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APPLIED MATHEMATICS AND STATISTICS LABORATORY  
STANFORD UNIVERSITY  
CALIFORNIA

AN APPLICATION OF THE SCHIFFER VARIATION TO  
THE FREE BOUNDARY PROBLEMS OF  
HYDRODYNAMICS

By  
EDWARD B. McLEOD, JR.

TECHNICAL REPORT NO. 15

November 6, 1953

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1. Introduction.

It is demonstrated in the literature which deals with the physical applications of the calculus of variations [32] that the static equilibrium position of a mechanical configuration is one in which the potential energy has a stationary value. There are many problems in mechanics and electrostatics that can be formulated either as problems involving static forces or as variational problems. It should not be startling to discover that equivalences similar to these also occur in hydrodynamics. The variational principles of hydrodynamics were first noticed by Lord Kelvin [19].

This paper will be devoted to existence proofs and solutions of some variational problems which have hydrodynamical analogues. We use a technique which has been devised by M. Schiffer. As the principal problem, we shall treat the problem of finding an extremum logarithmic capacity (other terminology referring to this functional or related functionals is Robin's constant, or outer mapping radius) when the length of all competing curves is held fixed; in this problem we also add the constraint that each competing curve contain, in the domain complementary to its exterior, a particular fixed segment. We shall also discuss the hydrodynamical interpretation of this problem. The logarithmic capacity is a functional analogous to the virtual mass of a two-dimensional vortex. We shall treat this problem thoroughly with regard to existence of the solution and the justification of the formal processes which are involved. We shall obtain, finally, a differential equation which defines the solution of the problem. The plan

of the dissertation will be as follows:

In Section 2 we shall introduce a technique devised by Hadamard [14], [15]. Though this variation of Hadamard is not as mathematically rigorous as that conceived by Schiffer, it will be of interest from an historical point of view and will develop for the reader a physical intuition for the hydrodynamical interpretation of the variational problems. We shall use this method to demonstrate formally the variational equivalents of some specific free boundary problems.

In Section 3 we define the variation of Schiffer and prove some theorems which justify its use in conjunction with Lagrange multipliers. In Section 4 we prove some theorems which are useful in establishing the existence of a solution of the principal problem, justifying rigorously each formal step.

The final section will be devoted primarily to solving some important hydrodynamical free boundary problems by considering their variational equivalents. Before discussing variational problems let us first discuss free boundary problems.

An example of a classical free boundary problem is the following: Consider the motion of a fluid issuing through an aperture in one of the walls of an infinite reservoir (Figure 1). The fluid will emerge as a jet which is in contact only with empty space. The boundary of this jet must be determined along with the rest of the streamlines of the flow. This problem, when simplified to the two-dimensional case, is treated in standard textbooks on classical hydrodynamics (see, e.g., [19] and [21]). By comparing various conformal mappings, one is able to arrive at a differential equation involving the complex velocity potential of the flow. We shall incorporate some of these techniques in the solutions of the problems in this dissertation. In general a fluid flow problem, where part of the boundary is

deformable and its shape is not known beforehand, is called a free boundary problem and the undetermined surface is called the free boundary.

The problems which we shall encounter will be of a two-dimensional nature, since we shall be applying the methods of complex variables. A two-dimensional flow is one where the streamlines and the boundary of the region of flow are the same in every cross-sectional plane perpendicular to the same direction (which, of course, can be chosen as one of the coordinate axes); consequently, only two of the coordinates are involved and the flow may be represented by considering a particular cross-section in one plane. If the flow is irrotational, the velocity vector is derivable from a scalar quantity known as the velocity potential; if, in addition, the fluid is incompressible, it can be shown that the velocity potential satisfies Laplace's equation and that there exists a stream function which is conjugate to the velocity potential. By forming the analytic function having the velocity potential as its real part and the stream function as its imaginary part, we have what is called the complex velocity potential; the use of this function enables us to solve many two-dimensional problems with the aid of the theory of functions of a complex variable. We shall adopt the symbol  $w(z)$  to denote the complex velocity potential. The complex vector representing the conjugate of the velocity can be shown to be  $w'(z)$ . All flows which we shall consider will be assumed to be two-dimensional, incompressible, and irrotational, thus enabling us to introduce the complex velocity potential.

A classical formula is the two-dimensional, irrotational, incompressible external force-free equation of Bernoulli which states

$$\frac{1}{2} \left| \frac{dw}{dz} \right|^2 + \frac{P(z)}{\rho} = C,$$

where  $P(z)$  is the real function which represents the pressure at a point  $z$ ,  $C$  is a constant, and  $\rho$  is the density of the fluid. If  $P_{\infty}$  represents the

pressure at infinity, if the density of this incompressible fluid is unity, and if  $w$  represents the complex potential of a uniform flow of unit velocity about a body, then Bernoulli's equation becomes

$$(1.1) \quad \frac{1}{2} \left| \frac{dw}{dz} \right|^2 + P(z) = P_{\infty} + \frac{1}{2}$$

As an example of a free boundary problem let us determine the shape of a bubble filled with gas which is placed in a uniform stream of unit velocity; the gas, together with the effects of the depth of submerision, creates within the bubble a pressure  $P_b$ ; there is a force  $T$ , caused by the surface tension, which acts on a unit length of this two-dimensional, cylindrical bubble. Let us consider the forces acting on a surface element of this bubble which is of unit length along the axis perpendicular to the coordinate plane and has as its other dimension an element of arc  $\Delta S$ . Let  $z_1$  and  $z_2$  be the end-points of this arc and let  $\theta_1$  and  $\theta_2$  represent the respective arguments of the tangent vectors at these points. Let us use the principles of elementary mechanics and construct a free body force diagram for this element of arc and consider especially the forces acting along the normal at  $z_1$ .

The pressure  $P_b$  contributes a force  $P_b \Delta S$ , while the hydrodynamical pressure force is  $p(z_1) \Delta S$ . The force due to the surface tension directed along the normal is  $T \sin(\theta_2 - \theta_1)$ . In order that these forces be in equilibrium the following equation should hold:

$$(1.2) \quad P_b \Delta S - P(z_1) \Delta S - T \sin(\theta_2 - \theta_1) = 0$$

Let us now pass to the limit  $z_2 \rightarrow z_1$ ; we have then, for any point  $z$  on the surface of the bubble,

$$P_b - P(z) = T \frac{d\theta}{ds} = T K(z),$$

where  $K$  represents the curvature at  $z$ . Recalling (1.1), this equation is expressible in the form

$$(1.3) \quad P_b + \frac{1}{2} \left| \frac{dw}{dz} \right|^2 = P_\infty + \frac{1}{2} + TK$$

It is for smaller bubbles that the surface tension forces become significant. An important special case, which we shall discuss in detail in the last section, is that for which  $P_b = P_\infty + \frac{1}{2}$ . In this case (1.3) becomes

$$(1.4) \quad \left| \frac{dw}{dz} \right|^2 = 2T$$

This is a simplified mathematical idealization of such naturally occurring problems as those described in the following examples:

1. When a bottle of champagne is uncorked, gases which have previously been dissolved in the liquid are released in the form of rising bubbles.
2. When a driller has struck oil, the top of the derrick is uncapped and because of this sudden change in pressure bubbles of the above type are formed.

In the next section we shall show that these equations, which we obtained by the use of Bernoulli's equation, can also be derived by certain variational formulations.

The author wishes to express his indebtedness to Professor P. R. Garabedian and Professor M. Schiffer, whose guidances have greatly contributed to the successful completion of this work; however, the author is individually responsible for the accuracy and correctness of the results.

## 2. The Variation of Hadamard and the Equivalence of Variational and Free Boundary Problems.

Let us recall the formal procedure which is employed in the calculus of variations. We wish to find within a family  $y^*(x)$  a function  $y(x)$  which extremizes an integral of the form

$$I = \int f(y^*, y^*, y^*, \dots, x) dx$$

We assume that an extremizing function actually exists. Let us now set

$$y^* = y(x) + \epsilon \eta(x) ,$$

where  $\epsilon$  is an arbitrary real parameter and  $\eta(x)$  is assumed to be as many times differentiable as is necessary for the discussion. At  $\epsilon = 0$ ,  $y^*$  is equal to the extremizing function  $y$ . A necessary condition that a function  $y$  be an extremizing function is that  $I$ , when considered as a function of the parameter  $\epsilon$ , satisfy

$$\frac{dI}{d\epsilon} = 0 \quad \text{at} \quad \epsilon = 0 .$$

Application of this condition leads us to a set of differential equations involving our extremizing function.

The types of problems to be discussed in this discussed in this paper will include the extremization of a functional  $J$  which depends on the form of a domain  $D^*$ . Throughout this paper we shall adopt the convention of allowing letters in their unaugmented form to refer to our supposed extremal domain  $D$ , and letters which are super-scripted with an asterisk to refer to any domain  $D^*$  admitted to competition. Let  $r$  denote a vector drawn from the origin to a point  $z$  on our extremal curve  $C$ , and let  $n$  denote the unit vector extending along the outer normal at  $z$  (i.e., away from  $D$ ). The variation devised by Hadamard is to define a point on a curve of comparison  $C^*$  by the vector

$$\vec{r}^* = \vec{r} + \epsilon \rho(z) \vec{n} ,$$

where  $\rho(z)$  is an arbitrary real function depending on the point  $z \in C$ , and  $\epsilon$  is an arbitrary real parameter.  $J(D^*)$  becomes a functional which depends on  $\epsilon$ ; if  $J$  is to serve as an extremizing functional, a necessary condition which must hold is that

$$\frac{dJ(\epsilon)}{d\epsilon} = 0 \quad \text{at} \quad \epsilon = 0 .$$

If a side condition

$$G(D^*) = 0$$

also holds, we may extremize our functional J by constructing the functional

$$H = \kappa J + \lambda G \quad ,$$

where  $\kappa$  and  $\lambda$  are Lagrange multipliers. Our necessary condition is

$$\frac{dH}{d\epsilon} = 0 \quad \text{at} \quad \epsilon = 0 \quad ;$$

this result follows from the classical calculus of variations. The method of Lagrange multipliers will be used formally in this section and explained completely in Section 3.

We shall now develop some specific expressions which relate functionals of the domain of comparison to the corresponding functionals of the extremal domain. We adopt the notation:

D -- the domain; this symbol will usually represent the domain exterior to a simple closed contour;

C -- The boundary of D;

$l$  -- the length of the curve C, if C is rectifiable;

A -- the area of the complement of D; and

$\gamma$  -- the logarithmic capacity of D.

We use the variation of Hadamard to find these functionals which are associated with  $D^*$ , the domain of comparison.

From geometrical considerations the area  $A^*$  is observed to be

$$(2.1) \quad A^* = A - \epsilon \int \rho \, ds + o(\epsilon) \quad .$$

To derive an expression for the length  $l^*$ , we let  $\vec{r}$  denote the vector position of any point on the curve C. Then

$$\vec{r}^* = \vec{r} + \epsilon \rho \vec{n}$$

$$\frac{d\vec{r}^*}{ds} = \frac{d\vec{r}}{ds} + \epsilon \frac{d(\rho \vec{n})}{ds}$$

$$\left| \frac{d\vec{r}^*}{ds} \right|^2 = 1 + 2\epsilon \frac{d\vec{r}}{ds} \cdot \frac{d(\rho \vec{n})}{ds} + o(\epsilon) = 1 + 2\epsilon \frac{d\vec{r}}{ds} \cdot \left[ \rho \frac{d\vec{n}}{ds} + \vec{n} \frac{d\rho}{ds} \right] + o(\epsilon) \quad .$$

But since  $\frac{d\vec{r}}{ds} \cdot \vec{n} = 0$  and  $\frac{d\vec{n}}{ds} = -\kappa \frac{d\vec{r}}{ds}$  [4], we have

$$\left| \frac{d\vec{r}^*}{ds} \right| = \frac{ds^*}{ds} = 1 - \epsilon \rho \kappa + o(\epsilon) ,$$

where  $\kappa$  denotes the curvature.

On integrating both sides of this last equation with respect to  $s$  over the full length of the curve, we obtain the result

$$(2.2) \quad l^* = l - \epsilon \int_C \rho \kappa ds + o(\epsilon) .$$

In a two-dimensional flow a vortex of unit strength has a complex velocity potential

$$w = i \log z .$$

If a body is placed in this vortex flow and  $\zeta(z)$  transforms the region of flow  $D$  about the body onto the exterior of the unit circle, then the complex velocity potential of such a flow is given by

$$w = i \log \zeta .$$

A function which is very useful in the representation of functions is the Green's function; it is defined as follows:

The Green's function  $g$  of a region  $D$  is a function of two points  $z$  and  $\zeta$  with the following properties: it is regarded as a function of  $z$  with  $\zeta$  fixed in  $D$ , is continuous and vanishes on the boundary, and is regular at all points of  $D$  except at  $\zeta$ , where it becomes logarithmically infinite in such a way that

$$g(z, \zeta) - \log \frac{1}{|z - \zeta|} = w(z, \zeta)$$

is regular and harmonic.

It can be shown that this function actually exists and, in addition, has the symmetric property that

$$g(z, \zeta) = g(\zeta, z) .$$

We shall be attacking problems which involve the region exterior to a curve bounding a simply-connected region. We introduce the notation  $g(z)$  to represent the function  $g(z, \infty)$ . Then  $g(z)$  has an expansion about infinity of the form

$$g(z) = \log |z| - \gamma + \frac{a_1}{z} + \frac{a_2}{z^2} \dots$$

The constant  $\gamma$  is called the logarithmic capacity or Robin's constant; sometimes the quantity

$$R = e^{-\gamma}$$

is called the outer mapping radius. We shall find it convenient to introduce the analytic function

$$p(z) = g(z) + i h(z) \quad ,$$

where  $h(z)$  is the harmonic conjugate of the Green's function. It can be shown that

$$\zeta(z) = e^{p(z) + iH_0}$$

maps the region  $D$  onto the region exterior to the unit circle;  $H_0$  is an arbitrary constant of rotation. We have the following expansion which defines a mapping from the exterior of the unit circle onto  $D$ :

$$z = R\zeta + a_0 + \frac{a_1}{\zeta} + \dots$$

It is of interest to notice that  $g(z)$  in its hydrodynamical sense represents the stream function of a vortex of unit strength flowing about the boundary  $C$  in the region  $D$ .

To derive a formula for  $\gamma^*$ , we construct a circle  $S$  so large that it lies in  $D$  and includes all of  $C$  and  $C^*$  in its interior. We note that from (2.1)

$$\begin{aligned} \iint_{D^*-D} (\nabla g \cdot \nabla g^*) d\sigma &= \iint_{D^*-D_S} (\nabla g \cdot \nabla g^*) d\sigma - \iint_{D-D_S} (\nabla g \cdot \nabla g^*) d\sigma \\ &= \epsilon \int_C \rho (\nabla g \cdot \nabla g^*) ds + o(\epsilon) \quad , \end{aligned}$$

where  $D_g$  denotes the domain exterior to  $S$ . Since

$$\nabla g^* = \nabla g + o(\epsilon) ,$$

we may write

$$\epsilon \int_C \rho (\nabla g \cdot \nabla g^*) ds = \epsilon \int_C (\nabla g)^2 ds + o(\epsilon) ;$$

but from Green's theorem

$$\begin{aligned} \iint_{D^* - D_g} \nabla g \cdot \nabla g^* d\sigma - \iint_{D - D_g} \nabla g \cdot \nabla g^* d\sigma &= \int_C g^* \frac{\partial g}{\partial n} ds - \int_C g \frac{\partial g^*}{\partial n} ds \\ &+ \int_S (g^* \frac{\partial g}{\partial n} - g \frac{\partial g^*}{\partial n}) ds = \int_S (g^* \frac{\partial g}{\partial n} - g \frac{\partial g^*}{\partial n}) ds . \end{aligned}$$

By letting  $r$  denote the radius of  $S$ , we have

$$g^* = \log r - \gamma^* + \frac{x a_1}{r^2} + \dots$$

$$g = \log r - \gamma + \frac{x a_1}{r^2} + \dots$$

Our previous equation must hold for any choice of  $S$ ; and consequently must hold in the limit as  $S$  becomes infinite; therefore, we have

$$-2\pi(\gamma^* - \gamma) = \epsilon \int \rho (\nabla g)^2 ds + o(\epsilon) .$$

We have, finally,

$$(2.3) \quad \gamma^* = \gamma - \frac{\epsilon}{2\pi} \int_C \left(\frac{\partial g}{\partial n}\right)^2 \rho ds + o(\epsilon) .$$

Let us now investigate the problem of maximizing the logarithmic capacity within a family of curves all having a fixed length. We wish to extremize  $\gamma^*(\epsilon)$  defined by (2.3) under the side condition that

$$l^* = l - \epsilon \int \rho \kappa ds + o(\epsilon) = \text{constant} .$$

By applying the method of Lagrange multipliers, we construct the functional

$$J = \gamma^* + \lambda l^* ,$$

and if this functional is to satisfy the necessary condition that

$$\frac{dJ}{d\epsilon} = 0 \quad \text{at} \quad \epsilon = 0 ,$$

the following equation must hold:

$$\int_C [(\frac{\partial G}{\partial n})^2 - \lambda K] \rho ds = 0 .$$

Since  $\rho$  is arbitrary, we have

$$(2.4) \quad \frac{\partial G(z)^2}{\partial n} = \lambda K(z) , \quad z \in C ;$$

but we have

$$\frac{\partial p}{\partial s} = p'(z) \frac{dz}{ds} = \frac{\partial G}{\partial s} + i \frac{\partial h}{\partial s} = i \frac{\partial h}{\partial s} ;$$

and, according to the Cauchy-Riemann equations,

$$\frac{\partial G}{\partial n} = \frac{\partial h}{\partial s} ,$$

hence

$$\frac{\partial G}{\partial n} = ip'(z)z$$

where we use the dot to denote differentiation with respect to  $s$ . (2.4) becomes

$$(2.4)' \quad p'(z)^2 \dot{z}^2 = -\lambda K ,$$

more conveniently expressed in the form

$$p'(z)^2 dz = -\lambda \dot{z} K ds .$$

Making the substitution  $\frac{dz}{ds} = e^{i\psi}$ , where  $\psi$  is the argument of the tangent vector to  $C$ , we obtain

$$p'(z)^2 dz = -\lambda e^{-i\psi} d\psi .$$

By integration along  $C$ , we obtain

$$(2.5) \quad q(z) = K + i\lambda e^{-i\psi} \quad \text{for } z \in C ,$$

where  $K$  is a constant of integration and  $q(z)$  is an analytic function in  $D$  such that

$$q'(z) = p'(z)^2 .$$

(2.5) shows that the curve  $C$  maps into a circle in the  $q$ -plane. Noting that for large  $z$

$$p'(z)^2 = \frac{1}{z^2} + \dots$$

we see that its integral must have an expansion which begins  $-1/z$ . Since  $\zeta$

represents the transformation of  $D$  onto the exterior of the unit circle, and since the image of  $D$  in the  $q$ -plane is the interior of a circle, we are led to the equation

$$(2.6) \quad q = \frac{a}{\zeta} + K_1, \quad ,$$

where  $a$  is a complex constant and  $K_1$  is an arbitrary constant of translation. But since  $q'(z) = p'(z)^2 = \zeta'^2/\zeta^2$ , we obtain, on differentiating both sides of (2.6),

$$\frac{\zeta'^2}{\zeta^2} = -\frac{a\zeta'}{\zeta^2}$$

We have, finally,

$$(2.7) \quad \zeta'(z) = -a$$

By solving this differential equation it becomes clear that our extremal curve is a circle in the  $z$ -plane. This circle must have a radius  $\ell/2\pi$ , where  $\ell$  is the value of the fixed length.

The complex velocity potential of a body bounded by a curve  $C$  placed in a uniform stream of unit velocity has an expansion

$$w = z + \frac{\alpha}{z} \dots$$

Letting  $a = \text{Re } \alpha$ , it can be shown [13] that the following relation holds:

$$(2.8) \quad a = \frac{1}{2\pi} (M + A), \quad ,$$

where  $M$  is the virtual mass of the flow about  $C$  and  $A$  is the area enclosed by  $C$ .

In a manner very similar to the one in which we derived (2.3), we can also show [13] that

$$(2.9) \quad a^* = a - \frac{\epsilon}{2\pi} \int_C (\nabla\varphi)^2 \rho(z) ds + o(\epsilon), \quad ,$$

where  $\varphi$  represents the velocity potential of the flow about  $C$ . From (2.8) and (2.9) we obtain

$$(2.10) \quad M^* = M + \epsilon \int [1 - (\nabla\varphi)^2] ds + o(\epsilon) \quad .$$

Let us now consider the problem of maximizing the constant  $a$ , when the length  $\ell^*$  of each member of a family of curves  $C$  is held at a fixed value  $\ell$ .

We construct the functional

$$J = a^* + \lambda l^*$$

By repeating the procedure in which we solved the problem which led to (2.7), we use (2.8) and (2.9) to construct the functional  $J$ ; on applying the conditions for a minimum that  $\frac{dJ}{d\epsilon} = 0$ , for  $\epsilon = 0$ , as before we obtain

$$(2.11) \quad (\nabla \varphi)^2 = \left| \frac{d\mathbf{w}}{dz} \right|^2 = \lambda K$$

But this equation has the same form as (1.4), which we derived by Bernoulli's equation. This variational problem is equivalent to the special case of the problem of determining the shape of the bubble with surface tension and with  $P_b = P_\infty + 1$ . We shall treat this problem in detail from the variational point of view in Section 6. Prescribing the fixed length  $l$  is equivalent to prescribing the surface tension  $T$ .

It has also been shown that variational problems which involve logarithmic capacity are reasonable idealizations of free boundary problems in a two-dimensional vortex flow [13]. Since this functional has been studied widely in mathematical literature, we treat a problem in great detail which involves this functional. The proofs of existence and justification of formal procedures are easily extended to the hydrodynamical problems which we consider in Section 5. The problem considered in Section 5 is an idealization of the following hydrodynamical problem:

Find the shape of the cross section of an infinitely long inflated rubber sheet attached at both edges to an infinitely long rectangular plate of fixed width, when this configuration is placed in a unit two-dimensional vortex.

