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The Propagation of Elastic Pulses Through Rods and Plates

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THE PROPAGATION OF ELASTIC PULSES

THROUGH RODS AND PLATES*

by W. L. Pondrom, Department of Physics, the University of Texas, Austin, Texas.

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Abstract: The propagation of an elastic pulse through the simple geometric forms, rods and plates, is investigated from a theoretical point of view with the aid of the small-motion dynamic elastic equations of an isotropic, homogeneous, dissipationless solid.

The investigation is restricted to disturbances which are initially plane-wave pulses of dilatation, and formal solutions are developed by Fourier transform methods (symmetric and one sided). The resulting formal solutions are developed into infinite series, the terms of which represent the total contribution of wave-groups which can be associated with the paths of minimum transit time predicted by the methods of geometrical optics.

These paths, and the associated wave groups, are found to be characterized by two integers n_1 and n_2 which represent the number of times the thickness of the plate (or diameter of the rod) has been traversed as a dilatational wave and as a rotational wave respectively. The variety in these paths is found to result from conversions of dilatational wave energy to rotational wave energy at the free surfaces.

When the Poisson ratio, σ, is zero, this conversion effect does not exist for the disturbances considered, and all of the energy is carried by the direct dilatational wave ($n_1 = n_2 = 0$).

The terms of these series are simplified by contour deformation methods and are found to represent transients with the minimum transit times predicted by the methods of geometrical optics. In the case of the plate, the simplification enables one to carry out finite numerical integrations in obtaining quantitative data on the pulse shape and amplitude of the disturbances associated with any specific values of n_1 and n_2.

Various interference effects are found between these wave-groups as $t \to \infty$ and/or the distance of transmission $\to \infty$. It is shown that these wave-groups interfere in such a way that the total disturbance becomes asymptotically a plane wave travelling with the velocity predicted in the classical theories of thin plates and rods.

A comparison is made of these theoretical considerations and the experiments reported by Hughes, Pondrom and Mims,[1] and their failure to identify any wave groups other than those corresponding to the direct dilatational and critical angle paths (all of which have $n_1 = 0$) is explained.

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[1] D. S. Hughes, W. L. Pondrom, and R. L. Mims, Phys. Rev. 75, 1552 - 1556, (1949).

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I. INTRODUCTION

In a recent paper, Hughes, Pondrom, and Mims[1] have described a quick

[1] D. S. Hughes, W. L. Pondrom, and R. L. Mims, *Phys. Rev.* **75**, 1552-1556, (1949).

and accurate method of determining the dynamic elastic constants of isotropic, homogeneous solids from the transmission times of elastic pulses through samples in the shape of right circular cylinders.

In this method a longitudinal elastic pulse is delivered to one end of the rod by a piezo-electric driver (an X-cut quartz crystal), and the arrival of longitudinal pulses at the opposite end is detected by a piezo-electric detector (another X-cut quartz crystal). In general, many pulses arrive at the detector for each pulse delivered by the driver, and the transit time of each pulse is determined by an electronic system in which a crystal oscillator acts as the basic time standard or clock.

Although only longitudinal excitation and detection are employed, some of the pulses are transmitted part of the way by rotational waves, and the velocity of rotational waves as well as the velocity of dilatational waves may be determined from these transit times. From these velocities and the density of the material, all of its elastic constants may be determined.

A simple theory was devised with the methods of geometrical optics, which had been successfully applied to many similar problems in seismology.

Referring to Fig. 1, the pulse delivered by the driving crystal

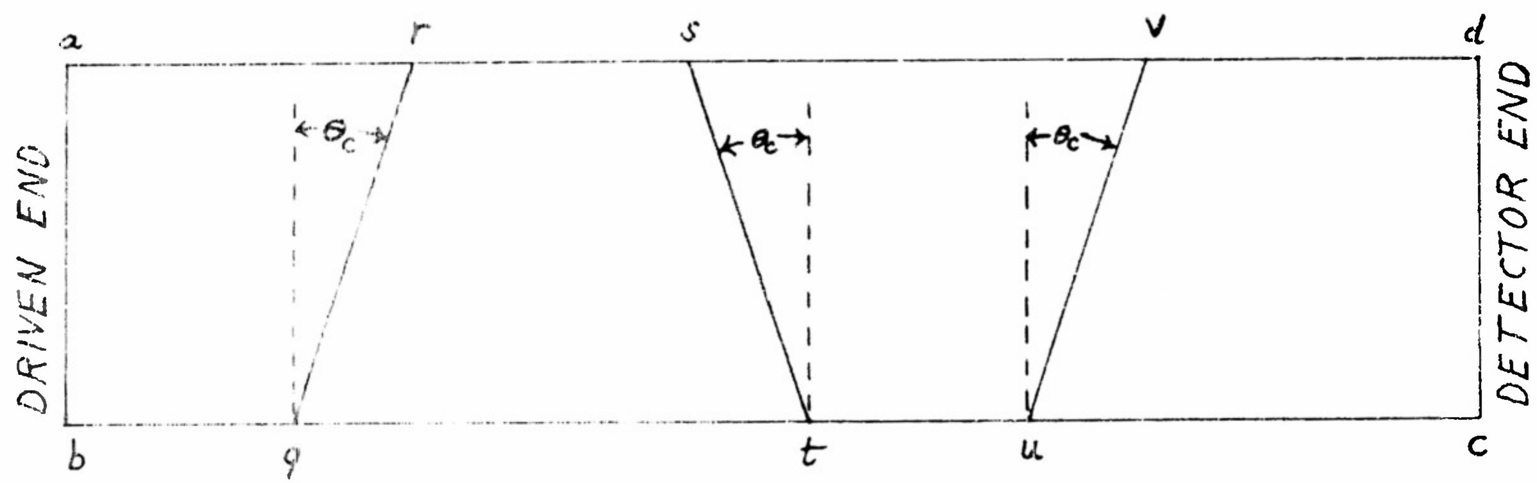

Fig. 1. Observed wave paths in a rod.

would be expected to generate a group of plane waves of dilatation having a continuous distribution of frequencies and traveling very nearly parallel to the free cylindrical wall of the rod. Part of the energy of these waves may be expected to be converted into rotational waves at the free cylindrical boundary[2] since dilatational waves alone cannot in general

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[2] H. Poincare, Lecons sur la theorie de l'elasticite, (Paris 1892), p. 124ff.
Handbuch der Physik, Bd. VI (Verlag, Julius Springer, Berlin, 1928), pp. 323-324.
See also reference 7.

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satisfy the free boundary conditions. On the other hand, a portion of the energy of these dilatational waves will reach the opposite end of the rod without modification. In terms of ray paths, this would correspond to such a path as $\overline{bc}$ or $\overline{ac}$. If L is the length of the rod and a is the velocity of dilatational waves in the rod, the pulse transmitted along such a path would arrive at the detector with a transmission time, t, given by

$$t = \frac{L}{a} \tag{1.1}$$

Returning to the rotational waves obtained by transformation at the free boundary, such waves must follow Snell's law as in optics. If b is the velocity of rotational waves in the rod, this requires that

$$\frac{\sin \theta_R}{\sin \theta_D} = \frac{b}{a} \tag{1.2}$$

where θ_D is the angle between the direction of travel and the normal to the boundary surface, and θ_R is the angle between the direction of motion of the rotational wave, obtained by transformation at the boundary, and the normal to the boundary surface.

Since $b \ll a$, and in the present case $\theta_D = 90°$, the rotational waves must travel approximately at the critical angle θ_C relative to the normal of the boundary surface where θ_C is given by

$$\sin \theta_C = b/a \qquad (1.3)$$

Such rotational waves, obtained by transformation at the upper boundary, would eventually strike the lower boundary and there give rise to a reflected rotational wave and a dilatational wave. In satisfying Snell's law, these dilatational waves obtained by transformation at the lower boundary must travel very nearly parallel to the boundary, and part of their energy would eventually reach the detector end of thr rod. This energy would then have been carried over such a path as $\overline{bqrd}$ where $\overline{bq}$ and $\overline{rd}$ were traversed as dilatational waves, and $\overline{qr}$ was traversed as a rotational wave.

If D is the diameter of the rod, it is obvious that the length of the path qr is $D \csc \theta_C$, while the distance advanced along the length of the rod is $D \tan \theta_C$. Consequently the time delay in transmission, Δt, incurred by taking the path $\overline{bqrd}$ instead of $\overline{bc}$ is given by

$$\Delta t = (D/b) \csc \theta_C - (D/a) \tan \theta_C = D(b^{-2} - a^{-2})^{1/2} \qquad (1.4)$$

where the last form is obtained by elimination of θ_C with equation (1.3).

It is apparent that such transformations can, in general, take place a number of times. Consequently, such paths as $\overline{bqrstc}$ and $\overline{bqrsturd}$ are possible and correspond to delays in transmission of $2\Delta t$ and $3\Delta t$ respectively, and each gives rise to a longitudinal pulse at the detector end of the rod.

The process of reflection at the ends of the rod can also give rise to delayed arrivals. In particular, part of the energy of the original dilatational wave group can be reflected without change of mode of transmission, once at each end of the rod, and thus travel the length of the rod three times. Similarly, it could be reflected twice at each end and travel the length of the rod five times etc.

As a result of these two processes, transit times corresponding to the equation

$$t = m(L/a) + nD(b^{-2} - a^{-2})^{1/2} \qquad (1.5)$$

are to be expected where m is any positive odd integer and n is any positive integer or zero. Here m is the number of times the length of the rod is traversed, and n is the number of delays incurred by the transfer of mode of transmission process. Since during each such delay the delayed disturbance moves the distance $D \tan \theta_C$ along the length of the rod, it is obvious that the integers m and n must satisfy the restriction

$$mL \geq nD \tan \theta_C \qquad (1.6)$$

The experimental results are in excellent agreement with equations (1.5) and (1.6). Each of the multiplicity of detected longitudinal pulses obtained for each driving pulse corresponds to a particular admissible combination of the integers m and n, and the appropriate variations are observed when the diameter or length of the rods is altered.[1] However, the relative amplitudes of the various delayed pulses are functions of L, D, m, n, a, b, and the tendency of the material in the rod to disperse the energy in such a disturbance into random elastic disturbances (heat motions), and many of the arrivals predicted by equations (1.5) and (1.6) are too feeble to be detected under some experimental conditions.

The exact wave form and duration of the driving pulse are obviously of considerable importance. The duration should be small compared to each of the times (L/a) and $D(b^{-2} - a^{-2})^{1/2}$ in order that the detected pulses be easily resolved in terms of arrival times, and the wave form should be simple but easily recognized against background disturbances. But, aside from these considerations, there appears to be no additional information obtainable from geometric optical methods.

In an effort to obtain more detailed information, the literature was searched for treatments of the transmission of pulses through a finite circular rod made of dissipationless, homogeneous, isotropic material. This search was entirely without success; there is not even an exact treatment of the free vibrations of a finite circular cylinder. Pochhammer[3] has given a reasonably complete treatment of some of the simpler modes of transmission of simple harmonic waves along an infinite circular cylinder, but he was unable to adjust his wave solutions to fit all of the boundary conditions at the ends of a finite circular cylinder. However, since his solutions do satisfy the boundary conditions on the free cylindrical surface, these should permit a theoretical treatment of the transfer of mode process dealt with above. This possibility is explored by means of integral transformations in Sections 2 and 4 of this paper.

Since this investigation requires the use of Bessel functions, and it was recognized from the start that some form of approximation might be required, it was found desirable to investigate pulse transmission through an infinite plate.

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[3] L. Pochhammer, J. f. Math (Crelle's Journal), Bd. 81, 324-336, (1876).

The simple harmonic modes of trans ion in an elastic plate were investigated by Lord Rayleigh[4] in 1889 and by H. Lamb[5] in 1891. Descriptions of the various modes were given, and the existence of certain types of nodal surfaces was discussed. Although Lamb[6] reconsidered the problem in 1916, very little additional progress was made. However, it was noted that the apparent "wave velocity" belonging to these modes could be infinite and was often greater than the velocity of either dilatational or rotational waves in an unbounded medium. These investigators were aware that the velocity of dilatational waves is the maximum velocity with which a transient could be propagated into an undisturbed region but offered no very clear picture of the relationship of these modes of transmission to the propagation of transients.

This problem was taken up by Prescott[7] in 1942 in an effort to throw some light on the behavior of rods. In a review of this article, Bourgin[8] suggested that these modal velocities be treated as phase velocities and that classical group velocity theory be used to rationalize the actual rate of energy flow.

[4]Lord Rayleigh, Proc. Math. Soc. London, 20, 225-234, (1889).

[5]H. Lamb, Proc. Math. Soc. London, 21, p 85ff, (1891).

[6]H. Lamb, Proc. Roy. Soc. London (A), 93, 114-128, (1916/17).

[7]J. Prescott, Phil. Mag. Ser. 7, 33, 703-754, (1942).

[8]D. G. Bourgin, Math. Rev., 4, p 121, (1943).

These modal velocities may properly be referred to as modal phase velocities but are certainly not phase velocities in the same sense as used in describing propagation in an unbounded medium. The group velocity calculated from these by Rayleigh's classical formula gives, as usual, the phase velocity of the modulation envelope for a disturbance which is simple-harmonically modulated at a modulation frequency small compared to the frequency of the carrier mode. These group velocities are not directly related to the fastest transmission of an abrupt signal. In fact Sommerfeld and Brillouin[9] clearly demonstrated in 1914 that the

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[9] A. Sommerfeld, Ann. der Physik, 44, 177-202, (1914).
L. Brillouin, ibid, 203, 240.

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fastest transmission of an abrupt signal takes place at the phase velocity corresponding to infinite frequency. These investigators actually dealt with the anamolous dispersion found in electromagnetic wave phenomena, but the results are capable of immediate generalization to all wave transmissions of signals.

In 1947 Cooper[10] gave a completely satisfactory demonstration that

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[10] J. L. B. Cooper, Phil. Mag. Ser. 7, 38, 1-22, (1947).

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the modes of simple harmonic propagation in a plate were in quantitative agreement with the expected maximum velocity of propagation, the velocity of dilatational waves in free space. His method was similar to that of Sommerfeld and Brillouin. It was also pointed out that the group velocity suggestion of Bourgin was defective in several respects. In elastic problems, a single velocity of energy transfer is at best some sort of an average, as there must be at least two rates of transfer corresponding to the dilatational and rotational methods of propagation. In the modes of propagation considered there are two plane waves of dilatation and two plane waves of rotation, all of which are individually traveling in different directions. In the classical group velocity theory all of the plane waves are traveling in the same direction. No mention is made of the fact that the maximum classical group velocity does not determine the velocity with which the first effects of a signal are transmitted.

Because of the complexity of his formal solution, Cooper did not obtain quantitative information on anything except the very first arrival time for a transient disturbance. In Section 3 there is considered a particular problem in pulse transmission through a plate which closely resembles in nature the pulse transmission problem in the rod as encountered by Hughes, Pondrom, and Mims. By the employment of a suitable series of mathematical manipulations, it is possible to break the disturbance into parts segregated according to the nature of the path (number and kind of reflections). The contribution of the wave groups making up each part is in turn analyzed in terms of a group transit time, and the total contribution of each wave group is then expressed in terms of a single integration which can be carried out by numerical means to obtain detailed quantitative information on the shape and amplitude of all the possible reflections from the free surfaces bounding the plate. The minimum group travel time is easily shown to determine the beginning of the disturbance carried by each wave group.

An extensive effort has been made by this author to break up into similar wave groups the formal solution obtained for the cylindrical rod obtained in Section 2. This effort was only partially successful, and a brief resume is given in Section 4.

In terms of physical experiments, only the rod problem can be set up in an exact manner. The plate problem could be approximated by a rod of rectangular cross section with one side of the rectangle large compared to the other side and to the distance from driver to detector. However, no quartz crystals of suitable shape are currently available for such an experiment. As a result, the concluding section is devoted to a comparison of the theoretical solution for the plate problem with some typical experimental data obtained from rods.

2. A FORMAL SOLUTION FOR THE PROPAGATION OF AN INITIALLY PLANE DILATATIONAL PULSE ALONG A CYLINDRICAL ROD OF INFINITE LENGTH

A direct theoretical approach to the experiments carried out by Hughes, Pondrom, and Mims is to consider the transmission of an initially plane dilatational pulse of energy along a rod of uniform circular cross-section. Since the terminal conditions at the piezo-electric driver and piezo-electric detector are both partially unknown and difficult (if not impossible) to handle by the present mathematical methods, we shall resort to a simplifying assumption as to the behavior of the driving crystal, take the rod to be infinite in length, and study the behavior of the average normal stress on a plane section of the rod, normal to its axis, located some distance along the rod from the driver. This should enable one to study the relative amplitudes of the various reflections and is approximately proportional to the response of the detecting crystal.

The simplifying assumptions regarding the action of the driving crystal are most easily understood from Fig. 2. The circular rod is actually assumed to extend to infinity in length along the Z axis in either direction from the origin of coordinates, and the driving crystal is thought of as occupying the plane section of the rod defined by $Z = 0$, $R \leq R_o$. This driving piezo-electric crystal will be formally replaced by a uniform surface distribution of double-sources[11] over this plane section,

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[11] A. E. H. Love, A Treatise on The Mathematical Theory of Elasticity (Cambridge 1927) 183-189 and 304-307.

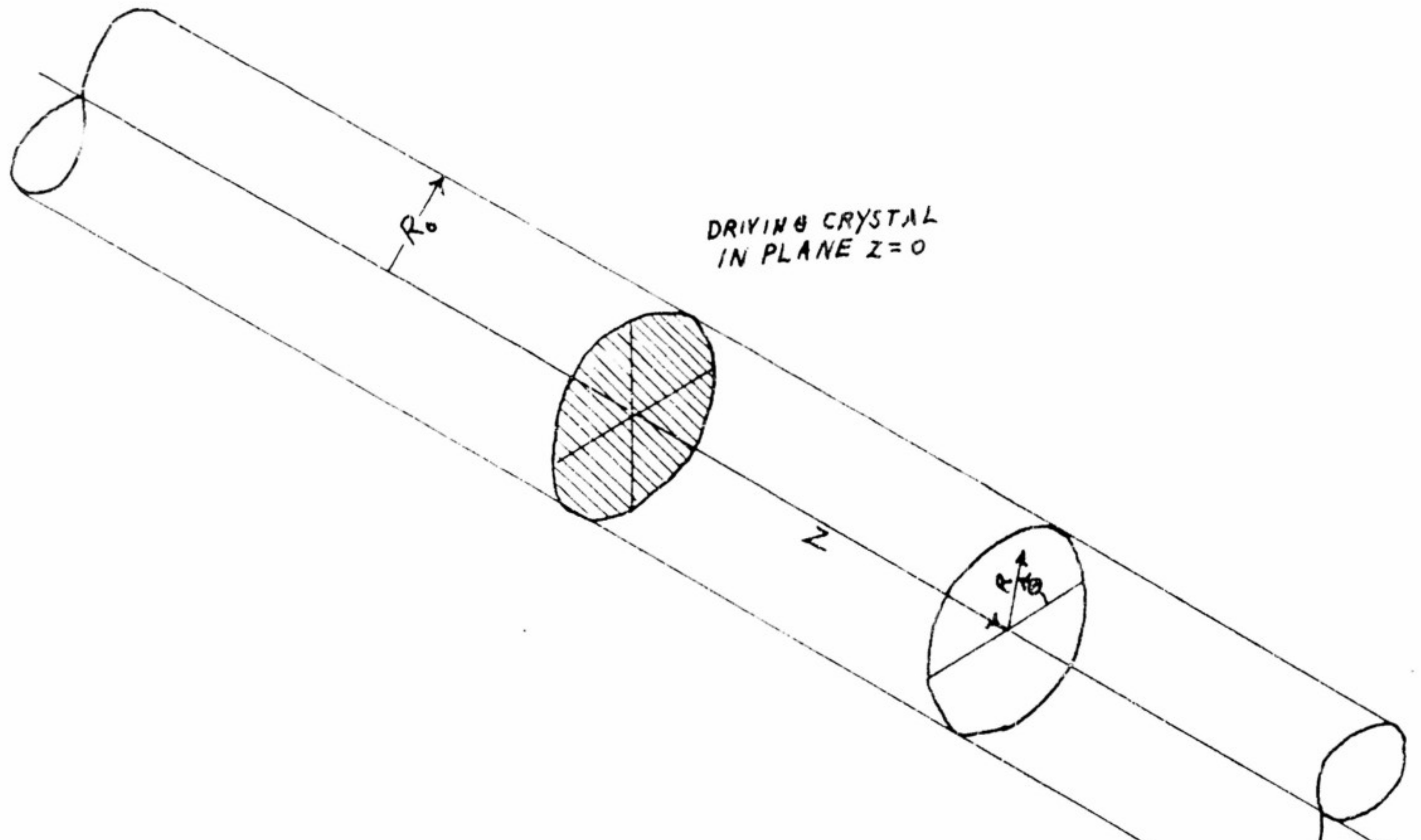

Fig. 2. Relationship of driving crystal and rod
to coordinate system.

and the strength of the double sources will be some suitable function of time after $t = 0$ and zero before $t = 0$.

