0^\infty \frac{E(k)}{k}\,dk = Av\left(\frac{3\pi}{4k}\right)\;, \qquad \ldots(29)$$

the average in (29) being weighted with respect to the energy spectrum.

The function $M(k/\kappa)$, which is plotted in Fig. 2 in its range $0 < k/\kappa < 2$, signifies the ratio of the energy scattered by an eddy to the energy scattered according to the formula which holds for large κ (i.e. small k/κ). Of course no energy is scattered by eddies with $k/\kappa > 2$, and Fig. 2 makes it clear that a not too poor approximation would be to say that eddies with $k < \kappa$ make their full contribution to P_s, namely

$$\frac{\pi I\kappa^2 E(k)}{a^2 k} \qquad \ldots(30)$$

per unit wave-number, and eddies with $k > \kappa$ do not scatter at all.

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For incident waves which are not pure tones it is easily seen that their intensity spectrum can be used directly to synthesize the intensity field of the scattered energy. In other words the energy scattered from two different pure tones may be added, since the corresponding fluctuations are uncorrelated. For the covariance of two fields like (10) with different values of κ takes the form (11) with different values of κ in the two cosines. The mean product of the two cosines is then zero. The same would be true of two sines with different κ, or of a sine with a cosine even with the same κ.

Thus if the incident wave has the intensity spectrum $I(\kappa)$ (with respect to wave-number), then

(i) if the values of κ for which $I(\kappa)$ is biggest exceed considerably the wave-numbers of the main energy-bearing eddies,

$$P_s \doteqdot 2L_1 \frac{\overline{v_1'^2}}{a^2} \int_0^\infty \kappa^2 I(\kappa)\, d\kappa; \qquad \ldots(31)$$

(ii) while, for isotropic turbulence, without other restriction,

$$P_s = \frac{\pi}{a^2} \int_0^\infty \kappa^2 I(\kappa) d\kappa \int_0^{2\kappa} \frac{E(k)}{k} M \frac{k}{\kappa}\, dk\,. \qquad \ldots(32)$$

To close this section it may be noted that, while to a first approximation the scattered sound has the same frequency $n = \kappa a/2\pi$ as the incident wave, a slight spread in the frequency of the scattered sound may be predicted if the analysis is observed more closely. For the frequencies in the scattered sound must be those in the quadrupole field (8), and these consist of sums and differences of n and the frequencies of fluctuation of v_i at a point. Thus the degree of spread would be a measure of the typical frequencies with which the turbulent velocities in the energy-bearing eddies fluctuate. It corresponds, of course, to nothing more than the "fading" which has been already discussed in §1.

4. Some Quantitative Conclusions from the Theory for Harmonic Waves.

If the wavelength $\Lambda = 2\pi/\kappa$ of the incident sound is smaller than the macro-scale of turbulence L_1, then by (29) κ will exceed by a factor of at least $\frac{8}{3}$ an average value of k for the energy-bearing eddies. Hence formula (20) should be a reasonable approximation, at least according to the indications of (26). Then the attenuation due to scattering, that is the proportion of the energy of the incident wave which is scattered in travelling through the turbulence for unit distance, is

$$\beta = 2\kappa_1^2 L_1 \frac{\overline{v_1'^2}}{a^2} = \frac{8\pi^2 L_1}{\Lambda^2} \frac{\overline{v_1'^2}}{a^2}\,, \qquad \ldots(33)$$

and the attenuation per wavelength, due to scattering, is

$$\beta\Lambda = \frac{8\pi^2 L_1}{\Lambda} \frac{\overline{v_1'^2}}{a^2}\,. \qquad \ldots(34)$$

This may be compared, for example, with the maximum attenuation per wavelength due to molecular causes, which (Knoser 1935, Knudsen 1935, and

references/

references there given) is 0.002 for air at 20°C. (being attained at a frequency which depends critically on the concentration of impurities). Since L_1 has been presumed to exceed Λ, the expression (34) will certainly exceed 0.002 if $(\overline{v_1'^2})^{\frac{1}{2}} > 0.005a = 1.7$m./sec., which is a fairly modest value.

However the scattered energy does not completely disappear, but merely adopts a new direction of propagation. This suggests an attempt to consider what will happen to a plane or spherical wave on travelling some distance through turbulent fluid, and in particular to find the distribution of direction of propagation of the sound, resulting from multiple successive scatterings.

To achieve this let $\frac{1}{2}p(\Lambda, \theta) \sin\theta d\theta$ denote the probability that after n scatterings the direction will make an angle between θ and $\theta + d\theta$ with the initial direction.* Then if a relatively large number of scatterings at relatively small angles ω are involved it may be shown that p satisfies approximately the partial differential equation (of diffusion)

$$\frac{\partial p}{\partial n} = \frac{1}{4}\overline{\omega^2}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial p}{\partial\theta}\right). \qquad \ldots(35)$$

Here $\overline{\omega^2}$ denotes the mean square deviation at each scattering. The solution of (35) such that the direction is initially $\theta = 0$ is

$$p = \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta)\, e^{-\frac{1}{4}\overline{\omega^2} m(m+1) n} \;. \qquad \ldots(36)$$

Note that, as $n \to \infty$, $p \to 1$, corresponding to a uniform directional distribution after an infinite number of scatterings.