### 3. The Variation of Schiffer.

In the previous section we were able to proceed in a formal manner directly to a differential equation when we used the Hadamard method of comparison. Very often, however, this method leads us only to an heuristic method of guessing an extremal solution to the problem. In order that the method of Hadamard be strictly applicable, one may admit to competition only curves with smooth boundaries. To generalize the class of admissible functions and to justify each formal process would require tedious arguments based on the theory of real variables. M. Schiffer has evaded this hopeless task by devising a method of comparison which employs a conformal mapping rather than a shift in the normal direction; consequently, this procedure is valid for any general sort of domain.

We let a point in  $D^*$ , our domain of comparison, be defined by

$$(3.1) \quad z^* = z + \frac{\epsilon e^{i\varphi}}{z - z_0},$$

where  $z_0$  is an arbitrary point of  $D$ ,  $\epsilon$  is an arbitrary real parameter which assumes values in some neighborhood of  $\epsilon = 0$ , and  $\varphi$  is an arbitrary member of  $[0, 2\pi)$ .  $z^*$  performs a Schlicht mapping of the exterior of a circle in the  $z$ -plane, having  $z_0$  as its center and having a radius  $\sqrt{|\epsilon|}$  onto the entire plane save for a slit extending from  $z_0 - 2\sqrt{|\epsilon|} e^{i\varphi/2}$  to  $z_0 + 2\sqrt{|\epsilon|} e^{i\varphi/2}$ . Since, for  $\epsilon = 0$ ,  $z^*$  performs the identity transformation on  $D$ , and since a circle which lies entirely within  $D$  may be drawn with  $z_0$  as center, provided that its radius is sufficiently small, there is some interval of values of  $\epsilon$ , say,  $[-\epsilon_1, \epsilon_1]$ , such that  $z^*$  performs a Schlicht mapping on the boundary of  $D$ . We now define  $D^*$  as the domain whose points comprise the exterior of the curve  $C^*$  on which  $C$  is mapped by (3.1). Consequently, if an extremal domain exists, the extremal value of the functional  $J(D^*)$  is given at  $\epsilon = 0$ , and  $J(D^*)$  is also defined in the

neighborhood  $[-\epsilon_1, \epsilon_1]$ . It is a necessary condition that

$$\frac{dJ}{d\epsilon} = 0 \quad \text{at} \quad \epsilon = 0$$

Let us now apply the Schiffer variation to the determination of the functionals associated with the domain of comparison. We shall find the length  $l^*$  of  $C^*$  the boundary of  $D^*$ . By differentiation of (3.1) with respect to arc length, we obtain

$$\frac{dz^*}{ds} = \frac{dz}{ds} \left( 1 - \frac{\epsilon e^{i\varphi}}{(z-z_0)^2} \right) ;$$

we have

$$\begin{aligned} \left| \frac{dz^*}{ds} \right|^2 &= \frac{dz^*}{ds} \overline{\frac{dz^*}{ds}} = 1 - \epsilon \left[ \frac{e^{i\varphi}}{(z-z_0)^2} + \frac{e^{-i\varphi}}{(\overline{z-z_0})^2} \right] + o(\epsilon) \\ &= 1 - 2\epsilon \operatorname{Re} \frac{e^{i\varphi}}{(z-z_0)^2} + o(\epsilon) = \left( \frac{dz^*}{ds} \right)^2 ; \end{aligned}$$

thus

$$\frac{ds^*}{ds} = 1 - \epsilon \operatorname{Re} \frac{e^{i\varphi}}{(z-z_0)^2} + o(\epsilon)$$

Integration of both sides of this equation over the full length of  $C$  produces

$$(3.2) \quad l^* = l - \epsilon \operatorname{Re} e^{i\varphi} \int \frac{ds}{(z-z_0)^2} + o(\epsilon)$$

Let us now develop an expression for the logarithmic capacity  $\gamma^*$  of  $D^*$ . With this goal in mind we construct the integral

$$\frac{1}{2\pi i} \oint_{C-C'+C''} (p^*(t^*) - p(t)) dp(t) ,$$

where  $C'$  represents a circle large enough to include  $C$  and the point  $z_0$  in its interior.  $C''$  represents a system of cuts connecting  $C$  and  $C'$  so that  $t$  traces  $C$  in a counter-clockwise direction,  $C'$  in a clockwise direction, and  $C''$  in such a way that the complete circuit encloses a simply connected region.

Noting that we may write  $dp = \frac{dp}{dt} dt$  on  $C'$  (not necessarily true for  $C$ , however), and that

$$\log \frac{z^*}{z} = \log \left( 1 + \frac{\epsilon e^{i\varphi}}{z(z-z_0)} \right),$$

we consider the expansion at infinity for the logarithmic function and for  $p'(t)$ ; since

$$p(z) = \log z + \gamma + \frac{a_1}{z} + \dots,$$

$\frac{\delta^* - \delta}{t}$  is the only term which contributes to the integral over  $C'$ ; by the residue theorem it becomes simply  $\delta^* - \delta$ . Now  $p^*(t^*)$  and  $p(t)$  are both imaginary for  $t \in C$ . This integral when viewed in the  $p$ -plane represents a contour integration with an imaginary  $dp$ . Considering the integration taken over the image of  $C$  in the  $p$ -plane, we see that the portion of our integral taken over  $C$  is imaginary. Thus, if  $z_0$  is a point of  $D$  and  $\Gamma$  is a circle whose circumference lies entirely within  $D$ , surrounding  $z_0$ , we have

$$\delta^* = \delta + \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma} [p^*(t^*) - p(t)] \frac{dp}{dt} dt.$$

(Note that here we are justified in writing  $dp = p'(t)dt$ .)

If  $z_0$  is an exterior point of  $D$ , then it is not enclosed in the circuit of integration and we have

$$\delta^* \equiv \delta,$$

meaning, of course, that  $\delta$  is conformally invariant under such a transformation.

To evaluate the integral over  $\Gamma$  we note that  $p(t)$  and  $p^*(t)$  are both regular inside  $\Gamma$  and hence

$$\frac{1}{2\pi i} \int_{\Gamma} [p(t) - p^*(t)] \frac{dp}{dt} dt = 0.$$

Thus,

$$\begin{aligned} \delta^* &= \delta + \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma} [p^*(t^*) - p^*(t)] p'(t) dt \\ &= \delta + \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma} [p^{*'}(t) \frac{\epsilon e^{i\varphi}}{z-z_0} + o(\epsilon)] p'(t) dt. \end{aligned}$$

Since

$$p^{*\prime}(t) = p'(t) + o(\epsilon) ,$$

we have

$$\gamma^* = \gamma + \operatorname{Re} \frac{e^{i\varphi}}{2\pi i} \int_{\Gamma} \frac{p'(t)^2}{t-z_0} dt + o(\epsilon) ;$$

by applying the residue theorem we have, finally,

$$(3.3) \quad \gamma^* = \gamma + \epsilon \sigma(z_0) \operatorname{Re} e^{i\varphi} p'(z_0)^2 + o(\epsilon) ,$$

where  $\sigma(z_0) = 1$ ,  $z_0 \in D$ ,

$$\sigma(z_0) = 0, z_0 \in \bar{D},$$

and where  $\bar{D}$  is the domain complementary to  $D$ .

Lastly, we shall develop an expression for the area  $A^*$  associated with  $D^*$ .

We use the following formula as a point of departure:

$$A = \frac{1}{2} \operatorname{Re} i \int_C z d\bar{z} .$$

It follows that

$$\begin{aligned} A^* &= \frac{1}{2} \operatorname{Re} i \int_C^* z^* d\bar{z}^* = \frac{1}{2} \operatorname{Re} i \int_C \left( z + \frac{\epsilon e^{i\varphi}}{z-z_0} \right) \left( 1 - \frac{\epsilon e^{-i\varphi}}{(z-z_0)^2} \right) d\bar{z} \\ &= A + \frac{\epsilon}{2} \operatorname{Re} i e^{i\varphi} \left[ \int_C \frac{d\bar{z}}{(z-z_0)} + \int_C \frac{\bar{z} d\bar{z}}{(z-z_0)^2} \right] + o(\epsilon) ; \end{aligned}$$

integration by parts provides us with the result

$$(3.4) \quad A^* = A + \epsilon \operatorname{Re} i e^{i\varphi} \oint_C \frac{d\bar{z}}{z-z_0} + o(\epsilon) .$$

It is possible in this manner to find the changes in many of the functionals associated with the domain of comparison. It should be noticed that formulas (3.2), (3.3), and (3.4) actually reduce to formulas (2.1), (2.2), and (2.3) when the boundary of the domain is a smooth curve. This is clear if we write the magnitude of the shift in the direction of the normal as  $\operatorname{Re} i \frac{e^{i\varphi} \bar{z}}{z-z_0}$ , where the normal is considered positive when pointing away from  $D$ . If we substitute this expression in (2.1), (2.2), and (2.3), we obtain

immediately (3.2), (3.3), and (3.4). Examination of these formulas shows that capacity and area are functionals which grow monotonically with the domain. Within families of twice differentiable convex curves this property is also true for length.

In most extremal problems which occur in conformal mapping, it is desired to extremize one functional while others are held constant. There are a number of ways in which this can be accomplished. One may either generalize the form of the Schiffer variation, apply a procedure which utilizes the Lagrange multipliers, or use a combination of the two methods. In the direction of generalizing the variation of Schiffer, one may extend the method of obtaining a comparison domain to include variations of the form

$$z^* = z + \sum_{i=1}^{i=\infty} \epsilon_i F_i(z, \bar{z})$$

$F_i(z, \bar{z})$  can generally be a non-analytic function of a point  $z$ , although it will usually involve some fixed points as parameters. We shall see that we will ultimately obtain a differential equation involving these points, whose solution defines an analytic function of the points.

Let us denote our functional which we wish to extremize by  $J_0(D^*)$ , which varies with the form of the domain  $D^*$  under certain constraining conditions. Let us now consider the hypothetical situation that a minimum value

$$J_0(D) = h$$

exists for the function  $J_0(D^*)$ , where  $J_i$  ( $i=1,2,\dots,m$ ) are functionals which satisfy the  $m$  constraining conditions

$$J_i(D^*) = 0 \quad (i=1,2,\dots,m)$$

Let  $z$  represent a point in the extremal domain and let a point in  $D^*$  be given by

$$z^* = z + \epsilon_0 F_0 + \epsilon_1 F_1 + \dots + \epsilon_i F_i + \dots$$

The  $F_1$  are chosen so that there is a one-to-one correspondence between the points of  $C$  and those of  $C^*$  for all  $\epsilon_1$  ( $i=0,1,\dots,n$ ) in a sufficiently small neighborhood of  $\epsilon_0 = \epsilon_1 = \epsilon_2 = \dots = 0$ . The theory of Lagrange multipliers tells us that if a minimum of  $J_0$  is to exist, a necessary condition to be satisfied is that there exist constants  $\lambda_1$  ( $i=0,1,2,\dots,m$ ), which we call Lagrange multipliers, such that

$$\frac{\partial J(D^*)}{\partial \epsilon_i} = 0 \quad \text{for} \quad \epsilon_i = 0 \quad (i=0,1,2,\dots),$$

where  $J(D^*) = \lambda_0 J_0 + \lambda_1 J_1 + \dots + \lambda_m J_m$ . There are an infinite number of equations for the  $m+1$   $\lambda$ 's. If these equations are to have a solution in the  $\lambda_1$ 's, then the infinite matrix

$$\begin{vmatrix} \frac{\partial J_0}{\partial \epsilon_0} & \frac{\partial J_1}{\partial \epsilon_0} & \dots & \frac{\partial J_m}{\partial \epsilon_0} \\ \frac{\partial J_0}{\partial \epsilon_1} & \cdot & \dots & \cdot \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial J_0}{\partial \epsilon_1} & \cdot & \dots & \frac{\partial J_m}{\partial \epsilon_1} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \end{vmatrix}$$

must be of rank less than  $m+1$ . To see that this is indeed the case, we note that if this matrix were of rank  $m+1$  it would be possible to choose a particular arrangement of the infinity of rows such that the determinant

$$\begin{vmatrix} \frac{\partial J_0}{\partial \epsilon_{k_0}} & \frac{\partial J_1}{\partial \epsilon_{k_0}} & \dots & \frac{\partial J_m}{\partial \epsilon_{k_0}} \\ \frac{\partial J_0}{\partial \epsilon_{k_1}} & \cdot & \dots & \cdot \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial J_0}{\partial \epsilon_{k_m}} & \cdot & \dots & \frac{\partial J_m}{\partial \epsilon_{k_m}} \end{vmatrix}$$

does not vanish. But the non-vanishing of this determinant implies, by the theorem of implicit functions, that the system of equations

$$\begin{aligned}
 J_0(D^*) &= J_0(\epsilon_{k_0}, \epsilon_{k_1}, \dots, \epsilon_{k_m}, 0, 0, \dots, 0) = h + U \\
 J_1(\epsilon_{k_0}, \epsilon_{k_1}, \dots, \epsilon_{k_m}, 0, 0, 0 \dots) &= 0 \\
 J_2(\epsilon_{k_0}, \epsilon_{k_1}, \dots, \epsilon_{k_m}, 0, 0, 0 \dots) &= 0 \\
 \vdots & \\
 J_m(\epsilon_{k_0}, \dots) &= 0
 \end{aligned}$$

can be solved so that the  $\epsilon_{k_i}$  are functions of  $U$  in some neighborhood of  $U=0$ . However, this means that  $J_0$  is defined for certain negative values of  $U$  in this neighborhood, and, consequently, there are values of  $U$  for which  $J_0 < h$ , even when the equations  $J_i(\epsilon_i) = 0$  are still satisfied. However, this situation is a violation of our original hypothesis that  $h$  is a solution of the minimum problem. Thus, the matrix is of rank less than  $m+1$  and must be solvable for the Lagrange multipliers.

#### 4. The Existence of Extremal Domains.

Once it has been established that a solution to an extremal problem exists, we may feel secure in applying the method of Lagrange multipliers to obtain a differential equation for our extremal domain. Because of the uniqueness of a solution of a differential equation with properly chosen boundary values we can be sure that the solution is actually the extremal solution. This section will be devoted to proofs of a general nature on the existence of solutions; we see in Section 5 that the conditions of our specific problem are special cases of the more general theorems in this section. Much of this discussion is a slight generalization of [31] and is designed to suit the needs of this paper.

We shall first demonstrate a property possessed by the length  $l$  of a curve belonging to a family of continuous, rectifiable curves  $C$  in the plane. Let there be a family  $C$  of such curves and let the complex position of a point on  $C$  be represented by

$$z = x + iy = f(t) \quad ,$$

where  $t$  is a real parameter which assumes values in an interval  $[0, T]$  and  $f(t)$  is periodic with a period  $T$ . We shall now prove the following theorem:

**Theorem 4.1.** Any particular member  $C_0$  of a family of curves  $C$  has the property that for any sequence of curves  $C_n$  such that  $f_n(t) \rightarrow f_0(t)$  for all  $t$  in  $[0, T]$ , given  $\epsilon > 0$ , there exists an  $N$  such that  $l(C_n) > l(C_0) - \epsilon$  for all  $n > N$ .

Let  $t_0, t_1, t_2, \dots, t_k$  denote a sequence of values of  $t$ , where  $0 = t_0 < t_1 < t_2 < \dots < t_k = T$ , and let  $f(t_0), f(t_1), \dots, f(t_k)$  represent the corresponding points on a curve  $C$ . If we now inscribe a polygon  $P$  in  $C$  with vertices at each of these points, then

$$l(P) = \sum_{i=1}^k |f(t_i) - f(t_{i-1})|$$

and

$$l(C) = \lim_{\substack{K \rightarrow \infty \\ \text{Max } |t_i - t_{i-1}| \rightarrow 0}} l(P) \quad .$$

If Theorem 4.1 were not true there would be a curve  $C_0$ , the length of which we shall denote by  $l_0$  where the following situation would hold: We could find a sequence  $f_n \rightarrow f_0$  such that for any  $\epsilon > 0$ , we would have

$$(4.1) \quad l(C_n) \leq l(C_0) - \epsilon$$

for every value of  $n$ .

Inscribe a polygon  $P_0$  in  $C_0$  with the  $f_1(t)$  as vertices.

Now, in view of the convergence of  $f_n$ , we may find an  $N_1$  so large that

$$|f_n(t_i) - f_0(t_i)| < \frac{\epsilon}{4K} \quad (i = 1, 2, \dots, K)$$

for  $n \geq N_1$ .

Let  $N = \text{Max } N_i$ .

Inscribe a polygon  $P_N$  in the corresponding curve  $C_N$ . On writing

$$f_0(t_i) - f_0(t_{i-1}) = f_0(t_i) - f_N(t_i) + f_N(t_i) - f_N(t_{i-1}) + f_N(t_{i-1}) - f_0(t_{i-1}),$$

we have

$$\begin{aligned} |f_0(t_i) - f_0(t_{i-1})| &\leq |f_0(t_i) - f_N(t_i)| + |f_N(t_i) - f_N(t_{i-1})| + |f_N(t_{i-1}) - f_0(t_{i-1})| \\ &< |f_N(t_i) - f_N(t_{i-1})| + \frac{\epsilon}{2K}; \end{aligned}$$

it follows that

$$l(P_0) < l(P_N) + \frac{\epsilon}{2} \leq l(C_N) + \frac{\epsilon}{2}.$$

But since  $C_N$  satisfies (4.1),

$$l(P_0) < L_0 - \frac{\epsilon}{2}.$$

Since this must hold for every possible choice of  $P_0$ , we have

$$l(C_0) \leq L_0 - \frac{\epsilon}{2}.$$

But this is a contradiction of our original hypothesis that

$$l(C_0) = L_0; \quad ;$$

hence, the only other alternative is that Theorem 4.1 must hold.

We shall find it convenient to use  $l(f)$  interchangeably with the symbol  $l(C)$ , where  $f(t)$  is the complex parametric function representing  $C$ . We shall say that a family of curves  $C$  is compact if it is possible to select from every sequence of  $C$  a sub-sequence  $C_n$  such that

$$\lim_{n \rightarrow \infty} f_n \rightarrow f_0, \quad ;$$

where  $f_0$  represents parametrically a curve  $C_0$  within the family  $C$ . No confusion should arise if we use interchangeably the terminology "the family  $f$ " and "the family  $C$ ". We now prove:

Theorem 4.2. If  $C$  represents a compact family of continuous, rectifiable curves, then the problem

$$l(C) = \text{Min}$$

has a solution within the family  $C$ .

Due to the obviously non-negative character of the length of a curve, there must certainly be a greatest lower bound  $L_0$ . In view of the definition of a greatest lower bound there must be a sequence of functions  $f_n$  such that

$$\lim_{n \rightarrow \infty} l(f_n) = L_0.$$

Since  $C$  is compact, we may select a sub-sequence  $f_{n_i}$  of  $f_n$  which converges to a function  $f_0$  within the family  $f$ .  $f_0$  must, of course, satisfy the inequality

$$l(f_0) \geq L_0,$$

since  $L_0$  is a greatest lower bound. Because of Theorem 4.1 we have

$$l(f_{n_i}) > l(f_0) - \epsilon$$

for sufficiently large values of  $i$ . But because of the convergence of  $l(f_{n_i})$  to  $L_0$ , there must be an  $i$  large enough that

$$l(f_{n_i}) < L_0 + \epsilon.$$

Then there is a value of  $i$  (say,  $I$ ) large enough so that

$$l(f_0) - \epsilon < l(f_{n_I}) < L_0 + \epsilon$$

and we must have

$$L_0 \leq l(f_0) < L_0 + 2\epsilon.$$

This inequality can only hold for all  $\epsilon$  if

$$l(f_0) = L_0.$$

Hence, the greatest lower bound is attained within the family.

It is clear from the manner in which we defined length that this terminology is also synonymous with the terminology, variation of the function  $f(t)$  in the interval  $[0, T]$ . If a curve  $C$  is rectifiable, then  $f(t)$  is necessarily a function of bounded variation. We shall have occasion in Section 5 to refer to the following theorems which we now state and prove:

Theorem 4.3. Let  $\alpha_n(t)$  ( $n = 0, 1, 2, \dots, \infty$ ) be a sequence of non-decreasing, bounded, real functions defined for  $t$  in an interval  $[0, T]$ ; then there exists a set of natural numbers

$$n_0 < n_1 < n_2 < \dots$$

and a non-decreasing, bounded function  $\alpha(t)$  such that  $\lim_{i \rightarrow \infty} \alpha_{n_i}(t) \rightarrow \alpha(t)$  for all  $t \in [0, T]$ .

Proof. Let us first form a sequence of values of  $t$  which is everywhere dense in  $[0, T]$ , say,  $t_m$  ( $m = 0, 1, 2, \dots$ ). Then since the sequence  $\alpha_n(t_0)$  is bounded at  $t_0$ , it follows from the Bolzano-Weierstrass theorem that it must have at least one limit point. We may select a sub-sequence  $\alpha_{n_0}(t_0), \alpha_{n_2}(t_0), \dots$  which converges to a limit which we shall call  $\alpha(t_0)$ . Since the sequence  $\alpha_{n_1}(t_1)$  is also bounded, we may select from it a sub-sequence  $\alpha_{n_0}(t_1), \alpha_{n_1}(t_1), \alpha_{n_2}(t_1), \dots$  for which  $\lim_{i \rightarrow \infty} \alpha_{n_i}(t_1) \rightarrow \alpha(t_1)$ . In the same manner we may select a subset of the integers  $n_0^1, n_1^1, n_2^1, n_3^1, \dots$ , say,  $n_0^2, n_1^2, n_2^2, n_3^2, \dots$ , such that

$$\lim_{i \rightarrow \infty} \alpha_{n_i^2}(t_2) = \alpha(t_2)$$

If we continue this diagonal process, we have for every  $t_m$

$$\lim_{i \rightarrow \infty} \alpha_{n_i^m}(t_m) = \alpha(t_m)$$

We thus have an  $\alpha(t)$  defined for all  $t_m$ . Furthermore, we are free to adopt the following definition for any  $t$  in the interval

$$\overline{\lim}_{i \rightarrow \infty} \alpha_{n_i^1}(t) = \overline{\alpha(t)}$$

$$\underline{\lim}_{i \rightarrow \infty} \alpha_{n_i^1}(t) = \underline{\alpha(t)}$$

where, of course,  $\alpha(t_m) = \overline{\alpha(t_m)} = \underline{\alpha(t_m)}$  ( $m = 0, 1, 2, \dots$ ).