If we employ the physical components[12] of displacement ξ_R, ξ_θ, ξ_Z

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[12]Ibid, 51-58, 89-91, and 287-288.

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in the directions of increasing R, θ, and Z respectively and the corresponding components ζ_R, ζ_θ, and ζ_Z of the body forces per unit of mass which act upon the material of the rod, the dynamic elastic equations of a homogeneous, isotropic material take the form

$$\frac{\partial^2\xi_R}{\partial t^2} - (a^2 - b^2)\frac{\partial\Delta}{\partial R} + \frac{2b^2}{R^2}\frac{\partial\xi_\theta}{\partial\theta} - b^2\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\left(\frac{\partial^2}{\partial\theta^2} - 1\right) + \frac{\partial^2}{\partial Z^2}\right]\xi_R = \zeta_R$$

$$\frac{\partial^2\xi_\theta}{\partial t^2} - \frac{(a^2 - b^2)}{R}\frac{\partial\Delta}{\partial\theta} - \frac{2b^2}{R^2}\frac{\partial\xi_R}{\partial\theta} - b^2\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\left(\frac{\partial^2}{\partial\theta^2} - 1\right) + \frac{\partial^2}{\partial Z^2}\right]\xi_\theta = \zeta_\theta \quad (2.1)$$

$$\frac{\partial^2\xi_Z}{\partial t^2} - (a^2 - b^2)\frac{\partial\Delta}{\partial Z} - b^2\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial Z^2}\right]\xi_Z = \zeta_Z$$

$$\Delta = \frac{1}{R}\frac{\partial}{\partial R}(R\xi_R) + \frac{1}{R}\frac{\partial\xi_\theta}{\partial\theta} + \frac{\partial\xi_Z}{\partial Z}$$

where <u>a</u> is the speed of dilatational waves, b is the speed of rotational waves, and t is the time

These must hold at all points within the body of the rod, that is $R \leq R_o$, if the rod is presumed to be without faults or other defects, and the solutions must be periodic in θ with the period 2π radians [i.e. $\vec{\xi}(R,\theta,Z,t) = \vec{\xi}(R,\theta+2\pi,Z,t)$] in order to be single valued.

Further, since no forces are applied to the outside wall of the rod, we must have the boundary conditions that the stress components S_{RR}, $S_{R\theta}$, and S_{RZ} vanish at the surface of the rod, $R = R_o$. These conditions are given by

$$\frac{S_{RR}}{\rho} = (a^2 - 2b^2)\Delta + 2b^2 \frac{\partial \xi_R}{\partial R} = 0$$

$$\frac{S_{R\theta}}{\rho} = b^2\left[\frac{1}{R}\frac{\partial \xi_R}{\partial \theta} + R\frac{\partial}{\partial R}\frac{\xi_\theta}{R}\right] = 0 \qquad (R = R_o) \qquad (2.2)$$

$$\frac{S_{RZ}}{\rho} = b^2\left[\frac{\partial \xi_R}{\partial Z} + \frac{\partial \xi_Z}{\partial R}\right] = 0$$

where ρ is the density of the rod material.

Now since the force actions produced by the driving crystal are independent of θ and have no component in the θ direction, it is desirable at the outset to restrict this study to those motions in which $\zeta_\theta = \xi_\theta = 0$, and the remaining components of $\vec{\zeta}$ and $\vec{\xi}$ are independent of θ. In this case the system of equations (2.1) reduces to

$$\frac{\partial^2 \xi_R}{\partial t^2} - (a^2 - b^2)\frac{\partial \Delta}{\partial R} - b^2\left[\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} - \frac{1}{R^2} + \frac{\partial^2}{\partial Z^2}\right]\xi_R = \zeta_R$$

$$\frac{\partial^2 \xi_Z}{\partial t^2} - (a^2 - b^2)\frac{\partial \Delta}{\partial Z} - b^2\left[\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right]\xi_Z = \zeta_Z \qquad (2.3)$$

$$\Delta = \frac{1}{R}\frac{\partial}{\partial R}(R\xi_R) + \frac{\partial \xi_Z}{\partial Z}$$

and the boundary conditions (2.2) reduce to

$$\frac{S_{RR}}{\rho} = (a^2 - 2b^2)\Delta + 2b^2\frac{\partial \xi_R}{\partial R} = 0$$

$$(R = R_o) \qquad (2.4)$$

$$\frac{S_{RZ}}{\rho} = b^2\left[\frac{\partial \xi_R}{\partial Z} + \frac{\partial \xi_Z}{\partial R}\right] = 0$$

Further, in order to study the transmission of pulses, it is convenient to take as initial conditions the simple situation in which ξ_R and ξ_Z and their first partial derivatives with respect to the time are zero throughout the rod at the time $t = 0$.

If the components, ζ_R and ζ_Z, of the body force per unit of mass, are taken to be zero for $t < 0$ and are assumed to be known after $t = 0$ throughout the rod, a solution of equations (2.3) and (2.4) is readily obtained by the application of Fourier integral transformations. The solution, subject to the assumed initial conditions, is unique, and an excellent account of the method is given by Titchmarsh.[13]

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[13] E. C. Titchmarsh, Introduction to The Theory of Fourier Integrals (Oxford 1937), particularly Chapter X.

We will employ the transformation indicated by the scheme

$$\vec{p}(R,\gamma,s) = \int_0^\infty \int_{-\infty}^{\infty} \vec{\xi}(R,Z,t)\, \epsilon^{i(st - \gamma Z)}\, dZdt \qquad (2.5)$$

where $\vec{\xi}$ is the vector body force whose components are some known functions of R,Z, and t, and $\vec{p}$ is the transform of $\vec{\xi}$. The components of $\vec{p}$ are obviously functions of R, and the parameters γ, and s. This integration process is assumed to converge for real values of γ, and any complex values of s such that the imaginary part of s is greater than some positive number δ. In symbolic form this will be written

$$\text{Im}(s) > \delta > 0 \qquad (2.6)$$

Similarly for the displacement vector $\vec{\zeta}$, we will have the transform, $\vec{u}$, given by

$$\vec{u}(R,\gamma,s) = \int_0^\infty \int_{-\infty}^{\infty} \vec{\zeta}(R,Z,t)\, \epsilon^{i(st - \gamma Z)}\, dZdt \qquad (2.7)$$

Now eliminating ζ_R and ζ_Z in equation (2.5) with equation (2.3) we find, on integrating by parts twice with respect to each of the variables Z and t, the result

$$\left[-s^2 + b^2\gamma^2 + \frac{a^2}{R^2} - \frac{a^2}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R}\right] u_R \quad - (a^2-b^2) i\gamma \frac{\partial u_Z}{\partial R} \quad = p_R$$

$$- (a^2- b^2) i\gamma \frac{1}{R}\frac{\partial (Ru_R)}{\partial R} + \left[-s^2 + a^2\gamma^2 - \frac{b^2}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R}\right] u_Z = p_Z \qquad (2.8)$$

where it has been assumed that the components of ξ and its first partial derivatives with respect to t and Z vanish at infinity as well as at t = 0. This additional assumption is in essence a boundary condition to which one is led by the physical reasoning, that the finite energy, stored in this system by the body forces during any driving pulse, must ultimately become diffused through the infinite rod so as to become negligible as $t \to \infty$ or $Z \to \pm \infty$. Thus the displacements, strains, and velocities must $\to 0$ as $t \to \infty$ or $Z \to \pm \infty$.

Similarly, regarding the system of equations (2.4) as the components of a vector, we find for its transform the result

$$\left[a^2 \frac{\partial}{\partial R} + \frac{(a^2- 2b^2)}{R}\right] u_R \quad + (a^2- 2b^2) i\gamma u_Z \quad = 0$$

$$(R = R_o) \qquad (2.9)$$

$$i\gamma b^2 u_R \quad + b^2 \frac{\partial u_Z}{\partial R} \quad = 0$$

When a solution of equation (2.8) for u is obtained which is regular for $R \leq R_o$, $\mathrm{Im}(s) > \delta$ and satisfies the boundary conditions (2.9), the displacement vector $\vec{\xi}$ can be found by the reciprocal transformation to that of equation (2.5), namely

$$\vec{\xi}(R,Z,t) = \frac{1}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} \vec{u}(R,\gamma,s)\, \epsilon^{i(st - \gamma Z)} d\gamma ds \qquad (2.10)$$

where

$$\tau > \delta > 0.$$

Although equations (2.8) and (2.9) can be solved for $\vec{u}$ when $\vec{p}$ is any given vector function of R, γ, and s, we need only consider such a vector p as would correspond to the particular longitudinal piezoelectric drive given the rod by the driving crystal. For this purpose we must take $\zeta_R = 0$ and $\zeta_Z = 0$ when $Z \neq 0$, but undefined when $Z = 0$. It is much easier for one to make definite assumptions about p and then interpret these physically, than to deal with the singular body force representing a surface distribution of sources. Consequently, we will take

$$p_R = 0$$

$$p_Z = \frac{-i\gamma g(s) a}{\pi s} \qquad (2.11)$$

Thus, since the only non-zero component of p is independent of R, equation (2.8) has the very simple particular solution

$$u_{RP} = 0$$

$$u_{ZP} = \frac{+2\gamma a g(s)}{\rho s(s^2 - \gamma^2 a^2)} \tag{2.12}$$

However, this does not in general satisfy the boundary conditions given by system of equations (2.9)

Since the systems of equations (2.8) and (2.9) are linear in u, one can generalise the above solution by adding as a complementary solution any linear combination of solutions of the homogeneous system of equations

$$\left[-s^2 + b^2\gamma^2 + \frac{a^2}{R^2} - \frac{a^2}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R}\right] u_R - (a^2 - b^2) i\gamma \frac{\partial u_Z}{\partial R} = 0$$

$$-(a^2 - b^2) i\gamma \frac{1}{R}\frac{\partial (R u_R)}{\partial R} + \left[-s^2 + a^2\gamma^2 - \frac{b^2}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R}\right] u_Z = 0 \tag{2.13}$$

The solutions of this equation are those considered by Pochhammer[3], and of these only the two solutions

$$u_R = -h J_1(hR)$$
$$u_Z = i\gamma J_0(hR) \tag{2.14}$$

$$u_R = -i\gamma J_1(kR)$$
$$u_Z = k J_0(kR) \tag{2.15}$$

and the linear combinations thereof are regular at $R = 0$.

In these solutions $J_0(z)$ and $J_1(z)$ denote Bessel functions of the first kind[14] of order zero and one respectively of the complex variable z,

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[14] G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge 1944), p.40.

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and

$$h = \left[(s/a)^2 - \gamma^2\right]^{1/2} \qquad Im(h) \geq 0$$
$$k = \left[(s/b)^2 - \gamma^2\right]^{1/2} \qquad Im(k) \geq 0 \tag{2.16}$$

Two other solutions can be found by replacing the Bessel functions of the first kind by those of the second kind, but these are singular at R = 0 and must be excluded since the displacements must be regular at R = 0 in order to satisfy equation (2.8) at R = 0.

Thus, for the vector body force given by equation (2.11), the most general solution of equation (2.8), regular at R = 0, is of the form

$$u_R = -C_1 h\, J_1(hR) - C_2 i\gamma\, J_1(kR)$$
$$u_Z = \frac{-2\gamma g(s)}{\rho a s h^2} + C_1 i\gamma\, J_0(hR) + C_2 k\, J_0(kR) \tag{2.17}$$

where C_1 and C_2 are arbitrary numbers independent of R.

Substituting this result in equation (2.9) and solving for C_1 and C_2, one finds that the boundary conditions are satisfied if

$$C_1 = \frac{2i(a^2-2b^2)\,\gamma^2\,(k^2-\gamma^2)\,J_1(kR_0)g(s)}{\rho b^2 a s h^2\,\Delta_{1/2}(h,k,R_0)}$$

and

$$C_2 = \frac{2(a^2-2b^2)\,\gamma^2\,(2\gamma h)\,J_1(hR_0)g(s)}{\rho b^2 a s h^2\,\Delta_{1/2}(h,k,R_0)} \tag{2.18}$$

where

$$\Delta_\nu(h,k,R_o)=(k^2-\gamma^2)^2 J_{\nu-\frac{1}{2}}(hR_o)J_{\nu+\frac{1}{2}}(kR_o)-\frac{4\nu hs^2}{R_o b^2}J_{\nu+\frac{1}{2}}(hR_o)J_{\nu+\frac{1}{2}}(kR_o)$$

(2.19)

$$+4\gamma^2 hk J_{\nu+\frac{1}{2}}(hR_o)J_{\nu-\frac{1}{2}}(kR_o)$$

By eliminating C_1 and C_2 in equation (2.17) with (2.18) and substituting the resulting values of u_R and u_Z into the reciprocal Fourier transormation equation (2.10), one readily obtains expressions for the displacements ξ_R and ξ_Z. In this connection, it is convenient to write

$$\xi_R = \xi_{RC}$$
$$\xi_Z = \xi_{ZP} + \xi_{ZC} \tag{2.20}$$

where ξ_{ZP} is the displacement produced by the particular solution of (2.8) and ξ_{RC}, and ξ_{ZC} are the components of displacement in the complementary solution which contains the influence of the boundary conditions. It is apparent that ξ_{ZP} is the displacement that would be produced if the rod were unbounded ($R_o = \infty$).

Upon carrying out the above process of elimination, it is clear that

$$\xi_{ZP} = \frac{1}{(2\pi)^2}\int_{-\infty+i\tau}^{\infty+i\tau}\int_{-\infty}^{\infty}\frac{2g(s)\,\epsilon^{-i(st-\gamma Z)}}{\rho\, ash^2}\gamma\, d\gamma\, ds \tag{2.21}$$

$$\tau > \delta > 0$$

$$\xi_{RC} = \frac{2(a^2 - 2b^2)}{(2\pi)^2 i} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{g(s)ds}{s} \int_{-\infty}^{\infty}$$

$$\cdot \frac{[(k^2 - \gamma^2) J_1(hR) J_1(kR_o) + 2\gamma^2 J_1(kR) J_1(hR_o)] \epsilon^{-i(st - \gamma Z)}}{\rho a b^2 h \Delta_{\gamma/2}(h,k,R_o)} \gamma^2 d\gamma \tag{2.22}$$

$$\xi_{ZC} = \frac{-2(a^2 - 2b^2)}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{g(s)ds}{s} \int_{-\infty}^{\infty}$$

$$\cdot \frac{[(k^2 - \gamma^2) J_0(hR) J_1(kR_o) - 2hk J_0(kR) J_1(hR_o)] \epsilon^{-i(st - \gamma Z)}}{\rho a b^2 h^2 \Delta_{\gamma/2}(h,k,R_o)} \gamma^2 d\gamma$$

In equation (2.21) the integration over γ is readily performed in terms of residues. If $Z > 0$, we may by Jordan's lemma[15] consider the

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[15] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge 1940), p. 115.

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contour of integration closed by an infinite semicircle in the upper half of the γ-plane, and the value of the integral is the sum of the residues at the poles in the upper half plane. If equation (2.21) is written as

$$\xi_{ZP} = \frac{-1}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} \frac{g(s)}{\rho s} \left[\frac{1}{\gamma a + s} + \frac{1}{\gamma a - s} \right] \epsilon^{-i(st - \gamma Z)} d\gamma ds$$

the evaluation by the residue at $\gamma = \frac{s}{a}$ gives

$$\xi_{ZP} = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{g(s)\,\epsilon^{-is(t - \frac{Z}{a})}}{\rho s a}\,ds$$

$$Z > 0$$

Similarly, we find

$$\hat{\xi}_{ZP} = \frac{-1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{g(s)\,\epsilon^{-is(t + \frac{Z}{a})}}{\rho s a}\,ds$$

$$Z < 0$$

Now, since the tensile stress along the axis of the rod S_{ZZ} is given by

$$\frac{S_{ZZ}}{\rho} = (a^2 - 2b^2)\Delta + 2b^2\,\frac{\partial \xi_Z}{\partial Z} \tag{2.23}$$

we find, by assuming the validity of differentiation under the integral sign, that the tensile stress produced by the particular solution, S_{ZZP}, is given by

$$S_{ZZP} = \frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} g(s)\,\epsilon^{-is(t - \frac{|Z|}{a})}\,ds \tag{2.24}$$

Now, taking

$$g(s) = \int_0^{\infty} F(t)\,\epsilon^{ist}\,dt \tag{2.25}$$

where $F(t) = 0$ for $t < 0$, we have the usual result for one-sided Fourier transforms[16],

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[16] E. C. Titchmarsh, op. cit., 4 - 5

$$\frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} g(s)\,\epsilon^{-ist}\,ds = \begin{array}{ll} F(t) & t > 0 \\ 0 & t < 0 \end{array} \tag{2.26}$$

and

$$\begin{aligned} S_{ZZP} &= 0 \qquad |Z| > at \\ S_{ZZP} &= F(t - \frac{|Z|}{a}) \qquad |Z| < at \end{aligned} \tag{2.27}$$

Thus, the primary disturbance generated by the singular body forces is a plane wave of dilatation propagated along the Z axis with speed **a** and a shape determined by the functional form of F(t) which we will leave unspecified for the present.

The contribution of the complementary solution to S_{ZZ} may likewise be computed by substituting ξ_{ZC} and ξ_{RC} into equation (2.23) and differentiating under the integral signs. This contribution S_{ZZC} is a function of R, and the receiving crystal will respond only to its average value. Since all displacements are independent of θ, we find, by averaging S_{ZZ} over a plane section normal to the axis of the rod, the result

$$\bar{S}_{ZZ} = \frac{2}{R_o^{\,2}}\int_0^{R_o} S_{ZZ}RdR = \frac{2\rho(a^2 - 2b^2)}{R_o^{\,2}}\Big[R\,\xi_R\Big]_0^{R_o} + \frac{2\rho a^2}{R_o^{\,2}}\int_0^{R_o}\frac{\partial\xi_Z}{\partial Z}\,RdR$$

Substituting the values of ξ_{RC} and ξ_{ZC} from equation (2.22) and performing the integration over R before those over γ we obtain

$$\bar{S}_{ZZC} = \frac{(1 - 2b^2/a^2)^2}{(2\pi)^2 i} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} \frac{4\gamma^2 as^3 g(s)\, J_1(hR_o)\, J_1(kR_o)\, \epsilon^{-i(st - \gamma Z)}}{b^4 h^3 R_o \Delta \chi(h,k,R_o)}\, d\gamma ds \qquad (2.28)$$

Since S_{ZZP} is independent of R, it is obvious that the average value of the entire stress, $\bar{S}_{ZZ}$, is given by

$$\bar{S}_{ZZ} = S_{ZZP} + \bar{S}_{ZZC} \qquad (2.29)$$

From equations (2.22) and (2.28), it is obvious that the complementary solution vanishes if $a^2 = 2b^2$. This situation corresponds to a zero Poisson ratio, σ, since

$$(1 - 2b^2/a^2) = \frac{\sigma}{1 - \sigma} \qquad (2.30)$$

An actual material having $\sigma = 0$ would be highly unusual and have interesting uses. It would not violate the conditions of physical stability. Rayleigh[17] has pointed out that an isotropic material is stable

[17] Lord Rayleigh, Proc. Math. Soc. of London, 17, 4-11, (1887)

if its Poisson ratio lies in the range $1/2 \geq \sigma \geq -1$. When $\sigma = 0$, it is obvious that no reflections of the primary dilatational disturbance considered here are produced by the surrounding walls. Such a material would be extremely useful for making certain types of solid acoustic delay lines.

The evaluation of such integrals as occur in equations (2.22) and (2.28) can be carried out by a number of different procedures. Since the integrands are even functions of h and k, they are single valued functions of s and γ. In addition, they are regular for $\mathrm{Im}(s) > \delta > 0$, $0 \leq R \leq R_o$, and $t < 0$. Thus, the integration over s is zero under these conditions. When $t > 0$, $0 \leq R \leq R_o$, the integrands are regular for $\mathrm{Im}(s) < \tau$ except for poles which are all located on the real axis of the s plane when γ is real. In this case the integration over s is obtained as the sum of the residues at these poles. Since the equation

$$\Delta_{1/2}(h,k,R_o) = 0 \qquad (2.31)$$

is actually the frequency wave-number condition obtained by Pochhammer[3] for the normal modes independent of θ for the infinite rod, it follows that this residue evaluation will express the integral as a series of these normal modes. Each of these modes must then be integrated over γ. These processes will converge very slowly and are incapable of showing the number of reflections which have given rise to the energy arriving at any given time and place.

