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RADIATION IN A DIFFUSING MEDIUM
WITH APPLICATION TO SNOW

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RADIATION IN A DIFFUSING MEDIUM
WITH APPLICATION TO SNOW

FACULTY INVESTIGATORS

R. V. Dunkle J. T. Gier

PERSONNEL

A. J. Test	Coordinator
J. T. Bevans	Project Engineer
M. Iwama	Engineering Aid
G. Lee	Engineering Aid

FOREWORD

The enclosed analysis is original as conceived, but is not original to the field fundamentally. The first reference in which the basic differential equations were established was that by A. Schuster (Astrophysical Journal, vol. 21, 1, (No. 5)) for the radiation through gaseous stars. Further elaboration of this method was performed by L. Silberstein (Philosophical Magazine, vol. 4, 129 (1927)), J. W. Ryde (Proceedings of the Royal Society of London, vol. 131A, 451, (1930)) and S. Q. Duntley (Journal of the Optical Society of America, vol. 32, 61 (1942)). Each succeeding author refined and extended the basic precepts until the analysis by Duntley contained eight constants describing various physical properties of the medium. Notwithstanding these facts, the application of the method has furnished an insight into correlating parameters for reflection and transmission of a snow cover. In addition, the transmission was shown to be related to the albedo of the cover and to be attenuated in an exponential manner.

RADIATION IN A DIFFUSING MEDIUM WITH APPLICATION TO SNOW

Introduction

The transmission and reflection characteristics of multiparticle media such as snow are normally presumed to be independent for purposes of correlating experimental measurements. If it is postulated that the medium is composed of homogeneous semi-infinite slabs of finite incremental thickness, the reflection and transmission of, n, unit thicknesses may be analyzed by the procedures described by Benford (1946). This method is satisfactory if only a few slabs are considered but as the number of slabs becomes very large, calculation of the transmission is difficult. In addition, the analysis presented by Benford is not satisfactory for determination of the transmission at various depths within the medium. Furthermore, no method of correlation of experimental measurements is indicated by this analysis.

In view of these difficulties, an attempt was made to approach the problem in a different manner. The basic assumptions of the analysis are:

(a) The medium is composed of a perfectly diffusing material with a uniform distributed reflectivity, r, which is defined as the reflectivity per unit distance, or the number of reflecting surfaces per unit distance times a constant reflectivity per surface.

(b) The absorption coefficient, k, is assumed uniform.

(c) Emission within the medium is negligible, i.e., the temperature is low compared to the energy source.

Analysis

The total radiation downwards at any depth, x, is given the symbol, Y, BTU/hrft^2, and the radiation upward is designated by, Z, BTU/hrft^2.

The energy passing downward through the medium, i.e., positive x direction, in a distance, dx, is decreased by absorption and reflection and increased by reflection of the upward component:

$$dY = - k\, Y\, dx - r\, Y\, dx + r\, Z\, dx \tag{1}$$

Similarly, the change in the energy passing upward is:

$$dZ = k\, Z\, dx - r\, Y\, dx + r\, Z\, dx \tag{2}$$

An illustrative diagram of the energy transfer is shown in Figure 1 below:

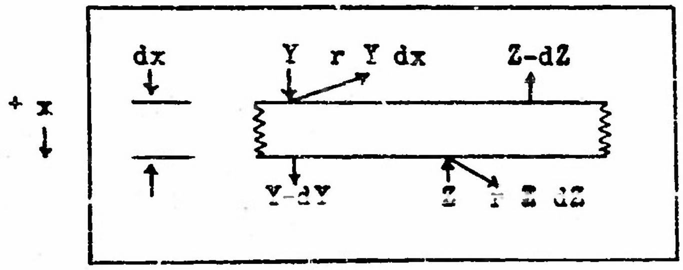


Figure 1.