If  $t$  is any point of continuity of  $\overline{\alpha}$  or  $\underline{\alpha}$ , we must have  $\overline{\alpha}(t) = \underline{\alpha}(t)$ , since one can find a sub-sequence  $t_{m_i}$  of  $t_m$  for which  $t_{m_i} \rightarrow t$ , and since  $\overline{\alpha}(t_{m_i}) = \underline{\alpha}(t_{m_i})$  for every member of this sub-sequence; we now have the

result that

$$\lim_{i \rightarrow \infty} \alpha_{n_i^i}(t) = \bar{\alpha}(t) = \underline{\alpha}(t)$$

for every member of  $t_n$  and for the points of continuity of  $\bar{\alpha}(t)$  and  $\underline{\alpha}(t)$ .

We are free to designate this common limit by  $\alpha(t)$ .

The remaining points comprise the set  $T_1, T_2, T_3, \dots, T_n$ , of points of discontinuity of the functions  $\bar{\alpha}(t)$  and  $\underline{\alpha}(t)$ , which, because  $\bar{\alpha}$  and  $\underline{\alpha}$  are non-decreasing, form a countable set. The sequence  $\alpha_{n_i^i}(T_0)$  is bounded, and from  $n_0^0, n_1^1, n_2^2, \dots$ , we may select a sub-sequence  $K_0^0, K_1^0, K_2^0$ , for which  $\alpha_{K_i^0}(T_0)$  approaches a limit, which we may designate as  $a_0$ . We may, in fact, apply the same diagonal process to the points of discontinuity that we applied to the set  $t_n$  and select a diagonal sequence  $K_i^0$  (where the subscripts and superscripts of  $K$  have the same meaning as those of  $n$ ) such that

$$\alpha_{K_i^0}(T_p) \rightarrow a_p \quad (p=0,1,2,\dots)$$

Let us now define  $\alpha(T_p) = a_p$ . Then  $\alpha(t)$  is defined at every point in the interval. Since  $K_i^0$  is a subset of  $n_i^i$ , we may set  $n_0 = K_0^0$ ,  $n_1 = K_1^0$ ,  $n_2 = K_2^0$ , and we have

$$\lim_{i \rightarrow \infty} \alpha_{n_i^i}(t) = \alpha(t)$$

defined for all points on the interval.

Furthermore,  $\alpha(t)$  is non-decreasing, for otherwise there would be a  $t_a < t_b$  such that  $\alpha(t_a) > \alpha(t_b)$ . Set  $\alpha(t_a) - \alpha(t_b) = 3\delta$ . Convergence of  $\alpha_{n_i^i}$  implies that we can find a positive integer  $N$  such that

$$|\alpha_N(t_a) - \alpha(t_a)| < \delta \quad \text{and} \quad |\alpha_N(t_b) - \alpha(t_b)| < \delta$$

for all  $n_i \geq N$ . We conclude that

$$\alpha(t_a) > \alpha(t_a) - \delta \quad ;$$

$$\alpha(t_b) < \alpha(t_b) + \delta \quad ;$$

$$\alpha_N(t_b) - \alpha_N(t_a) < \alpha(t_b) - \alpha(t_a) + 2\delta = -\delta < 0 \quad .$$

However, this last inequality is a contradiction of our hypothesis that  $\alpha_N$  is non-decreasing.

In view of the fact that a real function of bounded variation can be represented as the difference of two non-decreasing functions, Theorem 4.3 can be easily extended to apply to real and complex functions of bounded variation ([16] and [33]). This theorem, which we shall be using in conjunction with Theorem 4.2 to demonstrate compactness of our family, was originally proved by Helly [16]; we now state

**Theorem 4.4.** Let  $f_n(t)$  be a sequence of bounded, complex functions of uniformly bounded variation defined for the real parameter  $t$  in an interval  $[0, T]$ . Then there exists a set of positive integers

$$n_0 < n_1 < n_2 < \dots$$

and a function  $f(t)$  which is also bounded and of bounded variation for which

$$\lim_{i \rightarrow \infty} f_{n_i}(t) \rightarrow f(t) \quad .$$

Another result which we shall need, also due to Helly, is:

**Theorem 4.5.** Let the sequence of functions  $\alpha_n(t)$  be of uniformly bounded variation in  $[0, T]$ , suppose

$$\lim_{n \rightarrow \infty} \alpha_n(t) = \alpha(t) \quad ,$$

and let  $f(t)$  be continuous in  $[0, T]$ . Then

$$\lim_{n \rightarrow \infty} \int_0^T f(t) d\alpha_n(t) = \int_0^T f(t) d\alpha(t) \quad .$$

The hypothesis implies that  $\alpha(t)$  is of bounded variation. Hence there exists a number  $V$  which is not less than the variation of any of the functions  $\alpha_n(t)$  and  $\alpha(t)$ . Let  $\epsilon$  be an arbitrary positive number. Choose a sub-division of  $[0, T]$  so small that the oscillation of  $f(t)$  is less than in any of the sub-intervals. Let the points of the sub-division be

$$0 = t_0 < t_1 < t_2 < \dots < t_m = T,$$

and let us designate

$$\begin{aligned} H_n &= \int_0^T f(t) d\alpha_n(t) - \int_0^T f(t) d\alpha(t) \\ &= \sum_{i=1}^m \int_{t_{i-1}}^{t_i} [f(t) - f(t_{i-1})] d\alpha_n(t) - \sum_{i=1}^m \int_{t_{i-1}}^{t_i} [f(t) - f(t_{i-1})] d\alpha(t) \\ &\quad + \sum_{i=1}^m f(t_{i-1}) \int_{t_{i-1}}^{t_i} d[\alpha_n(t) - \alpha(t)]; \end{aligned}$$

then

$$|H_n| \leq \epsilon V + \epsilon V + M \sum_{i=1}^m \left| \int_{t_{i-1}}^{t_i} d(\alpha_n(t) - \alpha(t)) \right|,$$

where  $M$  is the maximum value of  $|f(t)|$  in  $[0, T]$ . Then

$$|H_n| \leq 2\epsilon T + M \sum_{i=1}^m [\alpha_n(t_i) - \alpha_n(t_{i-1}) + \alpha(t_i) - \alpha(t_{i-1})]$$

By selecting  $n$  sufficiently large we have

$$|\alpha_n(t_i) - \alpha_n(t_{i-1})| < \epsilon/m$$

and

$$|\alpha_n(t_{i-1}) - \alpha_n(t_{i-1})| < \epsilon/m;$$

hence,

$$\lim_{n \rightarrow \infty} |H_n| \leq 2\epsilon T + 2\epsilon M.$$

In order that this inequality should hold for all possible choices of  $\epsilon$ , we must have

$$\lim_{n \rightarrow \infty} H_n = 0,$$

and from this Theorem 4.5 follows. This result is also easily extended to complex functions.

5. An Extremum Problem with Capacity and Length.

We shall now extend the problem which we solved in Section 2 to the following problem with an added constraint, which we formulate in the following manner:

Find, within the family of rectifiable, continuous, closed curves  $C^*$ , all having an outer mapping radius  $R$ , and such that a fixed linear segment is contained in  $\bar{D}^*$ , the curve  $C$  which solves the problem

$$l(L^*) = \text{Min} .$$

We let the letter  $L$  refer to that part of the boundary  $C$  which excludes any part of the fixed segment which we specify to be the interval  $[-1,1]$ . It should be noticed that this segment can either be completely enclosed by an admissible curve  $C^*$ , or that part or all of this segment can be part of the boundary of  $C^*$ . We shall refer to any part of this segment which may happen to lie on the boundary of a curve  $C^*$  as the fixed boundary and to the remaining part of the curve  $L^*$  as the free boundary. We shall let the letter  $l$  by itself signify  $l(L)$ .

For each member of the family  $C^*$  there corresponds a Schlicht mapping function  $f^*(\zeta)$ , which maps the region exterior to the unit circle onto the region exterior to  $C^*$ . On setting  $\zeta = re^{i\theta}$ ,  $f^*(re^{i\theta})$  is expandable in a Fourier series. From the principles of Abel summability of Fourier series [24] it follows that  $\lim_{r \rightarrow 1} f^*(re^{i\theta}) = f^*(e^{i\theta})$ ; that is to say, the analytic function defined by the Fourier series defines the same analytic function on the boundary of the unit circle.

Because of the obviously non-negative character of length there will inevitably be a greatest lower bound  $m$  for which there would be a sequence  $L_n$  such that

$$\lim_{n \rightarrow \infty} l(L_n) = m .$$

Associated with each  $L_n$  is a curve  $C_n$  whose length is at most 2 more than  $L_n$ . Hence, each function  $f_n$  associated with  $C_n$  is of bounded variation on the unit circle, so that the result of Theorem 4.4 implies that from the family  $f_n(e^{i\theta})$ , which is defined for  $\theta$  in  $[0, 2\pi)$ , we may select a sub-sequence  $f_{n_1}(e^{i\theta})$  which converges to a function  $f_0(e^{i\theta})$  which must be of bounded variation. In the region exterior to the unit circle, each  $f_{n_1}$  has an

expansion of the form

$$f_{n_1} = R\zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \dots,$$

where

$$R = \frac{1}{2\pi i} \oint \frac{f_{n_1}(\zeta) d\zeta}{\zeta^2} = -\frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{df_{n_1}(\zeta)}{\zeta} = -\frac{1}{2\pi i} \int_{\theta=0}^{\theta=2\pi} \frac{df(e^{i\theta})}{e^{i\theta}}$$

Let  $f_{\infty}(\zeta)$  be the analytic function in  $D$  having as its expansion the coefficients

$$a_n = \lim_{i \rightarrow \infty} a_{n_1}^{i}$$

Then

$$a_n = -\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} df_{\infty}(e^{i\theta})$$

In view of Theorem 4.5 these coefficients must be equal to  $-\frac{1}{2\pi} \int e^{in\theta} df_0(e^{i\theta})$ ; hence,  $\lim_{r \rightarrow 1} f_{\infty}(re^{i\theta})$  is equal to  $f_0(e^{i\theta})$  and we may use  $f_0(\zeta)$  to denote the same analytic function exterior to the unit circle and also on its boundary. We have from Theorem 4.5,

$$R = \lim_{i \rightarrow \infty} -\frac{1}{2\pi} \int \frac{df_{n_1}(e^{i\theta})}{e^{i\theta}} = -\frac{1}{2\pi} \int \frac{df_0}{e^{i\theta}}$$

so that  $f_0(e^{i\theta})$  maps the unit circle onto a curve having the same outer mapping radius as each  $f_n(e^{i\theta})$ . Since  $f_0$  represents the limit of Schlicht functions, it must also be Schlicht.

We now show that the segment  $[-1,1]$  is included in the domain  $\overline{D}_0$  associated with  $f_0$ . Let us suppose that the contrary condition holds, namely, that there is a sequence which converges to a limit  $C_0$ , where a point  $z_0$  on  $[-1,1]$  belongs to  $D_0$ . Then by definition [22] there is an  $n$  (say,  $N$ ) so large that  $z_0$  may be surrounded by a neighborhood  $N_0$  containing points entirely in  $\overline{D}_N$ . This means that  $D_N$  must contain some point of  $[-1,1]$  which belongs to every member of the converging sequence  $\overline{D}_n$ . We have shown that the functions  $f_n$  form a compact family.

Let us now consider the family of free boundary curves  $L_n$  for which

$$\lim_{n \rightarrow \infty} \ell(L_n) = m$$

Let us now map each  $D_n$  by means of a mapping function  $t$  onto the half plane, so that  $L_n$  corresponds to the interval  $[0,1]$ . Then the coordinates of each  $L_n$  are represented by parametric functions  $h_n(t)$ , where  $t$  is a real parameter which assumes values in  $[0,1]$ . Since each  $h_n$  is of bounded variation, we may apply Helly's selection principle and find a subsequence  $h_{n_j}$  such that  $h_{n_j} \rightarrow h_0$ , where  $h_0$  is a function also of bounded variation. Remembering that the terminology  $\ell(L_n)$  is synonymous with the term variation of  $h_n$ , we have from Theorem 4.1 for any  $\epsilon$ , and sufficiently large  $j$ ,

$$\ell(L_{n_j}) > \ell(L_0) - \epsilon$$

where  $L_0$  represents a curve, which must be rectifiable, associated with the mapping  $z_0 = h_0(t)$ . But we also have for  $n_j$  sufficiently large

$$\ell(L_{n_j}) < m + \epsilon$$

from which it follows that

$$\ell(L_0) < m + 2\epsilon$$

Since  $\{D_n\}$  is a compact family, there must be associated with  $L_0$  a curve  $C_0$  having an outer mapping radius  $R$  and having  $[-1,1]$  in  $\overline{D}_0$ . If the last

inequality is to hold for every possible choice of  $\epsilon$ , we must have

$$\ell(L_0) \leq m .$$

Whether the equality or inequality sign holds,  $L_0$  solves our minimum problem, since it is associated with a curve  $C_0$  having the properties specified in the hypothesis which we used to formulate the problem.

It will be convenient to prove also that the extremal domain  $D$  must be bounded by a curve whose free boundary is convex. If it were not so, we would be able to find two free boundary points  $z_1$  and  $z_2$  which can be joined by a straight line which lies entirely within the domain  $D$ . The region  $\bar{D}$  enclosed by the boundary formed by replacing the arc  $\overline{z_1 z_2}$  by the straight line  $\overline{z_1 z_2}$  has grown monotonically. Consequently, the outer mapping radius  $R'$  associated with this new domain  $D'$  is greater than the  $R$  associated with  $D$ . Let us now select two free boundary points  $z_3$  and  $z_4$  which, unlike  $z_1$  and  $z_2$ , can be joined by a straight line lying entirely within  $\bar{D}'$ . The domain  $D''$ , bounded by the curve  $C''$ , formed by replacing the arc  $\overline{z_3 z_4}$  with the straight line  $\overline{z_3 z_4}$  will have an outer mapping radius  $R''$ , which is less than  $R'$ . By choosing the points  $z_1$  and  $z_2$  sufficiently close to each other, we may choose a location of  $z_3$  and  $z_4$  such that  $R'' = R$ . However, we decreased the total length when we constructed  $C'$  from  $C$  and decreased it again when we constructed  $C''$  from  $C'$ , so that we obtained  $\ell(L'') < \ell(L)$ . Since we have constructed a new curve with the same mapping radius,  $C''$  is eligible for membership in the family of curves admitted to competition, thus introducing a contradiction to our hypothesis that  $C$  is a curve whose free boundary has a minimum length. It can be seen from this argument that the curve having a minimum length must belong to the more exclusive sub-class of curves having convex free boundaries.

It should be noticed that there are four possibilities for the form of the extremum curve; they are listed as follows:

1. That the extremal curve degenerate to a segment along the real axis having a length greater than 2 (i.e., a point traces a total length which is greater than 4) and containing the segment  $[-1,1]$  (shown in Fig. 1a).
2. That the extremal curve completely surrounds  $[-1,1]$ , and no point of this interval lies on the boundary  $G$  (Fig. 1b).
3. That not all but only a part of  $[-1,1]$  lies on  $G$  (Fig. 1c).
4. That the entire interval be a part of  $G$  (Fig. 1d).

Let us consider the first of these situations. If the extremal curve has this sort of shape, we may apply a variation of Hadamard. Let us choose a point  $x_0$  on this segment and in an interval  $[x_0 - \alpha, x_0 + \alpha]$  which surrounds  $x_0$  apply a variation function  $\rho(x)$ , constructed so that  $\rho(x) = 0$  at all points outside the interval and so that  $\rho(x)$  is negative inside the interval; apply this variation to the upper branch of the segment and let  $\rho(x) = 0$  throughout the lower branch. In view of (2.2) we have for our curve constructed by this variation

$$l^* = o(\epsilon)$$

However, in view of (2.3) we have

$$\delta^* = \delta - \epsilon \int_{x_0 - \alpha}^{x_0 + \alpha} \left(\frac{\partial g}{\partial n}\right)^2 \rho dx + o(\epsilon)$$

resulting in a change having a magnitude of the first power of  $\epsilon$ . We have increased the logarithmic capacity by this variation, but we may decrease it to its original value by removing part of the end of this segment; we must, however, shorten this segment by a multiple of the first power of  $\epsilon$  in order to accomplish this result. Thus we have constructed a new curve also

eligible for membership in our class of competing curves by applying two operations. The first operation lengthened the curve by  $o(\epsilon)$  while the second reduced its length by according to the first power of  $\epsilon$ . By choosing  $\epsilon$  sufficiently small, we may make this newly constructed curve have a length which is less than that of the original, thus contradicting our hypothesis that this segment solves the minimum problem.

We shall postpone our discussion of Cases 2, 3, and 4 until we have constructed a certain auxiliary conformal mapping function  $q(z)$ .

We shall now construct a Schiffer variation of a form that is consistent with the constraints of our problem. Let  $z_n$  be a sequence of points in the domain  $D$  such that they have a limit point  $z_0$ . Let us denote a point in a domain of comparison  $D^*$  by the relation

$$(5.1) \quad z^* = z + e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{z-z_n} + e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{z-\bar{z}_n}$$

We have here an expression defined in a neighborhood of  $\epsilon_n = 0$ ,  $n=1,2,\dots,\infty$ ; we are, of course, obliged to restrict ourselves to such values of  $\epsilon_n$  such that these sums are convergent. This transformation retains the real axis and, in particular, transforms the segment  $[-1,1]$  into another real segment. This transformation will leave this segment totally invariant if we add the constraints

$$(5.2) \quad e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1-z_n} + e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1-\bar{z}_n} = 2 \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1-z_n} = f_1(\epsilon_n) = 0$$

$$(5.2') \quad e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1+z_n} + e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1+\bar{z}_n} = 2 \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{1+z_n} = f_{-1}(\epsilon_n) = 0$$

We now develop a formula for  $\ell^*$  with our extended Schiffer variation; differentiation of (5.1) yields

$$\frac{dz^*}{ds} = \frac{dz}{ds} \left[ 1 - e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{z-z_n} - e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{z-\bar{z}_n} \right],$$

$$\left(\frac{dz^*}{ds}\right)^2 = \frac{dz^*}{ds} \frac{dz^*}{ds} = 1 - e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{(z-z_n)^2} - e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{(z-\bar{z}_n)^2}$$

$$- \left( e^{-i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{(\bar{z}-z_n)^2} + e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{(z-\bar{z}_n)^2} \right) + o(\epsilon)$$

$$= 1 - 2 \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \epsilon_n \left[ \frac{1}{(z-z_n)^2} + \frac{1}{(\bar{z}-z_n)^2} \right] + o(\epsilon) \quad ;$$

$$\frac{dz^*}{ds} = 1 - \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \epsilon_n \left[ \frac{1}{(z-z_n)^2} + \frac{1}{(\bar{z}-z_n)^2} \right]$$

Integration of both sides produces

$$(5.3) \quad l^* = l - \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \epsilon_n \oint \left( \frac{1}{(z-z_n)^2} + \frac{1}{(\bar{z}-z_n)^2} \right) ds + o(\epsilon)$$

where we let  $\epsilon = \max \epsilon_n$ .

Each step in the derivation of  $\gamma^*$  in Section 3 is valid until we obtain an equation which must be modified to the following form:

$$\gamma^* = \gamma + \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \frac{\epsilon_n}{2\pi i} \int_{\Gamma} [p^*(t^*) - p^*(t)] dt,$$

where in this extended problem  $\Gamma$  must consist of a set of circles small enough to surround exclusively all  $z_n$ 's and  $\bar{z}_n$ 's. We have as a result

$$(5.4) \quad \gamma^* = \gamma + \operatorname{Re} \sum_{n=1}^{\infty} e^{i\varphi} \epsilon_n p'(z_n)^2 \sigma(z_n) + e^{-i\varphi} \epsilon_n p'(\bar{z}_n)^2 \sigma(\bar{z}_n) + o(\epsilon)$$

$$= \gamma + \operatorname{Re} \sum_{n=1}^{\infty} [e^{i\varphi} \epsilon_n p'(z_n)^2 \sigma(z_n) + e^{-i\varphi} \epsilon_n p'(\bar{z}_n)^2 \sigma(\bar{z}_n)] + o(\epsilon)$$

$$= \gamma + \operatorname{Re} e^{i\varphi} \sum_{n=1}^{\infty} \epsilon_n [p'(z_n)^2 \sigma(z_n) + \overline{p'(\bar{z}_n)^2 \sigma(\bar{z}_n)}] + o(\epsilon)$$

where  $\sigma(t)$  is a function defined such that

$$\begin{aligned} \sigma(t) &= 1, & t \in D, \\ \sigma(t) &= 0, & t \in \bar{D}. \end{aligned}$$

A necessary condition that a minimum to our problem exist is that the functional

$$J = \lambda_0 l^* - \lambda_1 \gamma^* - \lambda_2 \frac{f_1(\epsilon_n)}{2} - \lambda_3 \frac{f_{-1}(\epsilon_n)}{2}$$

have the property that

$$\frac{\partial J}{\partial \epsilon_n} = 0 \quad \text{for} \quad \epsilon_n = 0 \quad (n=1,2,\dots)$$

Therefore, we have

$$\begin{aligned} (5.5) \quad & \operatorname{Re} e^{i\varphi} \lambda_0 \left[ \oint_C \frac{ds_z}{(z-z_n)^2} + \oint_C \frac{ds_{\bar{z}}}{(\bar{z}-z_n)^2} \right] \\ & = \operatorname{Re} \left[ [e^{i\varphi} \lambda_1 (p'(z_n)^2 \sigma(z_n) + \overline{p'(z_n)^2} \sigma(\bar{z}_n))] + \frac{\lambda_2}{1-z_n} + \frac{\lambda_3}{1-\bar{z}_n} \right], \\ & \qquad \qquad \qquad n=1,2,\dots,\infty \end{aligned}$$

Since  $\varphi$  is arbitrary, we obtain

$$\begin{aligned} (5.6) \quad & \lambda_0 \left[ \oint_C \frac{ds_z}{(z-z_n)^2} + \oint_C \frac{ds_{\bar{z}}}{(\bar{z}-z_n)^2} \right] - \lambda_1 (p'(z_n)^2 \sigma(z_n) + \overline{p'(z_n)^2} \sigma(\bar{z}_n)) \\ & \qquad \qquad \qquad - \frac{\lambda_2}{1-z_n} - \frac{\lambda_3}{1+\bar{z}_n} = 0 \end{aligned}$$

We note that both  $p(z)^2$  and  $\overline{p'(\bar{z})^2}$  have expansions in  $1/z$  and we conclude that this is an analytic expression in  $z$  which vanishes for  $z = z_n$ . Since it vanishes at all points of this set  $z_n$ , which has a limit point, it must vanish everywhere [30]. We note also that

$$\begin{aligned} \frac{1}{1-z_n} &= -\frac{1}{z_n} \left[ 1 + \frac{1}{z_n} + \frac{1}{z_n^2} + \dots \right], \\ \frac{1}{1+\bar{z}_n} &= \frac{1}{z_n} \left[ 1 - \frac{1}{z_n} + \frac{1}{z_n^2} + \dots \right]; \end{aligned}$$

but since none of the other terms of (5.6) have terms in  $1/z_n$  in their expansion, we conclude that  $\lambda_2 = -\lambda_3$ . We have, finally, for all  $t$  in the complex plane

$$(5.7) \quad \lambda_0 \left[ \int_C \frac{ds_z}{(z-t)^2} + \int_C \frac{ds_{\bar{z}}}{(\bar{z}-t)^2} \right] = \lambda_1 [p'(t)^2 \sigma(t) + \overline{p'(\bar{t})^2} \sigma(\bar{t})] + \frac{\lambda_4}{1-t^2} ;$$

where  $\lambda_4 = 2\lambda_2 = -2\lambda_3$ .

