On the Stability of the Laminar Boundary Layer in a Compressible Fluid - Part I*

by

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Abstract

An order-of-magnitude analysis of the complete linearized equations for a three-dimensional disturbance superimposed on a two-dimensional boundary layer in a compressible fluid shows that they can be reduced to much simpler equations in the first approximation. The latter equations can then be transformed to equations of the same form as those for a two-dimensional disturbance, so that the mathematical theory is simplified considerably. In contrast to the situation for an incompressible fluid, however, the transformation gives no information directly regarding the relative importance of two-and three-dimensional disturbances.

First approximations to the viscous solutions of the disturbance equations have been obtained which are uniformly accurate over the whole boundary layer. In terms of these new solutions, the relation for the characteristic values again takes on a form very much like that in the previous theory of Lees and Lin. For small values of the non-dimensional wave-speed c, the present results when specialized to two-dimensional disturbances reduce to those of the earlier theory, so that the validity of the latter in such cases is verified.

Three-dimensional disturbances are found to be of little importance at low subsonic free-stream Mach numbers, since they usually have a higher minimum critical Reynolds number than two-dimensional ones, except possibly under conditions of extreme surface cooling. As the Mach number increases, three-dimensional disturbances become significant under conditions which are less and less extreme, until finally at a Mach number between one and

two they begin to play the leading role in many problems of practical interest.

At supersonic free-stream Mach numbers the boundary layer can never be completely stabilized with respect to all three-dimensional disturbances. There is always present a class of unstable three-dimensional disturbances no matter how low the temperature of the solid surface. However, for Mach numbers up to about two, surface cooling is still a very effective means of stabilizing the boundary layer. At higher Mach numbers the amount of cooling necessary to raise the minimum critical Reynolds number significantly appears to become practically prohibitive.

Numerical results illustrating these conclusions are presented for the case of the boundary layer on a flat plate.

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1. Introduction

In recent years there has been considerable interest in the theory of boundary-layer stability for compressible fluids. Most of the recent work on the subject has been based on the original theoretical investigations of Lees and Lin ([1] and [2]). As has been pointed out on several occasions, there are certain inadequacies in this basic theory which limit its significance and which in some cases make unreliable the results of numerical calculations based on it. The purpose of the present investigation is to clarify some of these problems, and in general to gain further insight into the stability theory for the laminar boundary layer in a compressible fluid.

One of the chief limitations of the theory up to now has been consideration of only two-dimensional disturbances, by analogy with the procedure for an incompressible fluid. Squire [3] has shown by means of a transformation of the equations of the problem that for an incompressible fluid every three-dimensional disturbance is equivalent to a two-dimensional one at a lower Reynolds number. Therefore, for an incompressible fluid the minimum critical Reynolds number for two-dimensional disturbances is the minimum critical Reynolds number for all disturbances, so that the restriction to two-dimensional disturbances is justified. If an attempt is made to prove a similar result for a compressible fluid, it is found that the equations obtained by making the usual approximations (see [1]) are too complicated to permit a transformation of the type used by Squire. In the present work an order-of-magnitude analysis of the terms in the complete equations for a three-dimensional disturbance is carried out. A consistent approximation scheme based on this now yields simplified equations of the same order of accuracy as the usual ones, but with fewer terms, so that a transformation to equations of the same form as for a two-dimensional disturbance is possible. However, these transformed two-dimensional equations are not the equations of a proper two-dimensional disturbance, so that no direct conclusion regarding the importance of three-dimensional disturbances can be made. For a compressible fluid the properties of three dimensional disturbances have to be found by means of a more elaborate investigation than for an incompressible fluid. The transformed two-dimensional equations lead to a considerable simplification mathematically, however.

It is found in the present investigation that for subsonic free-stream Mach numbers three-dimensional disturbances are not of much importance, but for supersonic Mach numbers they may play a leading role in certain circumstances. In particular, for supersonic Mach numbers there is always present a class of unstable three-dimensional disturbances no matter how low the temperature of the solid surface is, so that complete stabilization of the boundary layer in such cases, as predicted in [2], is impossible.

It is known that at high Mach numbers the usual form of the boundary layer theory becomes inaccurate ([4] and [5]), so that there is some question as to the proper form of the stability theory in such cases. An investigation of this problem [6] has shown that as long as the proper velocity and temperature profiles for the mean flow are used, the simplified stability equations derived in this report are still valid as a first approximation at high Mach numbers. However, in such cases the accuracy of a first approximation may not be very great, so that higher approximations and thus more complicated equations may have to be considered*.

Various objections to some of the approximations made in calculations based on the theoretical developments of [1] have been raised. In fact, it has been pointed out in [7] that the original theory of [1] has sometimes been used in situations in which it is not valid. In particular, the viscous solutions of [1] are accurate first approximations only in the immediate neighbourhood of the point where w = c. However, at high Mach numbers c is often quite large in the stability calculations, so that in order to satisfy conditions at the solid boundary the viscous solutions have to be accurate far away from this point. For an incompressible fluid Tollmien [8] has obtained viscous solutions which are uniformly accurate all across the boundary layer. This work has been extended to the case of a compressible fluid, and a new mathematical formulation of the stability problem in terms of the new viscous solutions has been obtained. In the present report the modified formulas for the calculation of the characteristic values are given. The detailed mathematical discussion will be presented elsewhere.

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*As an example of the increase in complication, in a second approximation terms must be retained whose coefficients involve the v-component of the basic flow velocity, so that to this order of approximation the flow cannot be considered to be parallel. This conclusion was reached independently in the thesis [6] and in an investigation by Cheng [16].

2. General formulation of problem

The basic differential equations for the motion of a compressible fluid, in Cartesian tensor notation, are as follows:

Momentum equations

$$(2.1)\qquad \frac{\partial u_i}{\partial t^*} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho^*}\frac{\partial \tau_{ij}}{\partial x_j}$$

Continuity equation

$$(2.2)\qquad \frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x_j}(\rho^* u_j) = 0$$

Energy equation

$$(2.3)\qquad \rho^* c_v\left(\frac{\partial T^*}{\partial t^*} + u_j \frac{\partial T^*}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(k^*\frac{\partial T^*}{\partial x_j}\right) + \tau_{ij} e_{ij}$$

Equation of state

$$(2.4)\qquad p^* = R^* \rho^* T^*$$

In these equations e_{ij} is the rate-of-strain tensor and τ_{ij} is the stress tensor, defined in the following way

$$(2.5)\qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$(2.6)\qquad \tau_{ij} = 2\mu^* e_{ij} + \left[\frac{2}{3}(\lambda^* - \mu^*)e_{kk} - p^*\right]\delta_{ij}$$

All quantities are dimensional, as defined in the List of Symbols. The coefficients of viscosity μ^* and λ^*, the coefficient of heat conductivity k^*, the specific heats C_p and C_v, and the Prandtl number σ are considered to be functions of the temperature alone. The gravitational force terms are omitted, since they can be shown to be negligible for the large values of the Froude number to be expected in most practical problems (see [1]).

We consider a time-independent basic flow and a small time-dependent disturbance. Thus, any quantity $Q(x_i, t^*)$ may be written as the sum of a part $\overline{Q}(x_i)$ and a part $Q'(x_i, t^*)$:

(2.7) $$Q(x_i, t^*) = \bar{Q}(x_i) + Q'(x_i, t^*)$$

We are interested in determining whether the disturbances Q' are stable or unstable, that is, whether they decrease or increase with time. Both the quantities Q in the disturbed motion and $\bar{Q}$ in the steady motion satisfy equations (2.1) to (2.4). Thus, if expressions of the form (2.7) for all the variables are inserted into these equations and quadratic terms in the small disturbance quantities are neglected, the following system of linearized equations for the fluctuations Q' is obtained:

(2.8) $$\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \rho^{*\prime} + \bar{\rho}^* \left(\frac{\partial u_i'}{\partial t^*} + \frac{\partial \bar{u}_i}{\partial x_j} u_j' + \bar{u}_j \frac{\partial u_i'}{\partial x_j} \right) = \frac{\partial \tau_{ij}'}{\partial x_j}$$

(2.9) $$\frac{\partial \rho^{*\prime}}{\partial t^*} + \frac{\partial}{\partial x_j} \left(\bar{u}_j \rho^{*\prime} + u_j' \bar{\rho}^* \right) = 0$$

(2.10) $$\bar{u}_j \frac{\partial \bar{T}^*}{\partial x_j} \rho^{*\prime} + \bar{\rho}^* \left(\frac{\partial T^{*\prime}}{\partial t^*} + \frac{\partial \bar{T}^*}{\partial x_j} u_j' + \bar{u}_j \frac{\partial T^{*\prime}}{\partial x_j} \right)$$
$$= \frac{1}{\bar{c}_v} \frac{\partial}{\partial x_j} \left(\frac{\bar{c}_p \bar{\mu}^*}{\bar{\sigma}} \frac{\partial T^{*\prime}}{\partial x_j} + \left[\frac{c_p \mu^*}{\sigma} \right]' \frac{\partial \bar{T}^*}{\partial x_j} \right) + \frac{1}{\bar{c}_v} \left(\bar{\tau}_{ij} e_{ij}' + \tau_{ij}' \bar{e}_{ij} \right)$$
$$+ \left[\frac{1}{c_v} \right]' \frac{\partial}{\partial x_j} \left(\frac{\bar{c}_p \bar{\mu}^*}{\bar{\sigma}} \frac{\partial \bar{T}^*}{\partial x_j} \right) + \left[\frac{1}{c_v} \right]' \bar{\tau}_{ij} \bar{e}_{ij}$$

(2.11) $$\frac{p^{*\prime}}{\bar{p}^*} = \frac{\rho^{*\prime}}{\bar{\rho}^*} + \frac{T^{*\prime}}{\bar{T}^*}$$

