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P. R. GARABEDIAN

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APPLIED MATHEMATICS AND STATISTICS LABORATORY  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA

# APPLICATIONS OF ANALYTIC CONTINUATION TO THE SOLUTION OF BOUNDARY VALUE PROBLEMS

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P. R. Garabedian

A better analysis of special solutions of particular boundary value problems is of considerable importance in the theory of partial differential equations, especially for equations of fourth order, because very few explicit solutions are known in this case. The present work was motivated by interest in the non-linear Navier-Stokes equation

$$\psi_{zz\bar{z}\bar{z}} = \psi_z \psi_{z\bar{z}\bar{z}} - \psi_{\bar{z}} \psi_{zz\bar{z}}$$

governing the steady two-dimensional flow of an incompressible viscous fluid, but the results have application in other fields, such as elasticity. We develop methods based on analytic continuation which yield for certain special plane domains the solution of the basic boundary value problems for the equations  $\Delta\phi = \phi$  and  $\Delta\Delta\psi = 0$ . The methods, which consist either in the introduction of independent complex variables  $z = x + iy$  and  $\bar{z}^* = x - iy$  or in a suitable exploitation of the analytic function  $G(z)$  such that the equation  $\bar{z} = G(z)$  represents the boundary of the domain, are of themselves interesting, since they provide formulas for the reflection of solutions of these partial differential equations across analytic boundaries. We are thus able, for example, to study the solution of the Oseen equation  $\Delta\Delta\psi = \Delta\psi_x$  near the end of a slit.

Let  $P(z, \bar{z})$  be a fixed real analytic function of  $x$  and  $y$ , and let  $\phi(z, \bar{z})$  be a real solution of the linear elliptic partial differential equation

$$(1) \quad \varphi_{z\bar{z}}(z, \bar{z}) = P(z, \bar{z}) \varphi(z, \bar{z})$$

in a plane domain  $D$ , where

$$\varphi_{z\bar{z}} = \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}, \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We denote by  $A(z, z^*; t, t^*)$  the Riemann function of the corresponding hyperbolic equation

$$(2) \quad \frac{\partial^2 \varphi}{\partial z \partial z^*} = P(z, z^*) \varphi$$

in two independent complex variables  $z$  and  $z^*$ . In other words,  $A$  is, as a function of  $z$  and  $z^*$ , a solution of (2) which satisfies the boundary conditions  $A(z, t^*; t, t^*) = A(t, z^*; t, t^*) = 1$ . If we continue the solution  $\varphi(z, \bar{z})$  of (1) analytically into the domain of complex values of  $x$  and  $y$ , it becomes a solution  $\varphi(z, z^*)$  of (2). It is clear by Stokes' theorem that because  $\varphi$  and  $A$  solve (2), the line integral

$$(3) \quad \int \left\{ \varphi \frac{\partial A}{\partial z} dz + A \frac{\partial \varphi}{\partial z^*} dz^* \right\}$$

is independent of the path of integration between given limits. We make use of this result to derive a dual pair of useful formulas expressing  $\varphi$  in terms of  $A$ .

The required formulas are

$$(4) \quad \varphi(t, t^*) = \varphi(t, \bar{t}) - \int_{\bar{t}^*}^t \left\{ \varphi(z, \bar{z}) \frac{\partial A(z, \bar{z}; t, t^*)}{\partial z} dz + A(z, \bar{z}; t, t^*) \frac{\partial \varphi(z, \bar{z})}{\partial \bar{z}} d\bar{z} \right\},$$

where the path of integration runs from  $\bar{t}^*$  to  $t$  in the real plane domain  $D$ , and [1]

$$(5) \quad \varphi(t, \bar{t}) = \varphi(\bar{t}^*, t^*) A(\bar{t}^*, t^*; t, \bar{t}) + \int_{\bar{t}^*}^t A(z, t^*; t, \bar{t}) \frac{\partial \varphi(z, t^*)}{\partial z} dz + \int_{t^*}^{\bar{t}^*} A(\bar{t}^*, z^*; t, \bar{t}) \frac{\partial \varphi(\bar{t}^*, z^*)}{\partial z^*} dz^*.$$