Now if  $\lambda_0$  were to vanish, (5.7) would assume the form

$$\lambda_1 [p'(t)^2 \sigma(t) + \overline{p'(\bar{t})^2} \sigma(\bar{t})] + \frac{\lambda_4}{1-t^2} = \sigma$$

Since these are analytic functions in  $t$  we may integrate this equation around a contour  $\Gamma$  which encloses a sub-arc of the free boundary  $L$ ; let  $\Gamma$  intersect  $L$  at points  $z_1$  and  $z_2$  and enclose a region  $\Delta$ , which consists partly of a portion of  $D$  and partly of a portion of  $\bar{D}$ . Since  $\overline{p'(\bar{t})^2} \sigma(\bar{t})$  is an analytic function in  $\Gamma + \Delta$  and  $\Delta + \Gamma$  contains neither 1 nor -1, and since  $p'(t) \sigma(t)$  is analytic in  $D$ , we have

$$U(z_2) = \lambda_1 \int_{z_1}^{z_2} p'(t)^2 dt = -\lambda_1 \int_{\Gamma} \overline{p'(\bar{t})^2} dt + \lambda_4 \oint \frac{dt}{1-t^2} = 0$$

This shows that  $U(z_2)$  is an analytic function that vanishes on the open free boundary. In view of the principles of analytic continuation, we have the absurd result

$$\frac{d}{dt} U(t) = \lambda_1 p'(t)^2 \equiv 0$$

Thus  $\lambda_0 \neq 0$ , and we may introduce

$$\lambda = \frac{\lambda_1}{\lambda_0}$$

It will be helpful in our investigation to construct a function  $r(z)$  having the property that

$$r''(z) = \lambda p'(z)^2$$

for  $z \in D$ . Toward this goal we write (5.7) in the form

$$(5.7') \quad \int_C \frac{ds_z}{(z-t)^2} + \int_C \frac{ds_{\bar{z}}}{(\bar{z}-t)^2} = \lambda [p'(t)^2 + \overline{p'(\bar{t})}^2 \sigma(\bar{t})] + \frac{\lambda_1}{1-t^2}$$

where  $t$  is in  $D$  and where  $\lambda' = \lambda_4/\lambda_0$ . Integration of (5.7') with respect to  $t$  yields

$$\lambda \int_{U_0}^U [p'(t)^2 + \overline{p'(\bar{t})}^2 \sigma(\bar{t})] dt + \lambda' \int_{U_0}^U \frac{dt}{1-t^2} = \int_{U_0}^U \left[ \oint_C \frac{ds_z}{(z-t)^2} + \oint_C \frac{ds_{\bar{z}}}{(\bar{z}-t)^2} \right] dt.$$

Integration with respect to  $U$  produces

$$\begin{aligned} \lambda \int_{t_1}^{t_2} \int_{U_0}^U [p'(t)^2 + \overline{p'(\bar{t})}^2 \sigma(\bar{t})] dt du + \lambda' \int_{t_1}^{t_2} \int_{U_0}^U \frac{dt}{1-t^2} &= \int_{t_1}^{t_2} \left( \oint_C \left[ \frac{1}{z-U} - \frac{1}{\bar{z}-U_0} \right] ds_z \right) du \\ &+ \int_{t_1}^{t_2} \oint_C \left( \frac{1}{z-U} - \frac{1}{\bar{z}-U_0} \right) ds_z du. \end{aligned}$$

Here  $t_1$  and  $t_2$  are points in  $D$ . Let  $v_0, v, w_1$ , and  $w_2$  denote points of  $\bar{D}$ ; then we obtain

$$\begin{aligned} \lambda \int_{t_1}^{t_2} \int_{U_0}^U \frac{dt du}{1-t^2} &= -\lambda \int_{t_1}^{t_2} \int_{U_0}^U \overline{p'(\bar{t})}^2 dt du + \lambda \int_{w_1}^{w_2} \int_{v_0}^v \overline{p'(\bar{t})}^2 dt dv \\ &- \lambda' \int_{t_1}^{t_2} \int_{U_0}^U \frac{dt du}{1-t^2} + \lambda' \int_{w_1}^{w_2} \int_{v_0}^v \frac{dt dv}{1-t^2} + \int_{t_1}^{t_2} \int_{U_0}^U \oint_C \frac{ds dt du}{(z-t)^2} \\ &+ \int_{t_1}^{t_2} \int_{U_0}^U \oint_C \frac{ds dt du}{(\bar{z}-t)^2} - \int_{w_1}^{w_2} \int_{v_0}^v \oint_C \frac{ds dt dv}{(z-t)^2} - \int_{w_1}^{w_2} \int_{v_0}^v \oint_C \frac{ds dt dv}{(\bar{z}-t)^2}. \end{aligned}$$

It may be inferred that

$$\begin{aligned}
 & \lambda \int_{t_1}^{t_2} \int_{U_0}^U p'(t)^2 dt + \int_{t_1}^{t_2} \oint_C \frac{ds du}{z-U_0} - \int_{w_1}^{w_2} \oint_C \frac{ds dv}{z-v_0} + \int_{t_1}^{t_2} \oint_{z \in C} \frac{ds-du}{(\bar{z}-U_0)} - \int_{w_1}^{w_2} \int_{z \in C} \frac{ds dv}{z-v_0} \\
 & = \lambda \int_{t_1}^{t_2} \int_{U_0}^U p'(t)^2 dt + (t_2-t_1) \oint \frac{ds}{z-U_0} + (t_2-t_1) \oint \frac{ds}{\bar{z}-U_0} + (w_2-w_1) \oint_{z \in C} \frac{ds}{z-v_0} + (w_2-w_1) \int_C \frac{ds}{z-v_0} \\
 & = \int_{t_1}^{t_2} \oint_C \frac{ds du}{(\bar{z}-U)} - \int_{w_1}^{w_2} \oint \frac{ds dv}{z-v} + \int_{t_1}^{t_2} \oint \frac{ds du}{\bar{z}-U} - \int_{w_1}^{w_2} \oint \frac{ds dv}{z-v} \\
 & \quad - \lambda' \int_{t_1}^{t_2} \int_{U_0}^U \overline{\sigma(\bar{t})} p'(\bar{t})^2 dt du + \lambda \int_{w_1}^{w_2} \int_{v_0}^v \overline{p'(\bar{t})}^2 dt dv - \lambda' \int_{t_1}^{t_2} \int_{U_0}^U \frac{dt du}{1-t^2} \\
 & \quad + \lambda \int_{w_1}^{w_2} \int_{v_0}^v \frac{dt dv}{1-t^2}
 \end{aligned}$$

has the desired property that  $r''(z) = \lambda p'(z)^2$ .

We shall find it a useful maneuver to evaluate  $\lim_{\substack{t_2 \rightarrow z_2 \\ t_1 \rightarrow z_1}} r(t_2)$ , where  $z_1$

and  $z_2$  are points on the free boundary of  $C$ . This is most easily accomplished by simultaneously allowing  $w_1 \rightarrow z_1$  and  $w_2 \rightarrow z_2$  within  $\bar{D}$ . We shall denote this passage to the limit by the symbol  $\lim_c$ . We have

$$\lim_{t_2 \rightarrow z_2} r(t_2) - \lim_{w_2 \rightarrow z_2} r(w_2) = \lim_{t_2 \rightarrow z_2} r(t_2) - 0 = \lim_c r(t_2)$$

We shall call this limit simply  $r(z_2)$ .

When no singularities are involved, this passage to the limit becomes a simple contour integral around a closed path  $\Gamma$ . As  $z$  traces  $\Gamma$ ,  $\bar{z}$  traces its reflected image in  $D$ , and since  $p'(\bar{t})$  is analytic in  $D$ , a double integration and passage to the limit nullifies this term. The same result is true for the term with  $\frac{1}{1-t^2}$  and also the integration and  $\lim_c$  performed

on  $\int_{z \in C} \frac{ds_z}{z-U_0}$ . Our remaining task is that of computing

$$r(z_2) = \lim_c \left[ \int_{t_1}^{t_2} \oint \frac{ds_z}{z-U} du - \int_{w_1}^{w_2} \oint \frac{ds_z}{z-V} dv \right]$$

$$= \lim_c [I_1 + I_2 + I_3 + I_4],$$

where

$$I_1 = \int_{t_1}^{t_2} \int_{z_1'}^{z_2'} \frac{ds_z}{z-U} du - \int_{w_1}^{w_2} \int_{z_1'}^{z_2'} \frac{ds_z}{z-V} dv$$

$$I_2 = \int_{t_1}^{t_2} \int_{z_2'}^{z_1''} \frac{ds_z}{z-U} du - \int_{w_1}^{w_2} \int_{z_2'}^{z_1''} \frac{ds_z}{z-V} dv$$

$$I_3 = \int_{t_1}^{t_2} \int_{z_1''}^{z_1'} \frac{ds_z}{z-U} du - \int_{w_1}^{w_2} \int_{z_1''}^{z_1'} \frac{ds_z}{z-V} dv$$

$$I_4 = \int_{t_1}^{t_2} \int_{z_2''}^{z_2'} \frac{ds_z}{z-U} du - \int_{w_1}^{w_2} \int_{z_2''}^{z_2'} \frac{ds_z}{z-V} dv$$

where all integrations are taken along  $C$  in a counter-clockwise sense (Fig. 2a). We let  $z_1'$  and  $z_1''$  be the endpoints of an arc of length  $\rho$  containing the point  $z_1$  and attach a corresponding meaning to  $z_2'$ ,  $z_2''$ , and  $z_2'''$ .

To evaluate  $I_1$ , we may pass to the limit without obstruction; since it is easily verified that this double integral is absolutely convergent; we may change the order of integration and write

$$\lim_c I_1 = \int_{\Gamma} \int_{z_1''}^{z_2''} \frac{ds_z dt}{z-t}$$

where  $\Gamma$  is a closed circuit enclosing the arc  $z_1' z_1''$  of  $C$  and passing through  $z_1$  and  $z_2$  (Fig. 2b). Hence

$$\lim_c I_1 = \int_{z=z_1''}^{z=z_2'} \int_{\Gamma} \frac{dt ds}{z-t} = 2\pi i (s_2' - s_1'') ,$$

where  $s_2'$  is equal to the arc parameter  $s$  at  $z_2'$  and  $s_1''$  has a similar meaning.

By analogous reasoning,

$$\lim_c I_2 = \int_{z_2''}^{z_1'} \oint_{\Gamma} \frac{dt ds}{z-t} = 0 .$$

To evaluate  $I_3$  (Fig. 2c), we find no obstruction in allowing  $t_2$  and  $w_2$  to approach  $z_2$  within  $D$  and  $\bar{D}$ , respectively; we obtain

$$\lim_{\substack{t_2 \rightarrow z_2 \\ w_2 \rightarrow z_2}} I_3 = \int_{t_1}^{w_1} \int_{z_1'}^{z_2'} \frac{ds_z}{z-t} dt ,$$

where  $t$  traces a curve in  $D$  between  $t_1$  and  $z_2$  and a continuation of this curve in  $\bar{D}$  from  $z_2$  to  $w_1$ . In view of the convexity and rectifiability of the free boundary, we are justified in writing

$$I_3 = \int_{z_1'}^{z_1''} [\log(z-t_1) - \log(z-w_1)] ds = \int_{z_1'}^{z_1''} [\log(z-t_1) - \log(z-w_1)] \frac{dz}{ds} ds .$$

Since convexity implies that  $\dot{z}(s)$  is of bounded variation,

$$\begin{aligned} I_3 &= \dot{z}_1'(z_1'' - t_1) \log \frac{z_1'' - t_1}{e} - \dot{z}_1'(z_1' - t_1) \log \frac{z_1' - t_1}{e} \\ &\quad - \dot{z}_1''(z_1'' - w_1) \log \frac{z_1'' - w_1}{e} - \dot{z}_1'(z_1' - w_1) \log \frac{z_1' - w_1}{e} \\ &\quad - \int_{z_1'}^{z_1''} (z-t_1) \log \frac{z-t_1}{e} dz + \int_{z_1'}^{z_1''} (z-w_1) \log \frac{z-w_1}{e} dz \end{aligned}$$

(The reader should note that  $\log \frac{z_1' - t_1}{e}$  is used to represent more concisely,  $\log (z_1' - t_1) - 1$ .)

We let  $\rho$  and  $|w_1 - t_1|$  be so small that

$$\begin{aligned} |I_3| \leq & |z_1' - t_1| \left( \log \frac{1}{|z_1'' - t_1|} + 1 + 2\pi \right) + |z_1' - t_1| \left( \log \frac{1}{|z_1'' - t_1|} + 1 + 2\pi \right) \\ & + |z_1'' - w_1| \left( \log \frac{1}{|z_1'' - w_1|} + 1 + 2\pi \right) + |z_1' - w_1| \left( \log \frac{1}{|z_1'' - w_1|} + 1 + 2\pi \right) \\ & + \left| \int_{z_1}^{z_1'} |z - t_1| \left( \log \frac{1}{|z - t_1|} + 2\pi + 1 \right) dz \right| + \left| \int_{z_1}^{z_1'} |z - w_1| \left( \log \frac{1}{|z - t_1|} + 1 + 2\pi \right) dz \right|. \end{aligned}$$

Now since  $\log x = \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{t^{1/2}} < 2x^{1/2} - 2 < 2x^{1/2}$ , for sufficiently small  $\rho$

we have  $\log \frac{1}{|z - z_1|} < \frac{2}{|z - z_1|^{1/2}}$  for all  $z$  along the closed arc  $z_1' z_2''$ . Thus

$$\begin{aligned} \lim_{\rho \rightarrow 0} |I_3| & < 2|z_1'' - z_1'| (2|z_1'' - z_1|^{-1/2} + 3\pi) + 2|z_1' - z_1| (2|z_1'' - z_1|^{-1/2} + 3\pi) \\ & \quad + 2 \left| \int_{z_1}^{z_2''} (|z - z_1|^{1/2} + 3\pi |z - z_1|) |dz| \right| \\ & < 8\rho^{1/2} + 2\pi\rho + 2 \int (\rho^{1/2} + 3\pi\rho) |dz| < 36\pi^2 \rho^2, \end{aligned}$$

since a corresponding result also holds for  $I_4$ ,

$$\lim_{\rho \rightarrow 0} |r(t_2) - I_1| = |r(z_2) - \lim_{\rho \rightarrow 0} I_1| < 72\pi^2 \rho^{1/2}$$

This result must also hold for the limiting case  $\rho \rightarrow 0$ ; hence for  $z \in G$ ,

$$r(z_2) = -2\pi i(s_2 - s_1) = \lambda \int_{z_1}^{z_2} \int_{v_0}^v p'(t)^2 dt dv$$

for all  $z$  on the free boundary  $L$ . We may deduce immediately that  $\lambda$  cannot vanish and that we may use it freely in either the numerator or denominator of an expression.

Since  $p(z)$  is an analytic function of  $z$  in  $D$ , save for a logarithmic pole at infinity, we have

$$\frac{z(z)}{2\pi i} = \frac{\lambda}{2\pi i} \iint p'(z)^2 dz dz = R(p)$$

where  $R(p)$  represents an analytic function of  $p$  in  $D$ . Let us now recall the definition of the Green's function  $g(z)$  and its analytic extension  $p(z)$ ; we deduce that the image of  $C$  in the  $p$ -plane is the imaginary axis. We also note that for  $z$  on the free boundary,  $R(p)$  is equal to the real quantity  $s$ . Now let  $\alpha$  denote an arc of the extremal free boundary and let  $H$  denote a region containing  $\alpha$  along with parts of  $D$  and  $\bar{D}$ , having  $\alpha$  as their common boundary. Then, in view of the Schwarz principle of reflection, we may define  $R(p)$  as an analytic function of  $p$  for all  $z \in H$ . If we write

$$R(p) = \frac{h}{2\pi i} \int \left( \int p'(z)^2 dz \right) \frac{dz}{dp} dp,$$

then

$$R'(p) = \frac{h}{2\pi i} \left( \int p'(z) dz \right) z'(p),$$

$$\frac{R'(p)}{z'(p)} = \frac{1}{2\pi i} \int \frac{dp}{z'(p)},$$

$$\frac{R''(p)}{z'(p)} = \frac{R'(p)z''(p)}{z'(p)^2} = \frac{1}{2\pi i R'(p)}.$$

Whence finally we have

$$(5.8) \quad z''(p) = z'(p) \left[ \frac{R''(p)}{R'(p)} - \frac{1}{2\pi i R'(p)} \right].$$

Now  $R''(p)$  and  $R'(p)$  are both analytic functions of  $p$  for  $z \in H$ , and since this is a linear equation in  $z$ , it is solvable uniquely for  $z$  as an analytic function of  $p$  at all points corresponding to  $z \in H$  ([6], [18]) except about the points corresponding to the zeros of  $R'(p)$ . These zeros, however, are isolated points; hence, if  $p_0$  is a zero of  $R'(p)$  which lies on the image of  $\alpha$  in the  $p$ -plane, there is some deleted neighborhood  $N_0$  of  $p_0$  on the image of  $\alpha$  where  $z$  is an analytic function of  $p$ . Let  $p_1$  and  $p_2$  be any two points in this neighborhood. Let us now write (5.8) in the form

$$(5.8') \quad \frac{R''(p)}{R'(p)} - \frac{z''(p)}{z'(p)} = \frac{1}{2\pi i R'(p)}.$$

Integration between the limits  $p_1$  and  $p_2$  yields for the left side of (5.8)

$$\log \frac{R'(p_2)}{z'(p_2)} + C,$$

where

$$C = \log \frac{z'(p_1)}{R'(p_1)}.$$

Now at all points  $p \in N_0$ ,  $z$  is an analytic function of  $p$ , which is purely imaginary on  $\alpha$ . Hence, it is an analytic function of  $ih$ , where  $h$  is a real parameter, and the arc corresponding to  $N_0$  in the  $z$ -plane is analytic. Thus at  $p = p_2$ ,  $R'(p_2) = \left| \frac{dz}{dz} \frac{dz}{dp} \right|_{p_2}$  and the integration side of (5.8') becomes simply

$$C + \log \frac{dz(p_2)}{ds} = C + i \arg \frac{dz}{ds} \Big|_{p_2}.$$

This expression remains bounded for any location of  $p_2$ , even near  $p_0$ . Since  $p_0$  is a zero of the analytic function  $R'(p)$ ,  $R'(p)$  is expressible in the form

$$R'(p) = (p - p_0)^m f(p);$$

here  $m$  is a natural number and  $f(p) \neq 0$ . However, the integrated right-hand side of (5.8') is

$$\frac{1}{2\pi i} \int_{p_1}^{p_2} \frac{dp}{R'(p)} = \frac{1}{2\pi i} \int_{p_1}^{p_2} \frac{dp}{(p - p_0)^m f(p)}.$$

Since  $p_0$  is a pole of this integral expression, we can make the absolute value as large as desired by bringing  $p_2$  sufficiently near  $p_0$ ; this, however, is a contradiction of the result concerning the integral of the left of (5.8'); hence the only other alternative is that  $z$  be solvable at any point on the boundary as an analytic function of  $p$ . Consequently, on the boundary  $z$  is an analytic function of a real parameter, and any sub-arc of the free boundary is an analytic arc. Therefore, the formal processes which follow are adequately justified.

