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**ATOMIC ENERGY  
RESEARCH ESTABLISHMENT**

**THE REPRESENTATION OF AN  
EMPIRICAL FUNCTION BY MEANS  
OF A POLYNOMIAL**

AN A. E. R. E. REPORT

by

**D.J. BEHRENS**

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MINISTRY OF SUPPLY,  
HARWELL, BERKS.  
1950

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THE REPRESENTATION OF AN EMPIRICAL FUNCTION  
BY MEANS OF A POLYNOMIAL.

by

D. J. BehrensAbstract

If the values of a function are given at  $n$  equally-spaced intervals of the argument, a polynomial of order  $n-1$  can be found which corresponds exactly to the given values. In general a polynomial of order  $r$  ( $r < n-1$ ) cannot be made to do this, and  $\mathfrak{J}_r(f)$ , the minimized sum of the squares of the deviations, will exceed 0.

It is shown in this paper how  $\mathfrak{J}_r(f)$ , and the leading coefficient of the polynomial in question, can be evaluated without having to solve any simultaneous equations.

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## 1. Introduction

It frequently happens in experimental work, that we can measure the value of some function  $f$  at several values of the argument  $x$ , and we wish to find a polynomial  $p(x)$  which is a sufficiently accurate measure of the given functional values.

We shall confine our attention to the case in which (1) equal errors are considered to be equally probable in each of the given functional values, so that the method of least squares may be used without weighting factors to give the "best" polynomial, and in which (2) the values of the function are given for equally spaced values of the argument  $x$ . We shall assume them to be given at  $x = 1, 2, \dots, n$ , and the corresponding functional values will be denoted by  $f_1, f_2, \dots, f_n$ .

The two reservations made in the previous paragraph can usually be ensured by suitable experimental technique. We make them, because on their validity depends the simple method developed below.

It is clear that, if  $n$  functional values are given, we can construct a polynomial  $p_{n-1}(x)$  of order  $n-1$  which will pass exactly through each point. In general, a polynomial of order  $r$ , less than  $n-1$ , will not pass exactly through all the points. We can however define an  $r^{\text{th}}$  order polynomial  $p_r(x)$  by adjusting its coefficients so as to minimize the sum of the squares

$$\sum_{j=1}^n \{p_r(j) - f_j\}^2.$$

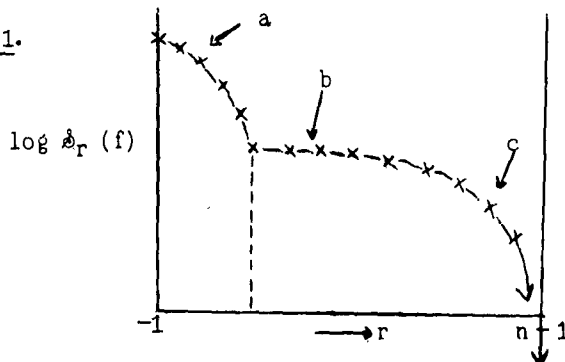
This minimized sum will be denoted by  $\delta_r(f)$ .

If now we examine the form of  $\delta_r(f)$  as  $r$  is increased from  $-1^*$  to  $n-1$ , we find that, if the functional values  $f_1, f_2, \dots, f_n$  really do lie close to a polynomial of comparatively low order, the general form of  $\delta_r(f)$  is as sketched in Fig. 1.

In the region shown as (a), we have too few parameters to represent adequately even the general outline of the curve, while in (c) we have so many that phoney "accuracy" is obtained by fitting a curve even through the random errors of observation. The region (b) is that in which we have enough parameters to represent the general form of the curve, but not so many as to purport spurious accuracy. The value of  $r$  in which we are interested lies at the lower end of this region, and is indicated by the dotted line in Fig. 1.

We shall show how to find this value of  $r$  without the tedious and heavy arithmetical work involved in actually calculating  $p(x)$  for each  $r$ .

Figure 1.



\*  $y = \text{constant}$  is a polynomial of order zero: -it is convenient to define  $y = 0$ , which has one fewer degrees of freedom, as a polynomial of order  $-1$ .

2. Determination of  $\delta_r(f)$ .

Let  $\Delta_j$  denote  $f_{j+1} - f_j$ ,  $\Delta_j^2 = \Delta_{j+1} - \Delta_j$ , and in general  $\Delta_j^r = \Delta_{j+1}^{r-1} - \Delta_j^{r-1}$ . This is the usual finite-difference notation for forward differences.

Let  $m_r$  denote the mean of the  $r^{\text{th}}$  order differences  $\Delta_j^r$ , weighted in accordance with the formula

$$m_r = \frac{\sum_{j=1}^{n-r} \binom{n-j}{r} \binom{r+j-1}{r} \Delta_j^r}{\binom{n+r}{2r+1}}$$

where the terms in round brackets ( ) denote binomial coefficients. [It will be proved in appendix (proof 1) that  $\sum_{j=1}^{n-r} \binom{n-j}{r} \binom{r+j-1}{r}$  is in fact equal to  $\binom{n+r}{2r+1}$ , so that  $m_r$  defined as above is in fact a weighted mean of the  $\Delta_j^r$ 's.]

$m_0$  is defined in the same way, remembering that  $\Delta_j^0 = f_j$ , and in fact  $m_0$  can be seen to be the arithmetic mean of the  $f$ 's. In general,  $m_r$  is the  $r^{\text{th}}$  difference of  $p_r(x)$ , the  $r^{\text{th}}$ -order polynomial giving the closest fit to the given  $f$ 's (See Appendix, Proof 2).

Then  $\delta_r(f)$  and  $\delta_{r-1}(f)$  are connected by the relation

$$\delta_{r-1}(f) - \delta_r(f) = \binom{n+r}{2r+1} m_r^2 \binom{2r}{r} \quad \text{See Appendix, proofs 3 and 4.}$$

Since we know that  $\delta_{n-1}(f) = 0$ , and that  $\delta_1(f) = \sum_{j=1}^n (f_j^2)$ , we can calculate the  $\delta_r$ 's beginning from either end. Alternatively the fact that both end-values are known serves as a check on the arithmetic if we calculate  $\delta_{r-1}(f) - \delta_r(f)$  for every  $r$  from 0 to  $n-1$ .

3. Probable error in  $m_r$ .

It remains to calculate the probable error of the  $m$ 's, in terms of that of the given functional values.

Expressing  $m_r$  in terms of the  $f$ 's themselves, instead of their  $r^{\text{th}}$  differences, we get

$$m_r = \frac{\sum_{j=1}^n f_j \sum_{m=0}^r (-1)^m \binom{r}{m} \binom{m+j-1}{r} \binom{n-j-m+r}{r}}{\binom{n+r}{2r+1}}$$

And thus the mean square error in  $m_r$  is given in terms of that in the  $f$ 's by

$$\delta^2 m_r / \delta^2 f = \frac{\sum_{j=1}^n \left\{ \sum_{m=0}^r (-1)^m \binom{r}{m} \binom{m+j-1}{r} \binom{n-j-m+r}{r} \right\}^2}{\binom{n+r}{2r+1}^2}$$

This reduces to

$$\delta^2 m_r / \delta^2 f = \binom{2r}{r} \left/ \binom{n+r}{2r+1} \right. \quad [\text{see Appendix, proof 5}]$$

But this ratio has already been shown to be equal to  $m_r^2 / (\delta_{r-1}(f) - \delta_r(f))$ , so we have the result that

$$\frac{\delta^2 m_r}{m_r^2} = \frac{\delta^2 f}{\delta_{r-1}(f) - \delta_r(f)}$$

Often, we do not know the mean square error  $\delta^2 f$ . We may however assume that, if  $r$  has been sensibly chosen, it is of the order of  $\delta_r(f)/(n-r-1)$  - we divide by  $(n-r-1)$  as this is the excess number of degrees of freedom over those required to fix the parameters of  $p_r(x)$ .