The quantities e_{ij}', τ_{ij}', $\left[\frac{c_p \mu^*}{\sigma}\right]'$, $\left[\frac{1}{c_v}\right]'$, $\lambda^{*\prime}$, $\mu^{*\prime}$ are given by

(2.12) $$e_{ij}' = \frac{1}{2} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)$$

(2.13) $$\tau_{ij}' = 2\left(\bar{\mu}^* e_{ij}' + \mu^{*\prime} \bar{e}_{ij} \right) + \left[\frac{2}{3} (\bar{\lambda}^* - \bar{\mu}^*) e_{kk}' + \frac{2}{3} (\lambda^{*\prime} - \mu^{*\prime}) \bar{e}_{kk} - p^{*\prime} \right] \delta_{ij}$$

and

(2.14) $$\begin{cases} \left[\frac{c_p \mu^*}{\sigma} \right]' = \overline{\frac{d}{dT^*} \left(\frac{c_p \mu^*}{\sigma} \right)} \, T^{*\prime} , & \left[\frac{1}{c_v} \right]' = \overline{\frac{d}{dT^*} \left(\frac{1}{c_v} \right)} \, T^{*\prime} \\ \lambda^{*\prime} = \overline{\frac{d\lambda^*}{dT^*}} \, T^{*\prime} ; & \mu^{*\prime} = \overline{\frac{d\mu^*}{dT^*}} \, T^{*\prime} \end{cases}$$

Equations (2.8) to (2.11) form a system of linear partial differential equations of the second order for the fluctuations. The coefficients are known functions given by the steady flow.

3. Stability of parallel and nearly parallel flows

When the fluid is incompressible, the basic steady flow can be strictly parallel, as in the case of two dimensional flow through a channel. The formulation of the stability problem is then very simple. For compressible fluids, however, these is no such simple basic flow. Flows satisfying the boundary-layer approximation are the closest to parallel flows. It will be shown that for such flows, the stability problem is essentially the same as that for a parallel flow which agrees locally with the basic flow of the boundary-layer type.

For proving the theorem in Section 4, and for application to boundary layers in three-dimensional flow, the development will be made in a somewhat more general manner than usual. We consider a semi-infinite flat plate in an oblique incoming stream. The free stream has components of velocity U_1 and U_3 parallel to the plate, and the flow inside the boundary layer is parallel to the free stream at each location (x_1, x_3). Indeed, the boundary layer can be described in terms of strips parallel to the direction of the resultant free stream. Each strip behaves like a flat plate with the leading edge normal to the resulting stream. The location of the leading edge coincides with that of the particular strip in question. Thus, if s is the distance of a point from the leading edge in the direction of the free-stream, the boundary-layer thickness δ at that point is given by

(3.1) $$\frac{\delta}{s} = O\left(R_s^{-1/2}\right) = O\left(R^{-1}\right)$$

where

(3.2) $$R_s = \frac{s\sqrt{U_1^2 + U_3^2}}{\nu_1^*}$$

(3.3) $$R = R_\delta = \frac{\delta\sqrt{U_1^2 + U_3^2}}{\nu_1^*}$$

and $\bar{\nu}_1^*$ is the kinematic viscosity in the free stream. As is well-known, the quantities $\bar{Q}$ associated with the basic flow satisfy the following order-of-magnitude relations (if the free-stream Mach number is not too large):

$$(3.4)\qquad \frac{\partial \bar{Q}}{\partial x_2} : \left(\frac{\partial \bar{Q}}{\partial x_1} \text{ or } \frac{\partial \bar{Q}}{\partial x_3} \right) = O(R_s^{1/2}) = O(R)$$

$$(3.5a)\qquad (\bar{u}_1 \text{ or } \bar{u}_3) : \bar{u}_2 = O(R_s^{1/2}) = O(R)$$

$$(3.5b)\qquad \bar{T}^* = O(\bar{T}_1^*) \quad , \quad \bar{\rho}^* = O(\bar{\rho}_1^*) \quad , \quad \bar{\mu}^* = O(\bar{\mu}_1^*)$$

We shall now simplify the equations for the small fluctuations by means of analogous considerations regarding the fluctuations and their derivatives. From physical considerations, and also from the mathematical nature of the boundary-value problem associated with the system of disturbance equations (2.8) to (2.11), it may be expected that the fluctuations have the typical boundary-layer type behaviour in a region adjacent to the solid boundary. That is, near to the boundary viscous effects and therefore the highest order derivatives in the direction normal to the solid surface are important, and changes in the normal direction are much more rapid than in any direction parallel to the surface.

It should first be recognized that the fluctuations have a general length scale of the order of the boundary-layer thickness, so that

$$(3.6)\qquad \frac{1}{Q'} \frac{\partial Q'}{\partial x_1} = \frac{1}{\bar{Q}} \frac{\partial \bar{Q}}{\partial x_1} O(R)$$

In the viscous region next to the surface, the characteristic length in the x_2 direction (normal direction) is much smaller, and is obtained by considerations analogous to the usual boundary-layer arguments for the basic flow. It is clear from (2.8) that the equation of motion for u_1' contains a term of the form $\bar{\rho}^* \bar{u}_1 \, \partial u_1'/\partial x_1$ from the inertial terms and a term of the form $\bar{\mu}^* \partial^2 u_1'/\partial x_2^2$ from the viscous terms. Thus, if these terms are to be of the same order, x_2-derivatives must be larger than x_1-derivatives. Now,

$$(3.7) \qquad \bar{\rho}^* \bar{u}_1 \frac{\partial u_1'}{\partial x_1} = O\left(\frac{\bar{\rho}_1^* U_1 u_1'}{\delta}\right)$$

$$(3.8) \qquad \bar{\mu}^* \frac{\partial^2 u_1'}{\partial x_2^2} = O\left(\bar{\mu}_1^* \frac{u_1'}{\delta^2} R^{2n}\right)$$

where R^n is a factor indicating the difference of order of magnitude between x_1 and x_2 derivatives:

$$(3.9) \qquad \frac{\partial Q'}{\partial x_2} : \left(\frac{\partial Q'}{\partial x_1} \text{ or } \frac{\partial Q'}{\partial x_3}\right) = O(R^n)$$

A comparison of (3.7) and (3.8) gives $n = 1/2$. The relation (3.9) is now comparable to (3.4) with R_s replaced by R_δ . The relation analogous to (3.5a) is

$$(3.10) \qquad (u_1' \text{ or } u_3') : u_2' = O(R^{1/2})$$

An examination of the equations also gives the following relations (which are used in Section 4):

$$(3.11) \qquad \frac{T^{*\prime}}{T_1^*} = O\left(\frac{u_1'}{U_1}\right), \qquad \frac{\rho^{*\prime}}{\bar{\rho}_1^*} = O\left(\frac{u_1'}{U_1}\right)$$

Therefore we have a boundary-layer phenomenon associated with the fluctuations within the boundary layer for the mean flow. In a thin layer next to the solid surface with thickness of the order of $\delta R^{-1/2}$ the fluctuations have the viscous behaviour characterized by rapid variation in the x_2-direction, while outside this layer they are usually independent of viscosity since the viscous terms in the equations are usually unimportant there. In general, the order-of-magnitude relations (3.9) and (3.10) are valid only in the viscous layer at the solid surface , while outside it the velocity fluctuations are all of the same order of magnitude and the characteristic length in all directions is the same (that is, of the order of the boundary layer thickness δ).

In all of these order-of-magnitude estimates $O(R^{1/2})$ may denote a quantity not actually of the order of magnitude of $R^{1/2}$ in the strict sense, but smaller. This happens when certain quantities involved in the comparison become zero. Such a situation occurs in certain layers of the fluid, where it is shown by a more complete analysis (see [6]) that $O(R^{1/2})$ is replaced by $O(R^{1/3})$. In fact, when the disturbance is analysed into periodic components (see Section 5) and the ordinary differential equations for the amplitudes of these components examined, it is found that such a layer ("critical layer") always occurs at the x_2-location in the boundary layer where the wave-speed in the x_1-direction of the particular component considered is equal to the component of the velocity of the basic flow in that direction. There is the possibility that in certain cases the critical layer (where viscosity is also always important) may occur outside the viscous layer at the solid surface described above. In such cases, relation (3.9) with $O(R^{1/2})$ replaced by $O(R^{1/3})$ will also be valid in the critical layer, so that variations through this layer in the x_2-direction will also be very rapid, just as in the layer at the wall. However, the order-of-magnitude analysis of the disturbance equations carries through in just the same way, so that it is not necessary for present purposes to consider the more complicated properties of the solutions of these equations.