With proper choice of the limits of integration we may write

$$r(z) = -2\pi i z$$

for  $z \in L$ . On differentiating with respect to  $z$ , we have

$$-\frac{r'(z)}{2\pi i} = \frac{i}{z}$$

Let us now denote  $-(r'(z))/(2\pi i)$  by  $q(z)$  and examine the conformal mapping properties of this function. We may infer that the free boundary is mapped onto an arc of the unit circle. At an earlier time we mentioned in this section that there were four possible forms that the extremal curve could assume. If  $C$  has the form mentioned in Case 2, the image of  $C$  in the  $q$ -plane must be the unit circle, whose boundary is traced exactly once in a clockwise sense as  $z$  traces  $C$  in a counter-clockwise sense. Since  $q(z)$  is an integral of a constant multiple of  $p'(z)^2$ , it has an expansion about infinity form

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

Thus the image of the point at infinity in the  $z$ -plane is located in the interior of the image curve in the  $q$ -plane. The region  $D$  is mapped by  $q$  onto the interior of the unit circle. We may conclude from the conformal mapping properties of  $q$ , which were discussed Section 2, that the extremal curve is a circle of radius  $R$ , which completely encloses the segment  $[-1,1]$ . This case is certainly impossible if  $R < 1$ . In the event that  $R \geq 1$ , let us set

$$R = \frac{1}{\left(1 + \frac{\rho}{\pi}\right) \sin \frac{\rho}{1 + \frac{\rho}{\pi}}} \quad (0 \leq \rho \leq \pi)$$

We shall see the significance of this substitution in the discussion which follows. Let us map the exterior of the unit circle onto the exterior of the configuration consisting of the segment  $[-1,1]$ , and the circular arc

having as its center the coordinates  $0, \cot \beta$  and having a radius  $\frac{1}{\sin \beta}$  such that the points at infinity correspond to each other. This configuration is mapped onto a wedge by the mapping  $z-1/z+1$  with the segment corresponding to the negative real axis and the arc corresponding to the radius vector in the direction  $e^{i\beta}$ . Multiplication by  $e^{-i\beta}$  makes the arc correspond to the positive real axis. The mapping  $h = (e^{-i\beta} \frac{z-1}{z+1})^{\pi/\pi+\beta}$  maps the exterior of this arc and segment onto the lower half plane. The unit circle is mapped onto the lower half plane, so that  $e^{i\alpha}$  corresponds to the origin and  $e^{-i\alpha}$  corresponds to the point at infinity by the transformation  $e^{-i\alpha} \frac{\zeta - e^{i\alpha}}{\zeta - e^{-i\alpha}}$ . The point at infinity in the  $z$ -plane will correspond to the point at infinity in the  $\zeta$ -plane if  $\alpha = \frac{\beta\pi}{\pi+\beta}$ . Thus the mapping from the  $\zeta$ -plane to the  $z$ -plane is accomplished by

$$z = \frac{\left(\zeta - e^{-\frac{i\beta\pi}{\pi+\beta}}\right)^{1+\frac{\beta}{\pi}} + \left(\zeta - e^{i\frac{\beta\pi}{\pi+\beta}}\right)^{1+\frac{\beta}{\pi}}}{\left(\zeta - e^{-\frac{i\beta\pi}{\pi+\beta}}\right)^{1+\frac{\beta}{\pi}} - \left(\zeta - e^{i\frac{\beta\pi}{\pi+\beta}}\right)^{1+\frac{\beta}{\pi}}}$$

The mapping radius of this transformation is given by

$$\begin{aligned} \lim_{\zeta \rightarrow \infty} \frac{1}{\zeta} &= \lim_{\zeta \rightarrow \infty} \frac{2}{\zeta \left(1 - \left(1 + \frac{\beta}{\pi}\right) e^{\frac{-i\beta\pi}{\pi+\beta}} + \dots\right) - \zeta \left(1 - \left(1 + \frac{\beta}{\pi}\right) e^{\frac{i\beta\pi}{\pi+\beta}} + \dots\right)} \\ &= \frac{1}{\left(1 + \frac{\beta}{\pi}\right) \sin \frac{\beta}{1 + \frac{\beta}{\pi}}} = R \end{aligned}$$

We have, however,

$$R = \frac{1}{\left(1 + \frac{\beta}{\pi}\right) \sin \frac{\beta}{\left(1 + \frac{\beta}{\pi}\right)}} > \frac{1}{\left(1 + \frac{\beta}{\pi}\right) \sin \beta}$$

But the length of the circular arc of the configuration which we have just shown to have an outer mapping radius  $R$  is  $2\frac{(\pi-\beta)}{\sin\beta}$ . We have

$$2\frac{\pi-\beta}{\sin\beta} = 2\pi\left(\frac{1-\frac{\beta}{\pi}}{\sin\beta}\right) < \frac{2\pi(1-\frac{\beta}{\pi})}{(1-\frac{\beta^2}{\pi^2})\sin\beta} = \frac{2\pi}{(1+\frac{\beta}{\pi})\sin\beta} < 2\pi R.$$

This circular arc, however, is eligible for membership in the family of free boundaries admitted to competition for solution of the extremal problem and has a length less than that of the circumference of the circle of radius  $R$ , and consequently this circle cannot be a solution to our minimum problem.

Let us now examine the manner in which  $q$  maps the fixed boundary. We have

$$p'(z) = \frac{\partial p}{\partial s} \frac{dz}{z} = i \frac{dz}{z} \frac{\partial g}{\partial n},$$

which is a purely imaginary quantity. Hence  $q(z)$ , which has the form

$$\frac{-\lambda}{2\pi i} \int_{z_0}^z p'(z)^2 dz,$$

must map the fixed boundary into a segment parallel to the imaginary axis, which, of course, must intersect the unit circle. In

view of the geometry of this mapping, we conclude that the tangent vector is discontinuous at the points of contact of the fixed boundary with the free boundary and that the limit of the argument of the tangent vector of the free boundary as it approaches one point of contact must be the negative of that at the other point of contact.

With this last fact in mind, let us consider the appearance of a possible extremal curve which satisfies the conditions specified in the third possible case. Let us suppose that our extremal problem is solved by a curve  $C$  having a fixed boundary which does not consist of the entire segment  $[-1,1]$ . If the free boundary does not intersect the fixed boundary

at 1 or -1, since it is convex it must enclose the ends of this segment. As we trace the point  $z$  around the free boundary, we see that the tangent vector must assume certain values twice. Hence  $q$  performs a mapping of  $C$  onto the complete unit circle and a chord parallel to the imaginary axis, where the chord separates the circle into an arc which is traced once and an arc which is traced twice. We may say with different wording that as  $z$  traces  $C$ ,  $q$  traces two closed circuits, one of them being the complete unit circle and the other being a crescent-shaped circuit  $\Gamma_q$  formed by the circular arc and chord.

According to the argument principle [22], the number of times  $q(z)$  assumes a certain value  $w_0$  is given by

$$\frac{1}{2\pi i} \oint_C \frac{q'(z) dz}{q(z) - w_0}$$

where the integration is taken around  $C$  in a clockwise sense. But this integral is equivalent to the following counter-clockwise integration around the image curve  $C_q$  in the  $q$ -plane:

$$\frac{1}{2\pi i} \int_{C_q} \frac{dq}{q - w_0}$$

By the residue theorem this integral is equal to 2 if  $w_0$  is inside  $\Gamma_q$ , since  $w_0$  is a point enclosed in two complete circuits equal to 1 if  $w_0$  is inside the unit circle but outside  $\Gamma_q$ , and 0 otherwise. We notice especially that certain values are assumed twice. This means that  $q(z)$  must have a branch point for a certain  $z \in D$  such that  $q'(z) = 0$ . This implies that  $p'(z) = 0$ , meaning, of course, that the derivative of Green's function along the normal of the level curve must vanish. This situation, however, is impossible as is shown in texts on potential theory and conformal

mapping ([17] and [29]). We are able to conclude from this argument that the extremal curve must satisfy Case 4.

Under the condition the image  $G_q$  of the extremal curve must be composed of a circular arc and a segment parallel to the imaginary axis and is traced exactly once in a clockwise sense as  $G$  is traced in a counter-clockwise sense. The number of times that  $q$  assumes a value  $w_0$  is as before

$$\frac{1}{2\pi i} \int_{G_q} \frac{dq}{q-w_0}$$

This integral is equal to 1 if  $w_0$  is inside  $G_q$ , and 0 otherwise. Hence, each value corresponding to a  $q(z)$  for  $z \in D$  is assumed exactly once and the mapping of  $D$  by  $q(z)$  onto the  $q$ -plane is Schlicht.

Let  $2\beta$  denote the angle subtended by the image of the fixed boundary in the  $q$ -plane. ( $\beta$  is also the angle which the tangent vector to the free boundary makes with the real axis at the point of contact with the fixed boundary.) On the fixed boundary  $q$  is expressible in the form

$$q(z) = \cos \beta + iv$$

where  $v$  is a real function of the position on the imaginary axis. Since  $q(z)$  has at infinity an expansion

$$q(z) = a_0 - \frac{\lambda i}{2\pi z} + \frac{a_2}{z^2} + \dots$$

we have as a result of the residue theorem

$$(5.9) \quad \int_G q(z) dz = 2\pi i \left(-\frac{\lambda i}{2\pi}\right) = \lambda$$

But this contour integration can be expressed as the sum of an integral  $I_1$  over the fixed boundary,  $G-L$ , and an integral  $I_2$  over the free boundary  $L$ .

We have

$$I_1 = \int_{C-L} (\cos \beta + iv) dz = 2 \cos \beta + i \int_{-1}^1 v(x) dx$$

$$I_2 = \int_L \frac{1}{z} dz = \int_L ds = l$$

From (5.9) we obtain

$$(5.9') \quad I_1 + I_2 = 2 \cos \beta + i \int_{-1}^1 v dx + l = \lambda$$

Since  $i \int v dx$  is the only imaginary term in this equation, it must vanish.

Also by the residue theorem we have

$$\oint q^2 dz = 2\pi i \left[ \frac{-a_0 \lambda i}{2\pi} \right] = a_0 \lambda$$

The portion of this integral over the fixed boundary we designate by

$$I_3 = \int_{-1}^1 [\cos^2 \beta - 2i \cos \beta v - v^2] dx,$$

which must be equal to a real constant  $K_1$ , since the imaginary part vanishes.

Over the free boundary we have

$$I_4 = \int_L \frac{1}{z^2} dz = \int_L d\bar{z} = [\bar{z}]_1^{-1} = -2$$

We obtain as a result

$$(5.10) \quad a_0 = \frac{K_1 - 2}{\lambda} = \lim_{z \rightarrow \infty} q(z)$$

Thus the point at infinity is mapped by  $q(z)$  onto the real axis.

A drawing which shows the image of  $C$  in the  $q$ -plane and in the  $\zeta$ -plane is shown in Fig. 3a and Fig. 3b, respectively. In the  $q$ -plane the fixed boundary corresponds to a straight line parallel to the imaginary axis and subtends an angle  $\beta$  with the origin. The image of the free boundary is the arc of the unit circle. Let us assume that the fixed boundary maps onto the unit circle, with  $e^{i\alpha}$  corresponding to 1 and  $e^{-i\alpha}$  corresponding to -1; these points correspond to  $e^{-i\beta}$  and  $e^{i\beta}$ , respectively, in the  $q$ -plane. It should be noted that by varying our fixed mapping radius  $R$ , we should

obtain a one-parameter family of extremal curves. Each different value of  $R$  should produce a certain  $\delta$ , which in turn depends on  $\alpha$ . We shall subsequently determine relations between these three parameters.

Let us now map the image of  $D$  in the  $q$ -plane onto the upper half plane. This operation is accomplished in the following three steps (See Fig. 4):

1. The  $q$ -plane is mapped onto a wedge with the negative real axis corresponding to the fixed boundary and the straight line oriented in the direction  $\beta$  corresponding to the free boundary  $z=1$  in the  $z$ -plane (or  $e^{-i\beta}$  in the  $q$ -plane) corresponds to the point at infinity, and  $z=-1$  corresponds to the origin. The mapping is accomplished by

$$h_1 = \frac{q - e^{i\beta}}{q - e^{-i\beta}}$$

2. 
$$h_2 = e^{-i\beta} h_1 = e^{-i\beta} \frac{q - e^{i\beta}}{q - e^{-i\beta}},$$

rotates the fixed boundary so that it corresponds to the positive real axis.

3. The wedge is spread apart so that it corresponds to the upper half-plane by the transformation

$$h = h_2^{1/k},$$

where

$$k = 1 - \frac{\beta}{\pi}.$$

The region corresponding to  $D$  in the  $\zeta$ -plane is mapped onto the upper half-plane so that  $e^{i\alpha}$  corresponds to the point at infinity and  $e^{-i\alpha}$  corresponds to the origin by a transformation of the form  $Ae^{i\alpha} \frac{\zeta - e^{-i\alpha}}{\zeta - e^{i\alpha}}$ , where these correspondences hold if  $A$  is any positive constant.

$q$  is related to  $\zeta$  by

$$(5.11) \quad \frac{q - e^{i\beta}}{q - e^{-i\beta}} = B e^{i\gamma} \frac{(\zeta - e^{-i\alpha})^k}{(\zeta - e^{i\alpha})^k},$$

where

$$\gamma = k\alpha + \beta$$

On solving for q, we have

$$(5.11') \quad q = \frac{e^{-i(\frac{\gamma}{2} - \beta)} (\zeta - e^{i\alpha})^k - B e^{i(\frac{\gamma}{2} - \beta)} (\zeta - e^{-i\alpha})^k}{e^{-i\frac{\gamma}{2}} (\zeta - e^{i\alpha})^k - B e^{i\frac{\gamma}{2}} (\zeta - e^{-i\alpha})^k},$$

where B is another positive parameter equal to  $A^k$ . We have

$$a_0 = \lim_{\zeta \rightarrow \infty} q = \lim_{z \rightarrow \infty} q = \frac{e^{-i(\frac{\gamma}{2} - \beta)} - B e^{i(\frac{\gamma}{2} - \beta)}}{e^{-i\frac{\gamma}{2}} - B e^{i\frac{\gamma}{2}}}$$

However, we learned from (5.10) that  $a_0$  is real; this can only occur if

$B=1$ . We have, finally,

$$(5.12) \quad q = \frac{e^{-i(\frac{\gamma}{2} - \beta)} (\zeta - e^{i\alpha})^k - e^{i(\frac{\gamma}{2} - \beta)} (\zeta - e^{-i\alpha})^k}{e^{-i\frac{\gamma}{2}} (\zeta - e^{i\alpha})^k - e^{i\frac{\gamma}{2}} (\zeta - e^{-i\alpha})^k}$$

On differentiation with respect to  $\zeta$ , we obtain

$$(5.13) \quad \frac{dq}{d\zeta} = \frac{k \sin \alpha \sin \beta (\zeta - e^{i\alpha})^{k-1} (\zeta - e^{-i\alpha})^{k-1}}{[e^{-i\frac{\gamma}{2}} (\zeta - e^{i\alpha})^k - e^{i\frac{\gamma}{2}} (\zeta - e^{-i\alpha})^k]^2}$$

$z$  must have the following expansion with respect to  $\zeta$ :

$$z = e^{i\phi} R \zeta + \frac{a_1}{\zeta} + \dots$$

where  $\phi$  is an undetermined constant of rotation and R is positive. But

$\zeta = e^{p(z)+iH_0}$ , where  $H_0$  is also a constant of rotation, since

$$\frac{dq}{dz} = \frac{-r''(z)}{2\pi i} = \frac{-\lambda}{2\pi i} p'(z)^2 = \frac{\lambda i}{2\pi} \frac{\zeta'^2}{\zeta^2},$$

from which we obtain

$$(5.14) \quad \frac{dz}{d\zeta} = \frac{i\lambda}{2\pi} \frac{1}{\zeta^2 \frac{d\zeta}{d\zeta}} = \frac{i\lambda}{2\pi} \frac{e^{-\frac{i\gamma}{2}(\zeta - e^{i\alpha})^k} - e^{\frac{i\gamma}{2}(\zeta - e^{-i\alpha})^k}}{8k\pi \sin \alpha \sin \beta (\zeta - e^{i\alpha})^{k-1} (\zeta - e^{-i\alpha})^{k-1} \zeta^2},$$

as the differential equation which defines our extremal curve. We have also

$$(5.15) \quad R = \lim_{\zeta \rightarrow \infty} \left| \frac{dz}{d\zeta} \right| = \Lambda \sin^2 \frac{\theta}{2},$$

where

$$\Lambda = \frac{\lambda}{2k\pi \sin \alpha \sin \beta};$$

by an algebraic manipulation let us write (5.14) in the form

$$(5.14') \quad \frac{dz}{d\zeta} = \frac{i\Lambda}{4} \left[ e^{-i\theta} \left(1 - \frac{e^{i\alpha}}{\zeta}\right)^{1+k} \left(1 - \frac{e^{-i\alpha}}{\zeta}\right)^{1-k} - 2 \left(1 - \frac{2 \cos \alpha}{\zeta} + \frac{1}{\zeta^2}\right) + e^{i\theta} \left(1 - \frac{e^{i\alpha}}{\zeta}\right)^{1-k} \left(1 - \frac{e^{-i\alpha}}{\zeta}\right)^{1+k} \right].$$

This equation, when expanded about infinity, becomes

$$(5.14'') \quad \frac{dz}{d\zeta} = \frac{i\Lambda}{4} \left[ e^{-i\theta} \left(1 - (1+k) \frac{e^{i\alpha}}{\zeta} \dots\right) \left(1 - (1-k) \frac{e^{-i\alpha}}{\zeta} \dots\right) - 2 \left(1 - \frac{2 \cos \alpha}{\zeta} + \frac{1}{\zeta^2}\right) + e^{i\theta} \left(1 - (1-k) \frac{e^{i\alpha}}{\zeta} \dots\right) \left(1 - (1+k) \frac{e^{-i\alpha}}{\zeta} \dots\right) \right].$$

In view of the form of the expansion of  $z$  in terms of  $\zeta$ , the term in  $1/\zeta$  should be absent when expanding  $\frac{dz}{d\zeta}$ ; on collecting the coefficients of  $1/\zeta$  in (5.14'') and nullifying this sum, we obtain

$$(5.16) \quad (1+k)\cos(\gamma - \alpha) + (1-k)\cos(\gamma + \alpha) - 2 \cos \alpha = 0,$$

or by a trigonometric rearrangement we have

$$(5.16') \quad \cos \gamma \cos \alpha + k \sin \gamma \sin \alpha = \cos \alpha.$$

We now have a method for evaluating the parameter  $R$  and a relation between  $\alpha$  and  $\beta$ . To get a method for evaluating the constant  $\Lambda$  which is related to the Lagrange multiplier, we use the following version of (5.14):

$$(5.14''') \quad \frac{dz}{d\zeta} = \frac{i\Lambda}{4} \left(1 - \frac{2 \cos \alpha}{\zeta} + \frac{1}{\zeta^2}\right) \left[ e^{-i\theta} \left( \frac{1 - \frac{e^{i\alpha}}{\zeta}}{1 - \frac{e^{-i\alpha}}{\zeta}} \right)^k - 2 + e^{i\theta} \frac{\left(1 - \frac{e^{-i\alpha}}{\zeta}\right)^k}{\left(1 - \frac{e^{i\alpha}}{\zeta}\right)^k} \right].$$

To evaluate  $dz$  or  $\frac{dz}{d\zeta} d\zeta$  on the unit circle for  $\theta$  in  $(-\alpha, \alpha)$ , we make the substitution  $\zeta = e^{i\theta}$ , from which we have  $d\zeta = ie^{i\theta} d\theta$ . We obtain from (5.14")

$$(5.17) \quad \frac{dz}{d\zeta} d\zeta = \frac{-\Lambda}{2} (\cos \theta - \cos \alpha) \left[ e^{-i\theta} \left( -e^{-i\alpha} \frac{\sin \frac{\alpha-\theta}{2}}{\sin \frac{\alpha+\theta}{2}} \right)^k - 2 + e^{i\theta} \left( e^{-i\alpha} \frac{\sin \frac{\alpha+\theta}{2}}{-\sin \frac{\alpha-\theta}{2}} \right)^k \right] d\theta$$

It should be noticed that the relation defined by (5.11) indicates that

$$e^{i\theta} \frac{(\zeta - e^{-i\alpha})^k}{(\zeta - e^{i\alpha})^k}, \text{ which reduces to } e^{i\theta} \left( -e^{-i\alpha} \frac{\sin \frac{\alpha+\theta}{2}}{\sin \frac{\alpha-\theta}{2}} \right)^k \text{ on the unit circle,}$$

performs a conformal mapping on the  $\zeta$ -plane such that a point in  $(-\alpha, \alpha)$ , which is an image of a fixed boundary point; is transformed into a point on the negative real axis. On substituting the values of  $k$  and  $\theta$ , we have

$$e^{i(k\alpha + \beta)} \left( -e^{-i\alpha} \frac{\sin \frac{\alpha+\theta}{2}}{\sin \frac{\alpha-\theta}{2}} \right)^k = e^{i\beta} \left( -1 \frac{\sin \frac{\alpha+\theta}{2}}{\sin \frac{\alpha-\theta}{2}} \right)^{1 - \frac{\beta}{\pi}};$$

and on replacing  $-1$  by  $e^{i\pi}$ , this expression assumes the obviously negative form

$$-\left( \frac{\sin \frac{\alpha+\theta}{2}}{\sin \frac{\alpha-\theta}{2}} \right)^{1 - \frac{\beta}{\pi}},$$

for values of  $\theta$  in  $(-\alpha, \alpha)$ , (5.17) becomes

$$(5.17') \quad \frac{dz}{d\zeta} d\zeta = \frac{\Lambda}{2} (\cos \theta - \cos \alpha) \left[ \left( \frac{\sin \frac{\alpha-\theta}{2}}{\sin \frac{\alpha+\theta}{2}} \right)^k + 2 + \left( \frac{\sin \frac{\alpha+\theta}{2}}{\sin \frac{\alpha-\theta}{2}} \right)^k \right] d\theta$$

This expression when integrated between  $-\alpha$  and  $\alpha$  must result in the length of the fixed boundary, which is equal to 2. Thus  $\Lambda$  is defined by the relation

$$(5.18) \quad 2 = \frac{\Lambda}{2} \int_{-\alpha}^{\alpha} (\cos \theta - \cos \alpha) \left[ \left( \sin \frac{\alpha - \theta}{2} \right)^k \left( \sin \frac{\alpha + \theta}{2} \right)^{-k} + 2 \right. \\ \left. + \left( \sin \frac{\alpha + \theta}{2} \right)^k \left( \sin \frac{\alpha - \theta}{2} \right)^{-k} \right] d\theta$$

We summarize our results with

**Theorem 5.1.** The solution of our minimum problem is defined by the differential equation

$$(5.14''') \quad \frac{dz}{d\zeta} = \frac{i\Lambda}{4\zeta^2} (\zeta^2 - 2\zeta \cos \alpha + 1) \left[ e^{-i\delta} (\zeta - e^{i\alpha})^k (\zeta - e^{-i\alpha})^{-k} - 2 \right. \\ \left. + e^{i\delta} (\zeta - e^{-i\alpha})^k (\zeta - e^{i\alpha})^{-k} \right]$$

The family of minimum domains depends on only one parameter, which is defined through elimination in the following equations involving the constants

$R$ ,  $\alpha$ ,  $\beta$ , and  $\Lambda$ :

$$(5.15) \quad R = \Lambda \sin^2 \frac{\gamma}{2}$$

$$(5.16') \quad \cos \gamma \cos \alpha + k \sin \gamma \sin \alpha = \cos \alpha$$

$$(5.18) \quad 2 = \frac{\Lambda}{2} \int_{-\alpha}^{\alpha} (\cos \theta - \cos \alpha) \left[ \left( \sin \frac{\alpha - \theta}{2} \right)^k \left( \sin \frac{\alpha + \theta}{2} \right)^{-k} + 2 \right. \\ \left. + \left( \sin \frac{\alpha + \theta}{2} \right)^k \left( \sin \frac{\alpha - \theta}{2} \right)^{-k} \right] d\theta$$

We recall that  $\Lambda = \frac{\lambda}{2k\pi \sin \alpha \sin \beta}$ ,  $k = 1 - \frac{\beta}{\pi}$ , and  $\gamma = k\alpha + \beta$ .