The first formula is derived by evaluating the integral (3) over a closed circuit consisting of a curve in the real plane  $z^* = \bar{z}$  from  $\bar{t}^*$  to  $t$ , plus a curve in the characteristic plane  $z = t$  from  $z^* = \bar{t}$  to  $z^* = t^*$ , plus a curve in the characteristic plane  $z^* = t^*$  from  $z = t$  to  $z = \bar{t}^*$ , for the integrals in the characteristic planes simplify by virtue of the boundary conditions satisfied by  $A$ . In a similar way, formula (5) follows by evaluation of (3) over a closed path made up of the four arcs which join, respectively, the points  $z^* = \bar{t}$  and  $z^* = t^*$  in the plane  $z = t$ , the points  $z = t$  and  $z = \bar{t}^*$  in the plane  $z^* = t^*$ , the points  $z^* = t^*$  and  $z^* = \bar{t}$  in the plane  $z = \bar{t}^*$ , and the points  $z = \bar{t}^*$  and  $z = t$  in the plane  $z^* = \bar{t}$ . Finally, since  $\varphi(z, \bar{z})$  and  $A(z, \bar{z}; t, \bar{t})$  are real, (5) reduces to

$$(6) \quad \varphi(t, \bar{t}) = \varphi(\bar{t}^*, t^*) A(\bar{t}^*, t^*; t, \bar{t}) + 2 \operatorname{Re} \left\{ \int_{\bar{t}^*}^t A(z, t^*; t, \bar{t}) \frac{\partial \varphi(z, t^*)}{\partial z} dz \right\}.$$

As a first application of the formulas (4) and (6) we consider the equation

$$(7) \quad \Delta \varphi = \varphi$$

in the wedge  $-\alpha < \arg z < \alpha$ . In this case  $P = 1/4$  and the Riemann function  $A$  has the form

$$A(z, z^*; t, t^*) = J[1 \sqrt{(z-t)(z^*-t^*)}] ,$$

where  $J$  is the Bessel function of order zero. We set  $t^* = 0$  and we take as path of integration the ray from the origin to  $z$ , whence (4) yields for the solution  $\varphi$  of (7)

$$(8) \quad \varphi(t, 0) = \varphi(t, \bar{t}) - \int_0^t \left\{ \varphi \frac{\partial}{\partial z} J[\sqrt{(t-z)\bar{z}}] dz + J[\sqrt{(t-z)\bar{z}}] \frac{\partial \varphi}{\partial \bar{z}} d\bar{z} \right\} .$$

The remarkable fact about this special case is that

$$(9) \quad \operatorname{Re}\{\varphi(t, 0)\} = \varphi(t, \bar{t}) - \int_0^t \left\{ \varphi \operatorname{Re}\left[\frac{\partial J}{\partial z} dz\right] + \frac{1}{2} J \frac{\partial \varphi}{\partial |z|} d|z| \right\},$$

since  $(t-z)\bar{z} > 0$  along our path of integration. Thus, along any ray through the origin,  $\operatorname{Re}\{\varphi(t, 0)\}$  is completely determined by  $\varphi(t, \bar{t})$  alone. Furthermore, for real arguments the Bessel function  $J$  is uniformly bounded, whereas  $\varphi(t, \bar{t})$  and its derivatives are required to vanish exponentially at infinity for solutions of boundary value problems in a wedge. Therefore, by (8),  $\varphi(t, 0)$  will be bounded for such solutions. It follows from this analysis that the first boundary value problem for (7) in a wedge

$-\alpha < \arg z < \alpha$  transforms by (9) into a Dirichlet problem for the harmonic function  $\operatorname{Re}\{\varphi(t, 0)\}$  in the same wedge. Thus  $\varphi(t, 0)$  can be found by standard procedures and the explicit solution  $\varphi(t, \bar{t})$  of (7) assuming the given boundary values can be obtained as an integral from the formula

$$(10) \quad \varphi(t, \bar{t}) = \varphi(0, 0)J(1|t|) + 2 \operatorname{Re} \int_0^t J[1\sqrt{(t-z)\bar{t}}] \frac{\partial \varphi(z, 0)}{\partial z} dz,$$

based on (6).