A second approach is to expand the integrand in a series, convergent for $\mathrm{Im}(s) > \delta$, and integrate term by term. This possibility is a consequence of the relations

$$\begin{aligned} \mathrm{Im}(h) &> \frac{1}{a}\,\mathrm{Im}(s) \\ \mathrm{Im}(k) &> \frac{1}{b}\,\mathrm{Im}(s) \end{aligned} \qquad (2.32)$$

which are readily obtained from equation (2.16) and are valid for all real values of γ. In this process, a type of series can be found in which the various types of geometrical reflected paths appear as separate terms distinguished by appropriate changes of phase according to the increased length of path. This method is pursued with some success in Section 4, but the method is much more successful in Section 3 in a similar problem involving pulse transmission in a plate.

This problem of pulse transmission in a flat plate is much more susceptible to formal integration and is to some extent an approximation to the present situation. Since it will serve as a guide in the more difficult problem of the rod, it is considered next.

3. THE PROPAGATION OF AN INITIALLY PLANE DILATATIONAL PULSE THROUGH AN INFINITE PLATE

As a parallel investigation, let us consider an infinite plate of uniform thickness, $2R_o$, which, if described in rectangular coordinates, is bounded by the two plane and parallel surfaces $X = R_o$, and $X = -R_o$, and extends to infinity in all directions perpendicular to the X axis. We will assume a uniform distribution of double sources, whose strength depends on the time in the same manner as in the previous example, along the infinite rectangular strip defined by $Z = 0$, $|X| \leq R_o$.

The effect of the initially plane dilatational pulse, and the corresponding reflections form the stress-free bounding surfaces, can again be readily studied by calculating the average stress $\bar{S}_{ZZ}$ on another infinite strip defined by Z = constant and $|X| \leq R_o$ which is parallel to the first strip containing the double sources and located a distance, Z, away.

Proceeding in the same manner as in the case of the infinite rod, the dynamic elastic equations, when expressed in rectangular Cartesian coordinates, take the form

$$\begin{aligned}
\frac{\partial^2 \xi_X}{\partial t^2} - (a^2 - b^2)\frac{\partial \Delta}{\partial X} - b^2\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right]\xi_X &= \zeta_X \\
\frac{\partial^2 \xi_Y}{\partial t^2} - (a^2 - b^2)\frac{\partial \Delta}{\partial Y} - b^2\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right]\xi_Y &= \zeta_Y \qquad (3.1) \\
\frac{\partial^2 \xi_Z}{\partial t^2} - (a^2 - b^2)\frac{\partial \Delta}{\partial Z} - b^2\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right]\xi_Z &= \zeta_Z
\end{aligned}$$

$$\Delta = \frac{\partial \xi_X}{\partial X} + \frac{\partial \xi_Y}{\partial Y} + \frac{\partial \xi_Z}{\partial Z}$$

for an isotropic and homogeneous solid, where ξ_X, ξ_Y, and ξ_Z are the components of displacement, and ζ_X, ζ_Y, and ζ_Z are the components of the body forces per unit mass in the di ections of increasing X, Y, and Z respectively.

Since the planes $X = \pm R_o$ which bound the plate must be stress free, the boundary conditions $S_{XX} = S_{XY} = S_{XZ} = 0$ must be satisfied on these surfaces. These may be written as

$$\frac{S_{XX}}{\rho} = (a^2 - 2b^2)\Delta + 2b^2 \frac{\partial \xi_X}{\partial X} = 0$$

$$\frac{S_{XY}}{\rho} = b^2\left[\frac{\partial \xi_Y}{\partial X} + \frac{\partial \xi_X}{\partial Y}\right] = 0 \qquad (X = \pm R_o) \qquad (3.2)$$

$$\frac{S_{XZ}}{\rho} = b^2\left[\frac{\partial \xi_Z}{\partial X} \quad \frac{\partial \xi_X}{\partial Z}\right] = 0$$

where the symbols ρ, a, and b are those previously defined.

For the present circumstances, we wish to consider the case in which $\xi_Y = \zeta_Y = 0$, and all of the other components of $\vec{\xi}$, and $\vec{\zeta}$ are independent of Y. This is the two-dimensional case discussed at length by Rayleigh[4] and Lamb[5,6] in which the above general equations reduce to the system

$$\frac{\partial^2 \xi_X}{\partial t^2} - a^2 \frac{\partial^2 \xi_X}{\partial X^2} - b^2 \frac{\partial^2 \xi_X}{\partial Z^2} - (a^2 - b^2) \frac{\partial^2 \xi_Z}{\partial X \partial Z} = \xi_X$$

$$\frac{\partial^2 \xi_Z}{\partial t^2} - b^2 \frac{\partial^2 \xi_Z}{\partial X^2} - a^2 \frac{\partial^2 \xi_Z}{\partial Z^2} - (a^2 - b^2) \frac{\partial^2 \xi_X}{\partial X \partial Z} = \xi_Z \qquad (3.3)$$

with the boundary conditions

$$\frac{S_{XX}}{\rho} = a^2 \frac{\partial \xi_X}{\partial X} + (a^2 - 2b^2) \frac{\partial \xi_Z}{\partial Z} = 0$$

$$(X = \pm R_0) \qquad (3.4)$$

$$\frac{S_{XZ}}{\rho} = b^2 \frac{\partial \xi_X}{\partial Z} + b^2 \frac{\partial \xi_Z}{\partial X} = 0$$

Applying the Fourier transformations indicated by equations (2.5) and (2.7) to this system with the assumptions $\xi_X = \xi_Z = 0$ for $t < 0$, and ξ_X, ξ_Z, and their first partial derivatives are zero at $t = 0$, one obtains the transformed system

$$(b^2\gamma^2 - s^2)u_X - i\gamma(a^2 - b^2) \frac{\partial u_Z}{\partial X} - a^2 \frac{\partial^2 u_X}{\partial X^2} = p_X$$

$$(a^2\gamma^2 - s^2)u_Z - i\gamma(a^2 - b^2) \frac{\partial u_X}{\partial X} - b^2 \frac{\partial^2 u_Z}{\partial X^2} = p_Z \qquad (3.5)$$

with the boundary conditions

$$a^2 \frac{\partial u_X}{\partial X} + i\gamma(a^2 - 2b^2)u_Z = 0$$

$$(X = \pm R_0) \qquad (3.6)$$

$$b^2 \frac{\partial u_Z}{\partial X} + i\gamma b^2 u_X = 0$$

Making the assumption

$$p_X = 0$$

$$p_Z = \frac{-2\gamma g(s)a}{\rho s} \tag{3.7}$$

corresponding to equation (2.11) of the previous analysis, we find that the system of equations (3.5) possesses a very simple particular solution independent of X which does not in general satisfy the boundary conditions (3.6).

This solution is easily seen to be

$$u_{XP} = 0$$

$$u_{ZP} = \frac{2\gamma g(s)}{a\rho sh^2}$$

To this solution we may add any linear combination of solutions of the homogeneous system of equations

$$\begin{aligned} (b^2\gamma^2 - s^2)u_X - i\gamma(a^2 - b^2)\frac{\partial u_Z}{\partial X} - a^2\frac{\partial^2 u_X}{\partial X^2} &= 0 \\ (a^2\gamma^2 - s^2)u_Z - i\gamma(a^2 - b^2)\frac{\partial u_X}{\partial X} - b^2\frac{\partial^2 u_Z}{\partial X^2} &= 0 \end{aligned} \tag{3.8}$$

Since these equations are each of the second order with coefficients independent of X, four linearly independent exponential solutions can be found. It is convenient to write these as

$$u_{X1} = ih\epsilon^{ihX} \qquad u_{Z1} = i\gamma\epsilon^{ihX}$$

$$u_{X2} = -ih\epsilon^{-ihX} \qquad u_{Z2} = i\gamma\epsilon^{-ihX}$$

$$u_{X3} = i\gamma\epsilon^{ikX} \qquad u_{Z3} = -ik\epsilon^{ikX} \tag{3.9}$$

$$u_{X4} = i\gamma\epsilon^{-ikX} \qquad u_{Z4} = ik\epsilon^{-ikX}$$

where h and k are given by equation (2.16).

Thus we have as a general solution of equations (3.5), (3.6), and (3.7)

$$u_X = \sum_{m=1}^{4} C_m u_{Xm}$$

$$u_Z = \sum_{m=1}^{4} C_m u_{Zm} + \frac{2\gamma g(s)}{\mu a s h^2} \tag{3.10}$$

where the four numbers C_m are arbitrary but independent of X.

By substituting this solution into the boundary conditions (3.6) and solving for the numbers C_m, one finds

$$C_1 = C_2 = \frac{i(1-2b^2/a^2)\,\dfrac{s\gamma^2 g(s)}{\mu b^2 s h^2}\,(k^2-\gamma^2)\sin(kR_o)}{\left[(k^2-\gamma^2)^2\cos(hR_o)\sin(kR_o)+4\gamma^2 hk\,\sin(hR_o)\cos(kR_o)\right]} \tag{3.11}$$

$$C_3 = -C_4 = \frac{i(1-2b^2/a^2)\,\dfrac{s\gamma^2 g(s)}{\mu b^2 s h^2}\,(2\gamma h)\sin(hR_o)}{\left[(k^2-\gamma^2)^2\cos(hR_o)\sin(kR_o)+4\gamma^2 hk\,\sin(hR_o)\cos(kR_o)\right]}$$

Upon substituting the values of u_X and u_Z thus determined into equation (2.10), it is convenient to write

$$\xi_X = \xi_{XC} \qquad \xi_Z = \xi_{ZP} + \xi_{ZC} \tag{3.12}$$

where ξ_{ZP} is again given by equation (2.21),

$$\xi_{XC} = \frac{2(1-2b^2/a^2)}{(2\pi)^2 i} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \frac{\left[(k^2-\gamma^2)\sin(hX)\sin(kR_o)+2\gamma^2\sin(kX)\sin(hR_o)\right] s\gamma^2 g(s)\, \epsilon^{-i(st-\gamma Z)}}{\left[(k^2-\gamma^2)^2\cos(hR_o)\sin(kR_o)+4\gamma^2 hk\sin(hR_o)\cos(kR_o)\right]\rho b^2 sh}\, d\gamma\, ds$$

$$\tau > \delta > 0$$

and (3.13)

$$\xi_{ZC} = \frac{-2(1-2b^2/a^2)}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \frac{\left[(k^2-\gamma^2)\cos(hX)\sin(kR_o)-2hk\cos(kX)\sin(hR_o)\right] s\gamma^3 g(s)\, \epsilon^{-i(st-\gamma Z)}}{\left[(k^2-\gamma^2)^2\cos(hR_o)\sin(kR_o)+4\gamma^2 hk\sin(hR_o)\cos(kR_o)\right]\rho b^2 sh^2}\, d\gamma\, ds$$

Here it is again found that the complementary part of the solution ξ_{XC} and ξ_{ZC} are zero when the Poisson ratio σ is zero ($a^2 = 2b^2$). The general similarity to equation (2.22) is rather obvious.

In rectangular coordinates the tensile stress component S_{ZZ} is given by

$$\frac{S_{ZZ}}{\rho} = (a^2-2b^2)\Delta + 2b^2\frac{\partial \xi_Z}{\partial Z} = a^2\frac{\partial \xi_Z}{\partial Z} + (a^2-b^2)\frac{\partial \xi_X}{\partial X} \tag{3.14}$$

Since, in the present example the displacements are independent of Y, the average value $\bar{S}_{ZZ}$ of S_{ZZ} over an infinite strip defined by Z = constant $|X| \leq R_o$ is given by

$$\bar{S}_{ZZ} = \frac{1}{2R_o}\int_{-R_o}^{R_o} S_{ZZ}\,dX = \frac{\rho a^2}{2R_o}\int_{-R_o}^{R_o}\frac{\partial \xi_Z}{\partial Z}dX + \frac{\rho(a^2-2b^2)}{2R_o}\Big[\xi_X\Big]_{X=-R_o}^{X=R_o} \tag{3.15}$$

From this expression it is easily shown that

$$\bar{S}_{ZZ} = S_{ZZP} + \bar{S}_{ZZC} \tag{3.16}$$

where

$$\bar{S}_{ZZC} = \frac{(1-2b^2/a^2)^2}{(2\pi)^2 i}\int_{-\infty+i\tau}^{\infty+i\tau}\int_{-\infty}^{\infty} \frac{s^3\gamma^2 a g(s)\sin(hR_o)\sin(kR_o)\,\epsilon^{-i(st-\gamma Z)}}{\left[(k^2-\gamma^2)^2\cos(hR_o)\sin(kR_o)+4\gamma^2 hk\sin(hR_o)\cos(kR_o)\right]R_o b^4 h^3}\,d\gamma\,ds \tag{3.17}$$

$$\tau > \delta > 0$$

and S_{ZZP} is given by equation (2.27).

Since the integrads in equations (3.13) and (3.17) are even functions of h and k, they are single-valued in s and γ and permit an evaluation in terms of the residues at the poles which appear due to the vanishing of the denominators. For $t < 0$, $|X| \leq R_o$, the integrands are regular for $I_m(s) > \delta > 0$, and the integral over s is zero for all real γ. Thus, $S_{ZZC} = 0$ for $t < 0$. For $t > 0$, $|X| \leq R_o$, the integrands are regular for $I_m(s) < \tau$ except for poles which occur on the real axis in the s plane for real γ. The vanishing of the bracketed part of the denominator in these expressions is the frequency wave-number condition for the simple harmonic modes of the even type discovered by Rayleigh[4], and the evaluation by residues for $t > 0$ will result in an expression in terms of these normal modes. This process will be quite complicated and is not capable of showing what kind of geometrical path is involved in any part of the disturbance arriving at a given place at a given time.

This process becomes a practical expedient in the limiting case $R_o \rightarrow 0$, as all of the poles due to the simple harmonic modes degenerate into a simple pole. Letting $R_o \rightarrow 0$ in equation (3.17), it reduces to

$$\lim_{R_o \to 0} \bar{S}_{ZZC} = \frac{-1}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} \frac{2(1-2b^2/a^2)s\gamma^2 a g(s)\, \epsilon^{-i(st-\gamma Z)}}{(s^2-\gamma^2a^2)(s^2-\gamma^2C_o^2)}\, d\gamma\, ds \qquad (3.18)$$

where C_o is the classical velocity of longitudinal waves in a thin flat plate[18] and is given by

$$C_o^2 = \frac{4b^2(a^2 - b^2)}{a^2} \tag{3.19}$$

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[18] A. E. H. Love, Op. Cit., p 497-498.

This integral is readily evaluated by the same steps as employed in connection with the integral in equation (2.21) to obtain

$$\lim_{R_o \to 0} \bar{S}_{ZZC} = (a/C_o)F(t - |Z|/C_o) - F(t - |Z|/a)$$

which when combined with equations (2.27) and (3.16) yields

$$\lim_{R_o \to 0} \bar{S}_{ZZ} = (a/C_o)F(t - |Z|/C_o) \tag{3.20}$$

We will later see that this is an asymptotic result valid as $(Z/2R_o) \to \infty$. For the present, let us simply note that for $1/2 > \sigma > -1$, $a^2 > C_o^2 > 2b^2$ and that at $\sigma = 0$, $a^2 = C_o^2 = 2b^2$. Thus, for $\sigma = 0$, the disturbance in this limiting case reduces to the original dilatational disturbance.

For the practical study of the transients involved in the complementary solution, a different approach is much more successful. By expressing the sines and cosines in equation (3.17) with exponentials and multiplying numerator and denominator of the integrand by $\epsilon^{i(h+k)R_o}$, it can be put in the form

$$\bar{S}_{ZZC} = \frac{(1-2b^2/a^2)^2}{(2\pi)^2 R_o} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \frac{2s^3\gamma^2 sg(s)\left[1 - \epsilon^{2ihR_o}\right]\left[1 - \epsilon^{2ikR_o}\right]\epsilon^{-i(st-\gamma Z)}}{b^4h^3 f_1(1 - w_1)(1 - w_2)} d\gamma ds \qquad (3.21)$$

where

$$w_1 + w_2 = \frac{f_2}{f_1}\left[\epsilon^{2ikR_o} - \epsilon^{2ihR_o}\right] \qquad (3.22)$$

$$w_1 w_2 = -\epsilon^{2i(h+k)R_o}$$

$$f_1 = (k^2 - \gamma^2)^2 + 4\gamma^2 hk$$

and (3.23)

$$f_2 = (k^2 - \gamma^2) - 4\gamma^2 hk$$

From the identity

$$\frac{1}{(1-w_1)(1-w_2)} = \frac{1}{w_1 - w_2}\left[\frac{1}{1-w_1} - \frac{1}{1-w_2}\right]$$

and the series expansion

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

which is uniformly and absolutely convergent for $|w| < 1$ it is easily found that

$$\frac{1}{(1-w_1)(1-w_2)} = \sum_{n=0}^{\infty} \frac{w_1^{n+1} - w_2^{n+1}}{w_1 - w_2} \tag{3.24}$$

and this series is uniformly and absolutely convergent if $|w_1| < 1$ and $|w_2| < 1$.

From the equations (3.22) and the algebraic identity

$$\frac{w_1^{n+1} - w_2^{n+1}}{w_1 - w_2} = \sum_{q=0}^{n/2} \frac{(n-q)!(-1)^q}{(n-2q))!q!} (w_1 w_2)^q (w_1 + w_2)^{n-2q} \tag{3.25}$$

which is valid when n is a positive integer or zero, one readily obtains by collecting like powers of the exponentials the result

$$\frac{w_1^{n+1} - w_2^{n+1}}{w_1 - w_2} = \sum_{n_2=0}^{n} P^{0}_{n-n_2, n_2}(r_2/r_1) \left[-\epsilon^{2ihR_o}\right]^{n-n_2} \left[-\epsilon^{2ikR_o}\right]^{n_2} \tag{3.26}$$

where

$$\mathrm{P}^{m}_{n_1 n_2}(f_2/f_1) = \sum_{q=0}^{\substack{\leq n_1 \\ \leq n_2}} \frac{(n_1+n_2-m-q)!(-1)^q}{(n_1-q)!(n_2-q)!\,q!} (f_2/f_1)^{n_1+n_2-2q} \tag{3.27}$$

when n_1, n_2, and m are positive integers or zero and m does not exceed the larger of n_1 and n_2.

From the obvious elimination between equations (3.26) and (3.24), the result

$$\frac{1}{(1-w_1)(1-w_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathrm{P}^{o}_{n_1 n_2}(f_2/f_1)(-1)^{n_1} \epsilon^{2iR_o(n_1 h+n_2 k)} \tag{3.28}$$

is obtained. Then multiplying by the appropriate factors and collecting like powers of the exponentials, we have the expansion

$$\frac{\left[1 - \epsilon^{2ihR_o}\right]\left[1 - \epsilon^{2ikR_o}\right]}{(1 - w_1)\,(1 - w_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty}$$

$$\cdot \left[(n_1+n_2) + (f_1/f_2)(n_1-n_2)\right] \mathrm{P}^{1}_{n_1 n_2}(f_2/f_1)(-1)^{n_1} \epsilon^{2iR_o(n_1 h+n_2 k)} \tag{3.29}$$

with the stipulation that the leading term $(n_1=n_2=0)$ has the value unity.

The absolute convergence of these double series is easily studied with the aid of a comparison series. It is obvious that the terms of the series (3.28) are in magnitude equal to or less than the corresponding terms of the double series of positive terms

$$V = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{q=0}^{\substack{\leq n_1 \\ \leq n_2}} \frac{(n_1+n_2-q)!}{(n_1-q)!(n_2-q)!q!} |f_2/f_1|^{n_1+n_2-2q} \left| \epsilon^{2iR_o(n_1h+n_2k)} \right|$$

Forming the diagonal sum[19], V_n, of all terms in this series such that

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[19] W. L. Ferrar, A Text-Book of Convergence (Oxford 1938), p.138-145.

$n_1 + n_2 \leq n$, it is easily shown that

$$V_n = \frac{1 - (v_1^{n+2} - v_2^{n+2})/(v_1 - v_2)}{(1-v_1)(1-v_2)}$$

where

$$v_1 + v_2 = \left| (f_2/f_1)\, \epsilon^{2ihR_o} \right| + \left| (f_2/f_1)\, \epsilon^{2ikR_o} \right|$$

$$v_1 v_2 = -\left| \epsilon^{2ihR_o} \right| \left| \epsilon^{2ikR_o} \right|$$

Thus, since

$$\lim_{n \to \infty} V_n = \frac{1}{(1-v_1)(1-v_2)} = V$$

when $|v_1| < 1$ and $|v_2| < 1$, it is clear that the series (3.28) is absolutely convergent if $|v_1| < 1$ and $|v_2| < 1$. Since the smaller in magnitude of v_1 and v_2 is always negative, it is obvious that these two conditions are equivalent to $(1-v_1)(1-v_2) > 0$ or that

$$1 > |f_2/f_1| \left[\epsilon^{-2R_o \mathrm{Im}(h)} + \epsilon^{-2R_o \mathrm{Im}(k)} \right] + \epsilon^{-2R_o [\mathrm{Im}(h) + \mathrm{Im}(k)]} \qquad (3.30)$$

is a sufficient condition for the absolute convergence of the series (3.28) and (3.29).