The procedure for solving (5.14) is to solve first (5.16) for  $\alpha$  and  $\beta$ , one of which can be specified arbitrarily, determine  $\Lambda$  from (5.18), and then integrate (5.14) along the unit circle to determine a set of points on the minimum curve.

Substitution of  $\beta = 0$  in (5.16) yields the equation

$$2 - 2 \cos \alpha = 0,$$

whose solution is  $\alpha = 0$ . Substitution of  $\beta = \pi$  gives the result

$$\cos(\pi - \alpha) + \cos(\pi + \alpha) = 2 \cos \alpha$$

$$-2 \cos \alpha = 2 \cos \alpha$$

$$\alpha = \pi/2$$

For  $\beta = \pi/2$ , (5.16) becomes

$$\frac{3}{2} \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \frac{1}{2} \cos\left(\frac{\pi}{2} + \frac{3\alpha}{2}\right) = 2 \cos \alpha$$

$$\frac{3}{2} \sin \alpha - \frac{1}{2} \sin \frac{3\alpha}{2} = 2 - 4 \sin^2 \frac{\alpha}{2}$$

but

$$\sin \frac{3\alpha}{2} = 3 \sin \frac{\alpha}{2} - 4 \sin^3 \frac{\alpha}{2}$$

hence we have

$$3 \sin \frac{\alpha}{2} - 3 \sin \frac{\alpha}{2} + 4 \sin^3 \frac{\alpha}{2} = 4 - 8 \sin^2 \frac{\alpha}{2}$$

which becomes

$$\sin^3 \frac{\alpha}{2} + 2 \sin^2 \frac{\alpha}{2} - 1 = 0$$

$$\left(\sin \frac{\alpha}{2} + 1\right)\left(\sin^2 \frac{\alpha}{2} + \sin \frac{\alpha}{2} - 1\right) = 0$$

$$\sin \frac{\alpha}{2} = \frac{-1 + \sqrt{5}}{2} = .613$$

$$\frac{\alpha}{2} \approx 37.5^\circ \quad \alpha \approx 75^\circ = \frac{5\pi}{12}$$

To determine  $l$  we may write (5.9') in the form

$$(5.19) \quad 2 \cos \beta + l = 2\left(1 - \frac{\beta}{\pi}\right) \pi \sin \alpha \sin \beta \Lambda$$

When  $\beta \rightarrow \pi$  we have

$$-2 + l = 0$$

$$l = 2$$

This shows that in this case C has degenerated to a slit and that the free boundary coincides with the fixed boundary. This result is consistent with geometrical intuition.

Another interesting limiting case is when the outer mapping radius becomes increasingly large: the fixed boundary becomes less and less significant and geometrical intuition tells us that the extremal curve should resemble a circle. We now consider the ratio  $l/R$  as  $\alpha \rightarrow 0, \beta \rightarrow 0$ .

To evaluate this limit it will be necessary to first compute the limit

$$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha \sin \beta}{\sin^2 \frac{\gamma}{2}}$$

which we write in the form

$$\lim_{\alpha \rightarrow 0} \left( \frac{\sin^2 \alpha}{\sin^2 \frac{\gamma}{2}} \right) \left( \frac{\sin \beta}{\sin \alpha} \right)$$

and then consider the limit

$$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\sin \frac{\gamma}{2}}$$

We write (5.16') as

$$k \sin \gamma \sin \alpha = \cos \alpha (1 - \cos \gamma)$$

which becomes

$$k \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} \sin \alpha = \cos \alpha \sin^2 \frac{\gamma}{2}$$

Solving for  $\frac{\sin \alpha}{\sin \frac{\gamma}{2}}$  we have

$$\frac{\sin \alpha}{\sin \frac{\gamma}{2}} = \frac{\cos \alpha}{k \cos \frac{\gamma}{2}} ;$$

on passing to the limit this expression becomes unity. To evaluate

$\frac{\sin \beta}{\sin \alpha}$  we differentiate numerator and denominator with respect to  $\alpha$

and obtain

$$\frac{\cos \beta \frac{d\beta}{d\alpha}}{\cos \alpha}$$

This limit must be equal to  $\lim_{\alpha \rightarrow 0} \frac{d\beta}{d\alpha}$ . To compute  $\frac{d\beta}{d\alpha}$  we differentiate (5.16') with respect to  $\alpha$ , obtaining

$$\begin{aligned} & -\sin \gamma \cos \alpha \left[ 1 - \frac{\beta}{\pi} - \frac{\alpha}{\pi} \frac{d\beta}{d\alpha} + \frac{d\beta}{d\alpha} \right] - \cos \gamma \sin \alpha \\ & - \frac{1}{\pi} \sin \gamma \sin \alpha + k \cos \gamma \sin \alpha \left[ 1 - \frac{\beta}{\pi} - \frac{\alpha}{\pi} \frac{d\beta}{d\alpha} + \frac{d\beta}{d\alpha} \right] \\ & + k \sin \gamma \cos \alpha = -\sin \alpha \end{aligned}$$

We divide both sides of this expression by  $\sin \frac{\alpha}{2}$  and recall  $\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1$  and  $\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\sin \frac{\alpha}{2}} = 2$  and obtain

$$\lim_{\alpha \rightarrow 0} \frac{d\beta}{d\alpha} = 1 ;$$

hence we have, finally,

$$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha \sin \beta}{\sin^2 \frac{\alpha}{2}} = 1$$

To compute  $l/R$ , we write (5.18) in the form

$$\frac{2 \cos \beta}{R} + \frac{l}{R} = 2k\pi \frac{\sin \alpha \sin \beta}{\sin^2 \frac{\alpha}{2}}$$

and on passing to the limit we have

$$\lim_{R \rightarrow \infty} \frac{l}{R} = 2\pi ;$$

this result is also consistent with geometrical intuition.

## 6. The Use of Conformal Mapping Techniques for Hydrodynamic Free Boundary Problems

Problems of the type which were discussed in the previous section are of special interest to the mathematician. Problems which are of more interest to the hydrodynamicist are those which involve virtual mass. First, we return to the variational equivalent of the hydrodynamical problem of

the bubble with surface tension, and having internal gases with such compensating pressures as to lead to condition (1.5). From the variational point of view, we wish to consider the problem of extremizing the constant  $a$  when the length of  $C$  is held fixed. We recall that the velocity potential  $w$  has the expansion

$$w = z + \frac{\alpha}{z} + \dots ;$$

here the constant  $a$  is equal to the real part of  $\alpha$ . We shall show that it is sufficient to minimize the dimensionless ratio  $\ell^2/a$ .

We note that we may also formulate this problem by fixing a value for  $a$  and minimizing the length  $\ell$ . Since our result should not depend on the scale which we select for our unit, we should be able to magnify the length of a curve by a factor  $d$  and also magnify our flow by the same factor and obtain an extremal curve of the same shape. Let us make the change of variable

$$Z = dz ;$$

then the complex velocity potential is given by

$$w = \frac{Z}{d} + \frac{d\alpha}{Z} + \dots$$

Multiplying the velocity of the flow by  $d$  we obtain

$$w_1 = Z + \frac{d^2\alpha}{Z} + \dots ;$$

hence the constants of the new flow which correspond to  $\alpha$  and  $a$  are multiplied by  $d^2$ . Thus, the ratio  $\ell^2/a$  remains invariant no matter what we choose as a scale, and we may consider the problem of minimizing this functional. If we let  $\zeta$  represent the function which transforms  $D$  onto the exterior of the unit circle so that the points at infinity correspond to each other,  $\zeta$  takes the form

$$\zeta = Rz + \frac{a_1}{z} + \dots$$

We may fix a convenient scale by selecting  $R=1$ . We may now state the problem:

Find, within the family of closed rectifiable curves  $C^*$ , such that the region  $D^*$  associated with each of these curves is mapped onto the unit circle by a transformation

$$z = \zeta + \frac{a_1^*}{\zeta} + \dots$$

the curve  $C$ , for which there is a minimum value  $l^2/a$  for the functional  $l^{*2}/a^*$ .

The manner in which we proved that a solution exists and that the boundary is convex and analytic, etc., in Section 5 can be easily extended to apply to the problem just stated. We shall omit repetition of these proofs and assume that the same formal procedures which we employed before are justified. We can also establish, from the fact that  $a$  is a functional which grows monotonically with the domain, that the solution does not degenerate to a slit. Hence,  $a \neq 0$ .

For a Schiffer variation of the form

$$z^* = z + \frac{\epsilon e^{i\phi}}{z-t}$$

we have the extremal condition

$$\frac{d}{d\epsilon} \left( \frac{l^{*2}}{a^*} \right) = 2 \left[ \frac{l^*}{a^*} \frac{dl^*}{d\epsilon} \right]_{\epsilon=0} - \left( \frac{l^*}{a^*} \right)^2 \left[ \frac{da^*}{d\epsilon} \right]_{\epsilon=0} = 0$$

This equation becomes

$$\frac{dl^*}{d\epsilon} \Big|_{\epsilon=0} = \frac{l^*}{2a^*} \frac{da^*}{d\epsilon} \Big|_{\epsilon=0}$$

Let  $z_0$  be a point of  $\bar{D}$ . Since the velocity potential is conformally invariant we have for  $z$  in the neighborhood of infinity

$$\begin{aligned}
 w^*(z^*) &= z^* + \frac{\alpha^*}{z^*} + \dots = z + \frac{\epsilon e^{i\varphi}}{z-t_0} + \frac{\alpha^*}{z + \frac{\epsilon e^{i\varphi}}{z-t}} + \dots \\
 &= z + \frac{\epsilon e^{i\varphi}}{z(1-\frac{t}{z})} + \frac{\alpha^*}{z(1 + \frac{\epsilon e^{i\varphi}}{z^2(1-\frac{t}{z})})} + \dots \\
 &= z + \frac{\epsilon e^{i\varphi}}{z} + \frac{\alpha^*}{z} + \dots = w(z) \\
 &= z + \frac{\alpha}{z} + \dots ;
 \end{aligned}$$

thus  $\alpha^* = \alpha + \epsilon e^{i\varphi}$ ,

$$\frac{d\alpha^*}{d\epsilon} = \operatorname{Re} \frac{d\alpha^*}{d\epsilon} = \operatorname{Re} e^{i\varphi}$$

At  $\epsilon = 0$  we have

$$\operatorname{Re} e^{i\varphi} \oint \frac{ds}{(z-t)^2} = \operatorname{Re} \frac{l e^{i\varphi}}{2a}$$

Since  $\varphi$  is arbitrary, this can be written

$$(6.1) \quad \oint_C \frac{ds}{(z-t)^2} = \frac{l}{2a}$$

For  $z_0 \in \bar{D}$  we are justified in integrating. Under the integral sign, from which we have

$$\oint_C \frac{ds}{(z-t)} = \frac{l}{2a} t + k,$$

where  $k$  is a constant of integration.

Let us define  $f(t)$  in  $D$  to be the integral  $\oint_C \frac{ds}{z-t}$ . For  $t$  in the neighborhood of infinity we may write

$$f(t) = - \oint \frac{ds}{t(1-\frac{z}{t})} = - \oint \frac{ds}{t} - \frac{1}{t^2} \int z ds \dots = - \frac{l}{t} + \dots$$

However, the function  $\frac{kt}{2a} + k - f(t)$  is one which vanishes identically for  $t \in \bar{D}$ . Let us denote this function by  $q(t)$ . We wish to examine the mapping properties of  $q(t)$  for  $t \in D$ .

We note that for large  $t$ ,  $q(t)$  must have an expansion of the form

$$\frac{kt}{2a} + k + \frac{f}{t} \dots$$

and to examine its mapping properties we must evaluate

$$q(z_2) = \lim_{t \rightarrow z_2} q(t) \Big|_{t \in D}$$

where  $z_2$  represents any point on the boundary  $C$ . Since  $q(t)$  vanishes for  $z \in \bar{D}$ , we have

$$q(z_2) = \lim_{t \rightarrow z_2} q(t) \Big|_{t \in D} - \lim_{t \rightarrow z_2} q(t) \Big|_{t \in \bar{D}}$$

Let us now evaluate the limit

$$\lim_{t \rightarrow z_2} f(t) \Big|_{t \in D} - \lim_{t \rightarrow z_2} f(t) \Big|_{t \in \bar{D}} = f(z_2)$$

We construct the integral

$$\int_{t \in D}^{t_2} \oint_C \frac{ds dt}{(z-t)^2} - \int_{t \in \bar{D}}^{w_2} \oint_C \frac{ds dt}{(z-t)^2}$$

where  $t_1$  and  $t_2$  are points of  $D$  and  $w_1$  and  $w_2$  are points of  $\bar{D}$ . On integrating by parts we have

$$\int_{t_1}^{t_2} \oint_C \frac{dz}{z-t} dt - \int_{w_1}^{w_2} \oint_C \frac{dz}{z-t} dt$$

and passing to the limit  $t_2 \rightarrow z_2$ ,  $t_1 \rightarrow z_1$ ,  $w_2 \rightarrow z_2$ ,  $w_1 \rightarrow z_1$ , a formal process which has been justified in Section 5, we obtain

$$\begin{aligned} \lim_C \left[ \int_{t_1}^{t_2} \oint_{z \in C} \frac{dz dt}{z-t} - \int_{w_1}^{w_2} \oint_{z \in C} \frac{dz dt}{(z-t)} \right] &= \oint_{z \in C} \oint_{t \in \Gamma} \frac{dz dt}{z-t} = -2\pi i \int_{z_1}^{z_2} dz \\ &= -2\pi i z_2 + k_1 \end{aligned}$$

where  $k_1$  is a constant of integration, and  $\Gamma$  is a closed contour, enclosing the arc  $\widehat{z_1 z_2}$  on  $C$  and intersecting  $C$  at  $z_1$  and  $z_2$ . Consequently, we have for  $z \in D$

$$f(z_2) = \frac{\ell z_2}{2a} + k_2 - 2\pi i z_2, \quad ,$$

where  $k_2$  is a constant. We obtain, finally,

$$(6.2) \quad q(z_2) = 2\pi i z_2 + k, \quad ,$$

where  $k$  is another constant.

We see from (6.2) and the convexity of  $C$  that  $C$  is mapped onto a circle which is traced once in a counter-clockwise direction as  $C$  is traced in a clockwise direction. In view of the form of the expansion of  $q(z)$  for  $z \in D$ , we note that the point at infinity in the  $z$ -plane corresponds to the point at infinity in the  $q$ -plane; hence there are points of  $D$  corresponding to points outside the unit circle in the  $q$ -plane. But because of the reversal of sense in which the image of  $C$  is traced in the  $q$ -plane, there are also points of  $D$  corresponding to the points inside the circle of radius  $2\pi$  in the  $q$ -plane. Thus by the mapping  $q(z)$ ,  $D$  is mapped onto the entire  $q$ -plane, save for the set of points on the circle corresponding to  $C$ . We note that  $q(z)$  has one pole of the first order in  $D$ , located at infinity. The difference between the number of poles of  $q(z)$  in  $D$  and the number of times that  $q(z)$  assumes a certain value  $w_0$  (denoted by  $N_{w_0}$ ) for  $z \in D$  is given by the integral

$$\frac{1}{2\pi i} \oint_{-C} \frac{q'(z) dz}{q(z) - w_0} = N_{\infty} - N_{w_0} = 1 - N_{w_0}, \quad ,$$

where the integration is taken around  $C$  in a clockwise sense. But this is equal to the integral

$$\frac{1}{2\pi i} \oint \frac{dq}{q-w_0}$$

where  $C_q$  is the image of  $C$  in the  $q$ -plane and the integration is in the counter-clockwise sense. The integral is equal to 1 if  $w_0$  is in the region enclosed by  $C_q$  and zero otherwise. Hence,  $q$  assumes each value inside the circle  $C_2$  twice and each value in the region exterior to this circle only once. Hence,  $D$  is mapped onto the  $q$ -plane so that the exterior region to  $C_q$  is covered once and the interior region is covered twice. With  $\zeta$  representing the mapping of  $D$  onto the exterior of the unit circle, the image of  $D$  in the  $\zeta$ -plane is mapped onto the region bounded by  $C_q$  in the  $q$ -plane so that  $q$  has two zeros at the designated points  $z=a$  and  $z=b$ .  $\log \frac{|q(z)|}{\lambda} - q(z, \infty)$  is a harmonic function which vanishes on the boundary  $C$ , where the constant  $\lambda$  is given a value such that the logarithmic terms cancel at infinity. This harmonic function, however, has negative logarithmic poles at the zeros of  $q(z)$ . However,  $g(z,a)+g(z,b)$  is a harmonic function having the same poles and boundary values. Hence, in view of the uniqueness of a solution to the Dirichlet problem, we must have

$$\log \frac{|q(z)|}{\lambda} - g(z, \infty) = -[g(z,a) + g(z,b)]$$

Letting  $A$  and  $B$  represent the respective images of  $a$  and  $b$  in the  $\zeta$ -plane, we have

$$\log |q(z)| = \log \lambda + \log |\zeta| - \log \left| \frac{1-\bar{A}\zeta}{\zeta-A} \right| - \log \left| \frac{1-\bar{B}\zeta}{\zeta-B} \right|$$

$$(6.3) \quad q(\zeta) = \frac{\lambda \zeta (\zeta-A)(\zeta-B) e^{iH}}{(1-\bar{A}\zeta)(1-\bar{B}\zeta)}$$

where  $H$  is a constant of rotation. We note that  $q'(\zeta)$  has the form

$$q'(\zeta) = \frac{P_4(\zeta)}{(1-\bar{A}\zeta)^2(1-\bar{B}\zeta)^2}$$

where  $P_4(\zeta)$  is a 4<sup>th</sup> degree polynomial in  $\zeta$ .

Let us now exhibit a relation between  $q$  and  $w$ . Remembering the expansion

$$q(t) = \frac{\ell}{2a} + k + \frac{\ell}{t} \dots,$$

we have

$$q'(t) = \frac{\ell}{2a} \left(1 - \frac{2a}{t^2} \dots\right)$$

$$\begin{aligned} \sqrt{q'(t)} &= \sqrt{\frac{\ell}{2a}} \left(1 - \frac{2a}{t^2} \dots\right)^{1/2} \\ &= \sqrt{\frac{\ell}{2a}} \left(1 - \frac{a}{t^2} \dots\right) \end{aligned}$$

Integration with respect to  $t$  yields

$$\int_{z_0}^z \sqrt{q'(t)} dt = \sqrt{\frac{\ell}{2a}} \left(z + \frac{a}{z} \dots\right) = r(z)$$

For  $z \in C$  we obtain by differentiating (6.2) with respect to  $s$

$$q'(z)z = 2\pi \frac{d}{ds} (iz) = 2kz,$$

and it follows that

$$\sqrt{q'(z)} = \sqrt{2\pi k(z)} \frac{1}{z}$$

Thus  $r = \int_{z_0}^z \sqrt{q'(t)} dt$  is a real quantity for  $z \in C$ . The same is also true for the velocity potential. But  $r(z) - \sqrt{\frac{\ell}{2a}} w(z)$  is regular at infinity. Its imaginary part vanishes on the boundary, and in view of the maximum principle for harmonic functions the imaginary part of  $r(z) - \sqrt{\frac{\ell}{2a}} w(z)$  must vanish everywhere. From the Cauchy-Riemann equations the real part of  $r(z) - \sqrt{\frac{\ell}{2a}} w(z)$  must be a constant. We have the result

$$(6.4) \quad r'(z) = \sqrt{q'(z)} = \sqrt{\frac{\ell}{2a}} w'(z) ;$$

this shows, incidentally, that  $\alpha = a$ . Thus

$$q'(z) = \frac{l}{2a} w'(z)^2 ;$$

$$\frac{dq}{d\zeta} \frac{d\zeta}{dz} = \frac{l}{2a} \left(\frac{dw}{d\zeta}\right)^2 \left(\frac{d\zeta}{dz}\right)^2 ;$$

$$\left(\frac{dz}{d\zeta}\right) = \frac{l}{2a} \left(\frac{dw}{d\zeta}\right)^2 / \frac{dq}{d\zeta}$$

Since

$$z = \zeta + \frac{a_1}{\zeta} \dots ,$$

and

$$w = z + \frac{a}{z} \dots ,$$

and because of the mapping properties of  $w$ , we have

$$w = \zeta + \frac{1}{\zeta}$$

and

$$\frac{dz}{d\zeta} = \frac{l}{2a} \frac{(1 - \frac{1}{\zeta^2})}{q'(\zeta)} = \frac{l}{2a} \frac{(\zeta^2 - 1)^2}{\zeta^4} \frac{P_4(\zeta)}{(1 - \bar{A}\zeta)^2(1 - \bar{B}\zeta)^2}$$

If  $\frac{dz}{d\zeta}$  were to vanish at a point on  $C$ , there would be a cusp on  $C$  at such a point, thus contradicting the results we have established concerning the convexity of  $C$ . Thus  $\frac{dz}{d\zeta}$  has no zeros on  $C$ , so that  $P_4(\zeta)$  must take the form  $\Lambda(\zeta^2 - 1)^2$ , where  $\Lambda$  is a complex constant.