Division of equation (1) by (2) eliminates dx and Y may be found in terms of Z by integration, giving:

$$(k + r)\, ZY - \frac{r}{2} Y^2 - \frac{r}{2} Z^2 = 0 \tag{3}$$

The value of the integration constant is to be evaluated from the boundary conditions after obtaining Y and Z in terms of k and r_o. Solving for Z, one obtains

$$Z = Y \left[\frac{k + r}{r} \pm \sqrt{\left(\frac{k + r}{r}\right)^2 - 1} \right] \tag{4}$$

Substitution of (4) into (1) and integrating gives

$$Y = y_1 e^{- x \sqrt{(k + r)^2 - r^2}} + y_2 e^{x \sqrt{(k + r)^2 - r^2}}$$

or

$$Y = y_1 e^{-\beta x} + y_2 e^{\beta x} \tag{5}$$

where

$$\beta = \sqrt{(k + r)^2 - r^2} \tag{6}$$

From equations (4) and (5), the expression obtained for Z is:

$$Z = \left[\frac{k + r - \beta}{r}\right] y_1 e^{-\beta x} + \left[\frac{k + r + \beta}{r}\right] y_2 e^{\beta x} \qquad (7)$$

The boundary conditions are (see Figure 2) at x = o:

$$Y_o = (1 - r_o)\, Y_i + r_o Z_o \qquad (8)$$

and at x = b:

$$Z_b = (1 - r_b)\, Z_i + r_b Y_b \qquad (9)$$

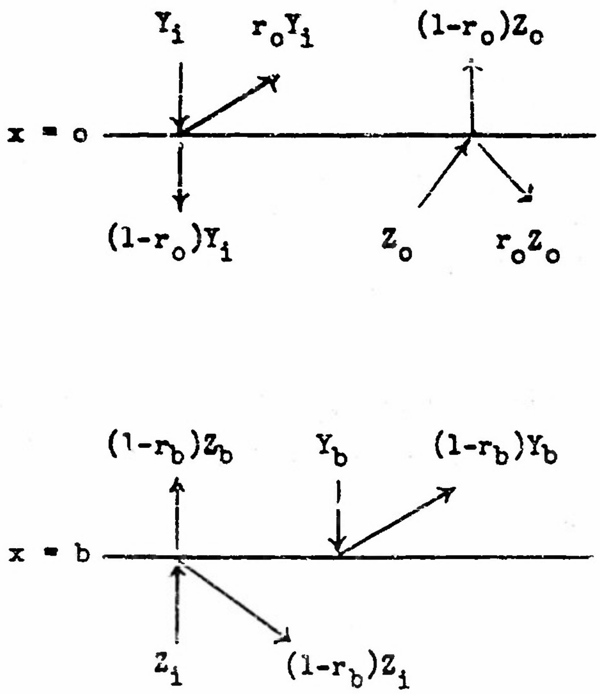


Figure 2.

The boundary conditions give

$$x = o: \quad y_1 + y_2 = (1 - r_o)\, Y_i + r_o Z_o$$

$$Z_o = \left(\frac{k + r - \beta}{r}\right) y_1 + \left(\frac{k + r + \beta}{r}\right) y_2$$

or

$$(1 - r_o)Y_i = \left[1 - \frac{r_o(k + r - \beta)}{r}\right] y_1 + \left[1 - \frac{r_o(k + r + \beta)}{r}\right] y_2 \qquad (10)$$

$$x = b: \quad Y_b = \left[y_1 e^{-\beta b} + y_2 e^{\beta b}\right] r_b$$

$$Z_b = \left[\frac{k + r - \beta}{r}\right] y_1 e^{-\beta b} + \left[\frac{k + r + \beta}{r}\right] y_2 e^{+\beta b}$$

or

$$(1 - r_b)Z_i = y_1 \left[\frac{k + r - \beta}{r} - r_b\right] e^{-\beta b} + y_2 \left[\frac{k + r + \beta}{r} - r_b\right] e^{\beta b} \qquad (11)$$