Thus we may say  $\delta^2 m_r / m_r^2 = \delta_r(f) / (n-r-1)(\delta_{r-1}(f) - \delta_r(f))$ , and the condition that  $m_r$  should be significantly different from zero is  $\delta_r(f) < (n-r-1)(\delta_{r-1}(f) - \delta_r(f))$ ,

$$\text{or} \quad \frac{\delta_r(f)}{(n-r-1)} < \frac{\delta_{r-1}(f)}{(n-r)}$$

It is therefore worth while, in tabulating  $\delta_r(f)$ , also to note the value of the quotient  $\delta_r(f)/(n-r-1)$ , since (a) when  $\delta^2 f$  is known, it is this quantity which must be kept down to the order of  $\delta^2 f$ , and (b) in any case if the quotient  $\delta_r(f)/(n-r-1)$  exceeds the corresponding value at  $(r-1)$ , this shows that no significant additional accuracy arises from taking an  $r^{\text{th}}$  order polynomial instead of  $p_{r-1}$ . This argument is used in Example 3 below.

#### Reference to other work

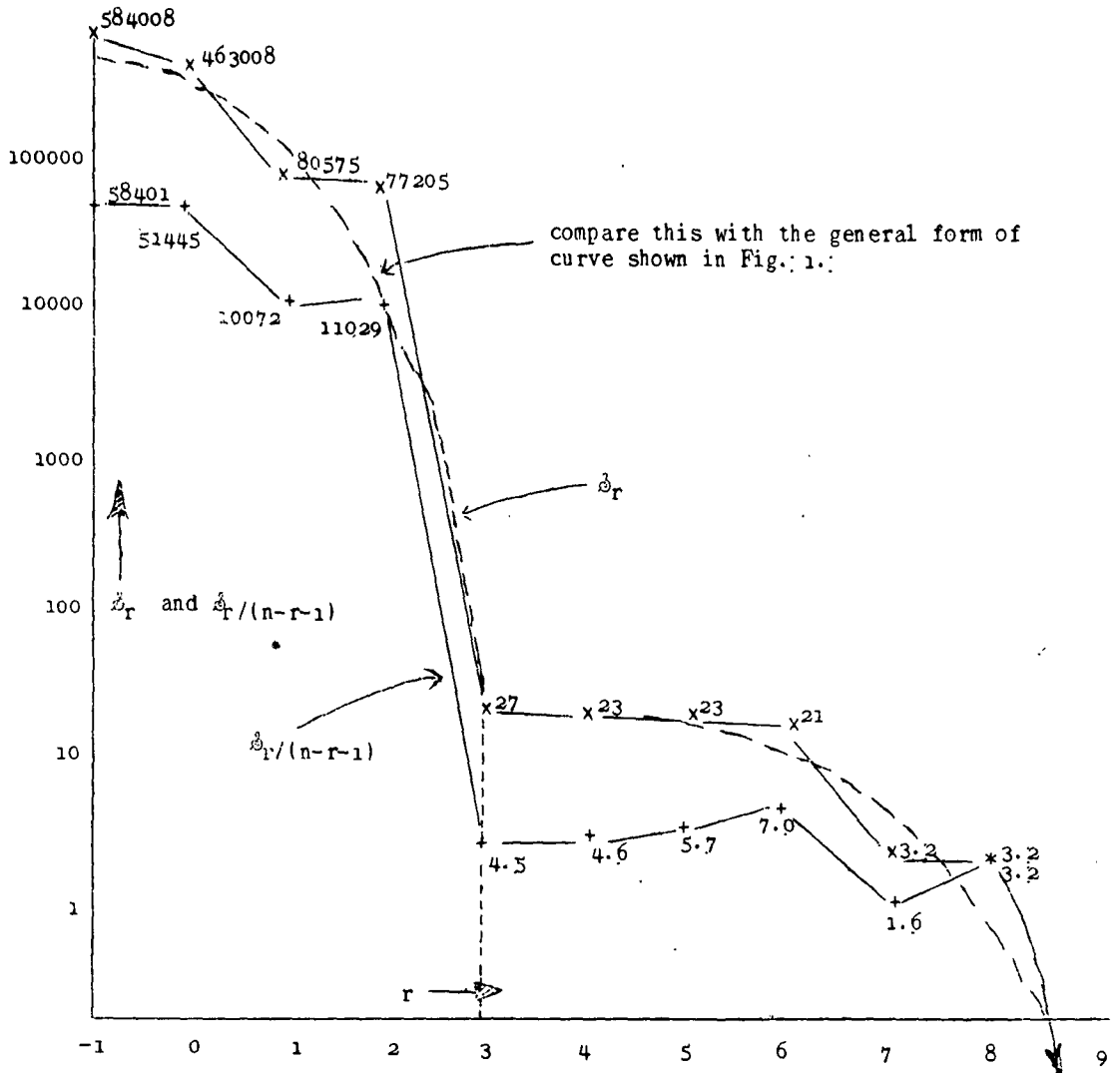
- M. G. Smith "The Lorentz Method of Analysis of Experimental Data using Orthogonal Polynomials" (A.R.E. Report No. 15/52). [This paper presents a method for calculating the best-fitting  $r^{\text{th}}$  order polynomial, having first obtained that of order  $r-1$ . It differs from the treatment of the present paper in computing first the polynomials themselves, and only afterwards the values of  $\delta(f)$ , whereas we have here treated the residual errors as being of prime interest - i.e. we determine the required order of polynomial without having first computed it. Either treatment can be more useful than the other, according to circumstances.]

**EXAMPLE I.** Functional values lying close to a polynomial of low order (cubic).

r	1	2	3	4	5	6	7	8	9	10	SUMS
fr	261	378	392	320	198	63	-69	-162	-181	-100	
w. f.	1	1	1	1	1	1	1	1	1	1	10
	261	378	392	320	198	63	-69	-162	-181	-100	1100
$\Delta$		117	14	-72	-122	-135	-132	-93	-19	81	
w. f.		9	16	21	24	25	24	21	16	9	165
		1053	224	-1512	-2928	-3375	-3168	-1953	-304	729	11234
$\Delta^2$			-103	-86	-50	-13	3	39	74	100	
w. f.			36	84	126	150	150	126	84	36	792
			-3708	-7224	-6300	-1950	450	4914	6216	3600	4002
$\Delta^3$				17	36	37	16	36	35	26	
w. f.				84	224	350	400	350	224	84	1716
				1428	8064	12950	6400	12600	7840	2134	51466
$\Delta^4$					19	1	-21	20	-1	-9	
w. f.					126	350	525	525	350	126	2002
					2394	350	-11025	10500	-350	-1134	735
$\Delta^5$						-18	-22	41	-21	-8	
w. f.						126	336	441	336	126	1936
						-2268	-7392	18081	-7056	-1008	357
$\Delta^6$							-4	63	-62	13	
w. f.							84	196	196	84	560
							-336	12348	-12152	1092	952
$\Delta^7$								67	-125	75	
w. f.								36	64	36	136
								2412	-8000	2700	-2888
$\Delta^8$									-192	200	
									9	9	18
									-1728	1800	+72
$\Delta^9$										392	
w. f.										1	1
										392	392

$\Delta_1 = \Sigma(f^2) = 584008$	$\frac{1}{10} \Delta_1 \approx 58401$
$\Delta_2 = 584008 - (1100)^2(10 \times 1) = 584008 - 121000 = 463008$	$\frac{1}{9} \Delta_2 \approx 51445$
$\Delta_3 = 463008 - (11234)^2 / (185 \times 2) = 463008 - 332432^{9/8} / 185 = 80575^{6/7} / 185$	$\frac{1}{8} \Delta_3 \approx 10072$
$\Delta_4 = 80575^{6/7} / 185 - (4002)^2 / (792 \times 6) = 80575^{6/7} / 185 - 3370^{4/9} / 132 = 77205^{2/3} / 600$	$\frac{1}{7} \Delta_4 \approx 11029$
$\Delta_5 = 77205^{2/3} / 600 - (51466)^2 / (1716 \times 20) = 77205^{2/3} / 600 - 77178^{4/8} / 8580 = 27^{2/6} / 858$	$\frac{1}{6} \Delta_5 \approx 4.5$
$\Delta_6 = 27^{2/6} / 858 - (735)^2 / (2002 \times 70) = 27^{2/6} / 858 - 3^{40/8} / 672 = 23^{2/3} / 132$	$\frac{1}{5} \Delta_6 \approx 4.6$
$\Delta_7 = 23^{2/3} / 132 - (357)^2 / (1335 \times 252) = 23^{2/3} / 132 - 200 / 780 = 22^{17/24} / 2145$	$\frac{1}{4} \Delta_7 \approx 5.7$
$\Delta_8 = 22^{17/24} / 2145 - (352)^2 / (530 \times 324) = 22^{17/24} / 2145 - 200 / 780 = 21^{11/2} / 2145$	$\frac{1}{3} \Delta_8 \approx 7.0$
$\Delta_9 = 21^{11/2} / 2145 - (2888)^2 / (136 \times 3432) = 21^{11/2} / 2145 - 17 \frac{101440}{118898} = 3^{17/3} / 935$	$\frac{1}{2} \Delta_9 \approx 1.6$
$\Delta_{10} = 3^{17/3} / 935 - (72)^2 / (18 \times 12870) = 3^{17/3} / 935 - 16 / 718 = 3^{105/3} / 12155$	$\Delta_{10} \approx 3.2$
$\Delta_{11} = 3^{105/3} / 12155 - (392)^2 / (1 \times 48820) = 3^{105/3} / 12155 - 3^{105/3} / 12155 = 0$	