Now when the sound travels a distance l, molecular and spherical attenuation will reduce its intensity, but of what energy remains a proportion independent of those effects, namely

$$e^{-\beta l}\,\frac{(\beta l)^n}{n!} \qquad \ldots(37)$$

(Poisson's distribution), will have been scattered n times. Summing the product of (36) and (37) from $n = 0$ to ∞, we obtain the distribution of direction after propagation through a distance l as

$$p = \sum_{m=0}^{\infty} (2m+1) P_m(\cos\theta) \exp[-\beta l(1 - e^{-\frac{1}{4}m(m+1)\overline{\omega^2}})]$$

$$= 1 + 3\cos\theta \exp[-\beta l(1 - e^{-\frac{1}{2}\overline{\omega^2}})] + \ldots \;. \qquad \ldots(38)$$

It is seen that if $\frac{1}{2}\overline{\omega^2}$ is small then the directional distribution tends to uniformity, as l increases, with a logarithmic decrement

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* The factor $\frac{1}{2}\sin\theta$ has been inserted to ensure that for a uniform directional distribution $p = 1$.

$$\tfrac{1}{2}\rho\,\overline{\omega^2} = \tfrac{1}{2}\int_0^\pi \rho\left(\frac{i_s}{I}\right) 2\pi x^2 \sin\theta d\theta\,. \qquad \ldots(39)$$

For homogeneous isotropic turbulence this becomes, by (25),

$$\frac{\pi\kappa^2}{8a^2}\int_0^\pi \theta^2\cos^2\theta\,\cot^2(\tfrac{1}{2}\theta)E(2\kappa\sin\tfrac{1}{2}\theta)\sin\theta d\theta$$

$$= \frac{\pi}{2a^2}\int_0^{2\kappa} kE(k)N\left(\frac{k}{\kappa}\right)dk\,, \qquad \ldots(40)$$

where

$$N(x) = (1 - \tfrac{1}{2}x^2)^2(1 - \tfrac{1}{4}x^2)(\sin^{-1}\tfrac{1}{2}x)^2/(\tfrac{1}{2}x)^2\,. \qquad \ldots(41)$$

The function $N(x)$, which is plotted in Fig. 3, is very like $M(x)$ (Fig. 2), and in practice an adequate approximation to (40) may be to replace N by 1 and the range of integration by $(0,\kappa)$. For very small wavelengths Λ, smaller even than the <u>micro-scale</u> of turbulence λ, the expression (40) becomes

$$\frac{\pi}{2a^2}\int_0^\infty kE(k)dk\,. \qquad \ldots(42)$$

This integral involves some of the energy-dissipating eddies as well as some of the energy-bearing eddies, and would be in practice rather less than

$$\left[\int_0^\infty E(k)dk\int_0^\infty k^2E(k)dk\right]^{\frac{1}{2}} = \left[\frac{15}{2}\frac{(\overline{v_1'^2})^2}{\lambda^2}\right]^{\frac{1}{2}}. \qquad \ldots(43)$$

Taking it as $2\overline{v_1'^2}/\lambda$, we obtain

$$\frac{3}{\lambda}\frac{\overline{v_1'^2}}{a^2} \qquad \ldots(44)$$

as the logarithmic decrement of the directional distribution. By (38) the distribution will be uniform to within 1 db when l times (44) is $\log_e[3/(1 - 10^{-0.1})] \doteqdot 3$, i.e., when

$$l \doteqdot \lambda\frac{a^2}{\overline{v_1'^2}}\,. \qquad \ldots(45)$$

In words, sound of wavelength less than the micro-scale λ of the turbulence should become directionally random in a distance of order λ divided by the mean square Mach number of the turbulence. (This is the distance in which the process becomes complete, but evidently very considerable changes of direction would occur in even a tenth of this distance.)

5. <u>Scattering of an Acoustic Pulse</u>.

For the scattering of a pulse it would perhaps be reasonable to assume without further discussion that, as stated in §1, the directional distribution of scattered energy per unit area, as a stochastic mean, is related to the spectrum of the incident pulse's energy exactly as in §3

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the distribution of intensity was found to be related to the spectrum of the incident wave's intensity. However a brief deduction of this result from the basic principles of §2 will be given.

For a plane pulse in which

$$v_i = af(x_1 - at)\delta_{i1} , \qquad \ldots(46)$$

the radiation field (9) of the scattering quadrupoles (3) is*

$$\rho - \rho_0 \sim \frac{\rho_0}{2\pi a}\frac{x_i x_1}{x^3}\int f''(y_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - at)v_i'\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{y}|}{a}\right)d\underset{\sim}{y} , \qquad \ldots(47)$$

provided that, as assumed (for the wave) in §3, the time-scale of the pulse is small compared with typical time-scales of the turbulence.

Now energy crosses unit area at a distant point $\underset{\sim}{x}$ at a rate $a^3(\rho - \rho_0)^2/\rho_0$ per unit time. This can be written as a double integral

$$\frac{\rho_0 a}{4\pi^2}\frac{x_i x_j x_1^2}{x^6}\int f''(y_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - at)f''(z_1 + |\underset{\sim}{x} - \underset{\sim}{z}| - at)$$

$$v_i'\left(\underset{\sim}{y}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{y}|}{a}\right)v_j\left(\underset{\sim}{z}, t - \frac{|\underset{\sim}{x} - \underset{\sim}{z}|}{a}\right)dydz . \qquad \ldots(48)$$

A stochastic mean of (48) is now taken, in the sense described in §1; this involves putting a bar over the product $v_i'v_j'$. The resulting covariance may then be approximated as a simultaneous covariance $v_i'(\underset{\sim}{y})v_j'(\underset{\sim}{z})$, for exactly the same reasons as in §3.