From equations (11) and (12)

$$y_1 = \frac{(1-r_o)\left[\frac{k+r+\beta}{r} - r_b\right] e^{\beta b} Y_i - (1-r_b)\left[1-r_o \frac{(k+r+\beta)}{r}\right] Z_i}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \qquad (12)$$

$$y_2 = \frac{(1-r_b)\left[1 - \frac{r_o(k+r-\beta)}{r}\right] Z_i - \left[\frac{k+r-\beta}{r} - r_b\right] (1-r_o) e^{-\beta b} Y_i}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \qquad (13)$$

where

$$D_1 = \left[\frac{k+r+\beta}{r} - r_b\right]\left[1 - \frac{r_o(k+r-\beta)}{r}\right] \qquad (14)$$

$$D_2 = \left[\frac{k+r-\beta}{r} - r_b\right]\left[1 - \frac{r_o(k+r+\beta)}{r}\right] \qquad (15)$$

Equations (12) through (15) and equations (5) and (7) determine the functions Y and Z. The majority of applications in practice would have $Z_i = 0$. For this condition:

$$Y = \frac{(1-r_o)\ Y_i}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \left\{ \left[\frac{k+r+\beta}{r} - r_b\right] e^{\beta b}\, e^{-\beta x} - \left[\frac{k+r-\beta}{r} - r_b\right] e^{-\beta b}\, e^{\beta x} \right\} \tag{16}$$

$$Z = \frac{(1-r_o)\ Y_i}{D_1 e^{\beta b} - D_2 e^{-\beta b}} \left\{ \left[\frac{k+r-\beta}{r}\right]\left[\frac{k+r+\beta}{r} - r_b\right] e^{\beta b}\, e^{-\beta x} - \left[\frac{k+r+\beta}{r}\right]\left[\frac{k+r-\beta}{r} - r_b\right] e^{-\beta b}\, e^{\beta x} \right\} \tag{17}$$

For the case of a thick layer, e.g. deep snow cover, equations (16) and (17) reduce to ($b \rightarrow \infty$):

$$Y = \frac{(1-r_o)\ Y_i\ e^{-\beta x}}{\left[1 - \frac{r_o(k+r-\beta)}{r}\right]} \tag{18}$$

$$Z = \frac{\left(\frac{k+r-\beta}{r}\right)(1-r_o)\ Y_i\ e^{-\beta x}}{\left[1 - \frac{r_o(k+r-\beta)}{r}\right]} \tag{19}$$

The value of the reflectance, R_o for a diffuse medium, is seen to be (refer to Figure 2):

$$R_o = \frac{r_o\ Y_i + (1 - r_o)\ Z_o}{Y_i} \tag{20}$$

The transmission factor may be defined as the net energy passing a plane at distance x:

$$T_x = \frac{Y_x - Z_x}{Y_i} \tag{21}$$

While the foregoing equations were developed for a distributed reflectivity, they can be applied with small error to a series of finite absorbing slabs. The distributed reflectivity in this case is taken as the number of reflections per unit distance times the reflectivity or:

$$r = \frac{2(n-1)r_s}{\Delta x} \tag{22}$$

where, n, is the number of slabs of thickness, Δx and surface reflectivity r_s.

The calculated reflectances and transmittances from these equations can be compared with Benford's equations as a check upon the error involved in the assumption of a distributed reflectivity.

The results by Benford (1946) gave for transmission and reflection of a finite number of similar slabs:

$$R_{2n} = R_{2n-1} + \frac{(R_{2n-1})(T_{2n-1})^2}{1 - (R_{2n-1})^2} = R_{2n-1}\left[1 + T_{2n}\right] \tag{23}$$

$$T_{2n} = \frac{T^2_{2n-1}}{1 - R^2_{2n-1}} \tag{24}$$

For an infinite (large) number of slabs, the expression for R_o derived by Benford was:

$$R_o = \frac{(1 + R_1^2 - T_1^2) - \left[(1 + R_1^2 - T_1^2)^2 - 4R_1^2\right]^{1/2}}{2R_1} \tag{25}$$