The general formula (6) for solutions  $\varphi$  of (1) in a domain  $D$  gives a convenient procedure [6] for continuing  $\varphi$  analytically across an analytic arc of the boundary of  $D$  along which  $\varphi$  satisfies, for example, the boundary condition  $\varphi = 0$ . We write the equation of the analytic arc in the form  $\bar{z} = G(z)$ , where  $G(z)$  is an analytic function. Such a representation can always be obtained by solving for  $\bar{z}$  the more usual equation of the arc. We suppose, without loss of generality, that the origin is the point on the boundary of  $D$  near which we wish to make the reflection. In  $D$ , near the origin, consider the analytic function

$$(11) \quad U(t) = \int_0^t A(z, 0; t, G(t)) \frac{\partial \varphi(z, 0)}{\partial z} dz$$

From (6) and the boundary condition  $\varphi = 0$ , we obtain along the curve  $\bar{z} = G(z)$  the identity

$$\begin{aligned} 0 &= \varphi(t, \bar{t}) = \varphi(0, 0)A(0, 0; t, \bar{t}) + 2 \operatorname{Re} \left\{ \int_0^t A(z, 0; t, \bar{t}) \frac{\partial \varphi(z, 0)}{\partial z} dz \right\} \\ &= 2 \operatorname{Re} \{ U(t) \} = U(t) + \bar{U}(\bar{t}) \end{aligned}$$

Hence the analytic function  $U(t)$  can be continued analytically across the curve  $\bar{z} = G(z)$  by the rule

$$U(t) = -\bar{U}(G(t))$$

since when  $t$  lies on one side of the curve and sufficiently near the origin, the point  $G(t)$  lies on the opposite side. We now derive from (11) the integral equation

$$(12) \quad \bar{U}(G(t)) + \int_0^t A(z, 0; t, G(t)) \frac{\partial \varphi(z, 0)}{\partial z} dz = 0$$

for the determination of the analytic function  $\varphi(t, 0)$  across the arc  $\bar{z} = G(z)$ . This equation is linear and of the Volterra type, so there is no difficulty in solving it by successive approximations. Thus the analytic continuation of  $\varphi(t, 0)$  is performed, and to obtain the continuation of the original solution  $\varphi(t, \bar{t})$  of (1) we have only to apply formula (6) again with  $t^* = 0$ .

The procedure just outlined can readily be extended to the case of fourth order elliptic equations. We illustrate this generalization with an application to the important special case of the Oseen equation

$$(13) \quad \Delta \Delta \psi = 2 \Delta \psi_x$$

This equation has a general solution

$$\psi = e^x \varphi + h$$

where  $\varphi$  is a solution of (7) and  $h$  is a harmonic function. By (10), the general solution can be written near the origin in the form

$$(14) \quad \psi = \operatorname{Re} \left\{ e^{(t+\bar{t})/2} \int_0^t J[i \sqrt{(t-z)\bar{t}}] g'(z) dz + C e^{(t+\bar{t})/2} J(1|t|) + f(t) \right\},$$

where  $C$  is an arbitrary constant and  $f(t)$  and  $g(t)$  are arbitrary analytic functions. We suppose that the solution  $\psi$  of (13) is defined in a region  $D$  whose boundary contains the analytic arc  $\bar{z} = G(z)$  passing through the origin, and along this arc we assume that the boundary conditions

$$(15) \quad \psi = \frac{\partial \psi}{\partial n} = 0$$

are fulfilled, where  $n$  denotes the inner normal of the arc. The problem is to continue  $\psi$  analytically across the arc  $\bar{z} = G(z)$ , and it is clearly sufficient by (14) to perform such a continuation on the analytic functions  $f(t)$  and  $g(t)$ .