In order to use the series representation (3.29) over the range of integration involved in equation (3.21) and integrate term by term, it is desirable that it be uniformly convergent over this range of integration. It is easy to demonstrate the uniformity of its absolute convergence for $\mathrm{Im}(s) > 0$ from the inequality (3.30). It is evident that the exponentials appearing in this expression have an appropriate behavior from equation (2.32) when $R_o > 0$, but the quantity $|f_2/f_1|$ requires some further examination.

From equation (2.16) it is readily established that

$$\gamma^2 = (b^2k^2 - a^2h^2)/(a^2 - b^2) \qquad (3.31)$$

and substituting this result into equation (3.23) one finds

$$\begin{aligned} f_1 &= 4(1-\sigma)^2(h-k)(h-m_1k)(h-m_2k)(h-m_3k) \\ f_2 &= 4(1-\sigma)^2(h+k)(h+m_1k)(h+m_2k)(h+m_3k) \end{aligned} \qquad (3.32)$$

where m_1, m_2, and m_3 are the roots of the cubic equati[illegible]

$$m^3 - m^2(1+\sigma)/(1-\sigma) - m - \sigma^2/(1-\sigma)^2 = 0 \qquad (3.33)$$

Investigation of the roots of this equation for finite real values of the Poisson ratio σ reveals that there is never more than one positive real root. If m_1 is taken to be this root, it is found that $m_1 > 1$, $Re(m_2) \leq 0$, and $Re(m_3) \leq 0$. When $\sigma = 0$, all of the roots are real, but $m_1 > 1$, one root is negative, and the third is zero.

From the definitions of h and k, equation (2.16), it is readily established that

$$Re(k/h) \geq 0 \tag{3.34}$$

and since $m_2 + m_3 < 0$ and $m_2 m_3 \geq 0$ it follows that

$$\frac{(h+m_2k)(h+m_3k)}{(h-m_2k)(h-m_3k)} = \frac{(h/k + m_2m_3k/h + m_2 + m_3)}{(h/k + m_2m_3k/h - m_2 - m_3)} \leq 1$$

Consequently, from the equations (3.32), we find

$$|f_2/f_1| \leq \left|\frac{h+k}{h-k}\right| \left|\frac{h+m_1k}{h-m_1k}\right|$$

Now with the aid of the equations (2.16)

$$\left|\frac{h+k}{h-k}\right| = \frac{a^2b^2}{a^2-b^2}\frac{|h+k|^2}{|s|^2} \leq \frac{a^2b^2}{a^2-b^2}\frac{(|h|+|k|)^2}{|s|^2} \leq \frac{a^2b^2}{a^2-b^2}\frac{(|s|/a+|\gamma|+|s|/b+|\gamma|)^2}{|s|^2}$$

or

$$\left|\frac{h+k}{h-k}\right| \leq \frac{a+b}{a-b}\left[1 + \frac{2ab|\gamma|}{(a+b)|s|}\right]^2$$

Similarly

$$\left|\frac{h+m_1k}{h-m_1k}\right| = \frac{|h+m_1k|^2}{|h^2-m_1^2k^2|} \quad \frac{a^2b^2\ |h+m_1k|^2}{(m_1^2a^2-b^2)|s^2-C_R^2\gamma^2|} \leq \frac{a^2b^2\ (|h_1|+m_1|k|)^2}{(m_1^2a^2-b^2)|s^2-C_R^2\gamma^2|}$$

or

$$\left|\frac{h+m_1k}{h-m_1k}\right| \leq \frac{a^2b^2(|s|/a+|\gamma|+m_1|s|/b+m_1|\gamma|)^2}{(m_1^2a^2-b^2)|s^2-C_R^2\gamma^2|} = \frac{(m_1a+b)\left[1+\frac{(m_1+1)ab|\gamma|}{(m_1a+b)|s|}\right]^2}{(m_1a-b)\left|1-C_R^2\frac{|\gamma|^2}{|s|^2}\right|}$$

where C_R is the speed of Rayleigh[20] surface waves and is the positive

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[20] Lord Rayleigh, Proc. London Math. Soc., 17, p. 4-11 (1887).

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root of

$$C_R^2 = \frac{b^2(m_1^2-1)}{m_1^2-(b/a)^2}. \tag{3.35}$$

Since $m_1 > 1 > b/a$ for physically real values of the Poisson ratio, it follows that $a > b > C_R$ for physically real situations.

Collecting the several inequalities, it is evident that

$$|f_2/f_1| \leq \left[\frac{a+b}{a-b}\right]\left[\frac{m_1a+b}{m_1a-b}\right]\frac{\left[1+\frac{2ab}{(a+b)}\left|\frac{\gamma}{s}\right|\right]^2\left[1+\frac{(m_1+1)ab}{(m_1a+b)}\left|\frac{\gamma}{s}\right|\right]}{1-C_R^2\frac{\gamma^2}{s^2}} \tag{3.36}$$

and that $|f_2/f_1|$ is thus bounded except as $\gamma/s \to \infty$ or $\pm 1/C_R$. As a function of γ/s, f_2/f_1 has a double pole at ∞ and simple poles at $\pm 1/C_R$.

If $Im(s) \geq 0$, it follows that $|f_2/f_1|$ is bounded so long as γ is restricted to any finite range of real values. The right-hand side of the inequality (3.30) is consequently bounded over all values of Re(s) and real γ when $Im(s) > 0$ and $R_o > 0$, because $Im(h) > 0$, $Im(k) > 0$ and both are functions which vary as γ when $\gamma \to \infty$ or when γ and $Re(s) \to \infty$ together in such a way that $\gamma/s \to C$ and $C < b$. Thus, the right-hand side of the inequality (3.30) has a least upper bound M which is a function of Im(s) but uniform over all values of Re(s) and real γ. Since Im(h) and Im(k) are functions which vary as Im(s), the right-hand side of the inequality $\to 0$ as $Im(s) \to \infty$ for all values of Re(s) and γ in the domain considered, and the least upper bound $M \to 0$ as $Im(s) \to \infty$. Consequently, there exists a positive number δ such that $M < 1$ for $Im(s) > \delta$. It is thus evident that the series (3.28) and (3.29) are uniformly absolutely convergent over the domain of integration in equation (3.21) for $Im(s) > \delta > 0$. The number $\delta \to \infty$ as $R_o \to 0$.

Substituting the series (3.29) into equation (3.21) and integrating term by term, one finds

$$\bar{S}_{ZZC} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1 n_2}(R_o, Z, t) \tag{3.37}$$

where

$$C_{n_1 n_2}(R_o, Z, t) = (-1)^{n_1} \frac{2(1-2b^2/\rho^2)^2}{(2\pi)^2 R_o} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \left[(n_1+n_2)/f_1 + (n_1-n_2)/f_2\right] P^1_{n_1 n_2}(f_2/f_1) \frac{\gamma^2 as^3 g(s)}{b^4 h^3} \epsilon^{-i(st-\gamma Z-2R_o n_1 h-2R_o n_2 k)} d\gamma ds \qquad (3.38)$$

$$\tau > \delta > 0$$

and the inversion of the order of summation and integration is justified by the uniform convergence of the integrals involved as well as the series.

Elementary inspection of the exponentials occurring in these integrals leads to the identification of each with the net contribution of all the waves which have traversed the thickness of the plate n_1+n_2 times, n_1 times as a dilatational wave and n_2 times as a rotational wave. This identification will be more completely justified in terms of the group transit time to be associated with each such wave group.

The integrals in equation (3.38) can be put into a much more useful form by means of a change of variables and certain contour deformations. Expressing the integrands in terms of the new variable $\bar{\gamma}$ instead γ by the transformation equations

$$\gamma = \bar{\gamma}s \qquad f_1 = \bar{f}_1 s^4$$

$$h = \bar{h}s \qquad f_2 = \bar{f}_2 s^4 \qquad (3.39)$$

$$k = \bar{k}s$$

and the group-transit time $\bar{t}_{n_1 n_2}$ defined by

$$\bar{t}_{n_1 n_2} = \bar{\gamma} Z + 2R_o n_1 \bar{h} + 2R_o n_2 \bar{k} \tag{3.40}$$

the above integral becomes

$$C_{n_1 n_2}(R_o, Z, t) = (-1)^{n_1} \frac{2(1-2b^2/a^2)^2}{(2\pi)^2 R_o} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{\bar{\Gamma}_o}$$

$$\cdot \left[(n_1+n_2)/\bar{f}_1 + (n_1-n_2)/\bar{f}_2 \right] P^1_{n_1 n_2}(\bar{f}_2/\bar{f}_1) \frac{\bar{\gamma}^2 a g(s) \epsilon^{-is(t-\bar{t}_{n_1 n_2})}}{b^4 \bar{h}^3 s} \, d\bar{\gamma} \, ds \tag{3.41}$$

where $\bar{\Gamma}_o$ is the straight line ($\arg \bar{\gamma} = -\arg s$) shown in Fig. 3.1 which corresponds to the real axis in the γ-plane. From equations (2.16), (3.23) and (3.39) it is obvious that

$$\begin{aligned}
\bar{h} &= (1/a^2 - \bar{\gamma}^2)^{1/2} & \qquad \mathrm{Re}\,(\bar{k}) &\geq 0 \\
\bar{k} &= (1/b^2 - \bar{\gamma}^2)^{1/2} & \qquad \mathrm{Re}\,(\bar{k}) &\geq 0 \\
\bar{f}_1 &= (\bar{k}^2 - \bar{\gamma}^2)^2 + 4\bar{\gamma}^2\, \overline{hk} \\
\bar{f}_2 &= (\bar{k}^2 - \bar{\gamma}^2)^2 + 4\bar{\gamma}^2\, \overline{hk}
\end{aligned} \tag{3.42}$$

Next, we consider the deformation of the contour $\bar{\Gamma}_o$ into a new contour $\bar{\Gamma}_{n_1 n_2}$ which is chosen such that the group-transit time $\bar{t}_{n_1 n_2}$ is real and positive at all points on the contour. Since there are poles of varied orders at [illegible], branch points at $\bar{\gamma} = \pm(1/b)$ and combined poles and branch points at $\bar{\gamma} = \pm(1/a)$, it will be convenient to introduce a cut in the $\bar{\gamma}$ plane extending from $\bar{\gamma} = 1/a$ out the positive real axis to ∞ and another cut extending from $\bar{\gamma} = -1/a$ out the negative real axis to $-\infty$. All of the singularities lie on these cuts as shown in Fig. 3.1 and the system of equations (3.42) is valid throughout the plane cut in this manner. This process of cutting the plane selects one sheet of a four sheeted Riemann surface. An alternative definition of the sheet selected is the conditions $Re(\bar{h}) \geq 0$ and $Re(\bar{k}) \geq 0$.

The construction of the required contour presents no very serious problem as there are never more than two roots of equation (3.40) in $\bar{\gamma}$ consistent with equation (3.42). As equation (3.40) is an equation expressing $\bar{t}_{n_1 n_2}$ as a function of $\bar{\gamma}$ it is convenient to define a reciprocal relationship expressing $\bar{\gamma}$ as a function of $\bar{t}_{n_1 n_2}$ when $\bar{t}_{n_1 n_2}$ is real. This relationship will be expressed as $\bar{\gamma}_{n_1 n_2}(\bar{t})$ and will be defined as the root lying in the upper half of the $\bar{\gamma}$ plane when these two roots are complex conjugates and as the root with the smaller magnitude when both roots are real. This provides a unique and continuous transformation from $\bar{t}_{n_1 n_2}$ to $\bar{\gamma}$ for real values of $\bar{t}_{n_1 n_2} \geq 2R_o[(n_1/a) + (n_2/b)]$. It is desirable to think of real values of $\bar{\gamma}_{n_1 n_2}(\bar{t})$ as $\lim_{\epsilon \to 0} \left[\bar{\gamma}_{n_1 n_2}(\bar{t}) + i\epsilon\right]$ because of the cut-plane.

From the equations (3.42) it is obvious that when $\bar{\gamma}$ is in the upper half-plane and much larger in magnitude than 1/a or 1/b, $\bar{h} \sim \bar{k} \sim -i\bar{\gamma}$. Applying this result to equation (3.40), it is obvious that when $\bar{t}$ and $\bar{\gamma}$ are large the asymptotic result

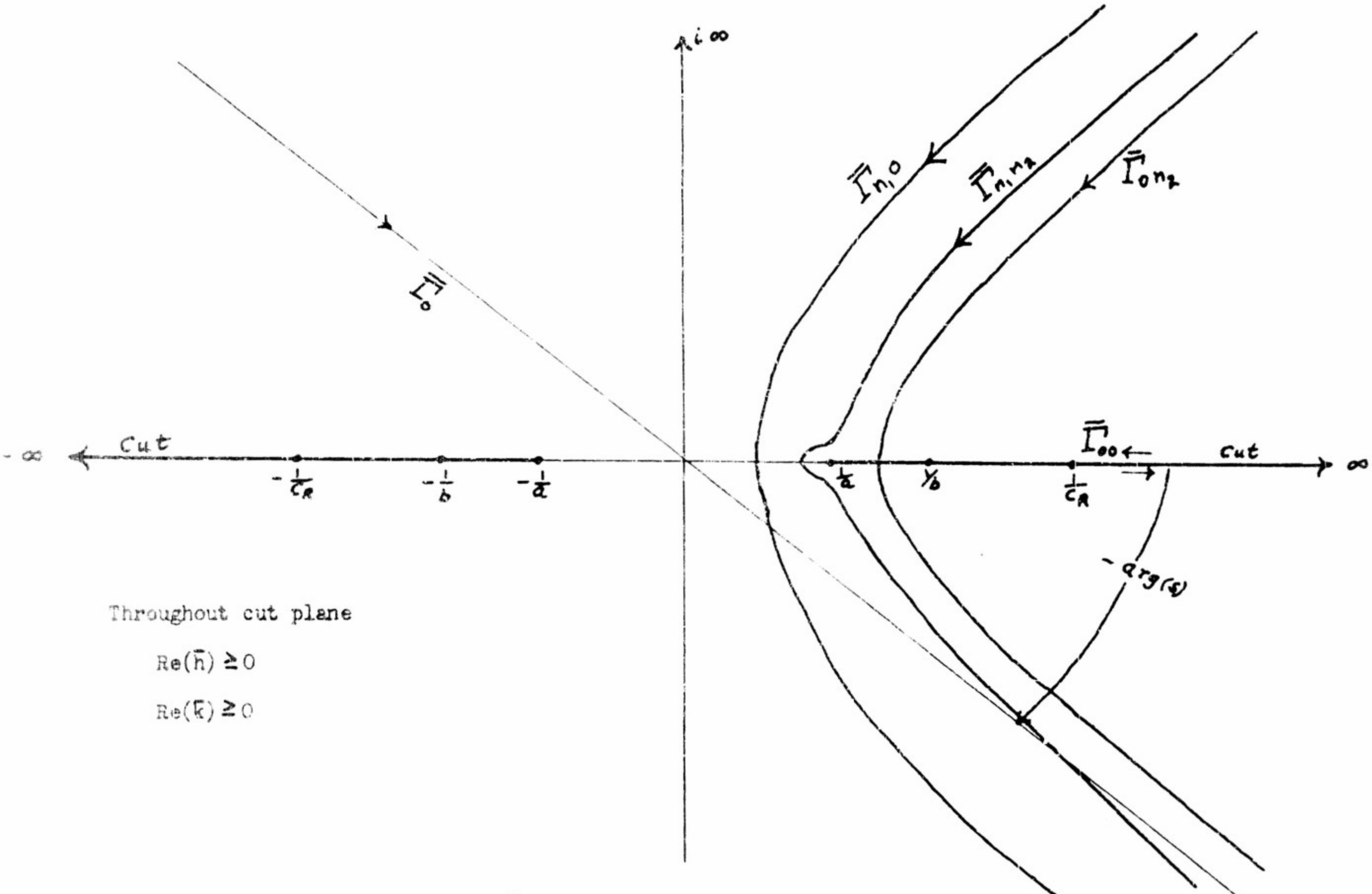


Fig. 3.1 Cut $\bar{\gamma}$-plane and deformed contours.

$$\gamma_{n_1 n_2}(\bar{t}) \sim \bar{t}/\left[Z - 2iR_o(n_1+n_2)\right] \tag{3.43}$$

is obtained, and, for the corresponding values of h and k, the similar expression

$$h \sim k \sim \bar{t}/\left[iZ + 2R_o(n_1+n_2)\right] \tag{3.44}$$

is obtained.

For large values of $\bar{t}$ the roots of equation (3.40) are thus complex. If we denote by $\bar{\gamma}^*_{n_1 n_2}(\bar{t})$ the complex conjugate of $\bar{\gamma}_{n_1 n_2}(\bar{t})$, these two numbers are the two roots of equation (3.40) consistent with equation (3.42). These roots decrease in magnitude as $\bar{t}$ decreases through real values and become real and equal at a value of $\bar{t}$ which we will denote by $\bar{t}^o_{n_1 n_2}$ when the corresponding real value of $\bar{\gamma}$ lies in the interval $-1/a \leq \bar{\gamma} \leq 1/a$. Under these circumstances we may obviously take $\bar{\Gamma}_{n_1 n_2}$ as the locus of $\gamma_{n_1 n_2}(\bar{t})$ as $\bar{t}$ decreases from $+\infty$ to $\bar{t}^o_{n_1 n_2}$ and the locus of $\bar{\gamma}^*_{n_1 n_2}(\bar{t})$ as $\bar{t}$ increases from $\bar{t}^o_{n_1 n_2}$ to $+\infty$. At the point where these two join, it is obvious that $\bar{t}_{n_1 n_2}$ must have a bend point minimum as $\bar{\gamma}$ varies in a continuous manner. At this point we must have

$$\frac{d\bar{t}_{n_1 n_2}}{d\bar{\gamma}} = Z - 2R_o\left[\frac{n_1}{\bar{h}} + \frac{n_2}{\bar{k}}\right]\bar{\gamma} = 0 \tag{3.45}$$

or

$$(Z/2R_o)^2 = \left[\frac{n_1}{\bar{h}} + \frac{n_2}{\bar{k}}\right]\bar{\gamma}^2$$

Eliminating $\bar{\gamma}^2$ with the equations (3.42), this takes the form

$$a^2\left[\left[(n_2\bar{h}/\bar{k})+n_1\right]^2+(Z/2R_o)^2\right] - b^2\left[\left[(n_1\bar{k}/\bar{h})+n_2\right]^2+(Z/2R_o)^2\right] = 0 \qquad (3.46)$$

This expression is obviously a quartic in the ratio $(\bar{h}/\bar{k})$ and can be shown to have only one real root $(\bar{h}/\bar{k}) \geq 0$ and no pure imaginary roots for all real values of $(Z/2R_o)$ if $n_1 \neq 0$, since $a^2 > b^2$. Since it is easily established that the left-hand side of equation (3.46) changes sign in the interval $0 < (\bar{h}/\bar{k}) < b/a$, it is apparent that this positive root in $(\bar{h}/\bar{k})$ lies in this interval, and this must correspond to the minimum group-transit time $\bar{t}^o_{n_1n_2}$ as it is the only root which satisfies $\mathrm{Re}\,(\bar{h}/\bar{k}) \geq 0$ which is valid throughout the cut plane.

Since it is easily established from the equations (3.42) that

$$\bar{\gamma}^2 = \frac{b^2 - a^2(\bar{h}/\bar{k})^2}{a^2b^2\left[1-(\bar{h}/\bar{k})^2\right]} \qquad (3.47)$$

it is apparent that for $n_1 \neq 0$, the contour described above crosses the real axis in the $\bar{\gamma}$ plane in the interval $-1/a < \bar{\gamma} < 1/a$ and thus lies entirely within the cut $\bar{\gamma}$-plane. Further study reveals that at this crossing point $\bar{\gamma}$ has the same sign as $(Z/2R_o)$ and that as $(Z/2R_o) \rightarrow \infty$, $(\bar{h}/\bar{k}) \rightarrow 0$, and $\bar{\gamma} \rightarrow 1/a$.

The case $n_1 = 0$ requires further study which is easily carried out for equation (3.46) simplifies to

$$(\bar{h}/\bar{k})^2 = (b/a)^2 - (1 - b^2/a^2)\,(Z/2R_o)^2 \qquad (3.48)$$

Since $a > b$, this will have a root in the interval $b/a \geqslant (\bar{h}/\bar{k}) > 0$ only if

$$(Z/2R_o)^2 < \frac{(b/a)^2}{1 - (b/a)^2} = \tan^2\theta_c \tag{3.49}$$

and from equation (3.47) the corresponding value of $\bar{\gamma}$, $\bar{\gamma}^o_{o\, n_2}$ is given by

$$\bar{\gamma}^o_{o\, n_2} = (1/b)(Z/2n_2R_o)\left[1 + (Z/2n_2R_o)^2\right] \tag{3.50}$$

Substituting this into equation (3.40) the corresponding minimum group-transit time $\bar{t}^o_{o\, n_2}$ is given by

$$\bar{t}^o_{o\, n_2} = (1/b)\left[Z^2 + (2n_2R_o)^2\right]^{1/2} \tag{3.51}$$

When $n_1 = 0$ and the inequality (3.49) is satisfied, the contour defined above in terms of the complex roots of equation (3.40) consistent with the equations (3.42) is an acceptable contour as it crosses the real axis in the interval $-1/a < \bar{\gamma} < 1/a$. This contour is easily shown to be a hyperbola.