The values of R_1 and T_1 are the reflectivity and transmissivity of a single slab. In terms of r_s and k this may be shown to be (see Appendix):

$$R_1 = r_r + \frac{r_s(1-r_s)^2 \, e^{-2k\Delta x}}{1 - r^2 \, e^{-2k\Delta x}} \tag{26}$$

$$T_1 = \frac{(1-r_s)^2 \, e^{-k\Delta x}}{1-r_s^2 \, e^{-2k\Delta x}} \tag{27}$$

An important fact to be noted is that as $\Delta x \rightarrow 0$, R_1 is not equal to zero. This is contrary to the hypothesis of Benford and results from the fundamental concept that reflectivity is a surface property. Also worthy of note is the lack in the work of Benford of a method for determining the transmission at any depth within the slabs, other than a backward reiteration process. Benford presented a method of calculation for this but the basis for the procedure was the postulate that R = 0 and T = 1.0 at zero thickness.

The work of Benford does furnish a means of establishing the validity of the analysis presented herein. The following values of k and r_s were selected ($r_s = r_o$):

k in^{-1}	r_s
1	.01
10	.1
100	.9

and a constant slab thickness of 0.005 inches was selected. The number of slabs selected were 2, 4 and ∞. The results of the calculations are shown in Table 1. The agreement of the two methods is within 2 percent or less of unity and the major portion of the differences may be attributed to significant figure errors in calculation.

TABLE I

Comparison of results obtained by the analysis of Benford and the method presented herein.

r_s	Benford T	Benford R	Present Analysis T	Present Analysis R	Benford R_o	Present Analysis R_o
	k = 1.0, x = 0.010" n = 2				k = 1.0	
0.01	.952	.038	.952	.041	.502	.503
0.10	.685	.305	.683	.291	.815	.804
0.90	.024	.967	.045	.945	.977	.958
	k = 10				k = 10	
0.01	.869	.035	.875	.025	.147	.153
0.10	.623	.282	.632	.273	.519	.526
0.90	.009	.939	.022	.925	.940	.931
	k = 100				k = 100	
0.01	.353	.018	.355	.020	.021	.029
0.10	.247	.162	.251	.170	.175	.220
0.90	.0004	.905	.0023	.907	.905	.908
	k = 1, x = 0.020" n = 4					
0.01	.906	.073	.907	.073		
0.10	.517	.463	.536	.444		
0.90	.009	.976	.033	.951		
	k = 10					
0.01	.757	.062	.759	.062		
0.10	.422	.402	.437	.388		
0.90	0	.940	.013	.925		
	k = 100					
0.01	.125	.021	.125	.024		
0.10	.063	.172	.065	.195		
0.90	0	.905	.001	.907		

Discussion

This analysis was initiated in an attempt to resolve a portion of the difficulties encountered in the measurement and correlation of transmission and reflection of a snow cover. While not specified, it must be remembered that r_s and k may vary with wavelength, and in general monochromatic values should be employed and an integration performed for specific sources.

The results obtained considered irradiation of both the top and bottom of the medium, but for a snow cover the primary source of irradiation will be from one side only. The equations allow the substitution of any reflectivity, r_b, which could be taken as the reflectivity of the ground in case of a snow cover. For a snow cover it has been shown (SIPRE Report No. 4, 1951) that for a depth greater than 12 to 24 inches, dependent upon the type and nature of the snow, that transmission of solar energy is negligible.

Consequently, the results expressed in equations (18) and (19) can be considered pertinent. Substitution of these expressions in equations (20) and (21) yields:

$$R_o = A = r_o + \frac{\left(\frac{k+r-\beta}{r}\right)(1-r_o)^2}{\left[1 - r_o\left(\frac{k+r-\beta}{r}\right)\right]} \quad (28)$$

and

$$T_x = \frac{(1-r_o)\, e^{-\beta x}}{\left[1 - r_o\left(\frac{k+r-\beta}{r}\right)\right]} \left[1 - \frac{k+r-\beta}{r}\right] \quad (29)$$

where

A = albedo

Equation (29) can be written

$$T_x = (1 - A)\, e^{-\beta x} \quad (30)$$

Thus, the net transmission of the medium is directly related to the albedo. Furthermore, the transmission of a particular snow cover will be exponential with depth (x) and an attenuation coefficient may be obtained. Wide variance between different observed values of this attenuation coefficient should be expected since k, r_s, and r_o influence the value obtained.