4. Simplified equations for small disturbances in flows of the boundary-layer type.

By making use of the boundary-layer approximations outlined above (corresponding to relations (3.4),(3.5),(3.9),(3.10),(3.11)) it can be shown that the equations for the small disturbances can be simplified to the following:

$$(4.1)\quad \bar{\rho}^*\left(\frac{\partial u_1'}{\partial t^*}+\bar{u}_1\frac{\partial u_1'}{\partial x_1}+\bar{u}_3\frac{\partial u_1'}{\partial x_3}+u_2'\frac{\partial \bar{u}_1}{\partial x_2}\right)=-\frac{\partial p^{*\prime}}{\partial x_1}+\bar{\mu}^*\frac{\partial^2 u_1'}{\partial x_2^2}$$

$$(4.2)\quad \bar{\rho}^*\left(\frac{\partial u_2'}{\partial t^*}+\bar{u}_1\frac{\partial u_2'}{\partial x_1}+\bar{u}_3\frac{\partial u_2'}{\partial x_3}\right)=-\frac{\partial p^{*\prime}}{\partial x_2}+\bar{\mu}^*\frac{\partial^2 u_2'}{\partial x_2^2}$$

$$(4.3)\quad \bar{\rho}^*\left(\frac{\partial u_3'}{\partial t^*} + \bar{u}_1\frac{\partial u_3'}{\partial x_1} + \bar{u}_3\frac{\partial u_3'}{\partial x_3}\right) = -\frac{\partial p^{*\prime}}{\partial x_3} + \bar{\mu}^*\frac{\partial^2 u_3'}{\partial x_2^2}$$

$$(4.4)\quad \frac{\partial \rho^{*\prime}}{\partial t^*} + \bar{u}_1\frac{\partial \rho^{*\prime}}{\partial x_1} + \bar{u}_3\frac{\partial \rho^{*\prime}}{\partial x_3} + u_2'\frac{\partial \bar{\rho}^*}{\partial x_2} = -\bar{\rho}^*\Delta'$$

$$(4.5)\quad \bar{\rho}^* c_v\left(\frac{\partial T^{*\prime}}{\partial t^*} + \bar{u}_1\frac{\partial T^{*\prime}}{\partial x_1} + \bar{u}_3\frac{\partial T^{*\prime}}{\partial x_3} + u_2'\frac{\partial \bar{T}^*}{\partial x_2}\right) = -\bar{p}^*\Delta' + \bar{k}\frac{\partial^2 T^{*\prime}}{\partial x_2^2}$$

$$(4.6)\quad \frac{p^{*\prime}}{\bar{p}^*} = \frac{\rho^{*\prime}}{\bar{\rho}^*} + \frac{T^{*\prime}}{\bar{T}^*}$$

where

$$(4.7)\quad \Delta' = \frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2} + \frac{\partial u_3'}{\partial x_3}$$

Equation (4.5) can also be written in the form

$$(4.8)\quad \bar{\rho}^*\left(\frac{\partial T^{*\prime}}{\partial t^*} + \bar{u}_1\frac{\partial T^{*\prime}}{\partial x_1} + \bar{u}_3\frac{\partial T^{*\prime}}{\partial x_3} + u_2'\frac{\partial \bar{T}^*}{\partial x_2}\right)$$

$$= \frac{1}{C_p}\left(\frac{\partial p^{*\prime}}{\partial t^*} + \bar{u}_1\frac{\partial p^{*\prime}}{\partial x_1} + \bar{u}_3\frac{\partial p^{*\prime}}{\partial x_3}\right) + \frac{\bar{\mu}^*}{\sigma}\frac{\partial^2 T^{*\prime}}{\partial x_2^2}$$

where $\sigma = C_p\bar{\mu}^*/\bar{k}^*$. These equations are correct except for terms $O(R^{-1/2})$.

A few typical steps in the process of simplification are discussed below:

Example 1. In the inertial terms of the equations of motion we have terms of the kind

$$\bar{u}_1\frac{\partial u_1'}{\partial x_1} + \bar{u}_2\frac{\partial u_1'}{\partial x_2}$$

Only the first term is included in (4.1). This is justified by the fact that

$$\bar{u}_2 \frac{\partial u_1'}{\partial x_2} : \bar{u}_1 \frac{\partial u_1'}{\partial x_1} = (\bar{u}_2/\bar{u}_1)\left(\frac{\partial u_1'}{\partial x_2}\Big/\frac{\partial u_1'}{\partial x_1}\right)$$

which is $O(R^{-1})$ outside the viscous layer and $O(R^{-1/2})$ inside the viscous layer, according to (3.4), (3.5), (3.9), (3.10).

<u>Example 2</u>. None of the terms arising from the dissipation function ($\tau_{ij}\ \epsilon_{ij} + p^* \partial u_i/\partial x_i$) are included in (4.5). This is justified as follows. The typical terms of the largest order which could be contributed to (4.5) are

$$\mu^{*\prime}\left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 + 2\bar{\mu}^* \frac{\partial \bar{u}_1}{\partial x_2}\frac{\partial u_1'}{\partial x_2} + \cdots$$

Their orders of magnitude can be estimated by comparing them with $\bar{\mu}^*(\partial \bar{u}_1/\partial x_2)^2$, which is the largest term in the dissipation function for the mean flow. Thus ,

$$\mu^{*\prime}\left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 = \frac{\mu^{*\prime}}{\bar{\mu}^*}\,\bar{\mu}^*\left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 \tag{4.10}$$

$$2\bar{\mu}^* \frac{\partial \bar{u}_1}{\partial x_2}\frac{\partial u_1'}{\partial x_2} = 2\left(\frac{\partial u_1'}{\partial x_2}\Big/\frac{\partial \bar{u}_1}{\partial x_2}\right)\bar{\mu}^*\left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 \tag{4.11}$$

On the other hand, the term $\bar{\rho}^* \bar{u}_1 C_v \partial T^{*\prime} / \partial x_1$ retained on the left-hand side of (4.5) can be estimated as follows:

$$(4.12) \quad \bar{\rho}^* \bar{u}_1 \, c_v \frac{\partial T^{*\prime}}{\partial x_1} = \bar{\rho}^* \bar{u}_1 c_v \left(\frac{\partial T^{*\prime}}{\partial x_1} \Big/ \frac{\partial \bar{T}^*}{\partial x_1} \right) \frac{\partial \bar{T}^*}{\partial x_1}$$

Now, from the energy equation for the basic flow, $\bar{\rho}^* \bar{u}_1 c_v \frac{\partial \bar{T}^*}{\partial x_1}$ is of the same order as the dissipation term $\bar{\mu}^* (\partial \bar{u}_1 / \partial x_2)^2$. Consequently, the various quantities given by (4.10), (4.11) and (4.12) are in the ratios 1:1:R outside the viscous layer and in the ratios $1:R^{1/2}:R$ within the viscous layer. We are therefore justified in neglecting the first two quantities, which are the largest contribution of the dissipation function to (4.5).

In addition to the approximations just described, we can also neglect the x_1 and x_3 dependence of the quantities in the basic flow, which determine the coefficients in equations (4.1) to (4.6). This approximation, which has been repeatedly discussed by various writers on the stability problem, depends on the boundary-layer nature of the mean flow and can be shown to be equivalent to neglecting terms of the order of R^{-1} (see [6]).

The system of equations (4.1) to (4.6) is essentially of the form expected from a strictly parallel flow. One important characteristic of the system is the absence of dissipative terms in the energy equation. This makes it possible, as we shall see in Section 5, to reduce the problem for a three-dimensional disturbance to a two dimensional problem. In previous investigations such a reduction was not possible because of certain terms retained in the equations for the disturbances which prevent a proper transformation. These terms are now shown to be negligible.

The discussion up to now has been restricted to the case of moderate Mach numbers, for which relations (3.1), (3.4) and (3.5) are valid. For very high supersonic Mach numbers the order-of-magnitude relations in the boundary layer corresponding to (3.1), (3.4), and (3.5) are different (see [5]), and consequently those for the disturbances, corresponding to (3.9), (3.10) and (3.11) are different. An order-of-magnitude analysis (see [6]) quite similar to that described for the case of low Mach numbers, shows that the disturbance equations are again, in the first approximation, of just the same form as (4.1) to (4.6). The proper velocity and temperature distributions given by a modified boundary-layer theory (see [4]) would have to be used, of course, at very high Mach numbers. The error estimate for the neglected terms in the disturbance equations is now different, however, being expressed not in terms of R but in terms of a Reynolds number

$$(4.13) \qquad R_o = \frac{\bar{\nu}_1^*}{\bar{\nu}_o^*} R$$

based on fluid properties at some representative temperature T_o^* inside the boundary layer. Since T_o^* is roughly proportional to M_1^2 at high Mach numbers, R_o is much less than R in such cases, so that the errors in the simplified equations (which are $O(R_o^{-1/2})$) might not be very small. A first approximation might not be very accurate at high Mach numbers, therefore.

The present analysis is carried out for any boundary layer over a flat surface, whether there is a pressure gradient or not. Thus, the results are valid, not only for the zero pressure-gradient flow over a flat plate, but also for the flows over wedge-shaped profiles. Although a pressure gradient is associated with the latter flows at subsonic Mach numbers, terms due to pressure gradient do not appear explicitly in the disturbance equations to the accepted order of approximation.*

Although the simplified equations (4.1) to (4.6) are obtained by rather crude order-of-magnitude arguments, an *a posteriori* check verifies that the approximations made are valid in the stability investigation of the boundary layer over a solid surface.**

*A more careful examination is necessary when there is surface curvature. Preliminary results indicate that for most three-dimensional disturbances curvature effects are negligible when the curvature is small, but that for those disturbances with propagation direction nearly normal to the free-stream direction (see Section 5) additional terms have to be retained in equation (4.2).

**Free boundary layers (such as wakes and jets) are excluded from all of the above discussion. The effect of viscosity on the stability of such flows is quite different. Further investigation of these cases is being carried out.

5. Periodic solutions of the equations for small disturbances

The equations (4.1) to (4.7) contain x_1, x_3 and t^* only through the partial derivatives, so that solutions of the form

$$(5.1) \qquad Q' = q(x_2)\exp\{i(\alpha_1 x_1 + \alpha_3 x_3 - \alpha_1 c^* t^*)\}$$

are possible. It may be expected that a disturbance of a fairly general nature can be analysed into normal modes of this type, with α_1 and α_3 real and c^* in general complex. A normal mode is damped if $c_i^* < 0$, neutral if $c_i^* = 0$, and self-excited if $c_i^* > 0$.

Evidently solutions of the form (5.1) may be regarded as waves propagating in a direction in the (x_1, x_3) plane specified by the numbers (α_1, α_3) (with changing amplitude in time if c^* is complex).