Thus

$$\begin{aligned} \frac{dz}{d\zeta} &= \Lambda_1 \frac{(1 - \bar{A}\zeta)^2(1 - \bar{B}\zeta)^2}{\zeta^4} \\ &= \Lambda_1 \frac{1 - (\bar{A} + \bar{B})\zeta + \zeta^2(\bar{A}^2 + \Lambda\bar{A}\bar{B} + B^2) - (\bar{A}^2\bar{B} + B^2\bar{A})\zeta^3 + \bar{A}^2\bar{B}^2\zeta^2}{\zeta^4} \end{aligned}$$

where  $\Lambda_1$  is also complex constant. But

$$z = \zeta + \frac{a_1}{\zeta} \dots ,$$

and

$$\frac{dz}{d\zeta} = 1 - \frac{a_1}{\zeta^2} \dots ,$$

and from this expression we conclude that the term in  $\frac{1}{\zeta}$  is absent so that

$$\bar{A}^2\bar{B} + \bar{B}\bar{A}^2 = 0 ,$$

or  $\bar{A} = -\bar{B}$  and consequently  $A = -B$ . Thus (6.3) has the form

$$(6.5) \quad q(\zeta) = \lambda e^{1H} \frac{\zeta(\zeta^2 - A^2)}{1 - \bar{A}^2 \zeta^2} ;$$

differentiation yields

$$q'(\zeta) = \frac{\lambda e^{1H} [-A^2 + \zeta^2(3 - A^2 \bar{A}^2) - \bar{A}^2 \zeta^4]}{(1 - \bar{A}^2 \zeta^2)^2} ,$$

and because of the form that the numerator must invariably have, we are led to the result

$$-\bar{A}^2 = -A^2 ,$$

so that  $q'(\zeta)$  takes the form

$$q'(\zeta) = \frac{\lambda e^{1H} [-A^2 + \zeta^2(3 - A^4) - A^2 \zeta^4]}{(1 - A^2 \zeta^2)^2} .$$

Also, because of the required form of the numerator, we have

$$2A^2 = 3 - A^4 ,$$

whose solution is

$$A^2 = -3 .$$

(The other solution  $A^2 = 1$  is absurd.) Thus

$$\frac{dz}{d\zeta} = \lambda_2 \frac{(1 + 3\zeta^2)^2}{\zeta^4} ,$$

where  $\lambda_2$  is a complex constant which, because of the form of the expansion of  $z$  about infinity, must be equal to  $1/9$ . We conclude that

$$(6.6) \quad \frac{dz}{d\zeta} = \frac{1}{9\zeta^4} (1 + 6\zeta^2 + 9\zeta^4) = 1 + \frac{2}{3\zeta^2} + \frac{1}{9\zeta^4} .$$

Integration and translation yields, finally,

$$(6.7) \quad z = \zeta - \frac{2}{3\zeta} - \frac{1}{27\zeta^3} .$$

It is interesting also to calculate the length  $l$ , the constant  $a$ , and also the virtual mass  $M$  of the flow about  $C$ . To calculate  $l$ , we write

$$l = \int_C |dz| = \int_{|\zeta|=1} \left| \frac{dz}{d\zeta} \right| |d\zeta|$$

Since

$$\frac{dz}{d\zeta} = \frac{1}{9\zeta^4} (1+3\zeta^2)^2$$

$$\left| \frac{dz}{d\zeta} \right| = \sqrt{\frac{dz}{d\zeta} \frac{d\bar{z}}{d\bar{\zeta}}} = \frac{(1+3\zeta^2)(1+3\bar{\zeta}^2)}{9|\zeta|^4};$$

$$(6.8) \quad l = \frac{1}{9} \int_0^{2\pi} [1 + 3e^{2i\theta} + 3e^{-2i\theta} + 9] d\theta = \frac{20\pi}{9}$$

To calculate  $a$  we write

$$(6.9) \quad a = \frac{1}{2\pi i} \oint_C w dz = \frac{1}{2\pi i} \oint_{|\zeta|=1} \left( \zeta + \frac{1}{\zeta} \right) \frac{dz}{d\zeta} d\zeta$$

$$= \frac{1}{2\pi i} \oint_{|\zeta|=1} \left( \zeta + \frac{1}{\zeta} \right) \left( 1 + \frac{2}{3\zeta^2} + \frac{1}{9\zeta^4} \right) d\zeta$$

$$= \frac{1}{2\pi i} \oint_{|\zeta|=1} \left[ \zeta + \frac{1}{\zeta} \left( 1 + \frac{2}{3} \right) \dots \right] d\zeta = \frac{5}{3}$$

from which we conclude

$$(6.10) \quad \frac{l^2}{a} = \frac{80\pi^2}{27}$$

To calculate the area of the curve  $C$  we evaluate

$$(6.11) \quad A = \frac{1}{2} \operatorname{Re} i \int_C \bar{z} dz = \frac{1}{2} \operatorname{Re} i \int_{|\zeta|=1} \left( \bar{\zeta} - \frac{2}{3\bar{\zeta}} - \frac{1}{27\bar{\zeta}^3} \right) \frac{(1+3\zeta^2)^2}{9\zeta^4} d\zeta$$

$$= \frac{1}{2} \frac{\operatorname{Re} i}{243} \int_{|\zeta|=1} (27\bar{\zeta}^4 - 18\bar{\zeta}^3 - \bar{\zeta})(9\zeta^4 + 6\zeta^2 + 1) d\zeta$$

$$= \frac{1}{486} \operatorname{Re} i \int_0^{2\pi} (24d - 168\zeta - 109\bar{\zeta} + 162\zeta^2 - 9\zeta^3 - 18\bar{\zeta}^3 + 27\bar{\zeta}^4) i e^{i\theta} d\theta$$

$$= 109\pi/243$$

In view of the expression

$$a = \frac{1}{2\pi} (M + A) ,$$

we obtain, finally,

$$(6.12) \quad M = 2\pi a - A = \pi \left( \frac{10}{3} - \frac{109}{243} \right) = \frac{711\pi}{243} .$$

We summarize:

**Theorem 6.1.** The family of curves  $C^*$  satisfies the inequality

$$\frac{\ell^{*2}}{a} \geq \frac{80\pi^2}{27} ,$$

where equality is obtained under the normalization  $R=1$  by the transformation

$$(6.7) \quad z = \zeta - \frac{2}{3\zeta} - \frac{1}{27\zeta^3} .$$

Under this normalization we have

$$(6.8) \quad \ell = \frac{20\pi}{9} ,$$

$$(6.9) \quad a = 5/3 .$$

A graph of the extremal curve is shown in Fig. 5. The curve is flattened along the real axis and elongated in the vertical direction. One may observe the rising bubbles in a newly uncapped bottle of champagne or the bubbles of boiling water in a silex coffee container and see that they are actually flattened in the direction of motion.

Having solved the problem in Section 5, and the problem at the beginning of this section, we wish to investigate the possibilities of solving very general free boundary problems. The problem of Section 5 in a more general form would be that of seeking a minimum length and shape for a free boundary when there is a fixed boundary of arbitrary shape in a class of curves, each member of which has the same outer mapping radius. One may also generalize the problem which was just solved in this section

by adding a fixed boundary in the formulation. One may conceive of this problem as a generalization of the following physical configuration: an airfoil, whose profile is partly rigid and consists in part of a canvas or rubber sheet, moves at a uniform velocity; this airfoil is inflated so that it has the same compensating pressure forces as in the first problem in this section. We shall again restrict ourselves to the case where the fixed boundary is a linear segment. We formulate the problem in the following manner:

To minimize the ratio  $\frac{\ell^*(L)^2}{a^*}$  in a family of closed rectifiable curves  $C^*$ , where  $a^*$  is fixed so that a point in a corresponding region  $D^*$  is given by the relation

$$z^* = \zeta + \frac{a^*}{\zeta} + \dots$$

and where a linear segment fixed symmetrically along the real axis belongs to  $\bar{D}^*$ .

$L$ ,  $\ell(L)$ , and  $\zeta$  have the same meaning as in the previous problems. We generate a one-parameter family of extremal curves by varying the length of the fixed segment. We assume that the free stream velocity is parallel to the x-axis. We may establish the existence, convexity, analyticity, etc., of the free boundary and also the fact that the fixed boundary must consist entirely of the linear segment, in a manner very similar to the one in which we established these properties in Section 5. In each of our previous problems we were led to some kind of condition regarding the symmetry of our extremal domain. It can be shown that the extremal curve for this extended problem must be symmetric in the y-axis. We shall use this result in order to simplify our construction procedure. The technique which has played the leading role in the solution of our problems is that

of constructing the function which we designated as  $q$  and of examining its conformal mapping properties. This function must be defined and constructed in such a way that we can ascertain the form of the fixed and free boundaries. For the problem which has just been formulated we shall see that it is convenient to define

$$q = \frac{1}{2\pi i} \int_{z_0}^z \frac{z'}{z^2} dz$$

By the same calculations which we performed in the similar problem without the fixed boundary, it can be shown that this function differs on the free boundary by an additive constant from  $z$ , and by proper selection of  $z_0$  this additive constant can be made to vanish. Because of the integral form of the function, the fixed boundary is mapped by  $q$  onto a segment parallel to the imaginary axis. The extremal domain is mapped onto the  $q$ -plane so that  $q$ , as before, is double-valued inside the image of  $C$  and single-valued outside. Let us designate the angle which the image of the fixed boundary subtends with the center of the unit circle as  $2\beta$ . The negative and positive end-points of the fixed boundary correspond to  $e^{i\beta}$  and  $e^{-i\beta}$ , respectively. Because of the conditions of symmetry, we may assume that the image of the fixed boundary in the  $\zeta$ -plane occupies an arc of the unit circle with end-points at  $-ie^{-i\alpha}$  and  $-ie^{i\alpha}$ , where  $\alpha$  is another parameter which must be determined.

We may place ourselves in a more convenient situation by mapping the  $q$ -plane by the transformation

$$t = e^{-i\beta} \frac{q - e^{i\beta}}{q - e^{-i\beta}}$$

This maps the  $q$ -plane into a wedge such that the fixed boundary corresponds to the positive real axis and the free boundary corresponds to the ray oriented in the direction  $\pi - \beta$ . Each value of  $t$  whose argument is in  $(0, \pi - \beta)$  is attained by two values of  $z$  in  $D$  and each value outside this wedge is attained by one value of  $z$ . On the free boundary there are two symmetrically located stagnation points which, of course, must correspond to  $-1$  and  $1$  in the  $\zeta$ -plane. (For certain values of the length of the fixed segment these points might lie on the fixed boundary, and the argument which follows would have to be modified; we shall treat only the case where they are on the free boundary. By making the fixed boundary very small we would have a configuration very nearly like the solution to the problem without a fixed boundary, so it is reasonable to expect that there would be lengths of the fixed boundary in some neighborhood of zero where the stagnation points would be on the free boundary.)

The zero streamline is mapped by  $q$  onto a curve, which extends to infinity, joins the circular arc portion of the image of  $C$  in the  $q$ -plane, and is symmetric in the  $x$ -axis. The portion of  $D$  which consists of the region  $D_1$ , located above the zero streamline, is mapped by  $q$  onto the region to the right of the curve corresponding to the zero streamline and the arc of the unit circle corresponding to the arc of the free boundary extending between the stagnation points. The remaining part of  $D$ ,  $D_2$ , is mapped onto the region at the left of the image of the zero streamline and the image of the part of the boundary consisting of the complement of the arc joining the stagnation points (See Fig. 6).

On applying the mapping function  $t$ , the stagnation points map into two points on the real axis. Let us designate these points by  $P_1$  and  $P_2$ . The

arc of the free boundary which lies above these stagnation points is mapped by  $t$  into the real segment  $(P_1, P_2)$ . The remaining part of the boundary consists of the remaining part of the wedge. The zero streamline is mapped by  $t$  into a contour terminating at  $P_1$  and  $P_2$  and extending in the lower half-plane. Let us designate the closed curve in the  $t$ -plane consisting of the image of  $\psi=0$  and the segment  $(P_1, P_2)$  by the symbol  $\Gamma$ . The domain  $D$  is mapped in a one-to-one manner onto the following two-sheeted Riemann surface: the first sheet, which corresponds to  $D_1$ , is the region exterior to  $\Gamma$ ; the second sheet,  $S_2$ , corresponding to  $D_2$  is the region interior to the wedge, plus the segment  $(P_1, P_2)$  and the interior of  $\Gamma$ . The two sheets are joined along the image of the zero streamline. The Riemann surface is illustrated in Fig. 6.

Let us now seek a relation between  $\zeta$  and  $t$ . Because of the form of  $S_1$ , it becomes apparent that there must be two points exterior to the unit circle in the  $\zeta$ -plane which correspond to the origin and infinity in the  $t$ -plane. In view of the symmetry we may take the liberty of designating these points by  $a$  and  $\bar{a}$ . We note that a general analytic function which maps the unit circle onto a slit along the real axis is infinite at  $a$  and vanishes at  $\bar{a}$  is the function

$$f(\zeta, a) = e^{i\phi} \frac{\zeta - \bar{a}}{1 - \bar{a}} + e^{-i\phi} \frac{1 - \bar{a}\zeta}{\zeta - a} - e^{i\phi} \frac{a + \bar{a}}{1 + a^2} - e^{-i\phi} \frac{1 + \bar{a}^2}{a + \bar{a}}$$

where  $\phi = -\arg \frac{1}{1 + a^2}$ . The mapping from the  $\zeta$ -plane to the  $t$ -plane is one in which

$$\arg t(\zeta) = \pi - \beta, \text{ for } \zeta \text{ on the image of the free boundary;}$$

$$\arg t(\zeta) = 0, \text{ for } \zeta \text{ on the image of the fixed boundary;}$$

$$t(a) = \infty;$$

$$t(\bar{a}) = 0.$$

In view of the uniqueness of solutions of the Dirichlet problem, we must have

$$\arg t = \arg f(\zeta, a) + \frac{\pi - \beta}{\pi} \operatorname{Im} \log e^{-i\alpha} \frac{i\zeta - e^{i\alpha}}{i\zeta - e^{-i\alpha}},$$

since this function has the required argument boundary values and singularities.

By forming the conjugate of our harmonic function  $\arg t$ , and then adding it to  $i \arg t$ , we obtain,

$$(6.8) \quad \log t = \log f(\zeta, a) + k \log e^{-i\alpha} \frac{i\zeta - e^{i\alpha}}{i\zeta - e^{-i\alpha}};$$

$t$  must be the exponential of the function.  $k$  is the quantity  $1 - \frac{\beta}{\pi}$  as in Section 5. On solving for  $q$ , we obtain

$$(6.9) \quad q = e^{-i\beta} \frac{f(\zeta, a) e^{-ik\alpha} (i\zeta - e^{i\alpha})^k - e^{i\beta} (i\zeta - e^{-i\alpha})^k}{f(\zeta, a) e^{-ik\alpha} (i\zeta - e^{i\alpha})^k - e^{-i\beta} (i\zeta - e^{-i\alpha})^k}.$$

We have a relation between  $\zeta$  and  $q$  which involves the parameters  $\alpha$ ,  $\beta$ , and  $a$ . We may reduce our results so that one less parameter is involved by utilizing the fact that the term in  $1/\zeta$  must be absent when  $\frac{dz}{d\zeta}$  is expanded in powers of  $1/\zeta$ . In the problem which we solved at the beginning of this section we determined a parameter from the fact that convexity on the boundary implied that  $\frac{dz}{d\zeta}$  cannot vanish on the stagnation points. A generalization of this principle will lead to another relation among the parameters.

There are many ways in which these free boundary problems can be generalized. In addition to varying the constraints and the form of the fixed boundary, there are many unsolved free boundary problems in multiply-connected domains. It is generally expected that this mathematical theory

which has been motivated by hydrodynamical problems will be greatly expanded, and it is also hoped that the worker in applied hydrodynamics will also find the techniques in conformal mapping a useful tool in some of the problems which he may encounter.

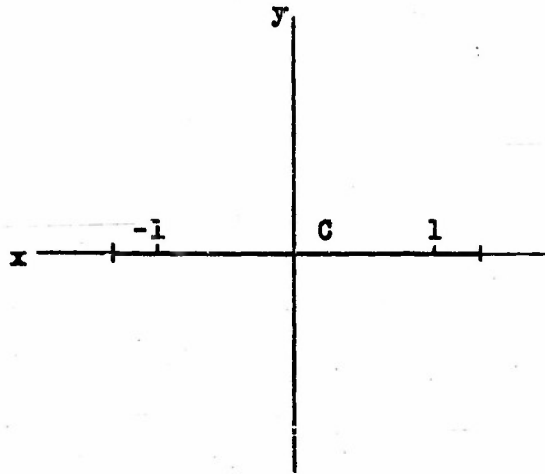


Fig. 1a

(Illustrates the possibility that the extremal curve might degenerate into a segment oriented along the real axis and containing the fixed boundary.)

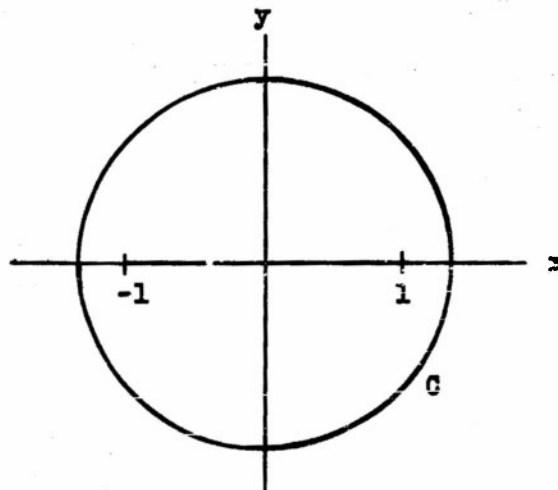


Fig. 1b

(Illustrates the possibility of having an extremal curve which completely encloses the fixed boundary. It is shown that if this case occurs, the curve must take the form of a circle.)

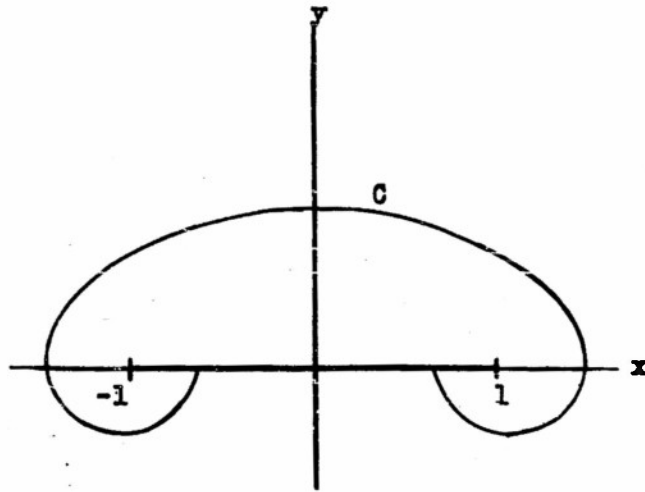


Fig. 1c

(Illustrates the possibility that the extremal curve  $C$  contains only a part of the fixed boundary.)

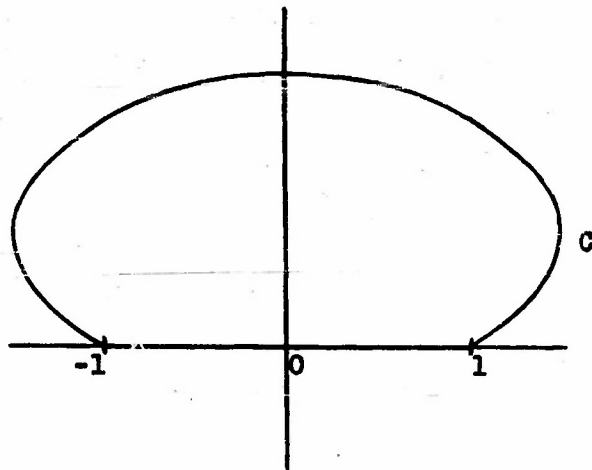


Fig. 1d

(Illustrates the remaining possibility that  $C$  contains the entire fixed boundary. It is shown that this is the only possible case.)

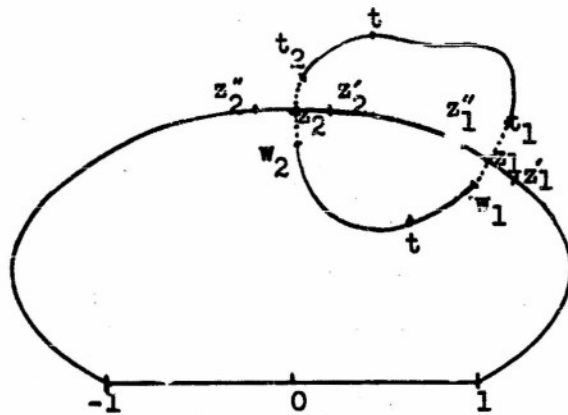


Fig. 2a

(Illustrating the manner in which the limit is reached.)