Next this mean rate at which energy crosses unit area at x is integrated with respect to time, to obtain the mean total scattered energy G_s which crosses unit area at $\underset{\sim}{x}$, in the form

$$G_s = \frac{\rho_0}{4\pi^2}\frac{x_i x_j x_1^2}{x^6}\iint C(s)\overline{v_i'(\underset{\sim}{y})v_j'(\underset{\sim}{z})}d\underset{\sim}{y}d\underset{\sim}{z} , \qquad \ldots(49)$$

where

$$s = y_1 - z_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - |\underset{\sim}{x} - \underset{\sim}{z}| \doteqdot (y_1 - z_1) - \frac{\underset{\sim}{x}\cdot(\underset{\sim}{y} - \underset{\sim}{z})}{x} , \qquad \ldots(50)$$

and

$$C(s) = \int_{-\infty}^{\infty} f''(r)f''(r + s)dr. \qquad \ldots(51)$$

But $C(s)$ can be written in terms of the energy spectrum of f. If $f(r)$ be written as a Fourier integral

$$f(r) = \int_{-\infty}^{\infty} e^{i\kappa r}F(\kappa)d\kappa, \qquad \ldots(52)$$

then/

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* The author hopes to be forgiven for an anomaly of notation (difficult to avoid where random functions are treated) throughout the next two pages whereby primes attached to v's signify departures from the mean, and primes attached to f's mean differentiation.

than the total energy of the pulse (per unit area) is

$$\rho_0 a^2 \int_{-\infty}^{\infty} f^2(r)dr = 4\pi\rho_0 a^2 \int_0^{\infty} F(\kappa)F(-\kappa)d\kappa \ , \qquad \ldots(53)$$

and hence

$$G(\kappa) = 4\pi\rho_0 a^2 F(\kappa)F(-\kappa) \qquad \ldots(54)$$

is the spectrum of its total energy per unit area per unit wave-number. But by (51) and (52)

$$C(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \kappa_1^2 F(\kappa_1)\rho^{i\kappa_1 r} d\kappa_1\right)\left(\int_{-\infty}^{\infty} \kappa_2^2 F(\kappa_2)\rho^{i\kappa_2(r+s)} d\kappa_2\right)dr$$

$$= 2\pi\int_{-\infty}^{\infty} \kappa^4 F(\kappa)F(-\kappa)\rho^{i\kappa s}d\kappa = \frac{1}{\rho_0 a^2}\int_0^{\infty} \kappa^4 G(\kappa)\cos \kappa s d\kappa \ . \qquad \ldots(55)$$

Inserting (55) in (49), and recalling the definitions (13) of $\underline{k}$ and (15) of $F_{ij}(\underline{k})$, we obtain the result

$$G_s = \int_0^{\infty} G(\kappa)d\kappa \left[\frac{2\pi\kappa^4}{a^2} \frac{x_i x_j x_1^2}{x^6} F_{ij}(\underline{k})\right] \ . \qquad \ldots(56)$$

Comparing this with (14) we obtain the principle enunciated at the beginning of this section.

This principle makes it possible to take over the whole of the rest of the theory without further labour. The general conclusions about directional distribution, and its dependence on the ratio of typical wave-numbers in the pulse and in the turbulence, are unaltered.

In particular, when the wave-numbers containing most of the pulse energy are large compared with those containing most of the turbulent energy, then most of the scattered energy is emitted at a small angle to the incident energy. Further the total energy e_s so emitted from unit volume of the turbulence*, by (20), is

$$e_s = 2L_1 \frac{\overline{v_1'^2}}{a^2}\int_0^{\infty} \kappa^2 G(\kappa)d\kappa = 2L_1\rho_0\overline{v_1'^2}\int_{-\infty}^{\infty} f'^2(r)dr \ . \qquad \ldots(57)$$

The form of (57), the principal result for an acoustic pulse, makes clear how the energy scattered from a pulse of given energy increases as its length is diminished. In the limiting case of a pulse containing a discontinuity in pressure (the idealization of a shock wave) it is evident from both expressions in (57) that e_s becomes infinite. In the case of the first expression this is because, if f is discontinuous, $G(k)$ is at least of order κ^{-2} as $\kappa \to \infty$, and in fact if f has only a single discontinuity, of amount ϵ,

$$G(\kappa) \sim \frac{\rho_0 a^2 \epsilon^2}{\pi\kappa^2} \ . \qquad \ldots(58)$$

(Note/

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* Here, as in §3, small letters (e_s and g_s) will be used for the values, per unit volume of turbulence, of the total energy scattered E_s and the energy which crosses unit area at a point G_s.

(Note that, since (57) was based on the approximation that κ is large, the use of that approximation cannot itself be the cause of the integral being infinite.)