Consider in particular the case $\alpha_3 = 0$. Then, the terms involving the x_3-derivatives drop out of the equations (4.1) to (4.7) and the equations for $u_1', u_2', p^{*'}, \rho^{*'}, T^{*'}$ are exactly the same as if the component $\bar{u}_3$ of the basic flow were zero. Thus, for waves propagating in the x_1-direction only the component $\bar{u}_1$ of the basic flow (besides the distributions of the scalar quantities $\bar{p}^*$ and $\bar{T}^*$) has any influence. Now, a rotation of the co-ordinate system in the (x_1, x_3) plane will not change the system of equations (4.1) to (4.7). Hence, we may conclude that for waves propagating in any direction only the component of the basic flow in that direction has to be considered, in addition to the distributions of the scalar quantities. Every periodic three-dimensional disturbance may be treated in terms of a corresponding two-dimensional problem. One need only take the proper component of velocity in the basic flow to carry out the simplification.

If we reduce the above equations to dimensionless forms, it is clear that the waves propagating in the direction of the resultant free-stream velocity are associated with the Reynolds number R and the Mach number M_1 of the basic flow. For a wave propagating at an angle θ relative to this direction, these numbers are reduced to

$$(5.2) \qquad \tilde{R} = R\cos\theta \quad , \quad \tilde{M}_1 = M_1 \cos\theta$$

A direct transformation verifying this statement will be given in the next section.

The advantage of the present treatment lies in its generality and simplicity. Whether the incoming stream is oblique or normal to the leading edge of the flat plate makes no difference in the discussion.

When specialized to the incompressible case, the present results cover both the original work of Squire [3] and the case of an oblique incoming flow discussed by Kuethe [9]. For two-dimensional parallel flows, Squire obtained a set of transformation formulas leading to a conclusion equivalent to (5.2). Supplementary discussions are required in Squire's method. Kuethe, dealing with the case of the oblique incoming stream by Squire's method, noted the variation of "effective" Reynolds number with different directions. He did not, however, arrive at the conclusion reached here that the maximum "effective" Reynolds number is reached by waves propagating in the direction of the free-stream. Consdquently, these waves first give rise to instability as the Reynolds number of the flow is increased.

6. Equations for the amplitude functions

Consider now the stability of a two-dimensional boundary layer over a flat plate coinciding with the (x_1, x_3) plane, and introduce the dimensionless variables given in the List of Symbols. The free-stream velocity is now U_1, since $U_3 = 0$. The free-stream reference quantities δ, U_1, $\bar{\rho}_1^*$, $\bar{T}_1^*$, $\bar{p}_1^*$, $\bar{\mu}_1^*$ are evaluated at a fixed point $x_1 = x_1^0$ (that is, at a definite distance from the leading edge of the plate) in the region of the boundary layer that is of interest.

The dimensionless velocity and temperature distributions, $w = \bar{u}_1/U_1$ and $T = \bar{T}^*/\bar{T}_1^*$, depend on a number of factors, in particular, the boundary conditions at the wall and the free-stream Mach number M_1.

We consider a small disturbance of the boundary layer in the manner discussed previously, with solutions of the form

$$(6.1) \qquad Q'(x,y,z,t) = q(y) \exp \left\{ i(\alpha x + \beta z - \alpha c t) \right\}$$

for the disturbance quantities, as indicated in the List of Symbols. When relations of the type (6.1) are inserted into equations (4.1) to (4.7), the following equations for the amplitude functions

f, $\alpha\,\phi$, h, π, r, θ are obtained:

$$(6.2) \qquad \rho \left\{ i\alpha\,(w-c)f + w'\alpha\phi \right\} = -\frac{i\alpha}{\gamma M_1^2}\pi + \frac{\mu}{R} f''$$

$$(6.3) \qquad \rho \left\{ i\,\alpha^2 (w-c)\phi \right\} = -\frac{1}{\gamma M_1^2}\pi' + \frac{\mu}{R}\alpha\,\phi''$$

$$(6.4) \qquad \rho \left\{ i\alpha\,(w-c)h \right\} = -\frac{i\beta}{\gamma M_1^2}\pi + \frac{\mu}{R} h''$$

$$(6.5) \qquad i\alpha\,(w-c)r + \rho'\alpha\phi + \rho(i\alpha f + i\beta h + \alpha\phi') = 0$$

$$(6.6) \qquad \rho\left\{ i\alpha(w-c)\theta + T'\alpha\phi \right\} = -(\gamma - 1)(i\alpha f + i\,\beta h + \alpha\phi') + \frac{\gamma\mu}{\sigma R}\theta''$$

$$(6.7) \qquad \frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T}$$

The combination of terms $\alpha f + \beta h$ suggests its replacement by a new variable $\tilde{\alpha}\,\tilde{f}$. Indeed, the following transformations convert equations (6.2) to (6.7) to those of an equivalent two-dimensional problem:

$$(6.8) \qquad \left\{ \begin{array}{ll} \alpha f + \beta h = \tilde{\alpha}\,\tilde{f} & c = \tilde{c} \\ \alpha\,\phi = \tilde{\alpha}\,\tilde{\phi} & \sqrt{\alpha^2 + \beta^2} = \tilde{\alpha} \\ \alpha\,\pi = \tilde{\alpha}\,\tilde{\pi} & \alpha R = \tilde{\alpha}\,\tilde{R} \\ \alpha\,r = \tilde{\alpha}\,\tilde{r} & \alpha M_1 = \tilde{\alpha}\,\tilde{M}_1 \\ \alpha\,\theta = \tilde{\alpha}\,\tilde{\theta} & \end{array} \right.$$

The two-dimensional equations are:

$$(6.9)\quad \rho\left\{i\tilde{\alpha}(w-c)\tilde{f} + w'\tilde{\alpha}\tilde{\phi}\right\} = -\frac{i\tilde{\alpha}}{\gamma\tilde{M}_1^2}\tilde{\pi} + \frac{\mu}{\tilde{R}}\tilde{f}''$$

$$(6.10)\quad \rho\left\{i\tilde{\alpha}^2(w-c)\tilde{\phi}\right\} = -\frac{1}{\gamma\tilde{M}_1^2}\tilde{\pi}' + \frac{\mu}{\tilde{R}}\tilde{\alpha}\tilde{\phi}''$$

$$(6.11)\quad i\tilde{\alpha}(w-c)\tilde{r} + \rho'\tilde{\alpha}\tilde{\phi} + \rho(i\tilde{\alpha}\tilde{f} + \tilde{\alpha}\tilde{\phi}') = 0$$

$$(6.12)\quad \rho\left\{i\tilde{\alpha}(w-c)\tilde{\theta} + T'\tilde{\alpha}\tilde{\phi}\right\} = -(\gamma-1)(i\tilde{\alpha}\tilde{f} + \tilde{\alpha}\tilde{\phi}') + \frac{\gamma\mu}{\sigma\tilde{R}}\tilde{\theta}''$$

$$(6.13)\quad \frac{\tilde{\pi}}{p} = \frac{\tilde{r}}{\rho} + \frac{\tilde{\theta}}{T}$$

Evidently the transformations (6.8) are equivalent to (5.2) with $\cos\Theta = \alpha/\sqrt{\alpha^2 + \beta^2}$, so that Θ does in fact give the direction of propagation of the waves (6.1). The dimensionless wave speed c is not changed, which indicates that the phase velocity in the x-direction is actually $\sec\Theta$ times the phase velocity in the direction of the wave normal.

To complete the analysis, it is necessary to verify the equivalence of the boundary conditions. In the stability problem the boundary conditions are homogeneous, so that there is no difficulty in this respect. The homogeneous boundary conditions of the three-dimensional problem clearly transform to the proper homogeneous boundary conditions of the equivalent two-dimensional problem. For forced oscillations, however, it is obvious that the equivalence would have to be more carefully examined.

We have now shown that the boundary value problem associated with the system of equations (6.2) to (6.7) for a three-dimensional disturbance is equivalent to that associated with the system of two-dimensional equations (6.9) to (6.13). Thus the stability problem for three-dimensional disturbances can always be formulated as a two-dimensional problem which is mathematically the same as that for two-dimensional disturbances. It should be realized, however, that the equivalence is purely mathematical since equations (6.9) to (6.13) do not represent a proper two-dimensional disturbance. There is a decisive difference between the present results and those of Squire for an incompressible fluid. In the latter case, because the Mach number does not enter, it is possible to transform the equations for a three-dimensional disturbance to equations which actually do represent a real two-dimensional disturbance at a lower Reynolds number. It is then possible to conclude that three-dimensional disturbances can be ignored when the minimum critical Reynolds number is being investigated. For a compressible fluid this is not true because of the necessity of transforming the free-stream Mach number. It is not even true that every three-dimensional disturbance is equivalent to a two-dimensional one with a modified Mach number as well as Reynolds number, because with a reduced Mach number the velocity and temperature distributions would also have to be modified. In the equivalence discussed above these distributions must remain the same.

To apply the results of a two-dimensional analysis to the three-dimensional problem we may use the transformations (6.8), with the quantities $\tilde{q}$ associated with the two-dimensional problem. It is convenient, therefore, to think of three-dimensional disturbances as waves propagating in various directions. For each value of $\tilde{M}_1$ (that is for each value of the direction angle of the waves) the stability problem is investigated by means of equations (6.9) to (6.13) with the usual methods of the two-dimensional theory. Thus, for each value of $\tilde{M}_1$ there is a neutral curve in the $(\tilde{\alpha}, \tilde{R})$ plane which transforms to one in the (α, R) plane by means of the transformation (6.8). The family of all such neutral curves for all possible values of $\tilde{M}_1$ in the range $0 \leq \tilde{M}_1 \leq M_1$ gives the stability characteristics of three-dimensional disturbances.

7. Formulation of the characteristic-value problem

Having established a two-dimensional formulation of the problem for three-dimensional disturbances, we may make use of the theory of Lees and Lin [1] for further investigations. However, some modifications are necessary when the wave-speed of the disturbance is not small compared with the free-stream speed. For supersonic free-stream velocities this is usually the most important case. Detailed mathematical discussions of the necessary modifications will be given in a separate report. Here we shall merely give a summary of the theoretical arguments and indicate how the final formulas differ from the existing ones.