The number of variables involved may be reduced by assuming, k, is a constant for all snows. This assumption is reasonably valid as the indices of refraction (directly related to k) for ice and water are essentially equal and the presumption that this is true for an ice crystal (snow) follows. Consequently, the primary variables are Δx and the reflectivities r_o and r_s. It is seen from equati n (23) that the distributed reflectivity is defined for the case of a deep snow ($x \to \infty$):

$$r = \frac{2r_s}{\Delta x} \tag{31}$$

It is thought that $(\frac{r_s}{\Delta x})$ will depend upon the shape and Δx upon the size of the snow particles.

As a snow ages, physical changes in the crystalline structure occur. Melting, compaction, etc. will alter the shape and size of the individual crystals causing changes in the transmission and reflection of the snow cover. In a new snow with small crystals Δx is small and hence r is large compared to k, resulting in a high albedo. As the snow ages and crystals grow larger, r will decrease relative to k resulting in a decrease in albedo. The limiting case would be a plane layer of homogeneous ice which would have a very low albedo. It must be remembered that this discussion applies only to clean snow.

From the foregoing discussion, the important variables affecting I_x and A, have been reduced to:

r - the distributed reflectivity

r_o - the first surface reflectivity

The first, r, is suggested as a parameter varying between types of snow and as a possible means of correlating the data. The latter variable r_o, is a surface condition which will also be dependent upon the type of snow but should have comparatively small effect upon the albedo (A) and transmission (T_x). The supposition that these may be treated separately is subject to question and is proposed as a first attempt to secure more satisfactory correlation of experimental measurements of snow transmission and reflection. Variations within the cover have been neglected and this may negate efforts to obtain descriptive parameters of various types of snow covers.

Conclusions

1. The transmission through an idealized snow cover has been shown to be directly related to the albedo and to be an exponentially decreasing function.

2. Two parameters have been proposed as possible correlating factors for transmission and albedo measurements. The first factor is a characteristic of the surface condition of the cover and the second is a characteristic of the snow beneath the surface.

Experimental Verification

The experimental verification of the applicability of this analysis to snow is contemplated. Measurements will be made in a snow cover utilizing the Solarimeter and Albedometer, described previously (Gier, Dunkle 1953).

Facilities are available locally for such work.

REFERENCES

Benford, F. (1946) Radiation in a diffusing medium, Journal of the Optical Society of America, vol 36.

University of Minnesota (1951) Review of the Properties of Snow and Ice. SIPRE Report 4. Institute of Technology, EES, Minneapolis, 156 pp.

Gier, J. T. and R. V. Dunkle (1953) Progress report for the year June 27, 1952 to June 27, 1953, Berkeley: The University of California, Institute of Engineering Research, Series No. 62, Issue No. 1, Contract No. DA-11-190-ENG-3, 73 pages.