The predicted infinity in the total energy scattered can be illuminated by a study of its directional distribution, although this is possible only for homogeneous isotropic turbulence. In this case the total energy g_s per unit area, scattered from unit volume turbulence at an angle θ to the direction of the incident pulse, by equation (25) of §3, is

$$g_s = \frac{\cos^2\theta \cot^2(\frac{1}{2}\theta)}{8a^2x^2}\int_0^\infty \kappa^2 G(\kappa)E(2\kappa\sin\tfrac{1}{2}\theta)\,d\kappa, \qquad \ldots(59)$$

and the contribution to this from a shock wave, by (58), is

$$\frac{\rho_0 a^2\epsilon^2}{\pi}\,\frac{\cos^2\theta\cot^2(\frac{1}{2}\theta)}{8a^2x^2}\,\frac{\frac{3}{2}\overline{v_1'^2}}{2\sin\frac{1}{2}\theta} = \frac{\epsilon^2}{16\pi x^2}\left(\frac{3}{2}\rho_0\overline{v_1'^2}\right)\frac{\cos^2\theta\cos^2(\frac{1}{2}\theta)}{\sin^3(\frac{1}{2}\theta)}. \qquad \ldots(60)$$

This becomes large so rapidly as $\theta \to 0$ that its integral e_s over a large sphere is infinite, in agreement with the result obtained above.

6. Scattering of a Weak Shock Wave

To get over the difficulty just mentioned, that a pulse with discontinuities in velocity produces according to the theory on infinite amount of scattered energy, consider now a single plane shock wave, at which the change in velocity is ϵa, incident upon turbulence. To accord with the assumptions of §2, ϵ must be small, that is, the shock must be weak.

To seek to improve the theory, and remove the infinity, by calling in to account the finite thickness of the shock wave, would leave e_s still very large indeed. And it will be seen below that there is another mechanism in operation which cuts off the large wave-number components in the shock wave spectrum (as far as their effect on scattering is concerned) at a lower level than is achieved by the internal structure of the shock wave.

The true cause of the theoretical infinite energy scattered straight ahead is in the perfect resonance (mentioned in §1) assumed between pulses emitted in this direction; and this is really absent because the shock speed is not that of sound, but is supersonic relative to the fluid ahead and subsonic relative to the fluid behind. Since (for this reason) the scattered energy must remain behind the shock, let a signify the speed of sound in this region; then the shock speed is $a/(1 + \alpha)$, where

$$\alpha = \frac{\gamma+1}{4}\epsilon + \left\{1 + \left(\frac{\gamma+1}{4}\epsilon\right)^2\right\}^{\frac{1}{2}} - 1 \doteqdot \frac{\gamma+1}{4}\epsilon, \qquad \ldots(61)$$

and γ is the ratio of the specific heats.

The velocity distribution resulting from the shock wave is therefore

$$V_i = \epsilon a\, H[-(1+\alpha)x_1 + at]\,\delta_{i1}, \qquad \ldots(62)$$

(where H signifies the unit function), rather than of the form (46). However the substitution of $(1+\alpha)x_1 - at$ for $x_1 - at$ in (46) makes only very slight differences in the subsequent work; these are as follows.

In/

In (47) and (48), $(1 + \alpha)y_1$ replaces y_1; in (48), $(1 + \alpha)z_1$ replaces z_1. Hence, in the definition (50) of s, $(1 + \alpha)(y_1 - z_1)$ replaces $(y_1 - z_1)$, and finally (56) is correct provided that $\underline{k}$ is given the new definition

$$k_i = \kappa\left(\frac{x_i}{x} - (1 + \alpha)\delta_{i1}\right) . \qquad \ldots(63)$$

From this, as in § §3 and 5, the value of G_s integrated over a large sphere (i.e. the total scattered energy) will be found first, for general turbulence, and secondly the directional distribution of G_s for homogeneous isotropic turbulence will be obtained.

Since $G(\kappa)$ is of order κ^{-2} for large κ it appears from (56) that the largest contribution to G_s will be from wave-numbers κ large compared with those of the energy-bearing eddies. Accordingly the integral of the term in square brackets in (56), over a sphere of radius x and centre the origin, is now found under this condition, following closely the method in §3. By (63) the integral is the same as x^2/κ^2 times the integral over a large sphere in k-space with centre $(-\kappa(1 + \alpha), 0, 0)$ and radius κ. If, as has been assumed, the integrand is significant only when k is much smaller than κ, then the only significant part of the sphere is approximately part of the plane $k_1 = -\kappa\alpha$. The integral is therefore approximately

$$\frac{2\pi\kappa^2}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{11}(-\kappa\alpha, k_2, k_3)dk_2dk_3 , \qquad \ldots(64)$$

(compare (18) of §3).

Substituting (64) for the term in square brackets in (56), and using the form (58) for $G(\kappa)$, which is exact for the simple plane shock wave here considered, we deduce that

$$E_s = 2\rho_0\epsilon^2\int_0^{\infty} d\kappa\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{11}(-\kappa\alpha, k_2, k_3)dk_2dk_3 . \qquad \ldots(65)$$

Since the integral of F_{11} over the whole of wave number space is $\overline{v_1'^2}$, per unit volume of turbulence, equation (65) may be written as

$$e_s = \frac{\epsilon^2}{\alpha}\rho_0\overline{v_1'^2} \doteqdot \frac{4}{\gamma + 1}\epsilon\rho_0\overline{v_1'^2} , \qquad \ldots(66)$$

where (61) has been used to show that in the case of a shock wave the energy scattered depends on the first power of the amplitude, not on the second as for continuous waves. It may be shown that the contribution to (56) from wave-numbers κ not large compared with those of the energy-bearing eddies, which has been neglected in the above, is small compared with (66) at least by a factor $O(\epsilon)$.