EXAMPLE I (continued)



EXAMPLE II.

Functional Values Not Lying Close to a Polynomial of Low Order.

Independent variable:-	0	1	2	3	4	5	6	SUMS
functional values:-	1	2	4	8	16	32	64	
weighting factors	1	1	1	1	1	1	1	$7 m_0 = \frac{127}{7}$
		1	2	4	8	16	64	127
first differences		1	2	4	8	16	32	
weighting factors		6	10	12	12	10	6	$56 m_1 = \frac{261}{28}$
		6	20	48	96	160	192	522
second differences			1	2	4	8	16	
weighting factors			15	30	36	30	15	$126 m_2 = \frac{233}{42}$
			15	60	144	240	240	699
third differences				1	2	4	8	
weighting factors				20	40	40	20	$120 m_3 = \frac{7}{2}$
				20	80	160	160	420
fourth differences					1	2	4	
weighting factors					15	25	15	$55 m_4 = \frac{25}{11}$
					15	50	60	125
fifth differences						1	2	
weighting factors						6	6	$12 m_5 = \frac{3}{2}$
						6	12	18
sixth differences							1	
weighting factors							1	$1 m_6 = 1$
							1	1

$$\phi_6 = 0$$

$$\phi_5 - \phi_6 = \frac{1}{924} m_6^2 = \frac{1}{924}$$

$$\phi_4 - \phi_5 = \frac{12}{252} m_5^2 = \frac{12}{252} \cdot \frac{9}{4} = \frac{3}{28} \therefore \phi_4 = \frac{1}{924} + \frac{3}{28} = \frac{100}{924} = \frac{25}{231}$$

$$\phi_3 - \phi_4 = \frac{55}{70} \times m_4^2 = \frac{55 \cdot 25}{70 \cdot 121} = \frac{625}{154} \therefore \phi_3 = \frac{25}{231} + \frac{625}{154} = \frac{1025}{462} = \frac{25}{6}$$

$$\phi_2 - \phi_3 = \frac{120}{20} \times \left(\frac{7}{2}\right)^2 = \frac{147}{2} \therefore \phi_2 = \frac{25}{6} + \frac{147}{2} = \frac{466}{6} = \frac{233}{3}$$

$$\phi_1 - \phi_2 = \frac{120}{6} \left(\frac{233}{42}\right)^2 = \frac{(233)^2}{84} \therefore \phi_1 = 233 \left(\frac{1}{3} + \frac{233}{84}\right) = \frac{20271}{28}$$

$$\phi_0 - \phi_1 = \frac{55}{2} \left(\frac{261}{28}\right)^2 = \frac{(261)^2}{28} \therefore \phi_0 = \frac{20271 + 68121}{28} = \frac{22098}{7}$$

$$\phi_{-1} - \phi_0 = \frac{7}{1} \left(\frac{127}{7}\right)^2 = \frac{(127)^2}{7} \therefore \phi_{-1} = \frac{16129 + 22098}{7} = 5461$$

$$\phi_6 = 0 \approx$$

$$\phi_5 = \frac{1}{924} \approx 1.1 \times 10^{-3}$$

$$\phi_4 = \frac{25}{231} \approx 1.1 \times 10^{-1}$$

$$\phi_3 = \frac{25}{6} \approx 4.2$$

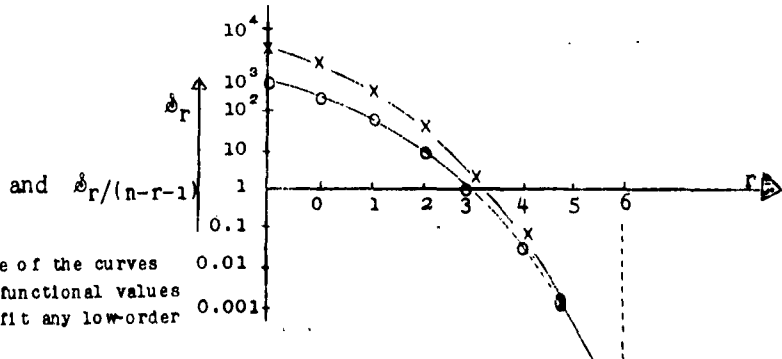
$$\phi_2 = \frac{233}{3} \approx 7.8 \times 10$$

$$\phi_1 = \frac{20271}{28} \approx 7.2 \times 10^2$$

$$\phi_0 = \frac{22098}{7} \approx 3.2 \times 10^3$$

$$\phi_{-1} = 5461 \approx 5.4 \times 10^3$$

Check:- Sum of  $f^2 = 1 + 4 + 16 + 64 + 256 + 1024 + 4096 = 5461$



And the shape of the curves suggest that the functional values certainly do not fit any low-order polynomial.

**EXAMPLE III**

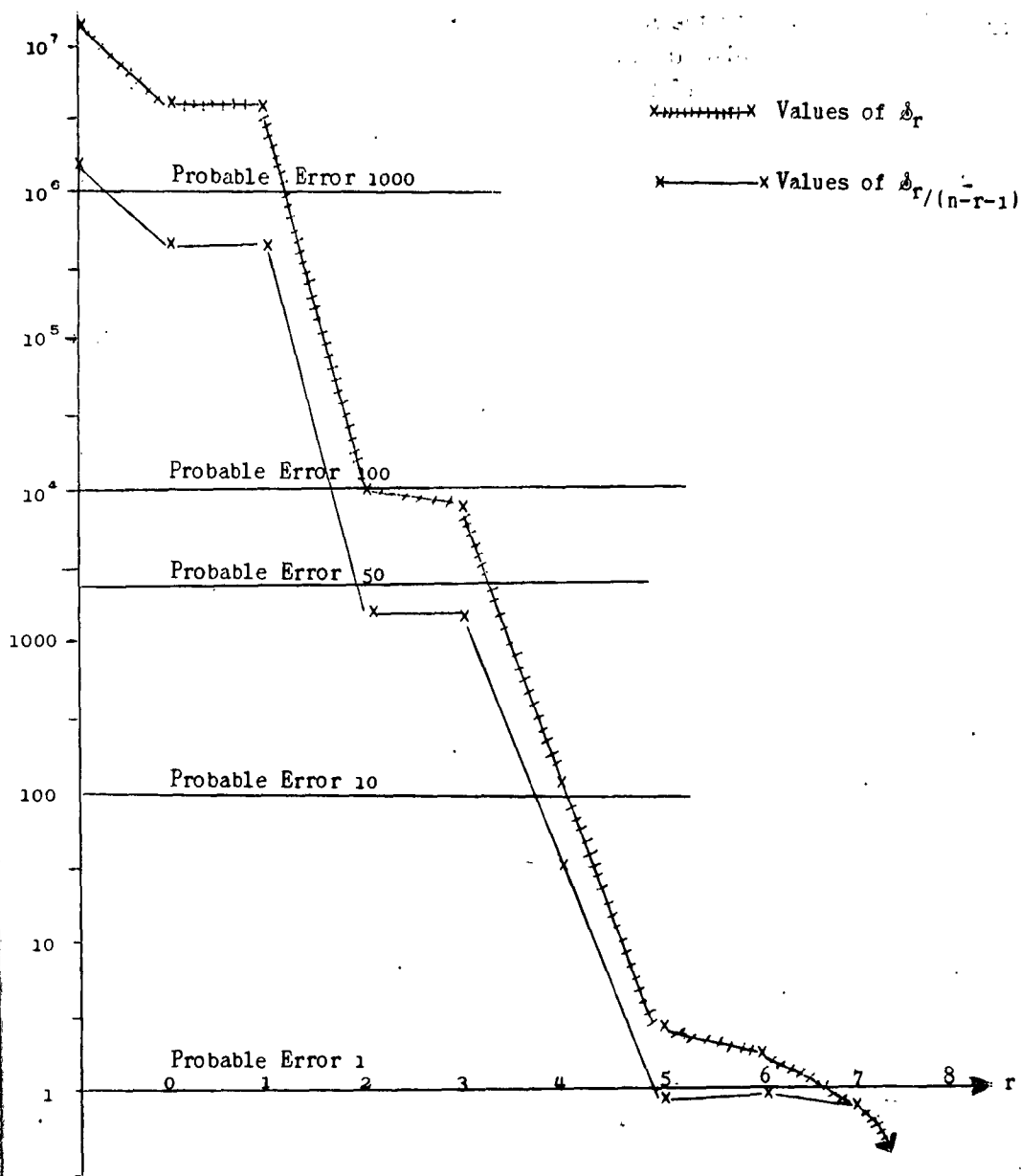
What is the lowest order of polynomial consistent with the following observations,