We denote by  $s$  the arc length along the curve  $\bar{z} = G(z)$ , and we verify that

$$(16) \quad \frac{\partial z}{\partial n} = i \frac{\partial z}{\partial s} = i G'(z)^{-1/2}.$$

Replacing (15) by

$$e^{-x} \psi = \partial(e^{-x} \psi) / \partial n = 0$$

and substituting (14) into the result, we obtain

$$(17) \quad \operatorname{Re} \{V(t)\} = \operatorname{Re} \{W(t)\} = 0$$

along the curve  $\bar{z} = G(z)$ , where  $V$  and  $W$  are the analytic functions defined by

$$(18) \quad V(t) = \int_0^t J[i \sqrt{(t-z)G(t)}] g'(z) dz + C J(1 \sqrt{tG(t)}) + f(t) e^{-(t+G(t))/2},$$

$$\begin{aligned}
 (19) \quad W'(t)G'(t)^{-1/2} &= iG'(t)^{-1/2} \left\{ g'(t) + iCJ'(i\sqrt{tG(t)}) \frac{G(t)-tG'(t)}{2\sqrt{tG(t)}} \right. \\
 &\quad + i \int_0^t J'(i\sqrt{(t-z)G(t)}) \frac{G(t)-(t-z)G'(t)}{2\sqrt{(t-z)G(t)}} g'(z) dz \\
 &\quad \left. + e^{-(t+G(t))/2} \left[ f(t) \frac{G'(t)-1}{2} + f'(t) \right] \right\}
 \end{aligned}$$

in D. Therefore these functions can be reflected across  $\bar{z} = G(z)$  by the rule

$$V(t) = -\bar{V}(G(t)) \quad , \quad W(t) = -\bar{W}(G(t))$$

It follows then from (18) and (19) that  $f(t)$  and  $g(t)$  can be continued analytically across  $\bar{z} = G(z)$  as solutions of the system of Volterra integral equations

$$\begin{aligned}
 (20) \quad iG'(t)\bar{W}'(G(t)) &= g'(t) + e^{-(t+G(t))/2} f'(t) + f(t) e^{-(t+G(t))/2} \frac{G'(t)-1}{2} \\
 &\quad + i \int_0^t J'(i\sqrt{(t-z)G(t)}) \frac{G(t)-(t-z)G'(t)}{2\sqrt{(t-z)G(t)}} g'(z) dz \\
 &\quad + iCJ'(i\sqrt{tG(t)}) \frac{G(t)-tG'(t)}{2\sqrt{tG(t)}}
 \end{aligned}$$

$$\begin{aligned}
 (21) \quad \frac{d}{dt} [\bar{V}'(G(t)) + i\bar{W}'(G(t))] &= e^{-(t+G(t))/2} f'(t) - f(t) e^{-(t+G(t))/2} \frac{G'(t)+1}{2} \\
 &\quad - i \frac{d}{dt} \int_0^t \frac{J'(i\sqrt{(t-z)G(t)}) \sqrt{t-z}}{\sqrt{G(t)}} g'(z) dz - iC \frac{d}{dt} \frac{J'(i\sqrt{tG(t)}) \sqrt{t}}{\sqrt{G(t)}}
 \end{aligned}$$

Once the solution of (20) and (21) has been obtained by successive approximations, the continuation of the stream function  $\Psi$  can be found from the expression (14).

We can apply the rule just described for reflecting  $\Psi$  to the case where  $\Psi$  is defined in a neighborhood of the origin slit along the positive real axis, on which the boundary conditions (15) are imposed. Here  $G(z) = z$

and it is found that  $V$  and  $\bar{W}$  are regular, single-valued functions of  $\sqrt{t}$  near the origin, whence by (20), (21) the same is true for  $f$  and  $g$ . Therefore  $\Psi$  is a regular function of  $\sqrt{t}$  and  $\sqrt{\bar{t}}$ , and since  $\Psi_x$  and  $\Psi_y$  must vanish at the origin, we conclude that the skin friction  $\Delta\Psi$  behaves like  $1/\sqrt{x}$  along the positive real axis [5].

Our method of reflection is valid for the equation  $\Delta\Delta\Psi = \lambda^4\Psi$  of the vibrating clamped plate, also, and, indeed, many further, more involved applications could be carried out.