Since, in terms of the "strength" $s = \Delta p/p$ of the shock wave, $\epsilon \doteqdot s/\gamma$, and since the turbulent energy per unit volume is approximately $\frac{3}{2}\rho_0\overline{v_1'^2}$, the energy scattered is approximately a fraction

$$\frac{8s}{3\gamma(\gamma + 1)} = 0.8s \qquad \text{(for air)} \qquad \ldots(67)$$

of/

of the kinetic energy of the turbulence traversed by the shock. The reader is reminded that this formula has been obtained on the assumption that s is small.

Next the directional distribution of the scattered energy G_s per unit area will be found, in the case of homogeneous isotropic turbulence. Then (56) becomes (compare (24) of §3)

$$g_s = \int_0^{\infty} G(\kappa)d\kappa \left[\frac{\kappa^4}{2a^2}\frac{x_1^2}{x^4}\frac{E(k)}{k^4}\left\{k^2 - \left(\underset{\sim}{k}.\frac{\underset{\sim}{x}}{x}\right)^2\right\}\right]. \qquad \ldots(68)$$

Writing as usual θ for the angle between the direction of emission and that of the incident shock, we have by (63)

$$k = 2\kappa\{(1+\alpha)\sin^2(\tfrac{1}{2}\theta) + \tfrac{1}{4}\alpha^2\}^{\frac{1}{2}} . \qquad \ldots(69)$$

In (69) the factor $(1+\alpha)$ is relatively unimportant, but the term $\frac{1}{4}\alpha^2$ is essential because it prevents the k^4 in the denominator of (68) from vanishing as $\theta \to 0$. Leaving only the latter in, we find after some reduction that

$$g_s = \int_0^{\infty} G(\kappa)d\kappa\left[\frac{\kappa^2}{2a^2x^2}\,\frac{\cos^2\theta\,\sin^2\theta}{16(\sin^2(\frac{1}{2}\theta)+\frac{1}{4}\alpha^2)^2}\,E\{2\kappa(\sin^2(\tfrac{1}{2}\theta)+\tfrac{1}{4}\alpha^2)^{\frac{1}{2}}\}\right]. \qquad \ldots(70)$$

Substituting for $G(\kappa)$ in (70) from (58), we deduce that

$$g_s = \frac{\epsilon^2}{64\pi x^2}\left(\tfrac{3}{2}\rho_0\overline{v_1'^2}\right)\frac{\cos^2\theta\,\sin^2\theta}{(\sin^2(\frac{1}{2}\theta)+\frac{1}{4}\alpha^2)^{\frac{5}{2}}} . \qquad \ldots(71)$$

This may be compared with the value (60) (its limit as $\alpha \to 0$) obtained by neglecting the difference between the speed of the shock and that of sound. By contrast (71) has a finite integral over the sphere, and the verification that (assuming α small) this integral is $\epsilon^2\rho_0 v'^2/\alpha$, as in (66), is straightforward.

It should now be observed that (see §1) energy scattered at a sufficiently small angle θ will almost immediately catch up with the shock. The condition for this is that the component, $a\cos\theta$, of its velocity in the direction of motion of the shock, shall exceed the shock speed $a/(1+\alpha)$, in other words that

$$\theta < \sec^{-1}(1+\alpha) \doteqdot \sqrt{(2\alpha)} . \qquad \ldots(72)$$

Hence the scattered energy which does not run into the shock is the integral of (71) over a sphere of radius x with the conical region (72) excluded. Since if $\theta > \sqrt{(2\alpha)}$ the $\frac{1}{4}\alpha^2$ in the denominator in (71) may be neglected, as in (60), this freely scattered energy is

$$\begin{aligned} e_{fs} &= \tfrac{1}{4}\epsilon^2\left(\tfrac{3}{2}\rho_0\overline{v_1'^2}\right)\int_{\sec^{-1}(1+\alpha)}^{\pi}\frac{\cos^2\theta\,\cos^3(\frac{1}{2}\theta)}{\sin^3(\frac{1}{2}\theta)}\,d\theta \\ &\doteqdot \tfrac{1}{4}\epsilon^2\left(\tfrac{3}{2}\rho_0\overline{v_1'^2}\right)2.83\alpha^{-\frac{1}{2}} - 8.27 + 10\alpha^{\frac{1}{2}} , \end{aligned} \qquad \ldots(73)$$

where three terms in the expansion for small α are given because the coefficients initially increase. In terms of the strength s, the freely

scattered/

scattered energy is a fraction

$$0.7s^{\frac{3}{2}} - 1.0s^{2} + 0.7s^{\frac{5}{2}} \qquad \ldots(74)$$

of the kinetic energy of the turbulence traversed by the shock wave. The fractions (67) and (74) for scattered and freely scattered energy are tabulated in Table I.

<u>Table 1</u>

s	Scattered	Freely scattered
0.05	0.04	0.006
0.1	0.08	0.014
0.15	0.12	0.024
0.2	0.16	0.035
0.25	0.20	0.047
0.3	0.24	0.060

The fate of the energy, which instead of being "freely scattered" is scattered at such an angle that it runs into the shock, has been discussed already in §1; it is probably mostly absorbed by the shock. A rough estimate of the distance it travels, on the average, before running into the shock, is the length-scale of an energy-bearing eddy. For evidently the square bracket in (70) is largest when κa is equal to the wave-number k of such an eddy.* Hence an average of the "effective wavelength" of the shock, is of order αk^{-1}, and pulses may be supposed to originate about this distance behind the shock. Since their speed exceeds that of the shock by a factor $1 + \alpha$, the distance travelled before catching up with it is of order k^{-1}.