(a) if the probable error of each observation is 50

(b) " " " " " " " " " 1

	x	1	2	3	4	5	6	7	8	9	
$f(x)$		27	676	1211	1576	1730	1640	1309	775	116	
f		27	676	1211	1576	1730	1640	1309	775	116	$\Sigma(\Delta wf)$
wf		1	1	1	1	1	1	1	1	1	9060
$\Delta$		649	535	365	154	-90	-331	-534	-659		
		8	14	18	20	20	18	14	8		1186
$\Delta^2$		-114	-170	-211	-244	-241	-203	-125			
wf		28	63	90	100	90	63	28			- 95271
$\Delta^3$			-56	-41	-33	3	33	78			
wf			56	140	200	200	140	56			- 5188
$\Delta^4$				15	8	36	35	40			
wf				70	175	225	175	70			19475
$\Delta^5$					-7	28	-1	5			
wf					56	126	126	56			3290
$\Delta^6$						35	-29	6			
wf						28	49	28			- 273
$\Delta^7$							-64	35			
wf							8	8			- 232
$\Delta^8$									-99		
wf									1		- 99

$\hat{s}_9 = 0$	$\hat{s}_7 = (99)^2/1 \times 12870 = 0.76$	$\hat{s}_8 = 0$	$\hat{s}_7 = 0.76$	$\hat{s}_7 = 0.76$	$= (.87)^2$
$\hat{s}_8 - \hat{s}_7 = (232)^2/16 \times 3432 = 0.98$	$\hat{s}_6 = 1.74$	$\hat{s}_6 = 1.74$	$\frac{1}{2} \hat{s}_6 = 0.87$	$= (.93)^2$	
$\hat{s}_7 - \hat{s}_6 = (273)^2/105 \times 924 = 0.77$	$\hat{s}_5 = 2.51$	$\hat{s}_5 = 2.51$	$\frac{1}{3} \hat{s}_5 = 0.84$	$= (.92)^2$	
$\hat{s}_6 - \hat{s}_5 = (3290)^2/364 \times 252 = 118$	$\hat{s}_4 = 121$	$\hat{s}_4 = 121$	$\frac{1}{4} \hat{s}_4 = 30$	$= (5.5)^2$	
$\hat{s}_5 - \hat{s}_4 = (19475)^2/715 \times 70 = 7578$	$\hat{s}_3 = 77 \times 10^2$	$\hat{s}_3 = 77 \times 10^2$	$\frac{1}{5} \hat{s}_3 = 15.4 \times 10^2$	$= 39^2$	
$\hat{s}_4 - \hat{s}_3 = (5188)^2/792 \times 20 = 1701$	$\hat{s}_2 = 94 \times 10^2$	$\hat{s}_2 = 94 \times 10^2$	$\frac{1}{6} \hat{s}_2 = 15.8 \times 10^2$	$= 39^2$	
$\hat{s}_3 - \hat{s}_2 = (95271)^2/462 \times 6 = 32.7 \times 10^5$	$\hat{s}_1 = 32.8 \times 10^5$	$\hat{s}_1 = 32.8 \times 10^5$	$\frac{1}{7} \hat{s}_1 = 4.7 \times 10^5$	$= (690)^2$	
$\hat{s}_2 - \hat{s}_1 = (1186)^2/120 \times 2 = 5861$	$\hat{s}_0 = 32.9 \times 10^5$	$\hat{s}_0 = 32.9 \times 10^5$	$\frac{1}{8} \hat{s}_0 = 4.1 \times 10^5$	$= (640)^2$	
$\hat{s}_1 - \hat{s}_0 = (9060)^2/9 \times 1 = 91.2 \times 10^5$	$\hat{s}_{-1} = 124.1 \times 10^5$	$\hat{s}_{-1} = 124.1 \times 10^5$	$\frac{1}{9} \hat{s}_{-1} = 13.8 \times 10^5$	$= (1200)^2$	



$\sqrt{\delta_r/(n-r-1)}$  should be of the same order of magnitude as the probable mean error of observation [for detailed treatment see any textbook of statistics, under the heading "X<sup>2</sup> distribution of goodness of fit"].

In the example given, it will be seen that a parabola ( $r=2$ ) is consistent with a probable error of 50, while if the probable error is only unity, a 5th order polynomial is required to give an adequate fit.

With a probable error of 50 in each  $f$ , however, the lower values of  $\sqrt{\delta_r/(n-r-1)}$  for  $r$  in excess of 3 would appear highly suspicious, and would lead very definitely to suppose that the errors are largely of a systematic rather than a random nature.

A P P E N D I X

Proof 1.

That 
$$\sum_{j=1}^{n-r} \binom{n-j}{r} \binom{r+j-1}{r} = \binom{n+r}{2r+1} \quad [n > r].$$

Let  $\delta(n, r)$  denote 
$$\sum_{j=1}^{n-r} \binom{n-j}{r} \binom{r+j-1}{r} - \binom{n+r}{2r+1}$$

Writing the binomial coefficients as the sums of coefficients of lower order, we may say

$$\delta(n, r) = \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} \left(1 + \frac{n-j-r}{r}\right) \binom{r+j-2}{r-1} \left(1 + \frac{j-1}{r}\right) - \binom{n+r-1}{2r} \left(1 + \frac{n-r-1}{2r+1}\right)$$

$$= \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} \binom{r+j-2}{r-1} \left(1 + \frac{n-r-1}{r} + \frac{(n-j-r)(j-1)}{r^2}\right) - \binom{n+r-2}{2r-1} \left(1 + \frac{n-r-1}{2r+1}\right) \left(1 + \frac{n-r-1}{2r}\right)$$

The term  $\frac{(n-j-r)(j-1)}{r^2}$  vanishes both at  $j=1$  and at  $j=n-r$ ; we may therefore shorten its range of summation by one at each end.

so that 
$$\delta(n, r) = \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} \binom{r+j-2}{r-1} \left(1 + \frac{n-r-1}{r}\right) + \sum_{j=2}^{n-r-1} \binom{n-j-1}{r} \binom{r+j-2}{r} - \binom{n+r-2}{2r-1} \left(1 + \frac{n-r-1}{r}\right) - \binom{n+r-2}{2r+1}$$