To determine the analytical nature of the system of equations (6.9) to (6.13) it is convenient to introduce the variables

$$(7.1) \quad \begin{cases} Z_1 = \tilde{f}, \quad Z_2 = \tilde{f}', \quad Z_3 = \tilde{\phi}, \\ Z_4 = \dfrac{\tilde{\pi}}{\tilde{M}_1^2}, \quad Z_5 = \tilde{\theta}, \quad Z_6 = \tilde{\theta}' \end{cases}$$

Then the equations transform to a system of six equations of the first order. The latter, when reduced to normal form, have coefficients analytic in the independent variable y and the parameters $\tilde{M}_1^2$, $\tilde{\alpha}\tilde{R}$, $\tilde{\alpha}^2$ and c.

Six homogeneous boundary conditions must be satisfied for a natural oscillation. These are given by

$$(7.2) \quad \begin{cases} \tilde{f}, \tilde{\phi}, \tilde{\theta} \text{ bounded as } y \to \infty, \\ \tilde{f}(0) = \tilde{\phi}(0) = 0, \text{ and either } \tilde{\theta}'(0) = 0 \text{ for an insulated} \\ \text{wall, or } \tilde{\theta}(0) = 0 \text{ for fixed temperature at the wall.} \end{cases}$$

The condition of boundedness for $y \to \infty$ can be put into an analytical form with the aid of the differential equations (6.9) to (6.13). For large y viscosity effects become negligible, so that the terms of order 1/R can be neglected, and the equations reduce to

$$(7.3) \qquad \frac{d}{dy}\left\{\frac{(w - c)\tilde{\phi}' - w'\tilde{\phi}}{T - \tilde{M}_1^2(w - c)^2}\right\} - \frac{\tilde{\alpha}^2(w - c)}{T}\tilde{\phi} = 0$$

For $y \to \infty$ this is easily seen to become

$$(7.4) \qquad \tilde{\phi}'' - \beta^2\tilde{\phi} = 0, \qquad \beta^2 = \tilde{\alpha}^2\left\{1 - \tilde{M}_1^2(1 - c)^2\right\}$$

Equation (7.4) has two solutions $\exp(\pm \beta y)$, where (for real c) β denotes the positive square root

$$(7.5) \qquad \beta = \tilde{\alpha}\sqrt{1 - \tilde{M}_1^2(1-c)^2}$$

Thus, for real c the condition of boundedness at infinity is automatically satisfied if

$$(7.6) \qquad c < 1 - \frac{1}{\tilde{M}_1}$$

In this case the characteristic values are continuous, and the solutions represent disturbances travelling at supersonic speeds relative to the component of the free-stream velocity in the direction of propagation of the disturbances. These supersonic disturbances correspond to sound waves propagating into or out of the boundary layer (see [1]).

For subsonic disturbances with

$$(7.7) \qquad c > 1 - \frac{1}{\tilde{M}_1}$$

the solution of (7.3) which is bounded at infinity is $\exp(-\beta y)$. Therefore the boundary condition for $y \to \infty$ is

$$(7.8) \qquad \frac{\tilde{\phi}'}{\tilde{\phi}} \to -\beta \text{ as } y \to \infty$$

In this case the characteristic values are discrete, and are determined from a secular equation which can be reduced to the form

$$E(\tilde{\alpha}, c, \tilde{M}_1^2) = F(z) \tag{7.9}$$

In this equation, F(z) is the function of Tietjens

$$F(z) = 1 + \frac{\int_{\infty}^{-z} \zeta^{3/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta}{z \int_{\infty}^{-z} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta} \tag{7.10}$$

where

$$z = (\tilde{\alpha} \tilde{R})^{1/3} \left[\frac{3}{2} \int_0^{y_c} \sqrt{\frac{c-w}{\nu}}\, dy\right]^{2/3} \tag{7.11}$$

The function $E(\tilde{\alpha}, c, \tilde{M}_1^2)$ is given by

$$\left(\frac{c-w}{\nu}\right)_w^{-1/2} \left[\frac{3}{2} \int_{y_c}^{0} \sqrt{\frac{c-w}{\nu}}\, dy\right] E(\tilde{\alpha}, c, \tilde{M}_1^2) = \begin{vmatrix} \tilde{\Phi}_{1w} & \tilde{\Phi}_{11}' + \beta \tilde{\Phi}_{11} \\ \tilde{\Phi}_{2w} & \tilde{\Phi}_{21}' + \beta \tilde{\Phi}_{21} \end{vmatrix} \div \begin{vmatrix} \dfrac{T_w \tilde{\Phi}_{1w}' + \tilde{M}_1^2 w_w' c \tilde{\Phi}_{1w}}{T_w - \tilde{M}_1^2 c^2} & \tilde{\Phi}_{11}' + \beta \tilde{\Phi}_{11} \\ \dfrac{T_w \tilde{\Phi}_{2w}' + \tilde{M}_1^2 w_w' c \tilde{\Phi}_{2w}}{T_w - \tilde{M}_1^2 c^2} & \tilde{\Phi}_{21}' + \beta \tilde{\Phi}_{21} \end{vmatrix} \tag{7.12}$$

The quantities $\tilde{\varphi}_{1w}, \tilde{\varphi}_{2w}$ are the wall values and $\tilde{\varphi}_{11}, \tilde{\varphi}_{21}$ the free-stream values of two linearly independent solutions of equation (7.3).

The principal differences between the present formulation and that given by Lees and Lin are in the formula (7.11) for z and the first factor in (7.12). The results of the previous theory are obtained by assuming c to be small and keeping only the first term in each of the power series expansions of w-c and ν about y_c. In the supersonic case, when c is not small, such a procedure does not represent a good approximation for our purposes.

8. <u>Determination of the curves of neutral stability for oblique waves.</u>

Consider oblique waves with direction angle arc cos $\tilde{M}_1/M_1$, where $\tilde{M}_1$ is some value in the range $0 \leq \tilde{M}_1 \leq M_1$. The condition (7.9) for the characteristic values $(\tilde{\alpha}, \tilde{R}, c)$ can be transformed to

(8.1) $$\Phi(z) = (1+\lambda)\frac{u+iv}{1+\lambda(u+iv)}, \quad \text{with} \quad \Phi(z) \equiv \frac{1}{1-F(z)},$$

where z, λ, and $u + iv$ are given by

(8.2a) $$\tilde{\alpha}\tilde{R} = z^3\left[\frac{3}{2}\int_0^{y_c}\sqrt{\frac{c-w}{v}}\,dy\right]^{-2}$$

(8.2b) $$\lambda(c) = \frac{3}{2}\,\frac{w_w'\,v_w^{1/2}}{c^{3/2}}\int_0^{y_c}\sqrt{\frac{c-w}{v}}\,dy - 1$$

(8.2c) $$u+iv = 1 + \frac{w_w' c}{T_w}\,\frac{\phi_{21}' + \tilde{\alpha}\sqrt{1-\tilde{M}_1^2(1-c)^2}\,\phi_{21}}{\phi_{11}' + \tilde{\alpha}\sqrt{1-\tilde{M}_1^2(1-c)^2}\,\phi_{11}}$$

For some purposes it is more convenient to solve (8.1) for $u + iv$, so that

(8.3) $$u+iv = \psi_r(z,c) + \psi_i(z,c)$$

where

(8.4 a) $$\psi_r(z,c) = \frac{(1+\lambda)\Phi_r - \lambda(\Phi_r^2+\Phi_i^2)}{[1+\lambda(1-\Phi_r)]^2 + \lambda^2\Phi_i^2}$$

(8.4 b) $$\psi_i(z,c) = \frac{(1+\lambda)\Phi_i}{[1+\lambda(1-\Phi_r)]^2 + \lambda^2\Phi_i^2}$$

Before discussing the determination of the neutral curve we shall write down the required formules and make certain transformations which are useful in the calculations. The quantities ϕ_{11} and ϕ_{21} are the free-stream values of two linearly independent solutions of the inviscid equation

$$(8.5)\qquad \frac{d}{dy}\left\{\frac{(w-c)\phi' - w'\phi}{T - \tilde{M}_1^2(w-c)^2}\right\} - \tilde{\alpha}^2\,\frac{w-c}{T}\,\phi = 0$$

These two solutions may be obtained as power series in $\tilde{\alpha}^2$, as described in [1], so that

$$(8.6a)\qquad \Phi_{11} = (1-c)\sum_{m=0}^{\infty} \tilde{\alpha}^{2m}\,\tilde{H}_{2m}$$

$$(8.6b)\qquad \Phi_{21} = (1-c)\sum_{n=0}^{\infty} \tilde{\alpha}^{2n}\,\tilde{K}_{2m+1}$$

$$(8.6c)\qquad \Phi_{11}' = (1-c)\left[\frac{1-\tilde{M}_1^2(1-c)^2}{(1-c)^2}\right]\sum_{m=1}^{\infty} \tilde{\alpha}^{2m}\,\tilde{H}_{2m-1}$$

$$(8.6d)\qquad \Phi_{21}' = (1-c)\left[\frac{1-\tilde{M}_1^2(1-c)^2}{(1-c)^2}\right]\sum_{m=0}^{\infty} \tilde{\alpha}^{2m}\,\tilde{K}_{2m}$$

where

$$(8.7a)\qquad \begin{cases} \tilde{H}_0 = 1\,, \qquad \tilde{H}_1 = \displaystyle\int_0^1 \frac{(w-c)^2}{T}\,dy\,, \\ \tilde{H}_2 = \displaystyle\int_0^1 \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy \int_0^y \frac{(w-c)^2}{T}\,dy\,, \\ \tilde{H}_3 = \displaystyle\int_0^1 \frac{(w-c)^2}{T}\,dy \int_0^y \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy \int_0^y \frac{(w-c)^2}{T}\,dy\,, \quad \ldots \end{cases}$$

$$(8.7b)\qquad \begin{cases} \tilde{K}_0 = 1\,, \qquad \tilde{K}_1 = \displaystyle\int_0^1 \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy\,, \\ \tilde{K}_2 = \displaystyle\int_0^1 \frac{(w-c)^2}{T}\,dy \int_0^y \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy\,, \\ \tilde{K}_3 = \displaystyle\int_0^1 \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy \int_0^y \frac{(w-c)^2}{T}\,dy \int_0^y \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy\,, \quad \ldots \end{cases}$$