A similar argument can be used to indicate the frequency spectrum of the freely scattered sound. Since components with wave-number κ in (70) are expected to produce sound with the same wave-number, it is seen that the wave-number spectrum of the freely scattered sound should reproduce, roughly, that of the turbulence, but with the wave-numbers scaled up by a factor $\frac{1}{2}\operatorname{cosec}(\frac{1}{2}\theta)$, which varies from $(2\alpha)^{-\frac{1}{2}}$ to $\frac{1}{2}$ in the range of directions of the freely scattered sound. (To obtain the frequency spectrum, one must insert the additional scaling-up factor $a/2\pi$).

<u>References</u>

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* It is evident from this, or even more clearly from (64), that the effective spectrum of the short cuts off at α^{-1} times the wave-number at which the spectrum of the turbulent energy does, say at $5/\alpha L_1$. But the interval structure of the shock wave introduces its cut-off at about $\frac{1}{2}\epsilon a/\nu$, so that if $\epsilon > 4\sqrt{(\nu/aL_1)}$, a very small value indeed, the latter will be unimportant.

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Appendix

It was pointed out in §2 that since the expression

$$\rho_0(v_i'V_j + v_j'V_i) \, , \qquad \ldots(75)$$

which is a first approximation to the strength per unit volume of the scattering quadrupoles, is uncorrelated with the incident wave V_i, the intensity fields of the two can be added. That of (75) is of order v^2V^2 for small v/a and V/a.

However there may be terms of smaller order in the scattering quadrupole strength density, namely terms of order v^2V, which are correlated with the incident wave, and the covariance of their field with the incident wave may provide contributions to the intensity field of the same order v^2V^2 as the intensity field of (75).

Such higher order terms will arise is the velocity field departs from $v_i + V_i$ as a result of interactions between the sound and the turbulence. When the wavelength is small compared with the size of the energy-bearing eddies, one may estimate this departure by means of geometrical optics to consist principally of a phase change in V_i, resulting from convection by the turbulent velocity field.

As a result of such convection a wave, which at the station $x_1 = x_0$ was plane and of the form (6) would become

$$V_i = \epsilon a \cos\left[\kappa\left(x_1 - at - \int_{x_0}^{x_1} \frac{v_1}{a} dx_1\right)\right]\delta_{1i}. \qquad \ldots.(76)$$

The fluctuating part of the phase change, namely

$$\frac{\kappa}{a}\int_{x_0}^{x_1} v_1' dx_1 \, , \qquad \ldots(77)$$

has/

has standard deviation approximately

$$\frac{\kappa}{a}(\overline{v_1'^2})^{\frac{1}{2}}(2L_1(x_1 - x_0))^{\frac{1}{2}} . \qquad \ldots(78)$$

Over a distance $x_1 - x_0$ of a few eddy-sizes (78) will be small, even though κL_1 has been assumed large, if the root mean square Mach number of the turbulence is small enough. (Note that to find what happens as the wave crosses a region which is a few eddy-sizes broad, one can imagine the wave as it enters the region split into plane harmonic waves, and treat each separately, as was shown to be permissible in §3.)

In consequence (75) is a valid first approximation to the quadrupole strength density and a second one is obtained by substituting (76) for V_i therein. This second approximation will differ from the first by approximately

$$\epsilon\kappa \sin[\kappa(x_1 - at)](\delta_{j1}v_j' + \delta_{j1}v_i')\int_{x_0}^{x_1} v_1 dx_1 . \qquad \ldots(79)$$

The radiation field of this extra quadrupole distribution, by (9), is

$$\rho - \rho_0 \sim -\frac{\rho_0\,\epsilon\kappa^3}{2\pi a^2}\frac{x_1 x_i}{x^3}\int \sin[\kappa(y_1 + |\underset{\sim}{x} - \underset{\sim}{y}| - at)]\left(v_i'\int_{x_0}^{y_1} v_1 dy_1\right)d\underset{\sim}{y} \ldots(80)$$

The intensity field, for large x, resulting from the interaction of (80) with the incident wave

$$\rho - \rho_0 = \rho_0\,\epsilon\cos[\kappa(x_1 - at)] \quad , \qquad \ldots(81)$$

is twice their covariance, multiplied by a^3/ρ_0, namely

$$-\frac{\rho_0 a\epsilon^2\kappa^3}{2\pi}\frac{x_1 x_i}{x^3}\int \sin[\kappa(|\underset{\sim}{x} - \underset{\sim}{y}| - (x_1 - y_1))]\left(v_i'\int_{x_0}^{y_1} v_1' dy_1\right)dy \ . \ldots(82)$$

Now expression (82) is negligible except at points x for which typical values of $x_2 - y_2$ and $x_3 - y_3$ are small compared with $x_1 - y_1$. For at other points the sine term fluctuates rapidly about zero and the net contribution from all points y in the turbulent field will be nearly zero. In consequence one may replace $x_i v_i'$ by $x_1 v_1'$, and then put

$$\overline{v_1'\int_{x_0}^{y_1} v_1' dy_1} = \overline{v_1'^2}L_1 . \qquad \ldots(83)$$

Note that equation (83) is correct independently of the lower limit x_0 provided that this is less than about $y_1 - 2L_1$. It is only the phase changes produced by the actual eddies containing a point which are correlated with the local turbulent velocity at the point, and which contribute to the intensity field (82).