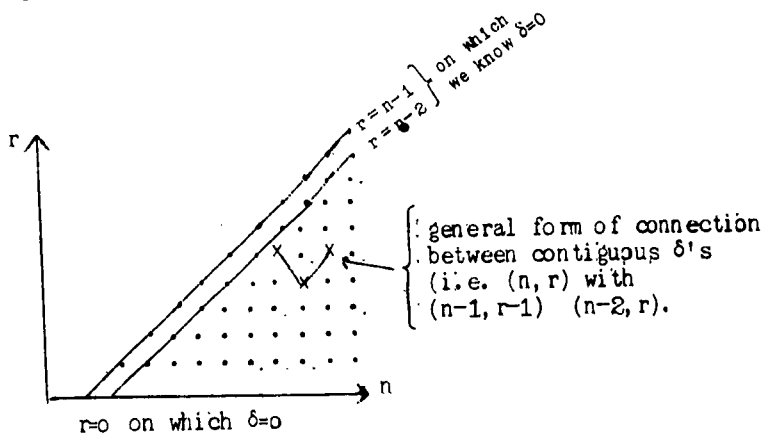
$$= \frac{n-1}{r} \delta(n-1, r-1) + \delta(n-2, r)$$

Now 
$$\delta(n, 0) = \sum_{j=1}^n \binom{n-j}{0} \binom{j-1}{0} - \binom{n}{1} = n - n = 0$$

Again, at  $r = n-1$ ,  $j$  can only be 1, and  $\delta(n, n-1) = \binom{n-1}{n-1} \binom{n-1}{n-1} - \binom{2n-1}{2n-1} = 1 - 1 = 0$

At  $r = n-2$ ,  $j$  can be 1 or 2, and  $\delta(n, n-2) = \binom{n-1}{n-2} \binom{n-2}{n-2} + \binom{n-2}{n-2} \binom{n-1}{n-2} - \binom{2n-2}{2n-2} = 2(n-1) - (2n-2) = 0$

This suffices to prove by induction that  $\delta(n, r) = 0$ .



Proof 2.

That  $\sum_{j=1}^{n-r} \binom{n-j}{r} \binom{r+j-1}{r} / \Delta_j^r \binom{n+r}{2r+1}$  is the  $r^{\text{th}}$  difference of the

$r^{\text{th}}$  order polynomial giving the closest fit to the points  $f_1, f_2, \dots, f_n$

-----

Let the  $f$ 's be chosen so that all but one of their  $r^{\text{th}}$  order differences vanish, and the other is unity. Let  $\Delta_j^r = 1$ ,  $\Delta_k^r = 0$  for  $k \neq j$ . Such a set of  $f$ 's is given by  $f_0 = f_1 \dots = f_{j+r-1} = 0$  and  $f_k = \binom{k-j-1}{r-1}$  for  $k \geq j+r$ .

$$\text{Let } p_r(x) = a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0.$$

We are to choose the  $a$ 's so as to minimize

$$\sum_{x=1}^{j+r-1} (a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0)^2 + \sum_{x=j+r}^n \left\{ a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0 - \binom{x-j-1}{r-1} \right\}^2$$

If we use a notation in which  $\binom{g}{h} = 0$  for all  $g < h$ , we may simplify this, and say that we have to minimize

$$\sum_{x=1}^n \left\{ a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0 - \binom{x-j-1}{r-1} \right\}^2$$

We obtain a set of simultaneous equations, from which  $a_{r-1}, a_{r-2}, \dots, a_1, a_0$  can be eliminated, leaving

$$a_r \begin{vmatrix} S_{2r} & S_{2r-1} & S_{2r-2} & \dots & S_r \\ S_{2r-1} S_{2r-2} & S_{2r-3} & \dots & S_{r-1} \\ S_{2r-2} S_{2r-3} & S_{2r-4} & \dots & S_{r-2} \\ \dots & \dots & \dots & \dots \\ S_r & S_{r-1} & S_{r-2} & \dots & S_0 \end{vmatrix} = \begin{vmatrix} \sum_1^n x^r \binom{x-j-1}{r-1} & S_{2r-1} & S_{2r-2} & \dots & S_r \\ \sum_1^n x^{r-1} \binom{x-j-1}{r-1} & S_{2r-2} & S_{2r-3} & \dots & S_{r-1} \\ \sum_1^n x^{r-2} \binom{x-j-1}{r-1} & S_{2r-3} & S_{2r-4} & \dots & S_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_1^n \binom{x-j-1}{r-1} & S_{r-1} & S_{r-2} & \dots & S_0 \end{vmatrix}$$

where  $S_k$  denotes  $(1^k + 2^k + 3^k + \dots + n^k)$ .

We have to calculate the values of these determinants.

It may be shown that

$$\begin{vmatrix} \frac{1}{2r+1} & \frac{1}{2r} & \frac{1}{2r-1} & \dots & \frac{1}{r+1} \\ \frac{1}{2r} & \frac{1}{2r-1} & \frac{1}{2r-2} & \dots & \frac{1}{r} \\ \frac{1}{2r-1} & \frac{1}{2r-2} & \frac{1}{2r-3} & \dots & \frac{1}{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{r+1} & \frac{1}{r} & \frac{1}{r-1} & \dots & \frac{1}{1} \end{vmatrix} = [\mathfrak{F}(r)]^4 / \mathfrak{F}(2r+1)$$

where  $\mathfrak{F}(r)$  denotes the factorial function  $1! \ 2! \ 3! \ 4! \ \dots \ r!$

From this it follows that

$$\begin{vmatrix} S_{2r} & S_{2r-1} & \dots & S_r \\ S_{2r-1} & S_{2r-2} & \dots & S_{r-1} \\ \dots & \dots & \dots & \dots \\ S_r & S_{r-1} & \dots & S_0 \end{vmatrix} = \mathfrak{F}(n+r) \mathfrak{F}(n-r-2) [\mathfrak{F}(r)]^4 / [\mathfrak{F}(n-1)]^2 \mathfrak{F}(2r+1)$$

It can also be shown that

$$\begin{vmatrix} \sum_1^n x^r \binom{x-j-1}{r-1} S_{2r-1} \dots S_r \\ \sum_1^n x^{r-1} \binom{x-j-1}{r-1} S_{2r-2} \dots S_{r-1} \\ \dots \\ \sum_1^n \binom{x-j-1}{r-1} S_{r-1} \dots S_0 \end{vmatrix} = \frac{\left\{ r! \binom{n-j}{r} \binom{r+j-1}{r} \binom{2r}{r} \right\} \begin{vmatrix} S_{2r-2} & S_{2r-3} & \dots & S_{r-1} \\ S_{2r-3} & S_{2r-4} & \dots & S_{r-2} \\ \dots & \dots & \dots & \dots \\ S_{r-1} & S_{r-2} & \dots & S_0 \end{vmatrix}}{[\mathfrak{F}(n-1)]^2 \mathfrak{F}(2r)} \\ = \frac{\binom{n-j}{r} \binom{r+j-1}{r} \mathfrak{F}(n+r-1) \mathfrak{F}(n-r-1) \mathfrak{F}(r-1) [\mathfrak{F}(r)]^4}{[\mathfrak{F}(n-1)]^2 \mathfrak{F}(2r)}$$

$$\text{So that } a_r = \binom{n-j}{r} \binom{r+j-1}{r} \frac{(n-r-1)! (2r+1)!}{(n+r)! r!}$$

And hence the  $m^{\text{th}}$  order difference of  $p_r(x)$ , which is  $r! a_r$ , is given by  $\binom{n-j}{r} \binom{r+j-1}{r} / \binom{n+r}{2r+1}$ . This is in fact the coefficient of  $\Delta_j^r$  in the expression for  $M_r$ . As this applies for any  $j$ , and the differences and  $M_r$ 's are all linear in the  $f$ 's, it follows that  $M_r$  is always the  $r^{\text{th}}$  difference of the best  $p_r(x)$ , whatever be the values of the  $f$ 's.

Proof 3. That  $\frac{\delta_{r-1}(f) - \delta_r(f)}{m_r^2}$  is a function only of  $n$  and  $r$ .

Let  $f$  denote a set of functional values  $f_1, f_2, f_3, \dots, f_n$  given at  $n$  equally-spaced intervals of the argument.