We turn our attention to the simplest fourth order equation, namely,

$$(22) \quad \Delta\Delta\Psi = 0 \quad ,$$

and we show that the above technique simplifies to such an extent that we are led to the explicit solution of specific boundary value problems. The general solution of the biharmonic equation (22) has the form

$$(23) \quad \Psi = \operatorname{Re} \left\{ \bar{z}f(z) + g(z) \right\} \quad ,$$

where  $f$  and  $g$  are arbitrary analytic functions of  $z$ . If we substitute this formula into the boundary conditions (15) along the arc  $\bar{z} = G(z)$  and take (16) into account, we find

$$(24) \quad \operatorname{Re} \left\{ G(z)f(z) + g(z) \right\} = 0 \quad ,$$

$$(25) \quad \operatorname{Re} \left\{ iG(z)f'(z)G'(z)^{-1/2} + ig'(z)G'(z)^{-1/2} - if(z)G'(z)^{1/2} \right\} = 0 \quad .$$

We define the two analytic functions

$$\Phi(z) = G(z)f(z) + g(z) \quad ,$$

$$\Psi(z) = i \int \left\{ G(z)f'(z) + \frac{1}{2} \left( \frac{1}{G'(z)} \right) - f(z)G'(z) \right\} dz \quad ,$$

and we note that (24) and (25) imply

$$(26) \quad \operatorname{Re} \left\{ \Phi(z) \right\} = \operatorname{Re} \left\{ \Psi(z) \right\} = 0$$

on the curve  $\bar{z} = G(z)$ . But also

$$f(z) = [\Phi'(z) + i \Psi'(z)] / (2G'(z)) \quad ,$$

$$g(z) = \Phi(z) - [\Phi'(z) + i \Psi'(z)] G(z) / (2G'(z)) \quad ,$$

whence the biharmonic function  $\Psi$  satisfying the boundary conditions (15) can be expressed in the form

$$(27) \quad \Psi = \operatorname{Re} \left\{ \Phi(z) + \frac{\bar{z} - G(z)}{2G'(z)} [\Phi'(z) + i \Psi'(z)] \right\}$$

in terms of the pair of analytic functions  $\Phi$  and  $\Psi$  satisfying the equivalent boundary conditions (26).

As in our earlier studies, the formula (27) gives a rule for continuing  $\Psi$  analytically across an arbitrary analytic arc  $\bar{z} = G(z)$ . Indeed, we have only to reflect  $\Phi$  and  $\Psi$  by the usual Schwarz principle, using (26). However, in the present situation, the reflection rule is so simple and elegant that for certain domains  $D$  it can be exploited in order to solve explicitly the first boundary value problem for (22). This can be done when  $D$  is bounded by a simple closed analytic curve  $\bar{z} = G(z)$  such that  $G(z)$  is single-valued in  $D$  and regular there except for a finite number of poles. The functions  $\Phi$  and  $\Psi$  defined by (27) are regular in  $D$  except for singularities at the poles of  $G$ , and on the boundary curve we find

$$(28) \quad \Psi = \operatorname{Re} \{ \Phi \} \quad ,$$

$$(29) \quad \int \frac{\partial \Psi}{\partial n} ds = \operatorname{Re} \{ \Psi \} \quad .$$

Hence the problem of determining a solution of (22) in  $D$  with prescribed values for  $\Psi$  and  $\partial \Psi / \partial n$  on the curve bounding  $D$  reduces to the problem of finding analytic functions  $\Phi$  and  $\Psi$  with given real parts on the boundary of  $D$  and with suitable singularities at the poles of  $G$ . The

solution therefore involves only the determination of a finite number of parameters associated with the singularities in such a way that  $\Phi' + i\Psi'$  vanishes at the zeros of  $G'$ . In particular, if  $G$  has only simple poles, then  $\Phi$  has only simple poles and  $\Psi' + i\Phi'$  has only simple poles.