Hence the intensity field from unit volume of turbulence, due to the second order terms in the quadrupole strength, is

$$i_s = -\frac{\rho_0 a\,\epsilon^2\kappa^3\overline{v_1'^2}L_1}{2\pi}\frac{x_1^2}{x^3}\sin\kappa(x - x_1) , \qquad \ldots(84)$$

and/

and the contributions of different parts of the turbulence are expected to cancel out except where $\kappa(x - x_1)$ is small.

To deduce the contribution to the total energy p_s radiated per unit time, one must integrate (84) over a large sphere of radius x, and take the mean over all large values of κx. Now the mean integral, in this sense, of $(x_1^2/x^2)\sin\kappa(x - x_1)$, is

$$\overline{2\pi x\int_0^\pi \cos^2\theta\,\sin[\kappa x(1 - \cos\theta)]\sin\theta d\theta} = \frac{2\pi}{\kappa}, \qquad \ldots(85)$$

as a partial integration of $\sin[\kappa x(1 - \cos\theta)]\sin\theta d\theta$ shows. Hence the net contribution to p_s is

$$-\rho_0 a\,\varepsilon^2\kappa^2\overline{v_1'^2}\,L_1 = -2I\kappa^2 L_1\frac{\overline{v_1'^2}}{a^2}\ . \qquad \ldots(86)$$

Comparing with (20), which also is obtained on the assumption of small wavelength, one sees that the two exactly balance.

The energy (86) is located principally where $\kappa(x - x_1)$ is less than about π, in other words in the paraboloidal region

$$x_2^2 + x_3^2 < \frac{2\pi}{\kappa}x_1\ . \qquad \ldots(87)$$

Since the reduced energy lies entirely in this region (for which $\theta \to 0$ as $x \to \infty$) and the scattered energy lies entirely outside a cone θ = constant > 0, one may conclude that ultimately they become quite separate, and that (86) represents an attenuation of the main wave directly corresponding to the scattered energy which was investigated in the main paper.

The argument given applies only asymptotically, for sound of small wavelengths, and it is possible that for a given wavelength the correspondence between attenuation and scattered sound may not be perfectly exact. But enough has been said to show that the sound wave is attenuated by an amount at least of the same order of magnitude as the energy scattered.

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FIG. 1.