Let  $p_r(x)$  denote the best  $r^{\text{th}}$  order polynomial through the functional values - i.e. the parameters are so chosen that  $\sum_{j=1}^n$  is minimized - and let  $\delta_r(f)$  be used to denote this minimized sum, and  $m_r(f)$  the  $r^{\text{th}}$  difference of the polynomial  $p_r(x)$ .

Now let  $f^*$  denote that set of functional values  $f_1^*, f_2^*, \dots, f_n^*$  such that the first  $(r+1)$  of them, viz  $f_1^*$  to  $f_{r+1}^*$ , are the same as those in the  $f$  set ( $f_1^* = f_1, f_2^* = f_2, \dots, f_{r+1}^* = f_{r+1}$ ), and the remaining  $n-r-1$  values are so chosen that they lie on the  $r^{\text{th}}$  order polynomial through  $f_1, f_2, \dots, f_{r+1}$ .

Then clearly  $\delta_r(f^*) = 0$ .

Next, define a new set  $f(\theta)$  of functional values, such that  $f_j(\theta) = \theta f_j + (1-\theta)f_j^*$  for any  $j$  from 1 to  $n$ . It is clear that all the  $f_j(\theta)$ 's must be linear in  $\theta$ , since  $f_j$  and  $f_j^*$  are constants already fixed. From this it follows that  $m_r(f(\theta))$  is also linear in  $\theta$ , as  $m$  is linear in all the  $f_j(\theta)$ 's.

Since  $m_r(f(\theta))$  is linear in  $\theta$ , there must exist a  $\theta$  for which  $m_r(f(\theta)) = 0$ . Let this value of  $\theta$  be denoted by  $\theta^{**}$ , and let  $f^{**}$  represent the corresponding set of functional values.

Then the statement that  $m_r(f^{**}) = 0$  means that the best  $r^{\text{th}}$  order polynomial through  $f_1^{**}, f_2^{**}, \dots, f_n^{**}$  has a leading coefficient of zero. I.e. it is of order  $(r-1)$  or lower. Thus it follows that  $\delta_r(f^{**}) = \delta_{r-1}(f^{**})$ .

Now the coefficients in  $p_r(x)$  [and likewise in  $p_{r-1}(x)$ ] can be seen to be linear in the functional values and hence in  $\theta$ . Thus it follows that  $\delta_r(f(\theta))$  and  $\delta_{r-1}(f(\theta))$  are both quadratic in  $\theta$ . The difference  $\delta_{r-1}(f(\theta)) - \delta_r(f(\theta))$  is also quadratic in  $\theta$ . But this difference can never be negative - (the best  $r^{\text{th}}$  order polynomial must always be at least as good a fit as the best  $(r-1)^{\text{th}}$  order). Also  $\delta_{r-1}(f(\theta)) - \delta_r(f(\theta))$  vanishes at  $\theta = \theta^{**}$ .

$$\therefore \delta_{r-1}(f(\theta)) - \delta_r(f(\theta)) = K_1(\theta - \theta^{**})^2 \text{ where } K_1 \text{ is some constant.}$$

Again,  $m_r(f(\theta))$  is as we have seen linear in  $\theta$ , and by definition vanishes at  $\theta = \theta^{**}$

$$\therefore m_r(f(\theta)) = K_2(\theta - \theta^{**}) \text{ where } K_2 \text{ is another constant.}$$

From this it follows at once that

$$\frac{\delta_{r-1}(f(\theta)) - \delta_r(f(\theta))}{m_r^2} = \frac{K_1}{K_2^2} = \text{a constant independent of } \theta.$$

As we could have defined  $f^*$  by any  $r+1$  points (not necessarily the first ones), we can see that this constant must be the same whatever the values of the functional values, and in fact  $(\delta_{r-1}(f) - \delta_r(f))/m_r^2(f)$  can depend only on the numbers  $n$  and  $r$ .

Proof 4. That  $\frac{\delta_{r+1}(f) - \delta_r(f)}{m_r^2}$  is equal to  $\binom{n+r}{2r+1} / \binom{r}{r}$

NOTE: In proof (3) above we have shown that  $\frac{\delta_{r-1}(f) - \delta_r(f)}{m^2(r)}$  depends only on  $r$  and  $n$ , and not on the  $f$ 's. Consequently we are at liberty to choose now what  $f$ 's we wish

\* \* \*

Let the  $f$ 's be chosen so that  $f_j = \frac{(-1)^{r-j+1} \binom{r}{j-1}}{r!}$  for  $1 \leq j \leq r+1$ ,

and  $f_j = 0$  for  $r+1 < j < n$ .

Then  $\sum_{j=1}^n j^k f_j = 0$  for  $k < r$  and  $= 1$  for  $k = r$ .

From this it follows that  $p_{r-1}(x) = 0$  and  $\delta_{r-1}(f) = \sum_{j=1}^n (f_j)^2$ .

For  $p_r(x)$  we have the conditions

$$\left. \begin{aligned} a_r S_{2r} + a_{r-1} S_{2r-1} + \dots + a_1 S_{r+1} + a_0 S_r &= 1 \\ a_r S_{2r-1} + a_{r-1} S_{2r-2} + \dots + a_1 S_r + a_0 S_{r-1} &= 0 \\ a_r S_{2r-2} + a_{r-1} S_{2r-3} + \dots + a_1 S_{r-1} + a_0 S_{r-2} &= 0 \\ \dots &\dots \\ a_r S_r + a_{r-1} S_{r-1} + \dots + a_1 S_1 + a_0 S_0 &= 0 \end{aligned} \right\} \dots (1)$$

$$\begin{aligned} \text{But } \delta_r &= \sum_{j=1}^n (a_r j^r + a_{r-1} j^{r-1} + \dots + a_1 j + a_0 - f_j)^2 \\ &= a_r(a_r S_{2r} + \dots + a_0 S_r) \\ &\quad + a_{r-1}(a_r S_{2r-1} + \dots + a_0 S_{r-1}) \\ &\quad + a_0(a_r S_r + \dots + a_0) \\ &\quad - 2a_r \sum f_j j^r - 2a_{r-1} \sum f_j j^{r-1} - \dots - 2a_0 \sum f_j \\ &\quad + \sum (f_j)^2 \\ &= a_r - 2a_r + \sum (f_j)^2 \end{aligned}$$

So that  $\delta_{r-1} - \delta_r = a_r$ .

But, from the conditions (1) above, it follows that

$$a_r = \frac{\begin{vmatrix} S_{2r-2} & \dots & S_{r-1} \\ \dots & \dots & \dots \\ S_{r-1} & \dots & S_0 \end{vmatrix}}{\begin{vmatrix} S_{2r} & \dots & S_r \\ \dots & \dots & \dots \\ S_r & \dots & S_0 \end{vmatrix}}$$

$$\begin{aligned}
&= \frac{\mathfrak{S}(n+r-1) \mathfrak{S}(n-r-1) [\mathfrak{S}(r-1)]^4 [\mathfrak{S}(n-1)]^2 \mathfrak{S}(2r+1)}{[\mathfrak{S}(n-1)]^2 \mathfrak{S}(2r-1) \mathfrak{S}(n+r) \mathfrak{S}(n-r-2) [\mathfrak{S}(r)]^4} \\
&= \frac{(n-r-1)! (2r)! (2r+1)!}{(n+r)! (r!)^4}
\end{aligned}$$

Again, the quantity  $\Delta_1^r$  for the f's as defined, is

$$\frac{1}{r!} \binom{2r}{r}; \quad \Delta_2^r \text{ is } -\frac{1}{r!} \binom{2r}{r+1}, \text{ and so on.}$$

$$\begin{aligned}
\text{Hence } m_r \text{ is } & \frac{\left\{ \binom{n-1}{r} \binom{r}{r} \binom{2r}{r} \right\} - \left\{ \binom{n-2}{r} \binom{r+1}{r} \binom{2r}{r+1} \right\} \dots \pm \left\{ \binom{r}{r} \binom{n-1}{r} \binom{2r}{n-r-1} \right\}}{r! \binom{n+r}{2r+1}} \\
&= \frac{2r!(2r+1)!(n-r-1)!}{(n+r)! (r!)^3} \sum_{\substack{j=r+1 \\ \text{or } j=n-r}}^{j=n-r} \frac{(-1)^{j+1} (n-j)!}{(n-j-r)!(j-1)!(r-j+1)!}
\end{aligned}$$

and the summation is in all cases unity.