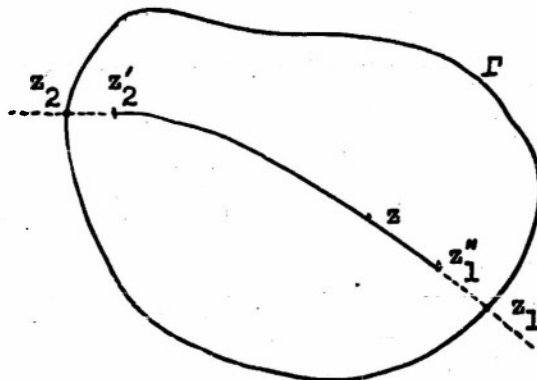


Fig. 2b

(Illustrates the passage to the limit in the evaluation of  $I_1$ .)

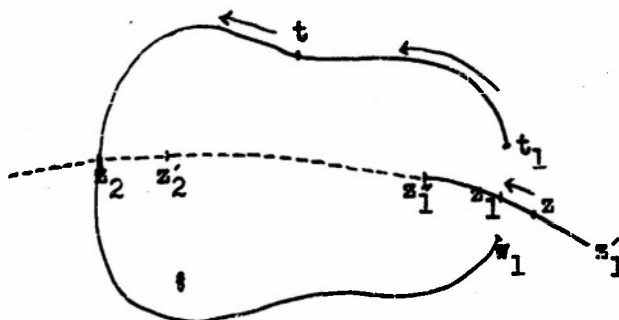


Fig. 2c

(Illustrates the paths of integrations used to evaluate  $I_3$ .)

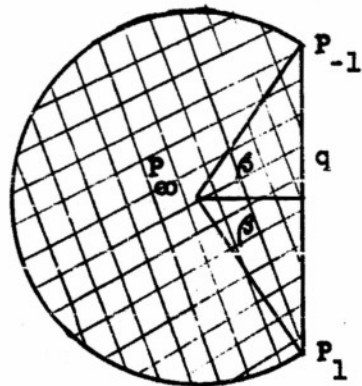


Fig. 3a

(The image of the extremal curve in the  $q$ -plane. The free boundary corresponds to the linear segment;  $P_1$  corresponds to 1 in the  $z$ -plane;  $P_{-1}$  corresponds to -1 in the  $z$ -plane.  $P_{\infty}$  corresponds to  $\infty$  in the  $z$ -plane. The image of the region  $D$  is the cross-hatched region.)

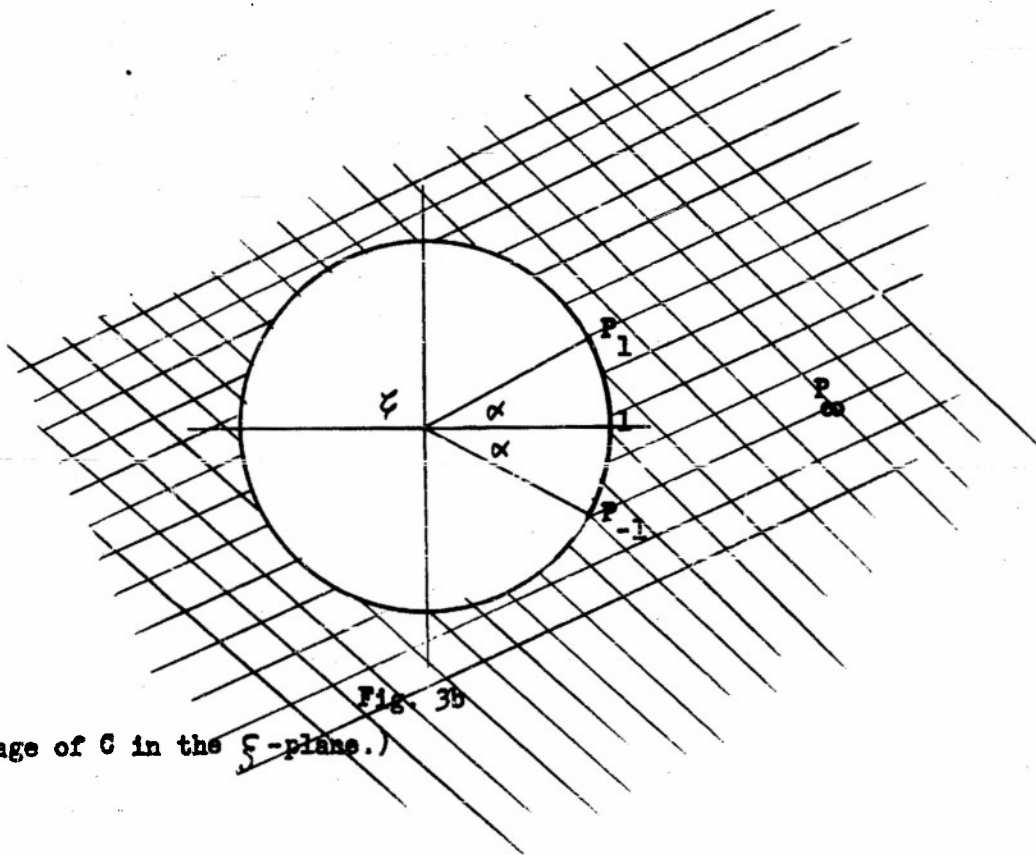


Fig. 3b

(Shows the image of  $C$  in the  $\xi$ -plane.)

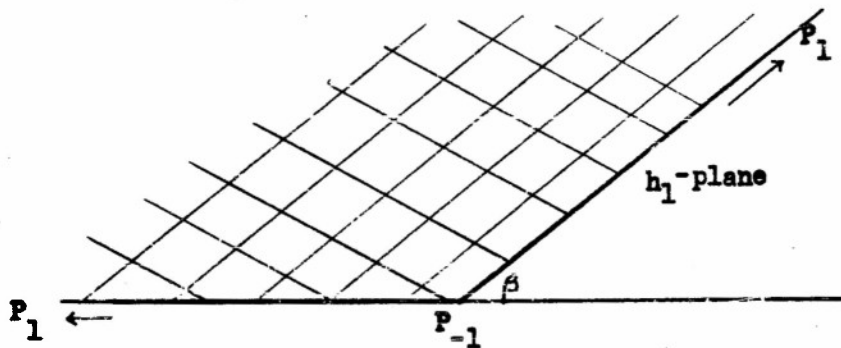


Fig. 4a

(The image of D under the mapping

$$h_1 = \frac{q - e^{-i\beta}}{q - e^{-i\beta}} .)$$

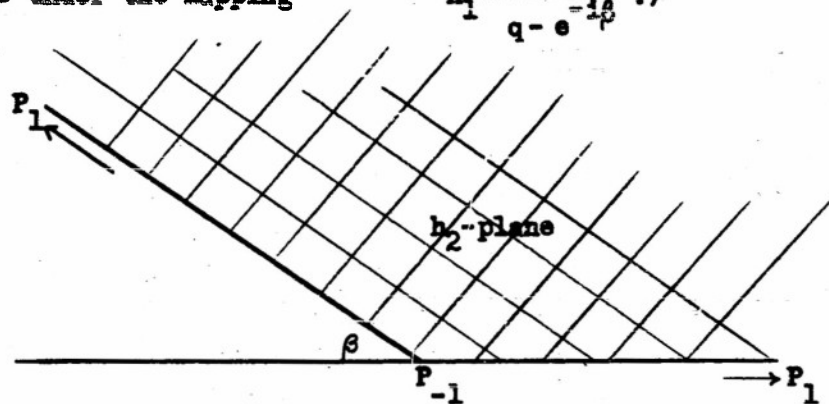


Fig. 4b

(Image of D for  $h_2 = e^{-i\beta} h_1$  .)

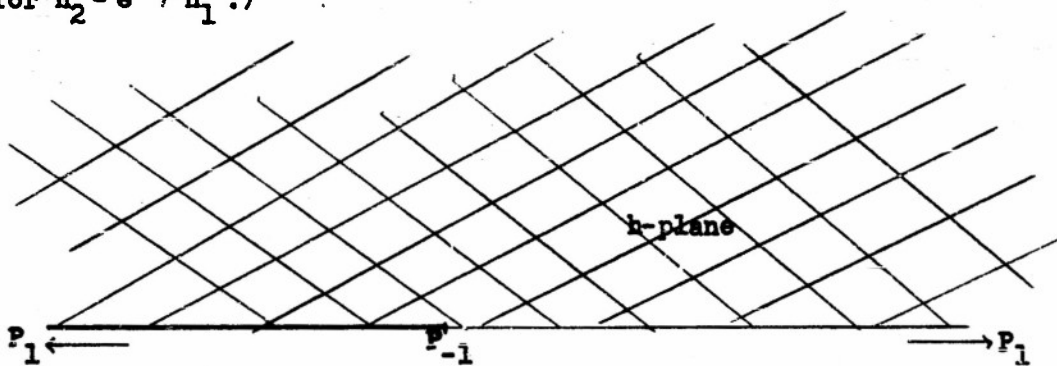
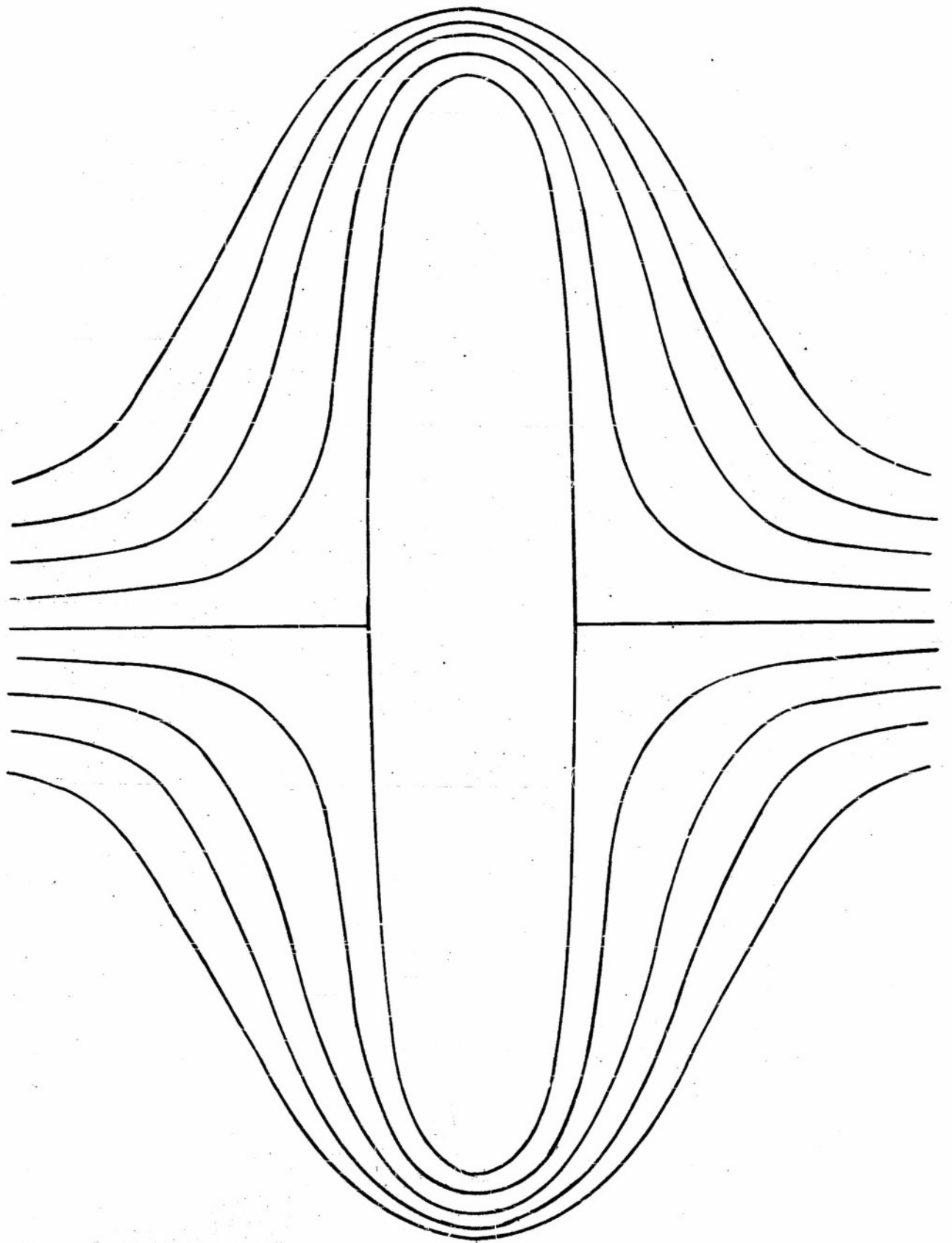


Fig. 4c

(Image of D in the plane  $h = h_2^{1/\alpha}$  .)

Fig. 5



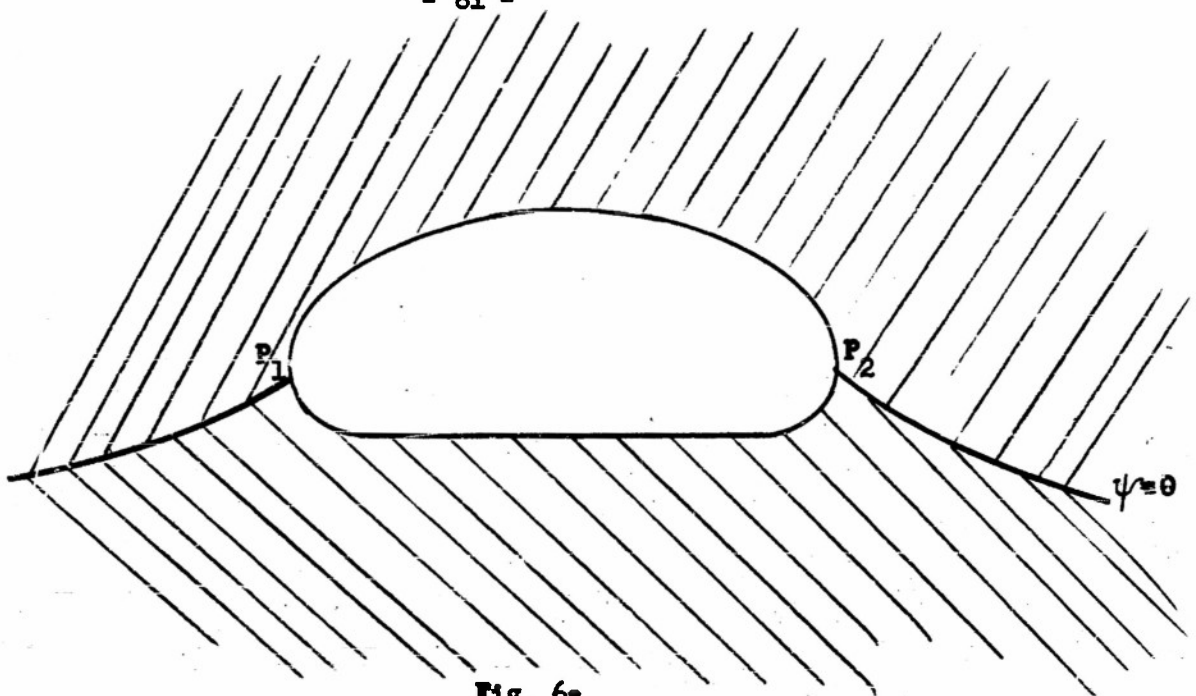
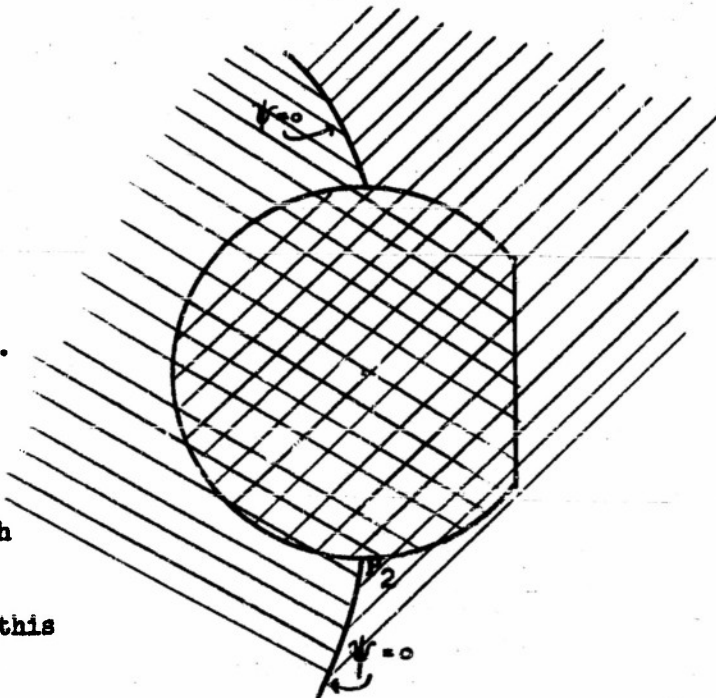


Fig. 6a

(The expected form of the extremal domain  $D$ ,  $D_1$ , and its images are designated with parallel lines leaning toward the right.  $D_2$  and its images are designated with lines which lean toward the left.)



(The image of  $D$  in the  $q$ -plane. Note that the inside of the image of  $C$  is drawn with both left and right leaning axes; this part is doubly covered.)

Fig. 6b

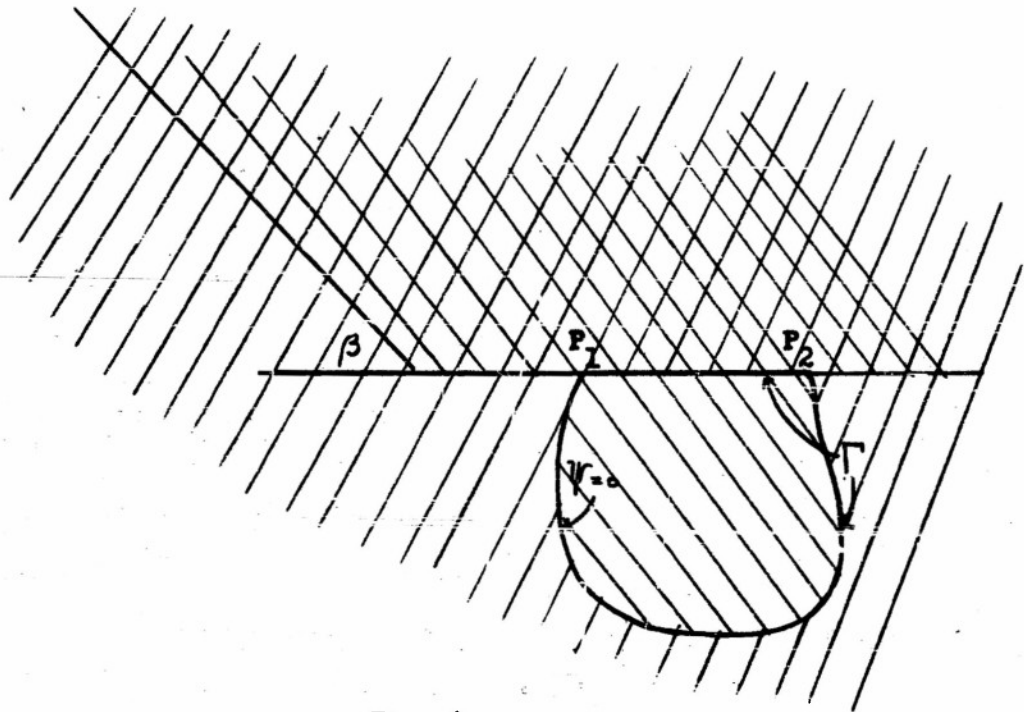


Fig. 6c

(The two-sheeted Riemann surface in the  $t$ -plane where  $S_1$  consists of region containing the lines leaning to the right and  $S_2$ , the lines leaning to the left.)

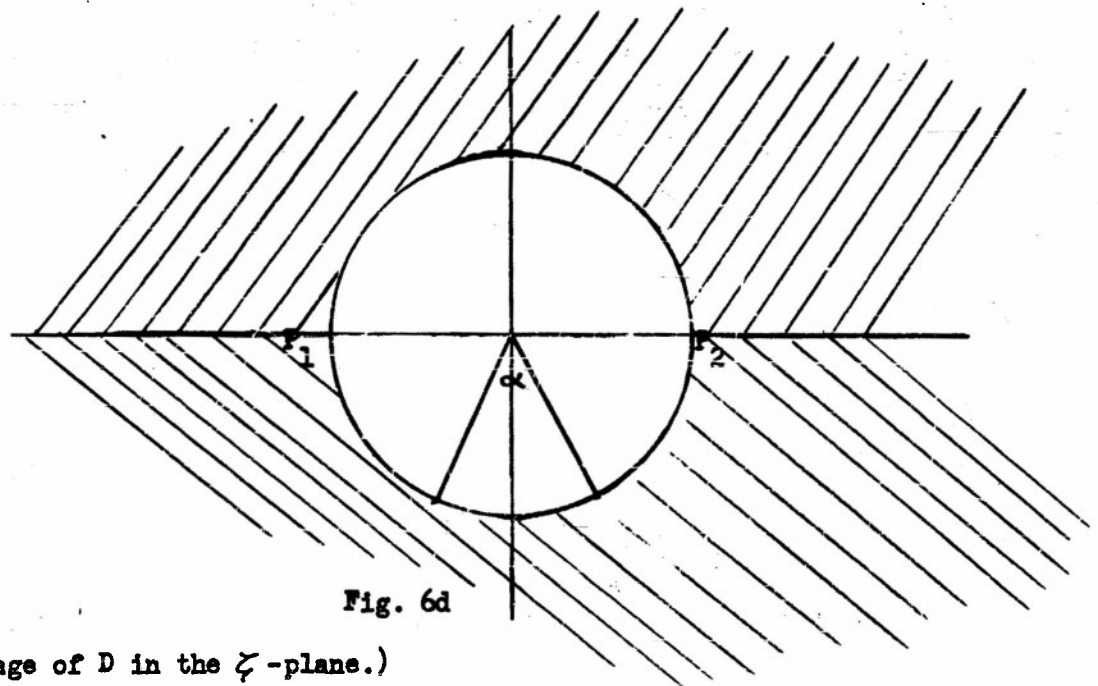


Fig. 6d

(The image of  $D$  in the  $\zeta$ -plane.)

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