For values of c that are not too large, the integrals (8.7a) and (8.7b) can be put into forms that are more suitable for calculation purposes. For example, in $\widetilde{K}_2$ we use the relation.

$$(8.8) \qquad \int_0^y \frac{T-\widetilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy = \int_0^1 \frac{T-\widetilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy - \int_y^1 \frac{T-\widetilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy$$

to transform $\widetilde{K}_2$ into

$$(8.9) \qquad \widetilde{K}_2 = \widetilde{H}_1\widetilde{K}_1 - \widetilde{N}_2 \quad , \text{ where } \quad \widetilde{N}_2 = \int_0^1 \frac{(w-c)^2}{T}\,dy \int_y^1 \frac{T-\widetilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy$$

Thus $\widetilde{N}_2$ has an integrand

$$(8.10) \qquad \frac{(w-c)^2}{T} \int_y^1 \frac{T-\widetilde{M}_1^2(w-c)^2}{(w-c)^2}\,dy$$

which is not numerically large for any value of y, so that $\widetilde{N}_2$ may be expected to be numerically smaller than $\widetilde{K}_2$. Similar transformations are made for the other integrals as follows:

$$(8.11) \qquad \begin{cases} \widetilde{K}_m = \widetilde{K}_1\widetilde{H}_{m-1} - \widetilde{N}_m \;, \; m \geq 3 \quad ; \quad \widetilde{N}_2 = \widetilde{H}_2 \quad ; \\ \widetilde{H}_m = \widetilde{H}_2\widetilde{H}_{m-2} - \widetilde{M}_m \quad , \; m \geq 3 \end{cases}$$

In general, $\widetilde{M}_n$ and $\widetilde{N}_n$ will be numerically smaller than the corresponding $\widetilde{H}_n$ and $\widetilde{K}_n$ for small c. The preceding transformations were first used in the stability theory for incompressible fluids [10], where c is usually fairly small, and then extended to the case of compressible fluids in [2]. Obviously they are not useful for reasonably large values of c, in which case other transformations may be more appropriate. This situation occurs for large enough Mach numbers - for M_1 greater than about two, perhaps. When M_1 becomes very large, in fact, c is often very close to unity.

Relation (8.2c) may now be written as

$$(8.12) \qquad u + i v = \frac{w_w' c}{T_w}\left[\tilde{K}_1 + \frac{T_w}{w_w' c} + \frac{1}{\tilde{\alpha}} G(\tilde{\alpha}, c)\right]$$

where

$$(8.13) \qquad G(\tilde{\alpha}, c) = (1 - \tilde{H}_2\tilde{\alpha}^2)\frac{\frac{\sqrt{1-\tilde{M}_1^2(1-c)^2}}{(1-c)^2}\left(1 - \sum_1^\infty \tilde{\alpha}^{2m}\tilde{N}_{2m}\right) - \sum_1^\infty \tilde{\alpha}^{2m+1}\tilde{N}_{2m+1}}{\left(1 - \sum_2^\infty \tilde{\alpha}^{2m}\tilde{M}_{2m}\right) + \frac{\sqrt{1-\tilde{M}_1^2(1-c)^2}}{(1-c)^2}\left(\tilde{\alpha}\tilde{H}_1 - \sum_1^\infty \tilde{\alpha}^{2m+1}\tilde{M}_{2m+1}\right)}$$

Now let $k = \frac{\sqrt{1-\tilde{M}_1^2(1-c)^2}}{(1-c)^2}$ and suppose that $k\tilde{H}_1\tilde{\alpha} < 1$. Then $G(\tilde{\alpha}, c)$ may be expanded as a power series in $\tilde{\alpha}$, so that on taking real and imaginary parts of (8.12) we find

$$(8.14a) \qquad u = \frac{w_w' c}{T_w}\left\{\tilde{K}_{1_r} + \frac{T_w}{w_w' c} + \frac{k}{\tilde{\alpha}}\left[1 - k\tilde{H}_1\tilde{\alpha} + (k^2\tilde{H}_1^2 - 2\tilde{H}_{2_r})\tilde{\alpha}^2 - +\right]\right\}$$

$$(8.15b) \qquad v = \frac{w_w' c}{T_w}\left\{\tilde{K}_{1_i} + \frac{k}{\tilde{\alpha}}\left[-2\tilde{H}_{2_i}\tilde{\alpha}^2 + \cdots\right]\right\}$$

The expansion of G as a power series is valid if

$$(8.15) \qquad \left| k\left(\tilde{\alpha}\tilde{H}_1 - \sum_1^\infty \tilde{\alpha}^{2m+1}\tilde{M}_{2m+1}\right) - \sum_2^\infty \tilde{\alpha}^{2m}\tilde{M}_{2m}\right| < 1$$

which is in general true if $k\tilde{H}_1\tilde{\alpha} < 1$. The latter inequality holds for all cases at subsonic and slightly supersonic Mach numbers, but at high supersonic Mach numbers it appears to hold only for cases of extreme cooling since otherwise k may be very large.

In the present work we shall neglect all terms of order $\tilde{\alpha}^2$ in (8.14a) and (8.14b), for the sake of simplicity, so that

(8.16a) $$u = \frac{w_w' c}{T_w}\left\{\tilde{K}_{1_r} + \frac{T_w}{w_w' c} + \frac{k}{\tilde{\alpha}}\left[1 - k\tilde{H}_1\tilde{\alpha}\right]\right\}$$

(8.16b) $$v = v_0(c) \equiv \frac{w_w' c}{T_w}\tilde{K}_{1_i}$$

In the calculations of [2] the same approximation (8.16b) for v is used, but terms of order $\tilde{\alpha}^3$ are retained in the relation (8.14a) for u. The results show, however, that very often the error corresponding to the neglect of terms of order $\tilde{\alpha}^2$ in (8.16a) is no larger than that in (8.16b). In any event, the approximations (8.16a) and (8.16b) appear to be reasonably adequate for all subsonic and slightly supersonic Mach numbers, with errors within the limits of error of the asymptotic theory. Moreover, they are very accurate at <u>any</u> Mach number <u>when $\tilde{\alpha}$ is very small</u> (which occurs, for example, when cooling at the solid surface is sufficient - see Section 10). For Mach numbers greater than about two, however, this is true only for cases of extreme cooling. The present approximations, therefore, are probably not accurate at such high Mach numbers in most cases of practical importance, and the method of numerical calculation may have to be revised*.

Now let us consider the procedure for finding the characteristic values $(\tilde{\alpha}, \tilde{R}, c)$ on the neutral curve. The various steps can be illustrated most clearly by geometrical considerations, with the aid of the condition for the characteristic values in the form (8.3) rather than (8.1).

For the values of c that occur in the calculations $\lambda(c)$ is always quite small, so that $\psi_r(z,c)$ and $\psi_i(z,c)$ are approximated fairly closely by $\Phi_r(z)$ and $\Phi_i(z)$. In fact, for $c = 0$, $\psi_r(z,0) = \Phi_r(z)$ and $\psi_i(z,0) = \Phi_i(z)$.

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*All existing calculations suffer from limitations of this kind. In some publications, numerical calculations were carried out by using methods adequate only for small c and α, even though the resulting values of these quantities were not small. Such results must be treated with reserve.

Therefore, for each value of c, the graphs of ψ_r and ψ_i against z have very much the same shapes as those of Φ_r and Φ_i respectively (see Figure 1). The maximum $\psi_{i_m}(c)$ on the graph of $\psi_i(z,c)$ against z is generally quite close to the maximum value $\Phi_i = 0.580$ at $z = 3.22$ on the Φ_i curve.

For Mach numbers that are not greater than about two, the graph of $v_o(c)$ against c has one of the typical forms shown in Figure 2(a) (which is plotted from the data given in [2]). In general, the greater the cooling (that is, the lower the wall temperature) the higher the v_o curve is in the (c,v_o) plane. For larger Mach numbers the v_o curves may have more complicated shapes, as indicated in Figure 2(b). For the present, we shall consider only Mach numbers for which the curves have the simple behaviour shown in Figure 2(a).

The pairs of values $(\tilde{\alpha},\tilde{R})$ are to be found for a series of real values of c. The minimum value of c for subsonic Mach numbers is $c = 0$, and for supersonic Mach numbers is either $c = 0$ or $c = 1-1/\tilde{M}_1$ if $\tilde{M}_1 > 1$.

For each value of c, relation (8.3) is solved for z and u with the aid of (8.16b). First, z is obtained by solving $v_o(c) = \psi_i(z,c)$, which is equivalent to finding the value of z corresponding to the point of intersection of the curve $\psi_i(z,c)$ against z with a straight line parallel to the z-axis at a distance $v_o(c)$ from it. From the behaviour of the graph of $\psi_i(z,c)$(which is similar to the graph of $\Phi_i(z)$ in Figure 1) we see that if $v_o(c) < 0$ there is one intersection point (one value of z), and if $v_o(c) > 0$ there are generally two intersection points (two values of z) if $v_o(c)$ is not too large. For some value of c large enough it is clear that the straight line will be just tangent to the corresponding graph of $\psi_i(z,c)$ at its maximum point, and for larger values of c there will be no intersection points at all. This maximum value of c, therefore, is given by the intersection of the graph of $\psi_{im}(c)$ with that of $v_o(c)$(see Figure 2). Corresponding to it there is just one value of z. With the value of z determined, $u = \Phi_r(z,c)$ is obtained from the graph of $\Phi_r(z,c)$ (which is similar to the graph of $\Phi_r(z)$ in Figure 1). Then $\tilde{\alpha}$ follows by solving (8.16a), since all the quantities in this equation besides $\tilde{\alpha}$ are now known functions of c, and finally $\tilde{R}$ is obtained from (8.2a). The corresponding values (α,R) are then determined by means of the transformations (6.8). The points (α,R) for all possible values of c give the curve of neutral stability in the (α,R) plane.