When the inequality (3.49) is reversed, equation (3.48) for $(\bar{h}/\bar{k})$ has pure imaginary roots which, when substituted into equation (3.47), show that the hyperbolic contour would cross the real axis in the interval $1/a^2 < \bar{\gamma}^2 < 1/b^2$ and would thus cross a cut in the $\bar{\gamma}$ plane. This must be avoided by a detour around the end of the cut involved. For $(Z/2R_o)$ positive, this means a detour around the point $\bar{\gamma} = 1/a$ and is readily accomplished.

Instead of crossing the real axis at the double root point given by equation (3.50), $\bar{t}$ is allowed to decrease below the value of $\bar{t}^{o}_{o\,n_2}$ given by equation (3.51) until the path of $\bar{\gamma}_{o\,n_2}(\bar{t})$ now real has followed along the top of the cut to the point $1/a$, and the path of $\bar{\gamma}_{o\,n_2}(\bar{t})$ is then followed along the bottom of the cut to return to the double root point as $\bar{t}$ goes through the same values in reverse order. The resulting contour is sketched in Fig. 3.1 as the contour $\bar{\Gamma}_{o\,n_2}$. It can be defined as the limit of the contour $\bar{\Gamma}_{n_1 n_2}$ as $n_1 \rightarrow 0$. For this indented contour it is obvious from equation (3.40) that the minimum values of $\bar{t}$ and $\bar{\gamma}$ are given by

$$\begin{aligned} \bar{t}^{o}_{o\,n_2} &= (Z/a) + 2n_2 R_o (1/b^2 - 1/a^2)^{1/2} \\ \bar{\gamma}^{o}_{o\,n_2} &= 1/a \end{aligned} \qquad (3.52)$$

which is valid when

$$(Z/2n_2 R_o) \geq \tan\,\theta_c$$

The minimum group-transit time is in this case not a bend point minimum.

The contour $\bar{\Gamma}_{oo}$ may be thought of as the limit of $\bar{\Gamma}_{n_1 n_2}$ as $n_1 \rightarrow 0$ and $n_2 \rightarrow 0$, and it obviously runs from $+\infty$ to $1/a$ along the top of the cut around the point $\bar{\gamma} = 1/a$ and back to $+\infty$ along the bottom of the cut when $Z > 0$. It is obvious that the equations (3.52) hold for this case for all $Z > 0$.

The process of deforming the original contour Γ_o to the new contour $\bar{\Gamma}_{n_1 n_2}$ can thus be accomplished without passing over any singular points in the cut $\bar{\gamma}$ plane. Further it is possible to show from equations (3.36) and (3.43) that the integrands vanish as $\left[|\bar{\gamma}|^{(2n_1+2n_2-3)}\,\epsilon^{-m|\bar{\gamma}|}\right]$ when $\bar{\gamma} \to \infty$ along any curve lying between $\bar{\Gamma}_o$ and $\bar{\Gamma}_{n_1n_2}$ and m is a positive number independent of $|\bar{\gamma}|$ which is never zero except when $n_1 = n_2 = 0$. Thus, the integrals over any curves joining $\bar{\Gamma}_o$ and $\bar{\Gamma}_{n_1n_2}$ at ∞ are zero, and the integrals over $\bar{\gamma}$ are absolutely convergent at all of the intermediate stages of the deformation process. The only difficulty with this process lies in letting the contours $\bar{\Gamma}_{oo}$ and $\bar{\Gamma}_{o\,n_2}$ touch some of the singular points. These difficulties may be treated by various limiting methods, and will be considered later as they occur in only a finite number of terms for finite values of $(Z/2R_o)$.

Having thus arrived at the expression

$$\mathcal{C}_{n_1n_2}(R_o,Z,t) = (-1)^{n_1}\frac{2(1-2b^2/a^2)^2}{(2\pi)^2R_o}\int_{-\infty+i\tau}^{\infty+i\tau}\int_{\bar{\Gamma}_{n_1n_2}}$$

$$\cdot\left[(n_1+n_2)/\bar{f}_1 + (n_1-n_2)/\bar{f}_2\right]P^{1}_{n_1n_2}(\bar{f}_2/\bar{f}_1)\,\frac{\bar{\gamma}ag(s)\,\epsilon^{-is(t-\bar{t}_{n_1n_2})}}{b^4\bar{h}^3 s}\,d\bar{\gamma}\,ds \qquad (3.54)$$

in which $t-\bar{t}_{n_1n_2}$ is real at all points in the range of integration, the order of integration can be reversed and the integration over s is easily accomplished. From equation (2.25) and (2.26) it is readily found that

$$\frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{-ist}{-is}\,g(s)ds = \int_0^t F(t)dt = G(t) \qquad (3.55)$$

By applying this result to equation (3.54) the single integral expression

$$C_{n_1n_2}(R_o,Z,t) = (-1)^{n_1}\frac{(1-2b^2/a^2)^2}{R_o\pi i}\int_{\bar{\Gamma}_{n_1n_2}}$$

$$\cdot\frac{\bar{\gamma}^2 a}{b^4\bar{h}^3}\left[(n_1+n_2)/\bar{\Gamma}_1 + (n_1-n_2)/\bar{\Gamma}_2\right]P^{1}_{n_1n_2}(\bar{\Gamma}_2/\bar{\Gamma}_1)G(t-\bar{t}_{n_1n_2})d\bar{\gamma} \qquad (3.56)$$

is obtained. Since $G(t) = 0$ for $t < 0$, this expression clearly represents a transient which is zero for $t < \bar{t}^{o}_{n_1n_2}$, the minimum group-transit time, for the wave groups represented. It is also evident that the range of integration may be terminated at the points $\bar{\gamma}_{n_1n_2}(t)$ and $\bar{\gamma}^{*}_{n_1n_2}(t)$ at which $\bar{t}_{n_1n_2} = t$, as $G(t-\bar{t}_{n_1n_2}) = 0$ for $\bar{t}_{n_1n_2} < t$.

In this expression all of the waves, which have traversed the thickness of the plate n_1 times as a dilatational wave and n_2 times as a rotational wave, have been segregated according to their transit time. This is a highly desirable representation from a physical point of view and throws some light on the more abstruse connections between wave theory an geometrical optics.

Considering for the moment only real values of $\bar{\gamma}$, $\bar{h}$ and $\bar{k}$, it is evident from equation (3.40) and (3.42) that the interpretation of these variables in terms of geometrical optics is

$$\bar{h} = (1/a) \cos \theta_D$$
$$\bar{k} = (1/b) \cos \theta_R \qquad (3.57)$$
$$\bar{\gamma} = (1/a) \sin \theta_D = (1/b) \sin \theta_R$$

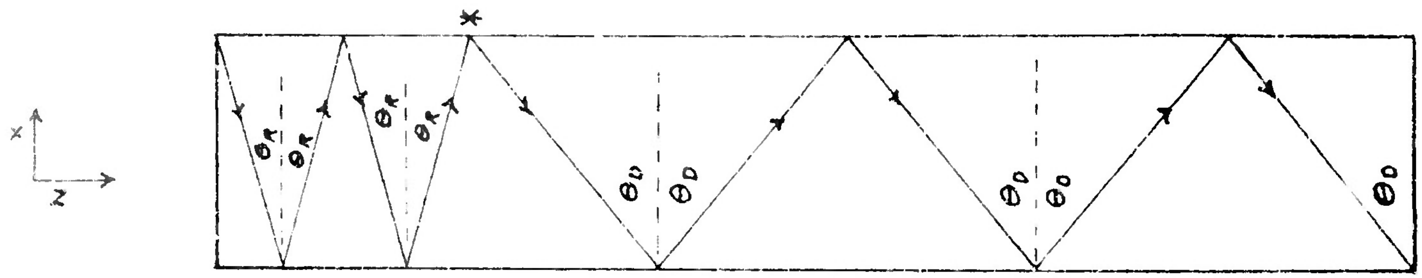
where the notation is that of Section 1. It is perfectly obvious that the last relationship is a restatement of Snell's law, equation (1.2).

A most striking feature of the representation in equation (3.56) is that the angles θ_D and θ_R are both real at only one point on the contour $\bar{\Gamma}_{n_1 n_2}$, and this point is the minimum transit time point. The representation is thus almost entirely made up of "waves" which have no geometrical meaning. The minimum transit time wave groups on the other hand have θ_R and θ_D real and correspond exactly with the geometrical minimum transit time paths to be expected from geometrical optics.

To demonstrate this equivalence, let us consider the possible geometric paths in which the disturbance travels the thickness of the plate n_1 times as a dilatational wave and n_2 times as a rotational wave obeying Snell's law where there is a transfer of mode and the law of reflection where there is no transfer of mode. Such paths are shown in Fig. 3.2.

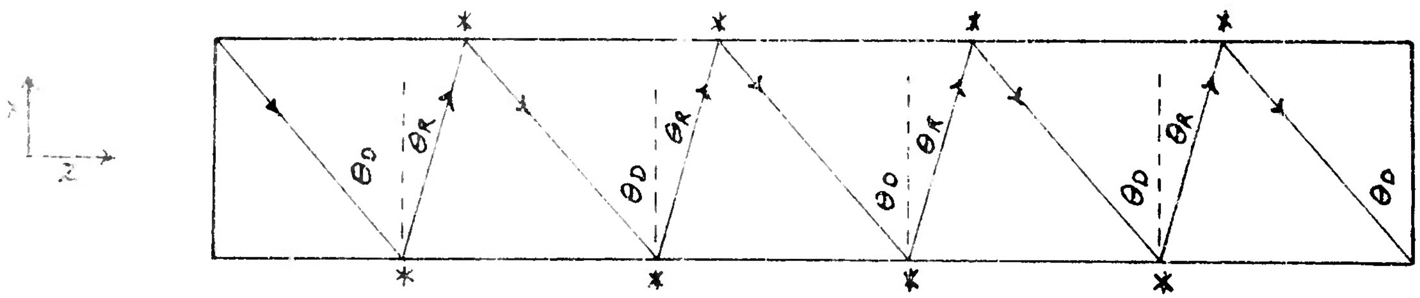
It is obvious that wherever the disturbance is rotational, it travels in a direction making the angle θ_R with the normal to the reflecting surfaces of the plate, and wherever it is dilatational with the angle θ_D.

Thus, since the thickness of the plate is $2R_o$ the distance moved along the plate is $2R_o \tan \theta_D$ for each dilatational trip and $2R_o \tan \theta_R$ for each rotational trip. The total distance traveled along the plate Z in the



$n_1=5$, $n_2=4$ with one transfer* of mode.

(a)



$n_1=5$, $n_2=4$ with eight transfers* of mode.

(b)

Fig. 3.2 Generalized wave paths in a plate.

total of $n_1 + n_2$ trips is thus

$$Z = 2R_o n_1 \tan \theta_D + 2R_o n_2 \tan \theta_R \qquad (3.58)$$

Making use of the transformation equations (3.57) this result is found to be identical with equation (3.47) which determines the minimum transit time point on the contour $\overline{\Gamma}_{n_1 n_2}$.

Since the total distance traveled in each rotational trip is $2R_o \sec \theta_R$, and that in each dilatational trip is $2R_o \sec \theta_D$, the transit time is given by

$$\bar{t}_{n_1 n_2} = \frac{2R_o n_1}{a \cos \theta_D} + \frac{2R_o n_2}{b \cos \theta_R} \qquad (3.59)$$

Multiplying equation (3.58) by $\bar{\gamma}$ and subtracting the result from equation (3.59) one finds

$$\bar{t}_{n_1 n_2} - \bar{\gamma} Z = 2R_o n_1 \left[\frac{1 - a\bar{\gamma} \sin \theta_D}{a \cos \theta_D} \right] + 2R_o n_2 \left[\frac{1 - b\bar{\gamma} \sin \theta_R}{b \cos \theta_R} \right]$$

which reduces to equation (3.40) upon using the transformation equations (3.57) to eliminate θ_D and θ_R.

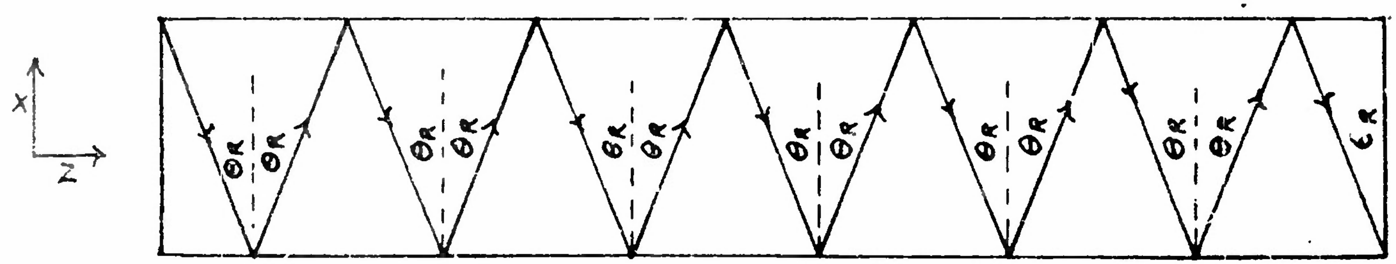
This general equivalence of the minimum transit times given by the two methods requires more study in the case $n_1 = 0$, for in this case the wave theory has given two possible minimum transit times. The correct one of these was found to be determined according to whether the inequality (3.49) was, or was not, satisfied. Setting $n_1 = 0$ in equation (3.58) and comparing with the inequality (3.49), one finds this inequality to be equivalent to

$$(\tan \theta_R)^2 \leq b^2/(a^2 - b^2) = (\tan \theta_C)^2 \tag{3.60}$$

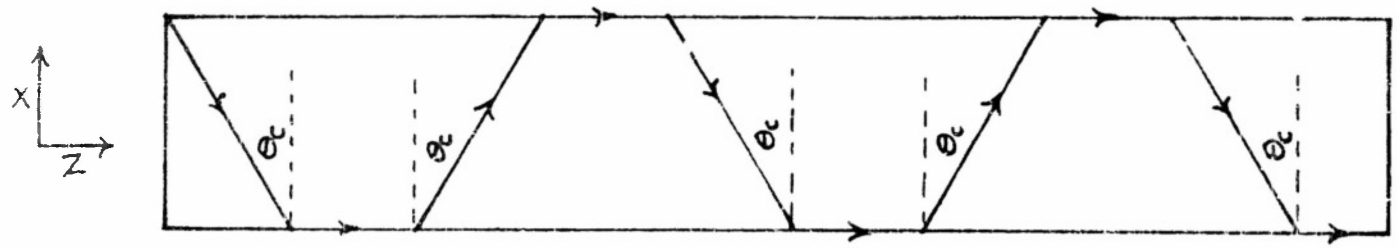
where θ_C is the critical angle of incidence for rotational waves and is given by equation (1.3). The minimum transit time, when the inequality (3.49) is satisfied, is given by equation (3.51) which is readily obtained setting $n_1 = 0$ into equations (3.58) and (3.59) and eliminating θ_R. Thus, when $\theta_R < \theta_C$, the minimum transit time corresponds to rotational waves travelling a path consisting of pure reflections as shown in Fig. 3.3a. Under these conditions, some interaction, (transfer of mode) is to be expected between the rotational and dilatational waves at the boundary where these reflections take place. However, these correspond to minimum transit time paths with $n_1 > 0$, and, thus, do not appear in the integral expression for $n_1 = 0$.

When $\theta_R > \theta_C$, the inequalities (3.49) and (3.60) are not satisfied, and the pure reflection path is not the quickest path of transmission that crosses the thickness of the plate n_2 times and as a rotational wave each time. In this case the critical angle paths, such as those shown in Fig. 3.3b, have a much smaller transit time. These paths were discussed in Section 1 in terms of geometric optics. Setting $m = 1$, $L = Z$, $n = n_2$, and $D = 2R_0$ into equation (1.5), one obtains equation (3.52) which was derived from the wave theory when the inequality (3.49) was not satisfied.

It is obvious that these two kinds of path are identical when $\theta_R = \theta_C$. However, it is not so obvious why such a transfer of mode process as this critical angle path is so important when the angle θ_R for the pure reflection path has the property $\theta_R > \theta_C$. In this situation the



Pure reflection path $n_1=0$, $n_2=13$.
(a)



Critical angle path $n_1=0$, $n_2=6$.
(b)

Fig. 3.3 Wave paths for $n_1=0$, and $n_2\neq 0$.

pure reflection path should lose no energy as it suffers total internal reflection and should then be expected to make a quite significant contribution to the overall disturbance. However, in their experimental work on the comparable case of pulse transmission in a rod, Hughes, Pondrom and Mims found that the critical angle paths explained all of the delayed arrivals which were apparently present.

Geometrical optics offers no relief from this apparent contradiction, but equation (3.56) and the nature of the contour $\Gamma_{o\,n_2}$ reveal mathematically the reason for this experimental result. When these critical angle paths give the minimum transit time, the contour $\bar{\Gamma}_{o\,n_2}$ encircles a pole at $\bar{\gamma} = 1/a$ at the minimum transit time, and the start of the corresponding disturbance is sharp and easily identified against the background.

On the other hand, when the pure reflection path, with $\theta_R < \theta_C$, gives the minimum transit time or for the general case $\bar{\Gamma}_{n_1 n_2}$, the contour is not close to this singularity, and the disturbance builds up at a much slower initial rate which is governed by the values of $d\bar{\gamma}/d\bar{t}_{n_1 n_2}$ in the neighborhood of the minimum transit time point on the contour, $\bar{\Gamma}_{n_1 n_2}$. These initial rises are much slower than those corresponding to a critical angle path and are much harder to detect in the presence of a considerable background of other disturbances.

This contrast in behavior has a distinct influence on the use of equation (3.56). When the contours touch the singularities as in the situations where a critical angle path gives the first arrival, some appeal to analyticity in the function $G(t)$ is required so as to admit small contour deformations to avoid an infinite integrand. This difficulty is easily avoided by the use of a double integral expression.

Returning to equation (3.54), and substituting the value of g(s) from equation (2.25), the expression

$$C_{n_1 n_2}(R_o,Z,t) = (-1)^{n_1} \frac{2(1-2b^2/a^2)^2}{(2\pi)^2 R_o} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{\bar{\Gamma}_{n_1 n_2}} \int_0^{\infty}$$

$$\cdot F(t') \left[(n_1+n_2)/\bar{f}_1 + (n_1-n_2)/\bar{f}_2\right] P^1_{n_1 n_2}(\bar{f}_2/\bar{f}_1) \frac{\bar{\gamma}^2 a \, \epsilon^{-is(t-\bar{t}_{n_1 n_2}-t')}}{b^4 \bar{h}^3 s} \, dt' \, d\bar{\gamma} \, ds \qquad (3.61)$$

is obtained. Then using the familiar result

$$\frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{\epsilon^{-ist}}{-is} = \begin{matrix} 1 & t > 0 \\ 0 & t < 0 \end{matrix} \qquad \tau > 0$$

the above expression can be reduced to the form

$$C_{n_1 n_2}(R_o,Z,t) = \int_0^{t-t^o_{n_1 n_2}} A_{n_1 n_2}(R_o,Z,t-t')F(t')dt' \qquad (3.62)$$

where

$$A_{n_1 n_2}(R_o,Z,t-t') = (-1)^{n_1} \frac{(1-2b^2/a^2)^2 a}{b^4 R_o \pi i} \int_{\bar{\gamma}_{n_1 n_2}(t-t')}^{\bar{\gamma}^*_{n_1 n_2}(t-t')}$$

$$\cdot \left[(n_1+n_2)/\bar{f}_1 + (n_1-n_2)/\bar{f}_2\right] P^1_{n_1 n_2}(\bar{f}_2/\bar{f}_1) \frac{\bar{\gamma}^2 d\bar{\gamma}}{\bar{h}^3} \qquad (3.63)$$

Since the functions involved in this integration over $\bar{\gamma}$ are defined for all values of $\bar{\gamma}$ in the cut $\bar{\gamma}$- plane except the actual singular points, it is easily seen that the contour of integration may be any curve joining the point $\bar{\gamma}_{n_1n_2}(t-t')$ to the point $\bar{\gamma}^*_{n_1n_2}(t-t')$ which does not cross a cut, and the value of the integral may be found for all values of R_o,Z, and $t-t'$ except those for which $\bar{\gamma}_{n_1n_2}(t-t')$ falls on a singular point. These must be found by an appropriate limiting process. In general, the numbers $\mathcal{A}_{n_1n_2}(R_o,Z,t-t')$, which determine the response of the complementary solution to a unit impulse delivered at the time t', can be found for all values of R_o,Z, and t that are of physical interest. It is also apparent that they are capable of analytic continuation into the realm of complex values of R_o,Z, and $(t-t')$.