As an example of this theory, we develop a representation for the biharmonic Green's function  $\Gamma(z, \zeta)$  of the exterior of the ellipse  $x^2 \text{ch}^{-2} \alpha + y^2 \text{sh}^{-2} \alpha = 1$ . The result should be compared with the earlier solution given by a related method [7,8]. We write the equation of the ellipse in the complex form

$$(z + \bar{z})^2 \text{ch}^{-2} \alpha - (z - \bar{z})^2 \text{sh}^{-2} \alpha = 4$$

and solve for  $\bar{z}$  as a function of  $z$  to obtain

$$(30) \quad G(z) = z \text{ch } 2\alpha - \sqrt{z^2 - 1} \text{sh } 2\alpha$$

Denote by  $p(z, \zeta)$  the analytic function of  $z$  whose real part is the Green's function for Laplace's equation in the exterior of the ellipse, with source point at  $z = \zeta$ . A fundamental solution of (22) is

$$\text{Re} \{ -|z - \zeta|^2 p(z, \zeta) \},$$

and we must add to this an expression of the type (27), adjusted so that the sum satisfies the homogeneous boundary conditions (15) on the ellipse. This gives on the ellipse

$$\text{Re} \{ \Phi \} = 0,$$

$$\text{Re} \{ \Psi \} = \text{Re} \left\{ i \int |z - \zeta|^2 p'(z, \zeta) dz \right\},$$

whence we have

$$(31) \quad \Phi = A(z^2 - G(z)^2),$$

$$(32) \quad \Psi = i \int (z - \zeta)(G(z) - \bar{\zeta}) p'(z, \zeta) dz + B(z - G(z)) + iC(z + G(z)) + Dp(z, \infty),$$

where the real parameters A, B, C, D are to be found from the condition that  $\Phi' + i\Psi'$  must vanish at the zeros  $\pm \operatorname{ch} 2\alpha$  of  $G'(z)$ . We thus obtain for A, B, C, D the equations

$$\begin{aligned} \pm 2A \operatorname{ch} 2\alpha \pm iDp'(\operatorname{ch} 2\alpha, \infty) + i(B + iC) \\ = (\operatorname{ch} 2\alpha \mp \zeta)(1 \mp \bar{\zeta})p'(\pm \operatorname{ch} 2\alpha, \zeta) \end{aligned} ,$$

and therefore

$$(33) \quad \Gamma(z, \zeta) = \operatorname{Re} \left\{ -|z - \zeta|^2 p(z, \zeta) + \Phi + \frac{\bar{z} - G(z)}{2G'(z)} [\Phi' + i\Psi'] \right\} ,$$

with G,  $\Phi$ , and  $\Psi$  given by (30), (31), and (32).

The above result, which is of some interest for the discussion of slow viscous flow around an ellipse, simplifies considerably in the limiting case when  $\alpha = 0$  and the ellipse degenerates into a slit from -1 to +1. Here we find, in fact, that the biharmonic Green's function is given by

$$(34) \quad \Gamma(z, \zeta) = \operatorname{Re} \left\{ -|z - \zeta|^2 p(z, \zeta) + iy(z - \zeta)(z - \bar{\zeta})p'(z, \zeta) \right\} ,$$

since  $G(z) = z$ . This same formula yields the biharmonic Green's function for the exterior of any finite number of slits along the real axis, provided that we interpret p to be that analytic function whose real part is the harmonic Green's function of the multiply-connected slit domain.

With our present method we can treat the first boundary value problem for (22) in domains bounded by slits along an analytic arc  $\bar{z} = G(z)$ . However, for the applications it is also desirable to study the behavior of biharmonic functions near a point of the boundary where two analytic arcs intersect at an arbitrary angle. In order to obtain a first analysis of problems of this type, we determine here the biharmonic Green's function  $\Gamma(z, \zeta)$  for an arbitrary crescent domain bounded by two circular arcs.

Without loss of generality, we can assume that the two circular arcs bounding the crescent intersect at the points 1 and -1. The conformal transformation  $z = thw$  therefore maps the crescent in the  $z$ -plane onto an infinite strip  $a < v < b$  in the  $w$ -plane,  $w = u + iv$ . The general solution (23) of (22) can be replaced in the  $w$ -plane by an expression

$$(35) \quad F(w, \bar{w}) = (\operatorname{ch} w \operatorname{ch} \bar{w}) \Psi = \operatorname{Re} \left\{ f^*(w) \operatorname{sh} \bar{w} + g^*(w) \operatorname{ch} \bar{w} \right\},$$

which is easily seen to be the general solution of the transformed differential equation

$$(36) \quad \left( \frac{\partial^2}{\partial w^2} - 1 \right) \left( \frac{\partial^2}{\partial \bar{w}^2} - 1 \right) F = 0.$$

The Green's function  $\Gamma(z, \zeta)$  of (22) for a crescent will be related to the Green's function  $G(w, \sigma)$  of (36) for the corresponding strip by the identity

$$(37) \quad G(w, \sigma) = |\operatorname{ch} w|^2 |\operatorname{ch} \sigma|^2 \Gamma(z, \zeta),$$

where  $\zeta = th \sigma$ ,  $\sigma = s + it$ . We proceed to determine  $G$  by the method of Fourier transforms, omitting details of a familiar nature [2,4,9].