$$\begin{aligned}
\therefore \frac{\mathfrak{S}_{r-1} - \mathfrak{S}_r}{m_r^2} &= \frac{(n-r-1)!(2r)!(2r+1)!}{(n+r)! (r!)^4} \div \left\{ \frac{2r!(2r+1)!(n-r-1)!}{(n+r)! (r!)^3} \right\}^2 \\
&= \frac{(n+r)! r!^2}{(2r+1)!(n-r-1)! 2r!} = \binom{n+r}{2r+1} \bigg/ \binom{2r}{r}
\end{aligned}$$

Q. E. D.

$$\text{Proof 5. That } \sum_j \left\{ \sum_m (-1)^m \binom{r}{m} \binom{m+j-1}{r} \binom{n-j-m+r}{r} \right\}^2 = \binom{2r}{r} \binom{n+r}{2r+1}.$$

Let  $m+j = k$  and  $n+r = s$

$$\begin{aligned}
\text{Then L.H.S.} &= \sum_j \left\{ \sum_k (-1)^k \binom{r}{k-j} \binom{k-1}{r} \binom{s-k}{r} \right\}^2 \\
&= \sum_j \sum_k \sum_l \left\{ (-1)^{k-l} \binom{r}{k-j} \binom{r}{l-j} \binom{k-1}{r} \binom{l-1}{r} \binom{s-k}{r} \binom{s-l}{r} \right\}
\end{aligned}$$

We can sum at once over all  $j$ , for

$$\sum_j \binom{r}{k-j} \binom{r}{l-j} = \binom{2r}{r+k-l}$$

$$\therefore \text{L.H.S.} = \sum_k \sum_l \left\{ (-1)^{k-1} \binom{2r}{r+k-l} \binom{k-1}{r} \binom{l-1}{r} \binom{s-k}{r} \binom{s-l}{r} \right\}$$

$$\text{Now } \binom{2r}{r+k-l} \binom{k-1}{r} \binom{l-1}{r} = \binom{2r}{r} \binom{k-1}{r+k-l} \binom{l-1}{r+l-k}$$

At this stage it is useful to put  $k=l+d$

$$\text{Thus L.H.S.} = \binom{2r}{r} \sum_l \sum_d \left\{ (-1)^d \binom{l+d-1}{r+d} \binom{l-1}{r-d} \binom{s-l-d}{r} \binom{s-l}{r} \right\}$$

Carrying out the summation first over all  $d$ , we get

$$\binom{2r}{r} \sum_l \binom{s-l}{r} \sum_d \left\{ (-1)^d \binom{l-1}{r-d} \binom{l+d-1}{r+d} \binom{s-l-d}{r} \right\}$$

which may be written as

$$\sum_l \binom{s-l}{r} \binom{l-1}{r} \sum_d \left\{ (-1)^d \binom{l+d-1}{r} \binom{s-l-d}{r} \binom{2r}{r+d} \right\}$$

$$\text{Now it may be shown that } \sum_d (-1)^d \binom{a+d}{r} \binom{b-d}{r} \binom{2r}{r+d} = \binom{2r}{r}$$

for all  $a > r$ ,  $b > r$ .

Putting  $a = l-1$ ,  $b = s-l$ , these inequalities are seen to apply. Consequently we are left with

$$\binom{2r}{r} \sum_l \binom{s-l}{r} \binom{l-1}{r} \{ 1 \}$$

$$\text{and this reduces at once to } \binom{2r}{r} \binom{s}{2r+1}$$

which = R.H.S.

[NOTE. We have omitted the limits of summation, since the summation is required for all values of the parameters giving nonzero contribution, and may in fact be taken from  $-\infty$  to  $+\infty$  for each parameter].

TABLES I and IA:

Weighting Factors to be Applied to  $\Delta_j^F$  in Calculating  $m_{r,j}$  and Sums of Weighting Factors.

Functional Values given at 3 points

w. f.	$m_0$	$m_1$	$m_2$
	1		
	1	2	1
	1	2	
SUMS	3	4	1

Functional Values given at 4 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$
	1			
	1	3	3	
	1	4	3	1
	1	3		
SUMS	4	10	6	1

Functional Values given at 5 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$
	1				
	1	4	6		
	1	6	9	4	
	1	6	6	4	1
	1	4			
SUMS	5	20	21	8	1

Functional Values given at 6 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$
	1					
	1	5	10			
	1	8	18	10		
	1	9	18	18	5	
	1	8	10	10	5	1
	1	5				
SUMS	6	35	56	38	10	1

TABLES I. IA Continued

Functional Values given at 7 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$
1		6					
1		10	15				
1		12	30	20	15		
1		12	36	40	25	6	1
1		12	36	40	25	6	
1		10	30	40	15		
1		10	15	20			
1		6					
SUMS	7	58	128	120	55	12	1

Functional Values given at 8 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$
1		7						
1		12	21					
1		15	45	35				
1		15	60	80	35	21	7	
1		16	60	100	75	36	7	1
1		15	45	90	75	36	7	
1		12	45	35	35	21		
1		12	21					
1		7						
SUMS	8	84	252	380	220	78	14	1

Functional Values given at 9 points.

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$
1		8							
1		14	28						
1		18	63	56	70				
1		18	90	140	175	56	28		
1		20	100	200	225	126	49	8	
1		20	100	200	225	126	49	8	1
1		18	90	140	175	126	28		
1		14	63	140	70	56			
1		14	28	56					
1		8							
SUMS	9	120	462	792	715	384	105	16	1

TABLES I & IA Continued

Functional Values given at 10 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$
1										
1	9									
1	16	36								
1	21	84	84		128					
1	24	126	224	350	128	84				
1	24	150	350	525	336	196	36		9	
1	25	150	400	525	441	196	64	9		1
1	24	150	350	525	336	84	36			
1	21	84	224	350	128					
1	16	36	84							
1	9									
1										
SUMS	10	165	792	1716	2002	1365	560	136	18	1

Functional Values given at 11 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	10										
1	18	45									
1	24	108	120		210						
1	28	168	336	210	252						
1	30	210	560	630	756	210		120			
1	30	225	700	1050	1176	588	268	45		10	
1	30	210	700	1225	1176	784	288	81	10		1
1	23	210	560	1050	1176	588	288	45	10		
1	24	168	336	630	756	210	120				
1	18	108	210	210	252						
1	10	45	120								
1											
SUMS	11	220	1287	3432	5005	4368	2380	816	171	20	1

Functional Values given at 12 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	11										
1	20	55									
1	27	135	165		330						
1	27	216	480	1050	462	462					
1	32	280	840	1990	1512	1470	320		165		
1	35	315	1120	2450	2646	2352	960	405	55		11
1	36	315	1225	2450	3136	2352	1296	405	100		11
1	35	280	1120	1990	2646	1470	960	165	55		
1	32	216	480	1050	1512	462	320				
1	27	135	165	330	462						
1	20	55	165								
1	11										
SUMS	12	286	2002	6435	11440	12376	8568	3876	1140	210	22

TABLES I. & IA Continued

Functional Values given at 13 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	12										
1	22	66									
1	30	165	220								
1	36	270	660	495							
1	40	360	1200	1650	792						
1	40	360	1200	1650	792	924					
1	43	420	1680	3150	2772	924	792				
1	42	441	1960	4410	7056	5880	4320	495			
1	42	441	1960	4900	7056	7056	4320	1485	220		66
1	40	420	1680	4410	7056	5880	4320	2025	550		121
1	36	360	1200	3150	5292	5880	2640	1485	550		66
1	30	270	660	1650	2772	3234	2640	495	220		
1	22	165	220	495	792	924	792				
1	12	66									
SUMS	18	364	3003	11440	24310	31824	27132	15504	5935	1540	253