Although the integral in equation (3.63) can be expressed in terms of algebraic processes and logarithms for all of the integral values of n_1 and n_2, these expressions are extremely complicated, and it is probable that any calculations will be more readily made by numerical methods of integration. As an example, for the simplest case $n_1 = n_2 = 0$, one finds

$$\mathcal{A}_{0\,0}(R_o,Z,t-t') = \frac{(1-2b^2/a^2)^2 a}{b^4 R_o \pi i}\int_{(t-t')/Z+0i}^{(t-t')/Z-0i} \frac{\bar{\gamma}^{-2}d\bar{\gamma}}{\bar{h}^{-3}\bar{f}_1}$$

$$= \frac{1}{R_o 2\pi i}\left[\frac{(1-2\sigma)^{3/2}}{2}\log\left\{\frac{\bar{\gamma}+\bar{k}(1-2\sigma)^{1/2}}{-\bar{\gamma}+\bar{k}(1-2\sigma)^{1/2}}\right\}+\frac{2\bar{\gamma}}{\bar{h}}\right. \tag{3.64}$$

$$\left.+i\sum_{q=1}^{3} N_q \log\left(\frac{\bar{\gamma}-i\bar{k}M_q}{\bar{\gamma}+i\bar{k}M_q}\right)\left(\frac{m_q\bar{\gamma}-i\bar{h}M_q}{m_q\bar{\gamma}+i\bar{h}M_q}\right)\right]_{\frac{t-t'}{Z}+0i}^{\frac{t-t'}{Z}-0i}$$

where

$$M_q = (2(1-\sigma)m_q^2 + 2\sigma - 1)^{1/2}$$

$$N_q = \frac{\sigma^2 M_q^3}{m_q^2\left[(m_q^2-m_1^2)(m_q^2-m_2^2)+(m_q^2-m_2^2)(m_q^2-m_3^2)+(m_q^2-m_3^2)(m_q^2-m_1^2)\right]}$$

σ is Poissons ratio, and m_1, m_2, and m_3 are the roots of equation (3.33). In addition to the obviously numerous steps in such a calculation, there is the further difficulty of having to compute a small difference of two quite large numbers when σ is nearly zero. This difficulty is present in nearly all of the integral expressions obtained in this analysis. It is due to the changing nature of the singularities located at $\bar{\gamma} = \pm(1/a)$. As $\sigma \to 0$ a pole, located on another sheet of the Riemann surface associated with the integrand for all real $\sigma \neq 0$, approaches the branch points $\pm(1/a)$ as a limit. However, it is readily observed that $\mathcal{A}_{n_1 n_2}(R_o, Z, t-t') \to 0$ as $\sigma \to 0$, and an approximate formula valid for small σ could be found in which the difficulties with large numbers are avoided. In the case of equation (3.64) the obvious approach is to expand in ascending powers of σ and use the leading term in this expansion.

The convergence of the series (3.37) is obvious for $R_o > 0$ and finite values of Z and t because it will consist of only a finite number of non-zero terms under these conditions. However, the number of terms may be quite large, and the individual terms are difficult to compute. In addition, there is a difficulty associated with the series as a whole. This stems from the fact that although any individual term has the property of beginning at a certain instant of time, it does not end or, in general, even become small as $t \rightarrow \infty$. Indeed, all of the terms except that for $n_1 = n_2 = 0$ become infinite as $t \rightarrow \infty$ even though the driving pulse, F(t), has a finite duration. Thus, the ultimate decay of the overall transient for any finite value of Z must be brought about by the destructive interference of the various wave groups which have different values of n_1 and n_2. It is thus necessary to consider all of the non-zero terms in order to get an accurate picture of what is happening at any particular place and time. A fair approximation can be had by considering only those wave groups whose minimum group-transit time is just less than the time under consideration when the driving pulse is of short duration. This follows from the behavior of the terms in which $n_1 + n_2$ is constant. For these terms we find as a result of equations (3.43) and (3.63) the asymptotic expressio

$$\sum_{n_1=0}^{n} A_{n_1, n-n_1}(R_o, Z, t-t') \sim (-1)^n \frac{2(1-2b^2/a^2)^2 a}{b^4 R_o \pi i} \tag{3.35}$$

$$\cdot \int_{(t-t')/(Z-2inR_o)}^{(t-t')/(Z+2inR_o)} \frac{\bar{\gamma}^{-2} d\bar{\gamma}}{\bar{f}_1 \bar{h}^3} \qquad n > 0 \tag{3.65}$$

which is valid when

$$t-t' >> (1/b)(Z^2 + 4n^2 R_o^2)^{1/2} \tag{3.66}$$

Thus, when t' is small as in a short driving pulse, we find that the sum of all of the terms, for which $n_1 + n_2$ is constant, remains finite as $t \rightarrow \infty$. Since these summed groups also interfere destructively as a result of the factor $(-1)^n$ is equation (3.65), it thus appears that a fair approximation can be had by neglecting all terms for which $n_1 + n_2 \leq n$ and n is such that the inequality (3.66) is satisfied. However, if any term is neglected, it is necessary to neglect all others having the same value of $n_1 + n_2$. A somewhat better approximation can be obtained by using equation (3.65) to calculate these terms.

A quite comprehensive interference effect is also to be expected as $Z \rightarrow \infty$, for under these circumstances the minimum group-transit time $\bar{t}^o_{n_1 n_2}$ approaches the minimum group-transit time $\bar{t}^o_{o\, n_2}$ and the contour $\bar{\Gamma}_{n_1 n_2} \longrightarrow \bar{\Gamma}_{oo}$ as a limit where n_1 and n_2 are any finite integers. Thus, as Z becomes large, any contour $\bar{\Gamma}_{n_1 n_2}$ effectively encircles the singularity at $\bar{\gamma} = (1/a)$ for values of $\bar{t}_{n_1 n_2}$ just greater than the minimum value $\bar{t}^o_{n_1 n_2}$ which are in turn just greater than the minimum value $\bar{t}^o_{o\,n_2}$ corresponding to a critical angle path. The corresponding contributions to $\hat{A}_{n_1 n_2}(R_o, Z, t)$ occur at times just slightly later than those of the critical angle paths included in $\hat{A}_{o\,n_2}(R_o, Z, t)$ and have a sign which is governed by the factor $(-1)^{n_1}$ appearing in equation (3.56). These successively later contributions are qualitatively of about the same size as those of the critical angle paths and as $Z \to \infty$ completely destroy the disturbance corresponding to the critical angle path. A similar destruction of the primary dilatational disturbance by terms of the form $\hat{A}_{n_1 o}(R_o, Z, t)$ takes place as $Z \to \infty$. In either case, this destructive interference by these slightly delayed but similar wave trains should be expected to ultimately shorten the wave trains corresponding to the direct dilatational and critical angle paths as Z becomes large. This is equivalent to a gradual elimination of the lower frequency components in these wave trains as Z increases. In order to maintain overall conservation of energy, these must reappear in new wave trains formed by constructive interference and appear at later times.

The very complicated undertaking of obtaining an asymptotic series and a remainder will not be attempted here, but it is possible to obtain the leading term in such an expansion in a fairly direct manner. Returning to equation (3.56) the integration was found to effectively involve only a finite part of the contour $\bar{\Gamma}_{n_1 n_2}$ as $G(t - \bar{t}_{n_1 n_2}) = 0$ for $t < \bar{t}_{n_1 n_2}$. The initial and terminal points were found to be $\bar{\gamma}_{n_1 n_2}(t)$ and $\vec{\gamma}^*_{n_1 n_2}(t)$ respectively, the values of $\bar{\gamma}$ for which $\bar{t}_{n_1 n_2} = t$.

For convenience, let us consider the case in which Z and t $\rightarrow \infty$ in such a way that t/Z approaches a limit somewhere in the interval $0 \leq (t/Z) \leq (1/b)$. We find from equation (3.40) that $\bar{\gamma}_{n_1 n_2}(t) \longrightarrow (t/Z) + 0i$ and $\vec{\gamma}^*_{n_1 n_2}(t) \longrightarrow (t/Z) - 0i$ for every finite value of n_1 and n_2. If the further assumption is made that $F(t) = 1(t)$ where $1(t)$ is Heavisides unit step function defined by

$$\begin{aligned} 1(t) &= 1 \quad t > 0 \\ 1(t) &= 0 \quad t < 0 \end{aligned} \tag{3.67}$$

the formal result

$$\lim_{Z \to \infty} \bar{S}_{ZZC} = \frac{(2i)(1-2b^2/a^2)^2}{2\pi R_o b^4} \int_{(t/Z)+0i}^{(t/Z)-0i} \frac{\bar{\gamma}^2 a}{\bar{h}^3} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1}$$

$$\cdot \left[(n_1+n_2)/\bar{f}_1 + (n_1-n_2)/\bar{f}_2 \right] P^1_{n_1 n_2}(\bar{f}_2/\bar{f}_1)\,(t-\bar{\gamma}Z-2n_1R_o\bar{h}-2n_2R_o\bar{k})\,d\bar{\gamma}$$

is readily obtained from equations (3.37), (3.54), and (3.55) in which the contour of integration may be the limiting contour or any curve joining the initial and terminal points which does not cross a cut in the $\bar{\gamma}$-plane except at the singular points. The double series involved here is easily summed to yield the result

$$\lim_{Z\to\infty} \bar{S}_{ZZC} = \frac{-(2i)(1-2b^2/a^2)^2}{2\pi} \int_{(t/Z)+0i}^{(t/Z)-0i} \frac{\bar{\gamma}^2 a^3}{(1-\bar{\gamma}^2a^2)(1-\bar{\gamma}^2C_o^2)}\, d\bar{\gamma}$$

which is readily integrated to yield

$$\lim_{Z\to\infty} \bar{S}_{ZZC} = -1(t-|Z|/a) + (a/C_o)1(t-|Z|/C_o)$$

Since this is a linear system, it is obvious that the general result is

$$\lim_{Z\to\infty} \bar{S}_{ZZC} = -F(t-|Z|/a) + (a/C_o)F(t-|Z|/C_o)$$

and that equation (3.20) is valid as $Z \to \infty$ as well as for $R_o \to 0$; and thus, it is valid as $(Z/2R_o) \to \infty$ in any way through real values.

While the preceding argument is satisfactory from an intuitive point of view, it is open to some mathematical objection as a result of exchanging the order of several limiting processes. In addition, if the limiting value of t/Z is greater than 1/b, the double series is always properly divergent over part of the integration contour. A somewhat more satisfactory argument can be made directly from equation (3.17) and will be considered in the next section.

4. THE TRANSFORMATION OF THE FORMAL SOLUTION OBTAINED FOR PULSE TRANSMISSION ALONG A ROD

The successful transformation of the formal solution obtained for pulse transmission along an infinite plate into a form which is closely related to the considerations of geometrical optics, leads one to consider the possibility of carrying out a similar transformation upon equation (2.28) which expresses the complementary solution for the similar case of pulse transmission along an infinite rod.

In so far as mathematical formalities are concerned, one might just as simply consider the more general expression

$$S_{ZZC}^{(\nu)} = \frac{(1-2b^2/a^2)^2}{(2\pi)^2 i} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \frac{(2\nu+1)2s^3\gamma^2 a g(s) J_{\nu+1/2}(hR_o) J_{\nu+1/2}(kR_o) \epsilon^{-i(st-\gamma Z)}}{b^4 h^3 R_o \Delta_\nu(h, k, R_o)} \, d\gamma \, ds \qquad (4.1)$$

$$\tau > \delta > 0$$

where $\Delta_\nu(h, k, R_o)$ is given by equation (2.19).

This expression reduces to equation (2.28) when $\nu = 1/2$, and to equation (3.17) when $\nu = 0$. More generally, when 2ν is a positive integer, this expression is the solution of the similar problem in a hyper-space having a total of $(2\nu + 2)$ space-like dimensions, and $(2\nu + 1)$ of these space-like dimensions making up the hyper-cylinder radius.

By expressing the Bessel functions of the first kind in terms of those of the third kind according to the equation

$$J_\nu(z) = (1/2)\left[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right] \qquad (4.2)$$

where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the two Bessel functions of the third kind, one obtains the relation

$$\frac{J_{\nu+1/2}(hR_o)\,J_{\nu+1/2}(kR_o)}{\Delta_\nu(h,\ k,\ R_o)} = \frac{\left(H^{(1)}_{\nu+1/2}(hR_o)+H^{(2)}_{\nu+1/2}(hR_o)\right)\left(H^{(1)}_{\nu+1/2}(kR_o)+H^{(2)}_{\nu+1/2}(kR_o)\right)}{\Delta_\nu^{(2,2)}(h,\ k,\ R_o)\,(1-w_1)(1-w_2)} \qquad (4.3)$$

where

$$w_1 + w_2 = -\frac{\Delta_\nu^{(1,2)}(h,\ k,\ R_o) + \Delta_\nu^{(2,1)}(h,\ k,\ R_o)}{\Delta_\nu^{(2,2)}(h,\ k,\ R_o)}$$

$$w_1 w_2 = \frac{\Delta_\nu^{(1,1)}(h,\ k,\ R_o)}{\Delta_\nu^{(2,2)}(h,\ k,\ R_o)} \qquad (4.4)$$

and

$$\begin{aligned}\Delta_\nu^{(m,n)}(h,\ k,\ R_o) = {} & (k^2-\gamma^2)^2\,H^{(m)}_{\nu-1/2}(hR_o)\,H^{(n)}_{\nu+1/2}(kR_o) \\ & - \frac{4\nu h s^2}{b^2 R_o}\,H^{(m)}_{\nu+1/2}(hR_o)\,H^{(n)}_{\nu+1/2}(kR_o) \qquad (4.5) \\ & + i\gamma^2 hk\,H^{(m)}_{\nu+1/2}(hR_o)\,H^{(n)}_{\nu-1/2}(kR_o)\end{aligned}$$

With the aid of the identity (3.25), it is easily shown from equation (4.4) that

$$\frac{w_1^{n+1} - w_2^{n+1}}{w_1 - w_2} = (-1)^n \sum_{n_2=0}^{n_2=n} Q^{(\nu)}_{n-n_2,n_2}(h, k, R_o) \qquad (4.6)$$

where

$$Q^{(\nu)}_{n_1 n_2}(h, k, R_o) = \sum_{q=0}^{\substack{\leq n_1 \\ \leq n_2}} \frac{(n_1+n_2-q)!(-1)^q}{(n_1-q)!(n_2-q)!q!}$$

$$\cdot \left[\frac{\Delta_\nu^{(1,2)}(h, k, R_o)}{\Delta_\nu^{(2,2)}(h, k, R_o)}\right]^{n_1-q} \left[\frac{\Delta_\nu^{(2,1)}(h, k, R_o)}{\Delta_\nu^{(2,2)}(h, k, R_o)}\right]^{n_2-q} \left[\frac{\Delta_\nu^{(1,1)}(h, k, R_o)}{\Delta_\nu^{(2,2)}(h, k, R_o)}\right]^{q} \qquad (4.7)$$

Thus, employing the series expansion (3.24), one obtains the result

$$\frac{1}{(1 - w_1)(1 - w_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} Q^{(\nu)}_{n_1 n_2}(h, k, R_o) \qquad (4.8)$$

which is readily substituted into equation (4.3) and rearranged slightly to obtain

$$\frac{J_{\nu+\chi_2}(hR_o)\, J_{\nu+\chi_2}(kR_o)}{\Delta_\nu(h, k, R_o)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} W^{(\nu)}_{n_1 n_2}(h, k, R_o) \qquad (4.9)$$

where

$$
\begin{aligned}
W^{(\nu)}_{n_1 n_2}(h, k, R_o) = {} & \frac{H^{(2)}_{\nu+\frac{1}{2}}(hR_o)\, H^{(2)}_{\nu+\frac{1}{2}}(kR_o)}{\Delta^{(2,2)}_{\nu}(h, k, R_o)}\, Q^{(\nu)}_{n_1 n_2}(h, k, R_o) \\
& + \frac{H^{(1)}_{\nu+\frac{1}{2}}(hR_o)\, H^{(1)}_{\nu+\frac{1}{2}}(kR_o)}{\Delta^{(2,2)}_{\nu}(h, k, R_o)}\, Q^{(\nu)}_{n_1-1, n_2-1}(h, k, R_o) \\
& - \frac{H^{(2)}_{\nu+\frac{1}{2}}(hR_o)\, H^{(1)}_{\nu+\frac{1}{2}}(kR_o)}{\Delta^{(2,2)}_{\nu}(h, k, R_o)}\, Q^{(\nu)}_{n_1 n_2-1}(h, k, R_o) \\
& - \frac{H^{(1)}_{\nu+\frac{1}{2}}(hR_o)\, H^{(2)}_{\nu+\frac{1}{2}}(kR_o)}{\Delta^{(2,2)}_{\nu}(h, k, R_o)}\, Q^{(\nu)}_{n_1-1, n_2}(h, k, R_o)
\end{aligned}
\qquad (4.10)
$$

with the stipulation that $Q^{(\nu)}_{n_1 n_2}(h, k, R_o) = 0$ when either or both of the integers, n_1 and n_2, is -1.

By using the same methods as employed in connection with the series (3.28), it is readily established that the series (4.7) and (4.8) are absolutely convergent if

$$
\begin{aligned}
|\Delta^{(2,2)}_{\nu}(h, k, R_o)| > {} & |\Delta^{(1,2)}_{\nu}(h, k, R_o)| + |\Delta^{(2,1)}_{\nu}(h, k, R_o)| \\
& + |\Delta^{(1,1)}_{\nu}(h, k, R_o)|
\end{aligned}
\qquad (4.11)
$$

This absolute convergence and its uniformity are readily studied with the aid of the asymptotic expressions for Bessel functions of the third kind which are valid for large values of z: [21]

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[21] G. N. Watson, op. cit., pp. 194-224.

$$H_\nu^{(1)}(z) \sim (2/\pi z)^{1/2} e^{i[z-\pi(2\nu+1)/4]} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\nu + m + 1/2)}{m!\,\Gamma(\nu - m + 1/2)\,(2iz)^m} \qquad -\pi < \arg z < 2\pi$$

$$H_\nu^{(2)}(z) \sim (2/\pi z)^{1/2} e^{-i[z-\pi(2\nu+1)/4]} \sum_{m=0}^{\infty} \frac{\Gamma(\nu + m + 1/2)}{m!\,\Gamma(\nu - m + 1/2)\,(2iz)^m} \qquad -2\pi < \arg z < \pi \tag{4.12}$$

Since z is either hR_o or kR_o, it is obvious from equation (2.32) that when $R_o > 0$, Im(s) is sufficiently large and positive, and γ is real, the leading terms in the above asymptotic expansions will become dominant and the simple expressions

$$H_\nu^{(1)}(z) = (2/\pi z)^{1/2}(1+\epsilon_1)\, e^{i[z-\pi(2\nu+1)/4]} \qquad -\pi < \arg z < 2\pi \quad |z| >> |\nu^2-1/4| \tag{4.13}$$

$$H_\nu^{(2)}(z) = (2/\pi z)^{1/2}(1+\epsilon_2)\, e^{-i[z-\pi(2\nu+1)/4]} \qquad -2\pi < \arg z < \pi \quad |z| >> |\nu^2-1/4| \tag{4.14}$$

will be useful. It is readily shown that

$$|\epsilon_1| \leq M \qquad -\pi/2 < \arg z < 3\pi/2$$
$$|\epsilon_2| \leq M \qquad -3\pi/2 < \arg z < \pi/2 \tag{4.15}$$

where

$$M = (\pi/2)\,(|\nu^2 - 1/4|/|z|)\,\epsilon^{(|\nu^2-1/4|/|z|)} \tag{4.16}$$

However, an examination of the range of values of hR_o and kR_o shows that it is necessary to use these simple expressions over the interval $0 < \arg z < \pi$ and a more critical examination must be made of the expression for $H_\nu^{(2)}(z)$. Applying the continuation formulae[22] for the

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[22] Ibid., p. 75.