Evidently z(c) and u(c) could be obtained graphically as just described by drawing graphs of $\Phi_r(z,c)$ and $\Phi_i(z,c)$ against z for each value of c. However, for the purpose of the numerical calculations it is more convenient and more accurate to solve the two real relations obtained from (8.1) for z and u by an iteration procedure with the aid of the tabulated values of $\Phi_r(z)$ and $\Phi_i(z)$ (see Table 1), as described in [2]. The maximum value of c and the corresponding values of z and u can also be found accurately by an iteration method (see Appendix).

Present calculations are carried out for the boundary layer on a flat plate at a Mach number of $M_1 = 1.6$ and ratio of wall to free-stream temperature of $T_w/T_1 = 1.073$. This temperature ratio is the critical ratio below which the boundary layer is completely stable with respect to two-dimensional disturbances (see Section 10). The Prandtl number is given the value 0.75, and for simplicity of calculation the viscosity coefficient is taken to be directly proportional to the absolute temperature, with the constant of proportionality chosen so as to give the correct result at the wall. This procedure for the viscosity coefficient is known to be a good approximation in the case of the boundary layer itself when the Mach number is not too large. Although it probably is not as accurate in the stability calculations, estimates shows that it does not significantly alter the general trend of the results. A detailed discussion of the calculations is given in the Appendix.

In Table 3 $\alpha, \alpha_x, \alpha_\theta$, R, R_x, R_θ are tabulated as functions of c for several values of $\tilde{M}_1$ in the range $0 < \tilde{M}_1 < 1.6$, where α_x and R_x are wave-number and Reynolds number based on a reference length x_1 (the distance from the leading edge of the plate) and α_θ and R_θ are those based on the momentum thickness as reference length.

In Figure 3 neutral curves in the $(\alpha_\theta, R_\theta)$ plane are shown for the three cases $\tilde{M}_1 = 1.4, 1.0, 0.6$ corresponding to direction angles of 29.0°, 51.3°, 68.0° respectively.

The approximation is now made that the minimum critical Reynolds number is very close to the value of R corresponding to the maximum value of c. This was found to be true in previous investigations for an incompressible fluid [10] and for a compressible fluid [2]. The neutral curves in Figure 3 show that in the present case it is also an excellent approximation, with only a very slight error. The minimum critical Reynolds number R_{x_c} approximated in this way is plotted against the direction angle of the oblique waves in Figure 4 and tabulated in Table 4. As might have been expected, R_{x_c} becomes infinite at direction angles of 0° and 90°. The minimum value of R_{x_c} occurs at $\tilde{M}_1 = 1.030$ (corresponding to a direction angle of 49.9°) and is given by $R_{x_c}{}^{(3)} = 1.64 \times 10^6$. This value $R_{x_c}{}^{(3)} = 1.64 \times 10^6$, therefore, is the minimum critical Reynolds number for all three-dimensional disturbances in this particular case.

9. The dependence of the minimum critical Reynolds number on the direction of propagation of the waves

For those cases in which the approximations (8.16a) and (8.16b) are sufficiently accurate, it is possible to derive fairly simple analytical results for the directional dependence of the minimum critical Reynolds number (that is, dependence on the parameter $\tilde{M}_1$). By solving (8.16a) for $\tilde{\alpha}$ we have

$$(9.1) \qquad \tilde{\alpha} = \frac{\tilde{\alpha}_0}{1 + k \tilde{H}_1 \tilde{\alpha}_0}$$

where

$$(9.2) \qquad \tilde{\alpha}_o = \frac{\frac{w_w' c}{T_w} \frac{\sqrt{1-\tilde{M}_1^2(1-c)^2}}{(1-c)^2}}{u - \frac{w_w' c}{T_w}\left(\tilde{K}_{1_r} + \frac{T_w}{w_w' c}\right)} = \frac{\frac{w_w' c}{T_w} k}{u - \tilde{L}}$$

Then the minimum value of R on the neutral curve in the (α, R) plane is given, with the aid of (6.8) and (8.2a), by

$$(9.3) \qquad R_c = \frac{M_1}{\tilde{M}_1} \tilde{R}_c = \frac{M_1}{\tilde{M}_1} z^3 \left[\frac{3}{2}\int_0^{y_c} \sqrt{\frac{c-w}{v}}\, dy\right]^{-2} \frac{1 + k\tilde{H}_1\tilde{\alpha}_o}{\tilde{\alpha}_o}$$

where c is approximated by the maximum permissible value of c described previously. Once the particular boundary-layer flow is specified (i.e. the functions $w(y)$ and $T(y)$), this value of c and therefore $z(c)$, $u(c)$ and all the other functions of c in (9.3) are determined. For small λ, previous discussions show that c is given approximately by $v_0(c) \approx 0.580$, and also that
$z(c) \approx 3.22$, $u(c) \approx 1.48$ (see Figure 1).

We now have

$$(9.4) \qquad R_c = K\left(\frac{1}{\tilde{M}_1\tilde{\alpha}_o} + \frac{k\tilde{H}_1}{\tilde{M}_1}\right)$$

where K depends on the given boundary-layer flow but is independent of $\tilde{M}_1$. In order to determine the dependence on $\tilde{M}_1$ of the other quantities entering into R_c, we note that

$$(9.5) \qquad \tilde{H}_1 = \int_0^1 \frac{(w-c)^2}{T}\, dy = H_1$$

is independent of $\tilde{M}_1$, and that

$$(9.6)\quad u-\tilde{L} = u - \frac{w_w' c}{T_w}\left\{\tilde{K}_{1r} + \frac{T_w}{w_w' c}\right\}$$

$$= u - \frac{w_w' c}{T_w}\left\{R_e\int_0^1 \frac{T-\tilde{M}_1^2(w-c)^2}{(w-c)^2}dy + \frac{T_w}{w_w' c}\right\}$$

$$= u - \frac{w_w' c}{T_w}\left\{R_e\int_0^1 \frac{T-M_1^2(w-c)^2}{(w-c)^2}dy + \frac{T_w}{w_w' c} + M_1^2-\tilde{M}_1^2\right\}$$

$$= u - \frac{w_w' c}{T_w}\left\{K_{1r} + \frac{T_w}{w_w' c} + M_1^2-\tilde{M}_1^2\right\}$$

$$= u - L - \frac{w_w' c}{T_w}\left(M_1^2-\tilde{M}_1^2\right)$$

Then, with the aid of the definition for k, (9.4) can be written as

$$(9.7)\quad R_c = \frac{A + B\tilde{M}_1^2}{\tilde{M}_1\sqrt{1-\tilde{M}_1^2(1-c)^2}}$$

where

$$(9.8a)\quad A = K\left\{(1-c)^2\left[\frac{T_w}{w_w' c}(u-L) - M_1^2\right] + \frac{H_1}{(1-c)^2}\right\}$$

$$(9.8b)\quad B = K\left\{(1-c)^2 - H_1\right\}$$

The quantities A,B,c are independent of $\tilde{M}_1$ and are determined from the distributions w(y) and T(y) of velocity and temperature, respectively, in the boundary layer.

From (9.7) we find

$$(9.9)\quad \frac{dR_c}{d\tilde{M}_1} = \frac{\tilde{M}_1^2\left[2A(1-c)^2+B\right]-A}{\tilde{M}_1^2\left[1-\tilde{M}_1^2(1-c)^2\right]^{3/2}}$$

Therefore

$$\frac{dR_c}{d\tilde{M}_1} = 0 \quad \text{when} \quad \tilde{M}_1 = \frac{1}{\sqrt{2(1-c)^2 + \frac{B}{A}}} \tag{9.10}$$

and

$$\frac{dR_c}{d\tilde{M}_1} < 0 \quad \text{when} \quad \tilde{M}_1 < \frac{1}{\sqrt{2(1-c)^2 + \frac{B}{A}}} \tag{9.11}$$

Since $0 \leq \tilde{M}_1 \leq M_1$ it follows that

$$M_1 \sqrt{2(1-c)^2 + \frac{B}{A}} < 1 \tag{9.12}$$

is the condition that all three-dimensional disturbances have a higher minimum critical Reynolds number than two-dimensional ones. Thus, if it can be shown that the parameter $C = M_1 \sqrt{2(1-c)^2 + \frac{B}{A}}$ for a specified boundary-layer flow is less than unity, then three-dimensional disturbances are more stable and are not very important in the stability problem.

In Table 5 and Figure 5 are given the results of calculations of C as a function of M_1 for an insulated flat plate from $M_1 = 0$ to $M_1 = 2.4$, and of C as a function of T_w/T_1 at a Mach number of $M_1 = 0.7$ from $T_w/T_1 = 0.70$ to $T_w/T_1 = 1.25$. For the insulated plate, it is clear that three-dimensional disturbances are more stable than two up to a Mach number of about 1.8 When they become slightly more unstable. The fact that C decreases and again becomes less than unity when M_1 increases beyond 1.8 may not be of much significance because of the large errors at such Mach numbers due to the approximations used in the deductions.

10. The influence of surface cooling on boundary-layer stability

The problem of boundary-layer stability at supersonic speeds is different from that at subsonic speeds in one important respect-there is always a class of oblique waves (those with $1 < \tilde{M}_1 \leq M_1$) for which the values of c are bounded away from zero (i.e. $c \geq 1 - 1/\tilde{M}_1 > 0$). This leads to interesting consequences in the case of surface cooling.