Functional Values given at 14 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	13										
1	24	78									
1	33	198	286								
1	40	330	880	715							
1	40	330	880	715	1287						
1	45	450	1650	2475	4752	1716					
1	45	450	1650	2475	4752	1716	1716				
1	48	540	2400	4950	9702	6468	1716	1287			
1	48	540	2400	4950	9702	6468	6336	1287	715		286
1	49	538	2940	7350	14112	12936	11880	4455	2200		726
1	49	538	2940	7350	14112	12936	11880	7425	2025		726
1	48	538	2940	7350	14112	12936	11880	7425	2200		726
1	48	540	2400	7350	14112	12936	11880	4455	2200		286
1	45	450	2400	4950	9702	6468	6336	4455	715		
1	40	450	1650	4950	4752	6468	6336	1287			
1	40	330	1650	2475	4752	6468	1716	1287			
1	33	198	880	715	1287	1716					
1	24	78	286								
1	13										
SUMS	14	455	4368	19448	48620	75582	77520	54264	26334	8855	2024

TABLES I & IA Continued

Functional Values given at 15 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	14										
1	28	91									
1	36	234	384								
1	44	396	1144	1001							
1	50	550	2200	3575	2002						
1	54	675	3300	7425	7722	3003					
1	54	756	4200	11550	16632	12012	3432				
1	53	784	4704	14700	25872	13728	3003				
1	56	756	4704	15876	31752	25872	11533	3002			1001
1	56	756	4704	14700	31752	38903	28512	22275	7150		3146
1	53	784	4704	15876	31752	39600	27225	12100			4356
1	56	756	4704	14700	31752	39600	22275	12100			3146
1	54	675	4200	11550	25872	25872	28512	11533	7150		1001
1	50	550	3300	7425	16632	12012	13728	3003	2002		
1	44	396	2200	3575	7722	3003	3432				
1	36	396	1144	1001	2002						
1	28	234	384								
1	14	91									
1											
SUMS	15	560	6188	31824	92378	167960	203490	170544	100947	42504	12350

Functional Values given at 16 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1											
1	15										
1	28	105									
1	39	273	455								
1	48	468	1456	1385							
1	55	660	2860	5005	3003						
1	60	825	4400	10725	12012	5005					
1	63	945	5775	17325	27027	21021	6435				
1	63	945	5775	23100	44352	48048	61776	6435	5005		
1	64	1008	6720	28460	58212	77616	95040	57915	20020	3003	
1	64	1008	7056	28460	58212	77616	97020	81675	39325	11011	
1	63	1008	6720	28460	58212	97020	109900	81675	48400	18876	
1	63	945	6720	23100	58212	77616	95040	57915	39325	18976	
1	30	825	5775	17325	44352	77616	95040	57915	20020	11011	
1	55	825	4400	17325	27027	48048	61776	27027	5005	3003	
1	48	660	4400	10725	27027	21021	27456	6435			
1	48	660	2860	5005	12012	5005	6435				
1	39	468	1456	1385	3003						
1	28	273	455								
1	15	105									
1											
SUMS	16	680	8568	50388	167960	352716	497420	490314	346104	177100	65780



TABLES I & IA Continued

Functional Values given at 19 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1	18										
1	34	153									
1	48	408	816								
1	60	720	2720	3060		8568					
1	70	1050	5600	11900	37128	18564					
1	84	1365	9100	27300	86632	31824	43758				
1	90	1638	12740	47775	91728	155584	48620	43758			
1	84	1638	16016	70070	168168	420420	411840	218790	243100	43758	
1	88	1848	18480	90090	252252	630630	772200	579150	629200	213928	
1	90	1980	19800	103950	324324	792792	1132560	1061775	1101100	528528	
1	90	2025	19800	108900	365904	792792	1359072	1486485	1431430	858858	
1	88	1980	19800	108900	365904	853776	1359072	1656369	1431430	1002001	
1	84	1848	18480	90090	324324	792792	1132560	1486485	1101100	858858	
1	84	1848	16016	90090	252252	630630	772200	1061775	1101100	528528	
1	84	1638	16016	70070	252252	420420	772200	579150	629200	213928	
1	78	1365	12740	47775	168168	411840	411840	218790	243100	43758	
1	70	1050	9100	47775	91728	224224	155584	43758	48620		
1	60	720	5600	27300	37128	86632	31824				
1	48	408	2720	11900	8568	18564					
1	34	153	816	3060							
1	18										
1	18										
SUMS 19	1140	20349	170544	817190	2496144	5200300	7726160	8436285	6906900	4292145	

Functional Values given at 20 points

w. f. for	$m_0$	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$	$m_{10}$
1	19										
1	36	171									
1	51	459	969								
1	64	816	3284	3876							
1	75	1200	6800	15300	11628						
1	84	1575	11200	35700	51408	27132					
1	91	1911	15925	63700	129948	254592	50388				
1	91	2184	15925	244608	346528	700128	75582	92378			
1	96	2376	20384	95550	672672	1093950	393822	486200	92378		
1	96	2376	24024	378378	872672	1372800	1093950	1337050	481338		
1	99	2475	24024	126126	504504	1051050	2123550	2123550	1283568		
1	100	2475	26400	150150	594594	1387386	2123550	3185325	2516800	2290288	
1	99	2475	26400	163350	594594	1585584	2718144	3864861	3578575	3006003	
1	96	2376	26400	163350	627264	1585584	2718144	3864861	4008004	3006003	
1	96	2376	24024	150150	594594	1387386	2718144	3185325	3578575	2290288	
1	91	2184	24024	150150	504504	1051050	2123550	2123550	2516800	1283568	
1	91	1911	20384	126126	378378	1372800	1372800	1093950	1337050	481338	
1	84	1575	15925	95550	244608	672672	700128	393822	486200	92378	
1	75	1200	11200	63700	129948	346528	254592	75582	92378		
1	64	816	6800	35700	129948	129948	50388				
1	51	459	3284	15300	51408	27132					
1	36	171	969	3876	11628						
1	19										
1	19										
SUMS 20	1330	26334	245157	1307504	4457400	10400600	17383860	21474180	20030010	14307150	

TABLE II

Summary Table of Sums of the Weighting Factors.

Number of points at which functional values are given.											
	Sum of w.f. for $m_0$	Sum of w.f. for $m_1$	Sum of w.f. for $m_2$	Sum of w.f. for $m_3$	Sum of w.f. for $m_4$	Sum of w.f. for $m_5$	Sum of w.f. for $m_6$	Sum of w.f. for $m_7$	Sum of w.f. for $m_8$	Sum of w.f. for $m_9$	Sum of w.f. for $m_{10}$
1											
2	1										
3	4	1									
4	10	6	1								
5	20	21	8	1							
6	35	56	36	10	1						
7	56	126	120	55	12	1					
8	84	252	330	220	78	14	1				
9	120	462	792	715	364	105	16	1			
10	165	792	1716	2002	1365	560	136	18	1		
11	220	1287	3432	5005	4368	2380	816	171	20	1	
12	286	2002	6435	11440	12376	8568	3876	1140	210	22	
13	364	3003	11440	24310	31824	27132	15504	5985	1540	253	
14	455	4368	19448	48620	75582	77520	54264	26334	8855	2024	
15	560	6188	31824	92378	167960	203490	170544	100947	42504	12650	
16	680	8568	50388	167960	352716	497420	490314	346104	177100	65780	
17	816	11628	77520	298930	705432	1144066	1307504	1081575	657800	293010	
18	969	15504	116280	497420	1352078	2496144	3268760	3124550	2220075	1184040	
19	1140	20349	170544	817190	2496144	5200300	7728160	8436285	6906900	4292145	
20	1330	26334	245157	1307504	4457400	10400600	17383860	21474180	20030010	14307150	

General form: The sum of the weighting factors for  $m_r$ , when functional values are given at  $n$  points, is the binomial coefficient  $\binom{n+r}{2r+1}$ .

TABLE III

$\delta_{r-1} - \delta_r$  is given by  $Km_r^2$ , where the constant  $K$  is equal to the sum of the weighting factors for  $m_r$ , given in the table above, divided by the binomial coefficient  $\binom{2r}{r}$ , a function of  $r$  only and independent of  $n$ . This is tabulated below for  $r$  up to 10.

$r$	0	1	2	3	4	5	6	7	8	9	10
$\binom{2r}{r}$	1	2	6	20	70	252	924	3432	12870	48620	184756



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