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multiple valued Bessel functions of the third kind to equations (4.13) and (4.14), one readily finds the result

$$|\epsilon_2| \leq M + 2(1+M)\ |\cos\nu\pi|\,\epsilon^{-2\mathrm{Im}(z)} \qquad 3\pi/2 < \arg z < \pi/2 \tag{4.17}$$

Since $|z|^{-1} \leq |\mathrm{Im}(z)|^{-1}$, it is obvious that $|\epsilon_1|$ and $|\epsilon_2|$ have upper bounds which are independent of $\mathrm{Re}(z)$ and are monotone decreasing with the limit zero as $\mathrm{Im}(z)$ increases through positive values to ∞. Thus, since $a > b$, equation (2.32) gives $\mathrm{Im}(z) \geq (R_o/a)\mathrm{Im}(s)$, and this implies that bounds independent of the real variable γ and $\mathrm{Re}(s)$ can be found which are monotone decreasing to a zero limit as $\mathrm{Im}(s)$ increases to ∞ through positive values. Thus, if $\mathrm{Im}(s)$ is chosen sufficiently positive, equation (4.11) is equivalent to equation (3.30), and the series (4.7) and (4.8) will be uniformly and absolutely convergent for $\mathrm{Im}(s) \geq \delta$ where δ is positive and $\rightarrow \infty$ as $|\nu| \rightarrow \infty$ and/or $R_o \longrightarrow 0$.

It is unfortunate that the similarity of the series (4.7) and (4.8) to the series (3.28) and (3.29) is not in general capable of much greater extension than this similarity of convergence. Whereas, each term of the series (3.28) and (3.29) is a single valued function of h and k having a finite number of poles and an exponential behavior at infinity, the corresponding terms of the series (4.7) and (4.8) have a finite number of poles and the same type of exponential behavior at infinity only if $-\pi < \arg(h) < \pi$ and $-\pi < \arg(k) < \pi$ and are, in general, multiple valued functions of h and k and have quite different characteristics when arg (h) and/or arg (k) lie outside the above open intervals. The only exceptions to this difference in behavior occur when ν has such a value that $\sin \nu\pi = 0$. In these exceptional cases the terms of the series (4.7) and (4.8) are single valued in h and k and differ in analytical nature from the corresponding terms of the series (3.28) and (3.29) only in the number of poles which increases as $|\nu|$ increases. When $\nu = 0$ these two pairs of series become identical.

In the contour deformations employed in Section 3 in connection with the transformation of equation (3.38) into equation (3.54), arg (h) and arg (k) are required to sweep through the range of avlues from $-\tan^{-1}[Z/2(n_1+n_2)R_o]$ to $\pi + \tan^{-1}[Z/2(n_1+n_2)R_o]$ where $(z/2R_o)$ is positive and the inverse tangent lies in the first quadrant. Thus, the values $\arg(h) = \pi$ and $\arg(k) = \pi$ are swept over in the process, and it cannot be employed in connection with the term by term integration of the series (4.9) for $\sin \nu\pi \neq 0$ as the changes in the behavior for large values |h| and |k| make the integrals over the deformed contours properly divergent.

Some relief from this difficulty is available as it is possible to obtain a second series based on the relationship

$$J_{\nu}(z) = -(1/2)\left[\epsilon^{2\pi\nu i}\, H_{\nu}^{(1)}(z) + H_{\nu}^{(2)}(z\,\epsilon^{-2\pi i})\right] \tag{4.18}$$

which is satisfied by the Bessel functions of the first kind and the multiple valued Bessel functions of the third kind. This series may be derived by the same processes as the series (4.9) and will be written as

$$\frac{J_{\nu+1/2}(hR_o)\; J_{\nu+1/2}(kR_o)}{\Delta_{\nu}(h,\, k,\, R_o)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(-1)^{n_1+n_2}\, W_{n_1 n_2}^{(\nu)\prime}(h,\, k,\, R_o) \tag{4.19}$$

The numbers $W_{n_1 n_2}^{(\nu)\prime}(h, k, R_o)$ are easily found by replacing the functions $H_{\nu}^{(2)}(z)$ which occur in $W_{n_1 n_2}^{(\nu)}(h, k, R_o)$, that is with $z = hR_o$ or kR_o, with the functions $\epsilon^{-2\pi\nu i}\, H_{\nu}^{(2)}(z\,\epsilon^{-2\pi i})$. This series is again uniformly and absolutely convergent for $Im(s) > \delta$, and all values of $Re(s)$ and real γ. However, whereas the asymptotic behavior of the terms of the series (4.9) is given by

$$W_{n_1 n_2}^{(\nu)} \sim (-1)^{n_2}\left[(n_1+n_2)/f_1 + (n_1-n_2)/f_2\right] P_{n_1 n_2}^{1}(f_2/f_1)$$

$$\cdot\, \epsilon^{i[n_1(hR_o+\nu\pi) + n_2(kR_o+\nu\pi)]} \tag{4.20}$$

for $|hR_o| >> |\nu|(|\nu|+1) \qquad -\pi < \arg (hR_o) < \pi$

and $|kR_o| >> |\nu|(|\nu|+1) \qquad -\pi < \arg (kR_o) < \pi$

the asymptotic behavior of the terms of the series (4.19) is given by

$$W^{(\nu)\prime}_{n_1 n_2} \sim (-1)^{n_2}\left[(n_1+n_2)/f_1 + (n_1-n_2)/f_2\right] P^{1}_{n_1 n_2}(f_2/f_1)$$

$$\cdot\, \epsilon^{i\left[n_1(hR_o-\nu\pi) + n_2(kR_o-\nu\pi)\right]} \qquad (4.21)$$

for $|hR_o| >> |\nu|(|\nu|+1) \qquad 0 < \arg (hR_o) < 2\pi$

and $|kR_o| >> |\nu|(|\nu|+1) \qquad 0 < \arg (kR_o) < 2\pi$

Each of these series (4.9) and (4.19) may be employed over half of the range of integration to yield

$$S^{(\nu)}_{ZZC} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C^{(\nu)}_{n_1 n_2}(R_o, Z, t) \qquad (4.22)$$

where the series have been integrated term by term and

$$C^{(\nu)}_{n_1 n_2}(R_o, Z, t) = -i(2\nu+1)(1-2b^2/a^2)^2/(2\pi)^2 \int_{-\infty+i\tau}^{\infty+i\tau} (-1)^{n_1+n_2}$$

$$\cdot \left[\int_0^{\infty} \frac{2s^3\gamma^2 a g(s)}{b^4 h^3 R_o} W^{(\nu)\prime}_{n_1 n_2}(h,\ k,\ R_o)\, \epsilon^{-i(st-\gamma Z)}\, d\gamma \right. \tag{4.23}$$

$$\left. + \int_{-\infty}^{0} \frac{2s^3\gamma^2 a g(s)}{b^4 h^3 R_o} W^{(\nu)}_{n_1 n_2}(h,\ k,\ R_o)\, \epsilon^{-i(st-\gamma Z)}\, d\gamma \right] ds$$

$$\tau > \delta$$

Upon changing to the new variables $\bar{\gamma}$, $\bar{h}$, and $\bar{k}$ defined by the system of equations (3.39), it is readily found th t the contours for the $\bar{\gamma}$ integration can be deformed to yield

$$C^{(\nu)}_{n_1 n_2}(R_o, Z, t) = -i(2\nu+1)(1-2b^2/a^2)^2/(2\pi)^2 \int_{-\infty+i\tau}^{\infty+i\tau} (-1)^{n_1+n_2}$$

$$\cdot \left[\int_{\bar{\Gamma}^+_{n_1 n_2}} \frac{2s^3\ \bar{\gamma}^2 a g(s)}{b^4\ \bar{h}^3 R_o} W^{(\nu)}_{n_1 n_2}(s\bar{h}, s\bar{k}, R_o)\, \epsilon^{-is(t-\bar{\gamma} Z)}\, d\gamma \right. \tag{4.24}$$

$$\left. + \int_{\bar{\Gamma}^-_{n_1 n_2}} \frac{2s^3\ \bar{\gamma}^2 a g(s)}{b^4\ \bar{h}^3 R_o} W^{(\nu)\prime}_{n_1 n_2}(s\bar{h}, s\bar{k}, R_o)\, \epsilon^{-is(t-\bar{\gamma} Z)}\, d\gamma \right] ds$$

where the contour $\bar{\Gamma}^+_{n_1 n_2}$ starts at $\bar{\gamma} = 0$ and runs along the positive real axis until it meets the contour $\bar{\Gamma}_{n_1 n_2}$ of Fig. (3.1). It then goes to ∞

along the branch of the contour $\bar{\Gamma}_{n_1 n_2}$ which lies in the upper half of the $\bar{\gamma}$-plane. The contour $\bar{\Gamma}^-_{n_1 n_2}$ is the mirror image of $\bar{\Gamma}^+_{n_1 n_2}$ in the real axis, but it begins at ∞ on the lower half-plane branch of $\bar{\Gamma}_{n_1 n_2}$ and ends at $\bar{\gamma} = 0$.

Difficulties are again encountered when $n_1 = 0$ as the contours may touch the singular points $\bar{\gamma} = \pm(1/a)$, $\pm(1/b)$, and $\pm(1/C_R)$ at which one or more of the poles of the integrands in equation (4.24), considered as a function of s, may recede to ∞. Stated in other words, when Im(s) is sufficiently large and positive, the integrands considered as a function of $\bar{\gamma}$ have a finite number of poles (the number dependent on ν) clustered about the branch points $\bar{\gamma} = \pm(1/a)$ and $\pm(1/b)$ and one in the immediate vicinity of each of the points $\bar{\gamma} = \pm(1/C_R)$. These poles approach the points mentioned as limits as $\mathrm{Im}(s) \rightarrow \infty$, and by choosing δ sufficiently large, the contour deformation can be accomplished without passing over these singularities so long as $n_1 \neq 0$. The integrals with $n_1 = 0$ cannot be handled in this way with a finite value of δ. Various methods of treating these integrals suggest themselves. The most obvious of these is to represent these integrals as the limit of the more general integrals as $n_1 \rightarrow 0$.

Substituting the value of g(s) from equation (2.25), and inverting the order of integration, one obtains the result

$$C^{(\nu)}_{n_1 n_2}(R_o, Z, t) = \int_{o}^{\infty} A^{(\nu)}_{n_1 n_2}(R_o, Z, t-t') \, F(t') \, dt' \qquad (4.25)$$

where

$$\mathcal{A}^{(\nu)}_{n_1 n_2}(R_o,Z,t-t') = -i(2\nu+1)(1-2b^2/a^2)^2/(2\pi)^2 \int_{-\infty+i\tau}^{\infty+i\tau} (-1)^{n_1+n_2}$$

$$\cdot\left[\int_0^{\bar{\gamma}_{n_1 n_2}(t-t')} \frac{2s^3\,\bar{\gamma}^2 a}{b^4\,\bar{h}^3 R_o} W^{(\nu)}_{n_1 n_2}(s\bar{h},\ s\bar{k},\ R_o)\,\epsilon^{-is(t-t'-\bar{\gamma}Z)}\ d\bar{\gamma}\right. \tag{4.26}$$

$$\left.+\int_{\bar{\gamma}^*_{n_1 n_2}(t-t')}^{\bullet} \frac{2s^3\,\bar{\gamma}^2 a}{b^4\,\bar{h}^3 R_o} W^{(\nu)\prime}_{n_1 n_2}(s\bar{h},\ s\bar{k},\ R_o)\,\epsilon^{-is(t-t'-\bar{\gamma}Z)}\ d\bar{\gamma}\right] ds$$

and the finite limits on the integration over $\bar{\gamma}$ are justified by the vanishing of the integral over s for $t-t' < \bar{t}_{n_1 n_2}$. This vanishing of the integral over s in turn follows from the fact that the integrand is regular as a function of s for any fixed $\bar{\gamma}$ and $\mathrm{Im}(s) > \delta$ when $t-t' < \bar{t}_{n_1 n_2}$. The integration contours for the variable $\bar{\gamma}$ can now be deformed so as to avoid the necessity, when $n_1 = 0$, of touching the points $\bar{\gamma} = \pm(1/a)$, $\pm(1/b)$, and $\pm(1/C_R)$ except when one of these points must be the terminal point on the contour. In such a case a limiting process may still be required.

Although the numbers $\mathcal{A}^{(\nu)}_{n_1 n_2}(R_o,Z,t-t')$ are easily shown to be zero for $t-t' < (n_1/a + n_2/b)R_o$, they do not appear to be zero for $t-t' < \bar{t}^o_{n_1 n_2}$ except for the situations when $\sin\nu\pi = 0$. In these special cases the two series (4.9) and (4.19) have identical terms, and the Bessel functions can be expressed in terms of algebraic functions and exponentials. Thus, equation (4.26) takes the form

$$A^{(\nu)}_{n_1n_2}(R_o,Z,t-t') = -i(2\nu+1)(1-2b^2/a^2)^2/(2\pi)^2 \int_{-\infty+i\tau}^{\infty+i\tau}$$

$$\cdot \int_{\bar{\gamma}^*_{n_1n_2}(t-t')}^{\bar{\gamma}_{n_1n_2}(t-t')} \frac{2s^3\,\bar{\gamma}^{-2}a}{b^4\,\bar{h}^3 R_o}\, T^{(\nu)}_{n_1n_2}(s\bar{h},\ s\bar{k},\ R_o)\,\epsilon^{-is(t-t'-\bar{t}_{n_1n_2})}\, d\bar{\gamma}\, ds \qquad (4.27)$$

$$\sin\nu\pi = 0,\ \tau > 0$$

where $T^{(\nu)}_{n_1n_2}$ is a rational function of $s\bar{h}$, $s\bar{k}$, and R_o, and the contour of integration over $\bar{\gamma}$ may be $\bar{\Gamma}_{n_1n_2}$ or any other curve joining the point $\bar{\gamma}^*_{n_1n_2}(t-t')$ to the point $\bar{\gamma}_{n_1n_2}(t-t')$ which does not cross the cuts in the cut $\bar{\gamma}$-plane as shown in Fig. 3.1. Since the integral over s vanishes for $t-t' < \bar{t}_{n_1n_2}$, it is obvious that $A^{(\nu)}_{n_1n_2}(R_o,Z,t-t') = 0$ for $t-t' < t^o_{n_1n_2}$ and may thus be associated with the same type of geometrical path as the impulsive response $A_{n_1n_2}(R_o,Z,t-t')$ was in Section 3 whenever $\sin\nu\pi = 0$.

The failure of the correspondence between the geometrical paths and the impulse responses, $A^{(\nu)}_{n_1n_2}(R_o,Z,t-t')$, when $\sin\ \pi \neq 0$, is rather disconcerting since the asymptotic properties displayed by equations (4.20) and (4.21) would lead one to expect these impulse responses to be associated with waves which have traveled the distance $2n_1R_o$ as a dilatational wave and $2n_2R_o$ as a rotational wave regardless of the value of ν.

The failure is readily traced to the integration over $\bar{\gamma}$ in the interval $0 \leq \bar{\gamma} \leq \bar{\gamma}^o_{n_1n_2}$ which occurs in the two integrals appearing in each of the equations (4.24) and (4.26) when the contours $\bar{\Gamma}^+_{n_1n_2}$ and $\bar{\Gamma}^-_{n_1n_2}$ are employed.

When $\sin\nu\pi = 0$, the contributions from this part of the range of integration are equal from the two integrals but opposite in sign and cancel out. When $\sin\nu\pi \neq 0$, the contributions are unequal and do not cancel each other.

A study of the steps taken in obtaining equations (4.24) and (4.26) reveals that the process can be generalized to the extent that the contours $\bar{\Gamma}^+_{n_1n_2}$ and $\bar{\Gamma}^-_{n_1n_2}$ do not have to be terminated at $\bar{\gamma} = 0$, and could have been terminated at any other common point $\bar{\gamma}_c$ lying in the open interval of real values $-(1/a)<\bar{\gamma}_c<(1/a)$. This follows from the uniform and absolute convergence of the series (4.9) and (4.19) to a common sum when $Im(s)> \delta$ and $\bar{\gamma}$ is on the real axis in the interval $-(1/a)<\bar{\gamma}<(1/a)$. Terminating the contours at such a common point $\bar{\gamma}_c$ will in general alter the values of $C^{(\nu)}_{n_1n_2}(R_o,Z,t)$ and $A^{(\nu)}_{n_1n_2}(R_o,Z,t-t')$ but will not alter $S^{(\nu)}_{ZZC}$ as the sum of the changes due to altering the contours $\bar{\Gamma}^+_{n_1n_2}$ will exactly cancel the sum of the changes due to altering the contours $\bar{\Gamma}^-_{n_1n_2}$ for any value of ν. Thus, in any calculation of $S^{(\nu)}_{ZZC}$ we may pick $\bar{\gamma}_c$ at any convenient point on the above interval. Since the minimum group-transit time on the thus generalized contours $\bar{\Gamma}^+_{n_1n_2}$ and $\bar{\Gamma}^-_{n_1n_2}$ will in general be a function of $\bar{\gamma}_c$, this choice may be used to advantage.

For example noting that

$$S^{(\nu)}_{ZZC} = \sum_{n_2=0}^{\infty} D^{(\nu)}_{n_2}(R_o,Z,t) \tag{4.28}$$

where

$$D_{n_2}^{(\nu)}(R_o,Z,t) = \int_0^\infty B_{n_2}^{(\nu)}(R_o,Z,t-t')\, F(t')dt' \tag{4.29}$$

and

$$B_{n_2}^{(\nu)}(R_o,Z,t-t') = \sum_{n_1=0}^{\infty} \hat{A}_{n_1n_2}^{(\nu)}(R_o,Z,t-t') \tag{4.30}$$

we find on letting $\bar{\gamma}_c \to (1/a) - 0$ that $B_{n_2}^{(\nu)}(R_o,Z,t-t')$ is 0 when $t-t' < \bar{t}^o_{o\,n_2}$ where $\bar{t}^o_{o\,n_2}$ is given by equation (3.52). It is thus apparent that although the correspondence with the general geometric paths is incomplete for $\sin\nu\pi \neq 0$, it is always possible to split S_{ZZC} up into parts which correspond to direct dilatational and critical angle paths such as are observed for rods.

The differences in the behavior of the series noted above lead one to consider the asymptotic properties of S_{ZZC} as $(Z/R_o) \to \infty$. This may be accomplished directly from equation (4.1)

As in Section 3, it is convenient to take $F(t) = 1(t)$ where the unit step-function is defined by equation (3.67). The corresponding transform g(s) is given by

$$g(s) = (1/s) \tag{4.31}$$

Substituting this value into equation (4.1) and changing from the variables γ and s to the variables γ' and s' defined by

$$\gamma' = Z\gamma$$
$$s' = Z s \tag{4.32}$$

one finds

$$S_{ZZC}^{(\nu)} = \frac{(1-2b^2/a^2)^2}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} \cdot$$

$$\cdot \frac{2(2\nu+1)(s'\gamma')^2 a J_{\nu+1/2}(h' R_o/Z) J_{\nu+1/2}(k' R_o/Z) \epsilon^{-[s'(t/Z) - \gamma']i}}{b^4 (h')^3 (R_o/Z) \Delta_\nu(h', k', R_o/Z)} d\gamma' ds' \tag{4.33}$$

where

$$h' = \left[(s'/a)^2 - (\gamma')^2\right]^{1/2}$$
$$k' = \left[(s'/b)^2 - (\gamma')^2\right]^{1/2} \tag{4.34}$$

It is then readily shown that

$$\lim_{(Z/R_o)\to\infty} S_{ZZC}^{(\nu)} = \frac{1}{(2\pi)^2} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty}$$

$$\cdot \left[\frac{2a}{(s')^2 - (a\gamma')^2} - \frac{2a}{(s')^2 - (c_\nu \gamma')^2}\right] \epsilon^{-i[s'(t/Z) - \gamma']} d\gamma' ds' \tag{4.35}$$

where c_ν is the positive root of

$$(c_\nu)^2 = 2b^2 \frac{a^2(\nu+1) - b^2(2\nu+1)}{a^2(\nu+1/2) - 2b^2} \tag{4.36}$$

or

$$(c_\nu)^2 = 2b^2 \frac{2\sigma\nu + 1}{2\sigma\nu + (1-\sigma)}$$

(when expressed in terms of b, ν, and the Poisson ratio, σ,) and it is assumed that t increases with Z in such a way that (t/Z) approaches a finite limit.

The integration is readily carried out in terms of residues with the aid of Jordan's lemma to obtain

$$\lim_{(Z/R_o)\to\infty} S^{(\nu)}_{ZZC} = -1(t - |Z|/a) + (a/c_\nu)\, 1(t - |Z|/c_\nu)$$

Since we are dealing with a linear system and such step-functions as 1(t) may be combined in a linear fashion to obtain a suitably arbitrary function F(t), it is obvious that the general result is

$$\lim_{(Z/R_o)\to\infty} S^{(\nu)}_{ZZC} = -F(t - |Z|/a) + (a/c_\nu)\, F(t - |Z|/c_\nu)$$

and that

$$\lim_{(Z/R_o)\to\infty} \bar{S}_{ZZ} = \lim_{(Z/R_o)\to\infty} (S_{ZZP} + \bar{S}^{(\nu)}_{ZZC}) = (a/c_\nu)\, F(t - |Z|/c_\nu) \tag{4.37}$$

for any real value of c_ν.