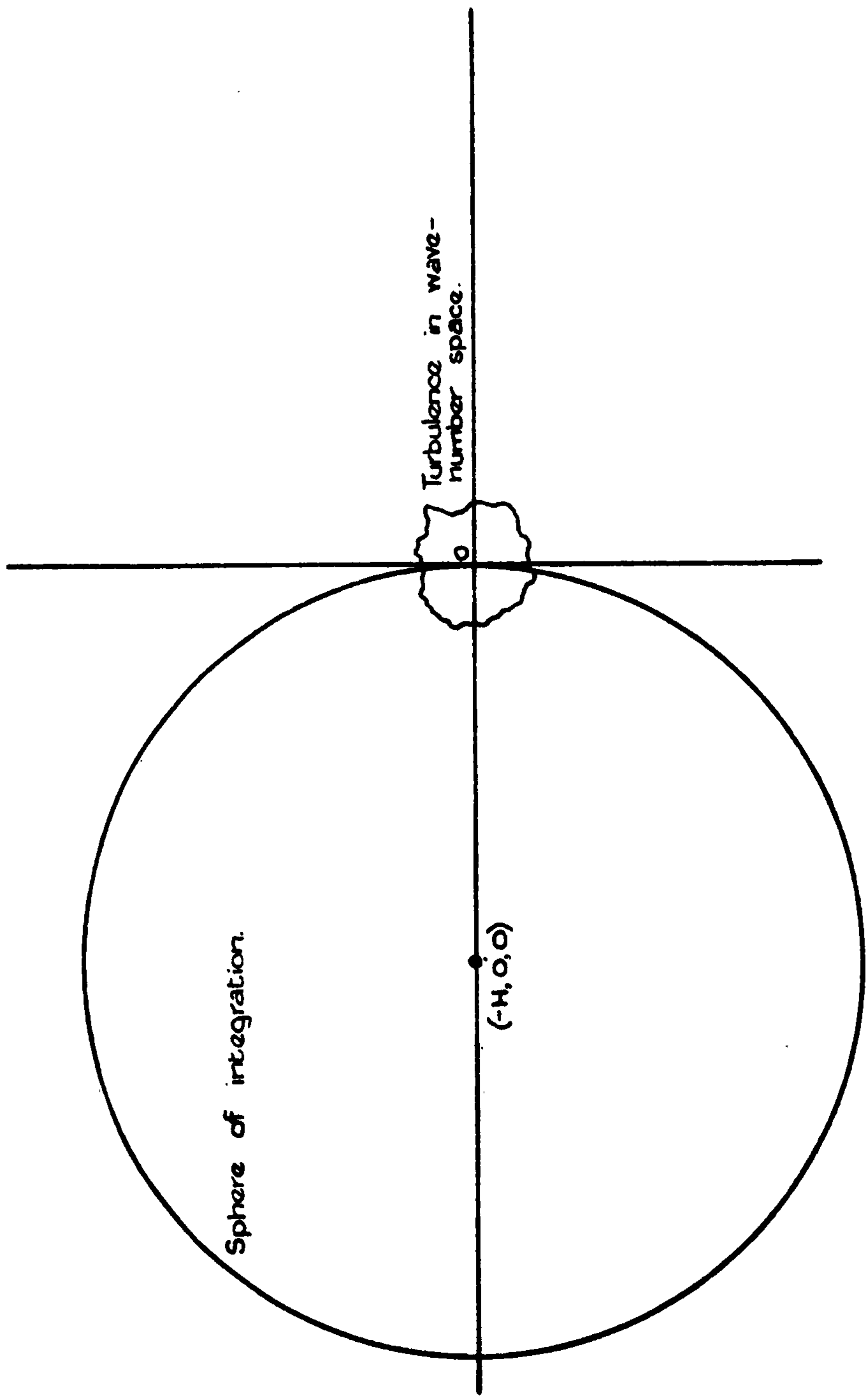

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Fig. 2 & 3.

Fig. 2.

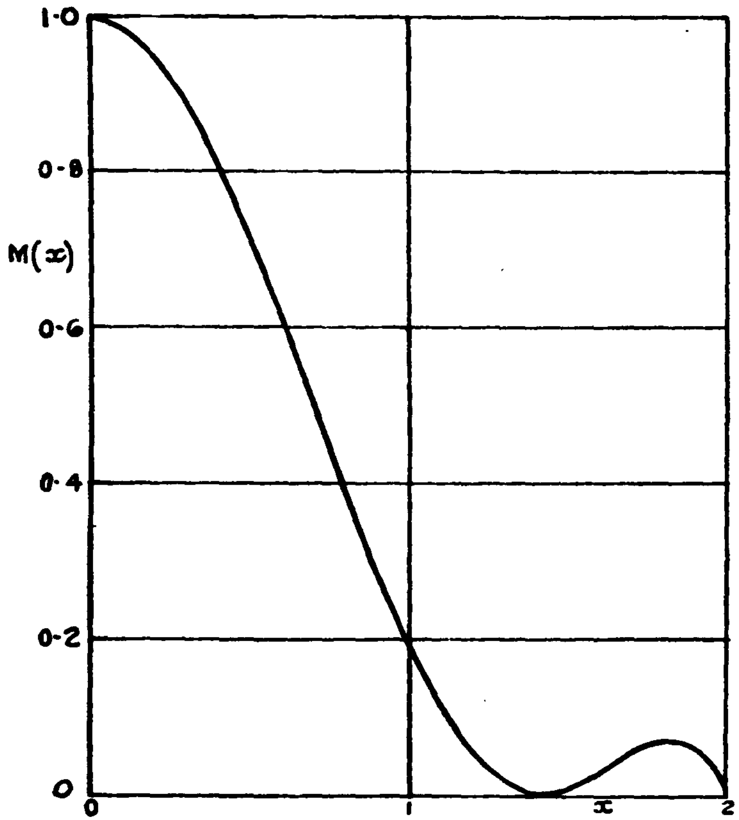

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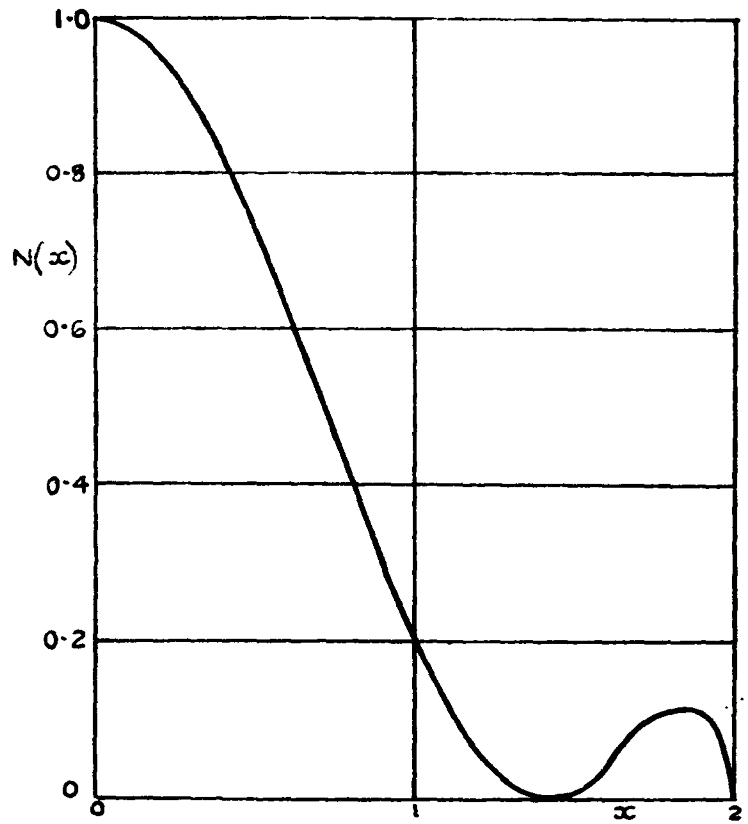
Fig. 3.



M.C

[dstl]

Information Centre
Knowledge Services
[dstl] Porton Down,
Salisbury
Wilts
SP4 0JQ
22060-6218
Tel: 01980-613753
Fax 01980-613970

Defense Technical Information Center (DTIC)
8725 John J. Kingman Road, Suit 0944
Fort Belvoir, VA 22060-6218
U.S.A.

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