We write  $G(w, \sigma)$  as the Fourier transform with respect to  $\theta$  of a quantity  $\gamma(v, t; \theta)$ , which turns out to be the Green's function with discontinuity at  $v = t$  of the ordinary fourth order differential equation

$$(38) \quad \frac{d^4 \gamma}{dv^4} + (8 - 2\theta^2) \frac{d^2 \gamma}{dv^2} + (16 + 8\theta^2 + \theta^4) \gamma = 0$$

in the interval  $a < v < b$ , since equation (36) has constant coefficients.

Thus we have

$$(39) \quad G(w, \sigma) = \frac{1}{\pi} \int_0^\infty \gamma(v, t; \theta) \cos \theta (u-s) d\theta,$$

where the symmetric Green's function  $\gamma(v, t; \theta) = \gamma(t, v; \theta)$  is given for  $a \leq v \leq t \leq b$  by the lengthy relation

$$\begin{aligned}
 (40) \quad & 4\theta(\theta^2 + 4)(4\operatorname{sh}^2 \ell \theta - \theta^2 \sin^2 2\ell) \gamma(v, t; \theta) \\
 & = (\theta^2 + 4)(\theta \operatorname{ch} \ell \theta \sin 2\ell - 2 \operatorname{sh} \ell \theta \cos 2\ell) [\operatorname{sh} \theta(v-a) \sin 2(v-a) \operatorname{sh} \theta(t-b) \sin 2(t-b)] \\
 & + (\theta^2 + 4) \operatorname{sh} \ell \theta \sin 2\ell \left\{ \operatorname{sh} \theta(v-a) \sin 2(v-a) [\theta \operatorname{ch} \theta(t-b) \sin 2(t-b) - 2 \operatorname{sh} \theta(t-b) \cos 2(t-b)] \right. \\
 & - \operatorname{sh} \theta(t-b) \sin 2(t-b) [\theta \operatorname{ch} \theta(v-a) \sin 2(v-a) - 2 \operatorname{sh} \theta(v-a) \cos 2(v-a)] \left. \right\} \\
 & - (\theta \operatorname{ch} \ell \theta \sin 2\ell + 2 \operatorname{sh} \ell \theta \cos 2\ell) [\theta \operatorname{ch} \theta(v-a) \sin 2(v-a) - 2 \operatorname{sh} \theta(v-a) \cos 2(v-a)] \cdot \\
 & \quad \cdot [\theta \operatorname{ch} \theta(t-b) \sin 2(t-b) - 2 \operatorname{sh} \theta(t-b) \cos 2(t-b)] ,
 \end{aligned}$$

in which we have replaced  $b-a$  by  $\ell$ .

The explicit representation of the biharmonic Green's function for a crescent given by formulas (37), (39), and (40) yields as limiting cases the solution of the first boundary value problem in a wedge of arbitrary angle [9], or in an infinite strip [4], or in a domain bounded by two tangent circles. For the special values  $a = -\pi/4$ ,  $b = \pi/4$  the representation reduces to the usual one for the unit circle, since the integral (39) can be evaluated in closed form when  $\ell = \pi/2$ . Another special case of particular interest is that of the semi-circle, for which  $a = 0$ ,  $b = \pi/4$ . We obtain, in general, a wide variety of explicit examples for the study of the multiple-valued character of biharmonic functions near a corner of a curve along which the boundary conditions (15) hold, and consequently we can analyze the viscous flow around a corner. Finally, since equation (36) has constant coefficients, it can be solved by means of the Fourier transform in any infinite strip, and hence we can obtain the solution of the biharmonic problem in domains bounded by logarithmic spirals.

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