NOMENCLATURE

A	=	albedo; reflectivity of a medium of large thickness (i.e. $x \rightarrow \infty$) defined by equation (28)
e	=	natural logarithm base
k	=	absorption or extinction coefficient; $(\text{inches})^{-1}$
n	=	number of slabs
r	=	distributed reflectivity; defined by either equation (22) or (31)
r_b	=	reflectivity of the lower boundary of the medium
r_o	=	reflectivity of the upper boundary of the medium
r_s	=	reflectivity of a single slab or lamina of the medium
R_o	=	total reflectivity of the medium at the upper boundary; defined by equation (20)
R_{2^n}	=	reflectivity of 2^n slabs; equation (23)
$R_{2^{n-1}}$	=	reflectivity of 2^{n-1} slabs; equation (23) and (24)
R_1	=	reflectivity of one slab; equation (25) and Appendix
T_x	=	effective transmission at depth x; defined by equation (21)
T_{2^n}	=	effective transmission of 2^n slabs; equations (23) and (24)
T_1	=	effective transmission of one slab; equation (25) and Appendix
t	=	transmission between boundaries of a unit slab; Appendix
x	=	thickness; inches
y_1 and y_2	=	constants; defined by equation (4)
Y	=	energy downward in medium; $BTU/hrft^2$
Y_b	=	energy incident upon lower boundary; $BTU/hrft^2$
Y_1	=	energy incident upon the upper surface of the medium at x = o, $BTU/hrft^2$
Y_o	=	energy incident upon upper boundary; $BTU/hrft^2$
Z	=	energy upward in medium; $BTU/hrft^2$
Z_b	=	energy incident upon the lower surface of the medium; $BTU/hrft^2$
Z_1	=	energy incident upon the lower surface of the medium at x = h, $BTU/hrft^2$

NOMENCLATURE (Cont'd)

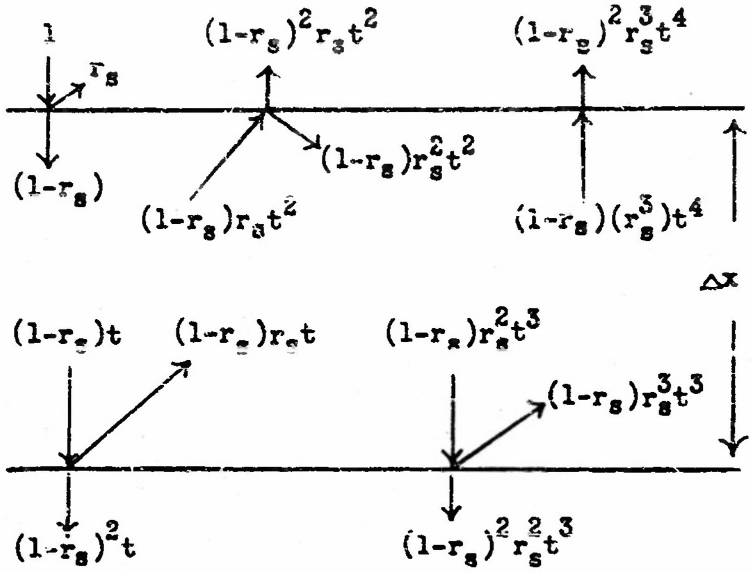
Z_o = energy incident upon the upper surface of the medium; BTU/hr-ft^2

β = variable; defined by equation (7)

Δ = increment

APPENDIX

Transmission and Reflection from a Single Slab



$$R_1 = r_s + (1-r_s)^2 r_s t^2 + (1-r_s)^2\, r_s^3\, t^4 + (1-r_s)^2\, r_s^5\, t^6 + - - -$$

$$= r_s + (1-r_s)^2 r_s t^2 \left[1 + r_s^2\, t^2 + r_s^4\, t^4 + - - -\right.$$

$$= r_s \left[1 + \frac{(1-r_s)^2 t^2}{1-r_s^2\, t^2}\right]$$

$$T_1 = (1-r_s)^2 t + (1-r_s)^2\, r_s^2\, t^3 + (1-r_s)^2\, r_s^4\, t^5 + - - -$$

$$= (1-r_s)^2 t \left[1 + r_s^2\, t^2 + r_s^4\, t^4 + - - -\right.$$

$$= \frac{(1-r_s)^2 t}{1-r_s^2\, t^2}$$

If $t = e^{-k\Delta x}$ the above equations become:

$$R_1 = r_s \left[1 + \frac{(1-r_s)^2\, e^{-2k\Delta x}}{1-r_s^2\, e^{-2k\Delta x}}\right]$$

$$T_1 = \frac{(1-r_s)^2\, e^{-k\Delta x}}{1-r_s^2\, e^{-2k\Delta x}}$$