This result implies that at large values of $(Z/2R_o)$ the disturbance appears to have travelled the entire distance Z with the speed C_ν. It is easily shown from equations (4.36) and (2.30) that $a^2 \geq (c_\nu)^2 \geq 2b^2$ for all real values of $\nu > -(1/2)$ and values of Poissons ratio σ in the range of physical stability $1/2 \geq \sigma \geq -1$. In addition for $\sigma = 0$ one finds $a^2 = c^2 = 2b^2$, and the complementary solution vanishes identically as would be expected.

For $\nu = 0$, equations (4.36) and (4.37) are identical with equations (3.19) and (3.20) respectively. For $\nu = 1/2$ we find

$$(c_{1/2})^2 = \frac{3a^2 - 4b^2}{a^2 - b^2} = 2b^2(1 + \sigma) \tag{4.38}$$

as the asymptotic speed of the disturbance in a cylindrical rod. This speed is exactly that predicted by Rayleigh's[23] approximate theory for

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[23]Lord Rayleigh, Theory of Sound (Cambridge, 1877), I. pp. 242-251.

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thin rods.

The simple expression, valid for all real values of C_ν, obtained for the asymptotic properties as $(Z/2R_o) \rightarrow \infty$, leads one to reexamine the processes leading to equations (4.26) and (4.27) with the idea of modifying these processes so as to obtain a segregation of the disturbance into parts which will correspond exactly with the geometrical paths of Section 3 regardless of the value of ν.

Having found the equation (4.27) which is consistent with such a segregation whenever $\sin \nu\pi = 0$, it is natural to consider the possibility of extending these results to other values of ν by interpolation between these values or some other such process of generalization.

The integration over s in equation (4.27) is readily carried out by evaluating residues with the aid of Jordan's lemma at $|2\nu + 1|$ poles of order $n_1 + n_2 + 1$. The results so obtained will give quite complicated combinations of exponential and algebraic functions which must then be integrated over $\bar{\gamma}$. In order to point out the fruitless nature of any effort at interpolation, it is only necessary to consider the values of $\mathcal{A}_{n_1 n_2}(R_o, Z, t-t')$ obtained from equation (4.27) for the initial rise just after $t = t' + \bar{t}^o_{n_1 n_2}$. Since the initial rise is controlled almost entirely by the Fourier components of high frequency, it is convenient to expand $\mathcal{T}_{n_1 n_2}(s\bar{h}, s\bar{k}, R_o)$ in inverse powers of s and use the leading term only in getting an approximation to the behavior in the neighborhood of the initial rise time. This is easily accomplished with the asymptotic formulae (4.20) and (4.21) which are identical when $\sin \nu\pi = 0$.

The result may be written as

$$\mathcal{A}^{(\nu)}_{n_1 n_2}(R_o, Z, t-t') = (2\nu + 1)\,(-1)^{\nu(n_1+n_2)}\,\mathcal{A}_{n_1 n_2}(R_o, Z, t-t') \qquad (4.39)$$

$$t' + \bar{t}^o_{n_1 n_2} < t < t' + \bar{t}^o_{n_1 n_2} + \epsilon$$

where ϵ is a small positive number, and $\mathcal{A}_{n_1 n_2}(R_o, Z, t-t')$ is given by equation (3.63).

From a physical point of view, the main feature of equation (4.39) is the phase changing factor $(-1)^{\nu(n_1+n_2)}$. There appears to be no way of generalization consistent with experiment in the case $\nu = 1/2$, which has been investigated.[1] Experimentally it appears that the initial rises of $A^{(1/2)}_{o\,n_2}(R_o,Z,t)$ are of the same sign as that of the direct dilatational wave and appear for all values of n_2 just as would be expected of a plate. If one attempts to generalize by replacing $(-1)^{\nu(n_1+n_2)}$ by $\cos\left[(n_1 + n_2)\nu\pi\right]$, rises could only be found for even values of n_2 when $n_1 = 0$ and $\nu = 1/2$, and these must oscillate in sign accordingly as n_2 increases through such even values.

Since the same factor $(-1)^{\nu(n_1+n_2)}$ is also contained in the neglected terms, it is apparent that some other approach is required when $\sin\nu\pi \neq 0$. However, the nature of this approach is not immediately obvious.

The complementary solution for $\nu = 1/2$ can be calculated from the equations developed in this section, but the effort and time required would be prohibitive.

The failure to obtain a solution like that for the plate, in which individual terms of the series correspond to each of the various types of geometrical paths, is certainly not to be construed as clouding the significance of such paths or the existence of the corresponding wave trains, for these wave trains, interpreted according to the principles of geometric optics, have been used in geophysical prospecting to map all kinds of curved surfaces in a consistent manner and in agreement with data obtained from well cores. The particular mathematical method employed has simply been inadequate to accomplish the splitting of the overall disturbance into separate pieces of the desired type.

The mathematical methods were adequate for the cases in which $\sin\nu\pi = 0$, as equations of the form

$$J_{\nu+1/2}(z) = (1/2\pi z)^{1/2}\Big[p_\nu(-z)\,\epsilon^{i(z - \pi(\nu + 1)/2)} + p_\nu(z)\,\epsilon^{-i(z - \pi(\nu + 1)/2)}\Big] \qquad (4.40)$$

exist where

$$p_\nu(z) = \sum_{m=0}^{|\nu + 1/2| - 1/2} \frac{(|\nu + 1/2| - 1/2 + m)!}{(|\nu + 1/2| - 1/2 - m)!\; m!\; (2iz)^m} \qquad (4.41)$$

when ν is a positive or negative integer or zero. The use of these expressions in the preceding analysis gives equation (4.27).

When ν is not a positive or negative integer or zero, there appears to be no function $p_\nu(z)$ satisfying equation (4.40) which has the property that the ratio $p_\nu(-z)/p_\nu(z)$ is single valued and regular at infinity (i.e. a rational function). However, it is in general possible to represent the Bessel functions of the first kind as the limit of a sequence of terms of the nature of equation (4.40) namely

$$J_{\nu+1/2}(z) = \lim_{N\to\infty} J^{(\mu,N)}_{\nu+1/2}(z) \qquad (4.42)$$

where

$$J^{(\mu,N)}_{\nu+\frac{1}{2}}(z) = (1/2\pi z)^{1/2}\,(z/2)^{\nu-\mu}\Big[p^N_{\nu\mu}(-z)\,\epsilon^{i(z-\pi(\mu+1)/2)} + p^N_{\nu\mu}(z)\,\epsilon^{-i(z-\pi(\mu+1)/2)}\Big] \qquad (4.43)$$

μ is an integer, and

$$p^N_{\nu\mu}(z) = \Gamma(\nu+1-\mu)\sum_{n=0}^{N} \cdot \frac{(\mu+2n+1/2)\;(\mu+n+1/2)(-1)^n}{n!\,\Gamma(\nu+1-\mu-n)\,\Gamma(\nu+n+3/2)}\,P_{\mu+2n}(z) \qquad (4.44)$$

in which the ratio $p^N_{\nu\mu}(-z)/p^N_{\nu\mu}(z)$ is obviously a rational function.

This infinite process is easily obtained from Sonine's expansion[24]

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[24] G. N. Watson, op. cit., pp. 139-140.

$$J_{\nu+\frac{1}{2}}(z) = (z/2)^{\nu-\mu}\,\Gamma(\nu-\mu+1)\sum_{n=0}^{\infty} \cdot \frac{(\mu+2n+1/2)\,\Gamma(\mu+n+1/2)}{n!\,\Gamma(\nu+1-\mu-n)\,\Gamma(\nu+n+3/2)}\,J_{\mu+2n+\frac{1}{2}}(z) \qquad (4.45)$$

which is valid when $\mu + 1/2$, $\nu + 1/2$, and $\nu - \mu$ are not negative integers. One merely defines $J^{(\mu,N)}_{\nu+\frac{1}{2}}(z)$ as the sum of the first N terms of the above series with the stipulation that μ is an integer and applies equation (4.40) to each term so considered.

Although these processes converge quite rapidly, one finds that $\lim_{N\to\infty} p^N_{\mu}(z)$ does not in general exist. Exceptions occur when ν, as well as μ, is an integer and $\nu \geq \mu$. In these cases the sequence terminates, and one finds

$$p^N_{\nu\mu}(z) = p_{\nu}(z) \qquad \nu \geq \mu,\ N \geq \nu - \mu \tag{4.46}$$

The possibility of representing $\bar{S}^{(\nu)}_{ZZC}$ as the limit of a sequence based on the above infinite sequence suggests itself. Defining $\bar{S}^{(\nu,\mu,N)}_{ZZC}$ by making the substitutions

$$\begin{aligned} S^{(\nu)}_{ZZC} &\longrightarrow S^{(\nu,\mu,N)}_{ZZC} \\ J_{\nu+\frac{1}{2}}(z) &\longrightarrow J^{(\mu,N)}_{\nu+\frac{1}{2}}(z) \\ J_{\nu-\frac{1}{2}}(z) &\longrightarrow J^{(\mu-1,N)}_{\nu-\frac{1}{2}}(z) \end{aligned} \tag{4.47}$$

into equation (4.1) and expanding the right-hand member in ascending powers of ϵ^{2ihR_o} and ϵ^{2ikR_o}, one readily obtains

$$\bar{S}^{(\nu,\mu,N)}_{ZZC} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_0^{t-\bar{t}^o_{n_1n_2}} A^{(\nu,\mu,N)}_{n_1n_2}(R_o,z,t-t')F(t')dt' \tag{4.48}$$

where

$$\mathcal{A}_{n_1 n_2}^{(\nu,\mu,N)}(R_o,Z,t-t') = -i(2\nu+1)(1-2b^2/a^2)^2(2\pi)^2 \int_{-\infty+i\tau}^{\infty+i\tau}$$

$$\cdot \int_{\bar{\gamma}^*_{n_1 n_2}(t-t')}^{\bar{\gamma}_{n_1 n_2}(t-t')} \frac{2s^3\bar{\gamma}^2 a}{b^4\bar{h}^3 R_o} \mathcal{T}_{n_1 n_2}^{(\nu,\mu,N)}(s\bar{h},s\bar{k},R_o)\, \epsilon^{-is(t-t'-\bar{t}_{n_1 n_2})}\, d\bar{\gamma} ds \qquad (4.49)$$

in which $\mathcal{T}_{n_1 n_2}^{(\nu,\mu,N)}(s\bar{h},s\bar{k},R_o)$ is a rational function of each of the variables $s\bar{h}, s\bar{k}$, and R_o. It has poles of order $n_1 + n_2 + 1$, considered as a function of s, and the number of these poles is dependent upon ν, μ, and N. It is easily demonstrated that $\mathcal{A}_{n_1 n_2}^{(\nu,\mu,N)}(R_o,Z,t-t') = 0$, for $t-t' < \bar{t}^o_{n_1 n_2}$ and for any value of ν, and that this result is compatible with equation (4.27), when $\sin\nu\pi = 0$, in the sense that

$$\lim_{N\to\infty} \mathcal{A}_{n_1 n_2}^{(\nu,\mu,N)}(R_o,Z,t-t') = \mathcal{A}_{n_1 n_2}^{(\nu)}(R_o,Z,t-t') \qquad (4.50)$$

as a result of equation (4.46).

If this process converges when $\sin\nu\pi \neq 0$, equation (4.50) would be a much more satisfactory definition of $\mathcal{A}_{n_1 n_2}^{(\nu)}(R_o,Z,t-t')$ than that provided by equation (4.26), as this impulse response would then always correspond to the appropriate geometrical path.

Although the integration over s in equation (4.49) could be carried out formally in terms of residues, the number and location of the poles depends upon N in such a way that the author has been unable to evaluate the limit involved in equation (4.50) or prove that it exists. The physical role of the geometric paths leads one to expect the limit to exist, but there is very little hope of reducing the calculation to processes of sufficient simplicity to warrant further consideration in this paper.

5. CONCLUSION

A theoretical investigation has been made of the propagation of elastic pulses through rods and plates. This investigation is by no means exhaustive as it has been limited to a plane longitudinal drive. Transverse drives and point-source drives are equally interesting from the point of view of making physical measurements of elastic constants, as are the parallel problems involving non-isotropic media.

In the case of the plate, these mathematical methods are capable of treating the transverse drive, point-source drives, and the non-isotropic media with minor alterations. The author intends to treat these and certain related geophysical problems in future papers.

In the case of the rod, it is clear that some other mathematical tool must be applied to the much more complicated functions which are involved. The primary objective of this research, a quantitative wave treatment of the experiments reported by Hughes, Pondrom, and Mims, has not been completely attained. At present the best approach is to consider the effects that appear in connection with rods in terms of an analogy with those that appear in connection with the plate. This is, at best, only a qualitative analysis.

For a longitudinal drive on a plate, it has been shown that the boundaries of the plate produce reflected or echo wave trains which correspond to each of the paths predicted by the methods of geometric optics. It is found that the critical angle paths should be the most evident experimentally as the associated wave trains have a much more abrupt start.

It is further found that these wave trains interfere with one another in such a way that, as the ratio of the distance of transmission to the thickness of the plate becomes infinite, the entire disturbance becomes asymptotically a disturbance traveling with the velocity c_0 which is classically the longitudinal velocity of propagation for a plate of zero thickness. These results are obtained independent of any assumptions regarding the frequency spectrum of the pulse.

By analogy, one should expect longitudinal drive on a rod to result in wave trains which correspond to each of the paths predicted by means of geometrical optics, but that those wave trains corresponding to the critical angle paths and the direct dilatational path will be the most easily detected. As the ratio of the distance of transmission to the diameter becomes infinite, interference of the above wave trains should produce the asymptotic result of a single wave train propagated with the speed, $c_{1/2}$, found by Rayleigh for a rod of zero radius. It is also to be expected that the frequency spectrum, or shape of the driving pulse, is of no importance to the theory. All of the conclusive findings of Section 4 are in accord with this analogy.

The experimental observation of the various wave trains is complicated by the multitude of ways in which interference can take place. This becomes particularly difficult when the duration of the driving pulse F(t) is greater than the differences between the minimum group-transit times of several of the various reflected wave trains. This condition existed in the experimental work reported by Hughes, Pondrom, and Mims[1] and effectively prevented the identification of any wave trains except those corresponding to the direct dilatational and critical angle paths which correspond to the contours $\overline{\Gamma}_{o\,n_2}$. The more general paths corresponding to $\overline{\Gamma}_{n_1 n_2}$ with n_1 small but not zero generally give rise to slow starting transients as pointed out in Section 3. However, as $(Z/2R_o)$ is made larger, these rises become more nearly like those corresponding to the critical angle paths and should then be more readily detected. Unfortunately, the minimum group transit times $\overline{t}^o_{n_1 n_2} \longrightarrow \overline{t}^o_{o\,n_2}$ as $(Z/2R_o) \rightarrow \infty$, and the wave trains corresponding to the more general paths interfere with other wave trains under the same conditions that they give fairly sharp rises unless F(t) is sufficiently short in duration to avoid this interference.

This point is illustrated by Fig. 5.1 which is a reproduction of the oscillographic traces for a cold-rolled steel rod 3.645 cm. in diameter and 5.08 cm. in length. For this sample of steel, a was found to be 5880 meters/sec., b was found to be 3203.5 meters/sec., and the value .289 was obtained for σ.

The primary dilatational disturbance begins at point A, 8.69 μsec. after the start of the driving pulse, which had an appreciable amplitude for some 4 μsec. Since the minimum group-transit times $\overline{t}^o_{n_1 o}$ associated with the contours $\overline{\Gamma}_{n_1 o}$ are given by

$$\overline{t}^o_{n_1 o} = (1/a) \left[Z^2 + (2n_1 R_o)^2 \right] \tag{5.1}$$

as the entire path is traversed as a dilatational wave, we thus find

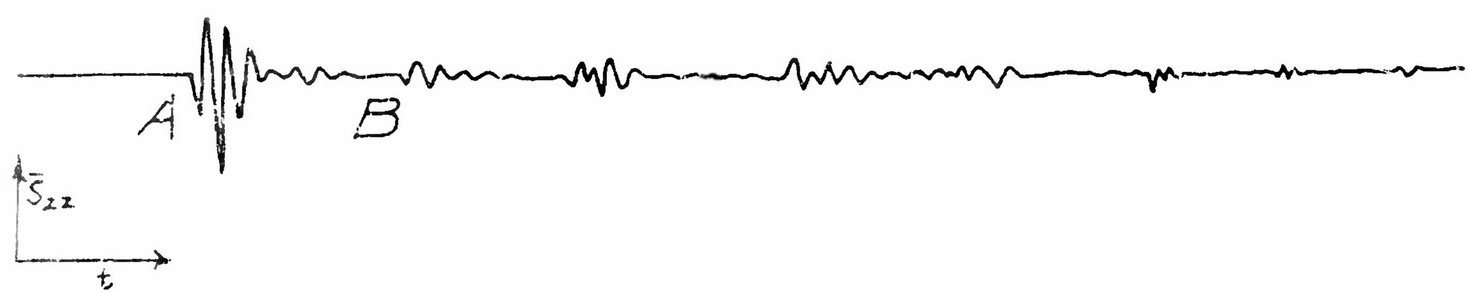

Fig. 5.1 Oscillographic trace obtained with cold-rolled steel rod 3.645 cm. in diameter and 5.08 cm. in length

$$\bar{t}^{o}_{n_1 o} / \bar{t}^{o}_{o\,o} = \left[1 + (2n_1 R_o/Z)^2\right]^{1/2}$$

Confining our attention to $\bar{t}^{o}_{1\,o}$, we find

$$\bar{t}^{o}_{1\,o} / 8.69 = \left[1 + (3.645/5.08)^2\right]^{1/2}$$

or

$$\bar{t}^{o}_{1\,o} = 10.696\ \mu\text{sec.}$$

Thus, the corresponding train of waves would interfere with the direct dilatational wave train over half of its length. As n_1 increases, some of the corresponding wave trains would be expected to be resolved in terms of time but would not be expected to have a sharp enough rise to be detected.

The obvious arrival at point B is due to a critical angle path corresponding to the contour $\bar{\Gamma}_{o\,1}$. The arrival time $\bar{t}^{o}_{0\,1}$ is found to be 18.61 μsec. after the s art of the driving pulse and is in agreement with calculations based upon equation (3.52). The wave train corresponding to the minimum group-transit time $\bar{t}^{o}_{1\,1}$, which is 21.23 μsec as calculated from equations (3.46), (3.47), and (3.40), and thus it should interfere with the wave train corresponding to $\bar{t}^{o}_{0\,1}$.

These conditions become much more aggravated as $(Z/2R_o)$ increases. Since the remainder of the rods considered have larger length to diameter ratios than the one considered above, it is apparent that there was little likelihood of identifying any wave trains except those corresponding to the direct dilatational and critical angle paths which always have sharply rising initials.

A more promising set of circumstances for the observation of the more general wave trains corresponding to the minimum group-transit times $\bar{t}^{o}_{n_1 n_2}$ with $n_1 \neq 0$ can be obtained by either shortening the duration of the driving pulse or by increasing both the length and the diameter of the rods to be considered by a factor of approximately ten so as to get a greater separation of neighboring values of $\bar{t}^{o}_{n_1 n_2}$.

The latter alternative is not very acceptable to this laboratory as the fundamental reason for developing this method of measuring velocities of elastic waves was to get a system in which the rod-shaped sample and the necessary driving and detecting crystals could be placed in a small volume inside of a high-pressure chamber. Thus, with one set-up, one can measure the velocities of rotational and dilatational waves for various hydrostatic pressures and temperatures. It is consequently desirable to pursue the former alternative, and efforts are now being made to reduce the duration of F(t) and increase its amplitude so as to increase the number of arrivals capable of being identified.

The question of precisely what is being measured in these experiments is closely tied in with the nature of the driving pulse. If $F(t) = 0$ for $t < 0$ and is non-zero for any short time interval, it is a well recognized fact that this function must be considered to have a continuous distribution of frequencies running from 0 to ∞. All elastic materials, for one or more reasons, are expected to become dispersive as the frequency of a simple harmonic disturbance becomes sufficiently large. Thus, the velocities $\underline{a}$ and b are complex functions of the frequency which is represented in the analysis of this paper by the symbol s. The velocities measured are apparently those associated with the first arrival of an abruptly initiated disturbance or "wave-front" velocities. Paralleling the reasoning of Sommerfeld and Brillouin[9], one might say that the velocities measured are really the limits of $\underline{a}$ and b as $s \rightarrow \infty$. Such a statement is rather naive since the results of any such measurement are very likely to be a function of the sensitivity of the detecting mechanism. Without a complete and experimentally acceptable theory of dispersion, one cannot say when the detector is sensitive enough to detect the very first arrival. There is some reason to expect that an extremely small part of the energy of an abruptly starting elastic disturbance is propagated with the speed of light.

In the absence of a complete theory of elastic dispersion, no exact description of the measured velocities can be given. One can only state that the measured velocities are the apparent "wave-front" velocities as observed with apparatus of specified characteristics. It is to be expected that an acceptable theory of dispersion will result from the study of the results of such experiments as these.

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