Let us first consider strictly two-dimensional disturbances (for which $\tilde{M}_1 = M_1$), since the situation is then somewhat simpler. The minimum value of c on the neutral curve is $1-1/M_1$, and the maximum value is given by $v_o(c) = \Psi_{1_m}(c)$, that is, it is the value of c corresponding to the intersection point of the graph of $v_o(c)$ with that of $\Psi_{1_m}(c)$ (see Figure 2). For sufficiently small T_w/T_1, $v_o(c) > 0$ for $c = 1-1/M_1$ and $v_o(c)$ is monotonically increasing for $c > 1-1/M_1$. Also, further decreases in T_w/T_1 raise the whole of the V_o curve higher in the (c, v_o) plane. This behaviour of the v_o curve is typical for Mach numbers up to $M_1 = 7.48$.* For Mach numbers greater than $M_1 = 7.48$ no amount of cooling is sufficient to raise the v_o curve sufficiently to make $v_o(c)$ positive at $c = 1-1/M_1$.

We see, therefore, that in this range of Mach numbers for T_w/T_1 small enough the range of values of c on the neutral curve is bounded below by $1-1/M_1$ and above by an upper limit given by $v_o(c) = \Psi_{1_m}(c)$, which can be made as close to $1-1/M_1$ as desired by making T_w/T_1 smaller. Thus, there is a critical temperature ratio $(T_w/T_1)_c$ given by $v_o(1-1/M_1) = \Psi_{1_m}(1-1/M_1)$ for which the whole neutral curve disappears. This can be seen by solving (8.16a) for $\tilde{\alpha}$ and noting that $\tilde{\alpha} \to 0$ as $c \to 1-1/M_1$. Therefore, when $T_w/T_1 \to (T_w/T_1)_c$, all values of c on the neutral curve approach $1-1/M_1$, so that all values of $\tilde{\alpha} \to 0$ and all values of $\tilde{R} \to \infty$.

It is to be noticed that in this limiting situation relations (8.16a) and (8.16b) becomes exact since the errors are $O(\tilde{\alpha}^2)$, so that the relation

$$v_o\left(1 - \frac{1}{M_1}\right) = \Psi_{1_m}\left(1 - \frac{1}{M_1}\right) \qquad (10.1)$$

*The actual value of this upper limit to the Mach number depends on the value of the Prandtl number and the particular viscosity law used. The value $M_1 = 7.48$ corresponds to a Prandtl number of 0.75 and a linear viscosity-temperature relationship.

for the critical temperature ratio $(T_w/T_1)_c$ is exact.

When $T_w/T_1 < (T_w/T_1)_c$ evidently $v_o(c) > \Phi_{im}(c)$ for $c \geq 1 - 1/M_1$ so that no solution is possible. Therefore for $T_w/T_1 \leq (T_w/T_1)_c$ the boundary layer is completely stabilized with respect to two-dimensional disturbances.

Calculations (described in detail in the Appendix) have been carried out to determine the variation of $(T_w/T_1)_c$ with M_1 for the boundary layer on a flat plate with Prandtl number equal to 0.75 and viscosity coefficient directly proportional to absolute temperature. The results are tabulated in Table 6 and the curve of critical temperature ratio against Mach number for two-dimensional disturbances is shown in Figure 6. Since the critical temperature ratio approaches zero at $M_1 = 7.48$, no amount of cooling is sufficient to completely stabilize the boundary layer with respect to two-dimensional disturbances when $M_1 \geq 7.48$. The present mumerical results do not differ appreciably from those of Van Driest [11], even though corrected viscous solutions (which enter into the determination of $\lambda(c)$ and thus of $\Phi_{im}(c)$) are used. The explanation is probably the fact that $v_o(c)$ is so extremely sensitive to variations in T_w/T_1 that small errors in Φ_{im} in (10.1) do not affect the solution T_w/T_1 very much.*

The conclusions depend, of course, very strongly on the value of the Prandtl number and the particular viscosity-temperature relationship used (see [11]). However, for the same reason as given above, it is not likely that the use of the present viscous solutions would significantly alter the results for these other cases obtained in [11].

*Van Driest's computations, in addition to the use of the inaccurate $\lambda(c)$ of reference [1], also involve what is essentially an approximation for $\Phi_{im}(c)$ in terms of $\lambda(c)$. It is interesting to observe that if the exact procedure of calculation is used, in combination with the inaccurate value of $\lambda(c)$ given by reference [1], several critical cooling ratios will be found for each value of $c = 1 - 1/M_1$. This has been pointed out by Professor Martin Bloom. However, with the present function $\lambda(c)$ whose influence is small, such difficulties do not arise.

Now consider oblique waves with direction angle arc cos $\tilde{M}_1/M_1$, where $1 < \tilde{M}_1 < M_1$. Just as in the case of strictly two-dimensional disturbances, there is again the possibility of completely stabilizing the boundary layer with respect to oblique waves propagating in this direction for small enough T_w/T_1. The arguments carry through very much as before except for one complication due to the fact that when the Mach number is large the curve $v_o(c)$ against c may have an inflection point and thus a maximum and minimum in the interval $0 < c < 1$ (see Figure 2(b)). Thus even when T_w/T_1 is small enough for the relation

$$v_o\left(1 - \frac{1}{\tilde{M}_1}\right) = \Psi_{1_m}\left(1 - \frac{1}{\tilde{M}_1}\right) \tag{10.2}$$

to hold, the corresponding $v_o(c)$ may not be monotonically increasing for $c > 1 - 1/\tilde{M}_1$, when $\tilde{M}_1$ is small enough*. There could still be values of $c > 1 - 1/\tilde{M}_1$ for which $v_o(c) < \Psi_{1_m}(c)$ (see Figure 2(b)), so that a neutral curve might still exist. The critical temperature ratio below which the boundary layer is completely stable with respect to oblique waves with direction angle arc cos $\tilde{M}_1/M_1$ is therefore the largest T_w/T_1 for which

$$v_o(c) \geq \Psi_{1_m}(c) \quad , \quad c \geq 1 - \frac{1}{\tilde{M}_1} \tag{10.3}$$

This value of T_w/T_1 is evidently the one for which the corresponding $v_o(c)$ curve is just tangent near its minimum point to the $\Psi_{1_m}(c)$ curve (see Figure 2(b)).

Obviously if $\tilde{M}_1 \leq 1$ the minimum value of c on the neutral curve is just $c = 0$. Therefore, it is not possible by any amount of cooling to completely stabilize the boundary layer with respect to oblique waves with direction angles greater than arc cos $1/M_1$. The limiting directions are thus the normals to the Mach lines in a plane parallel to the solid surface.

For the boundary layer on a flat plate at a Mach number $M_1 = 4$ with Prandtl number equal to 0.75 and a linear viscosity temperature relation, the critical temperature ratio T_w/T_1 has been calculated as a function of direction angle arc cos $\tilde{M}_1/M_1$ and plotted as a full-line curve in Figure 7. The calculations

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* This difficulty does not occur when $\tilde{M}_1$ is as large as M_1, at least for the boundary layer on a flat plate.

are carried out by first finding T_w/T_1 as a function of $\tilde{M}_1$ from (10.2). The resulting values of T_w/T_1 when plotted against arc cos $\tilde{M}_1/M_1$ give a curve with a maximum and a minimum for $\tilde{M}_1$ in the range $0 < \tilde{M}_1 < M_1$. The proper curve for the critical temperature ratio is the obtained by drawing a horizontal tangent to this curve at its minimum point (thus cutting off the dashed part around the maximum point in Figure 7)*. The calculation procedure is described more fully in the Appendix, and the numerical results are listed in Table 7.

Figure 7 shows that for a temperature ration $T_w/T_1 = 1.700$ all two-dimensional disturbances are completely stabilized, and for a somewhat smaller ratio $T_w/T_1 = 1.474$ the three-dimensional disturbances corresponding to oblique waves with direction angles from 0° to 74.3° are completely stabilized. The critical temperature ratio T_w/T_1 decreases to zero very rapidly, however, when the direction angle increases from 74.3° to arc cos $1/M_1 = 75.5^\circ$.

When the temperature ratio T_w/T_1 is as low as 1.474, therefore, all oblique waves are stabilized except those with direction angles greater than 74.3°. Estimates of the minimum critical Reynolds number of the latter (all of which have very small values of the wave-speed c) show that $R_c \sim 10^6$ or $R_{x_c} \sim 10^{12}$. Since such large values of the Reynolds number are far beyond the range of practical importance, it follows that if the wall temperature ratio can be reduced to as low a value as 1.474 the boundary layer will be very highly stabilized from the practical point of view. Such cooling is fairly extreme, however. The situation for larger values of T_w/T_1 is not clear, except that it seems likely the boundary layer is quite sensitive to

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*This is equivalent to the use of the condition mentioned in connection with relation (10.3).

changes in T_w/T_1. Even a small increase above $T_w/T_1 = 1.474$ might make the boundary layer quite unstable.

For each value of M_1 up to $M_1 = 7.48$ it is probable that the nature of the oblique waves is about the same as at $M_1 = 4$: that is, for a sufficiently small temperature ratio T_w/T_1 it is possible to completely stabilize oblique waves with direction angles less than arc cos $1/M_1$ but not those with larger direction angles. It is clear from Figure 6, however, that such temperature ratios would usually be so small as to be of little practical interest. At the present time, very little more can be said about the general problem of boundary-layer stability at such large Mach numbers. The complicated shape of the $v_o(c)$ curve in many cases of interest makes the problem difficult, as we have seen, and also the present methods of numerical calculation do not appear to be adequate except for the unimportant class of problems involving extreme cooling (when $\dot{\tilde{\alpha}}$ is very small).

For lower Mach numbers (up to about $M_1 = 2$), the picture is clearer. The theoretical results of Section 9, which are then satisfactory approximations, provide considerable useful information. The numerical calculations indicate that the parameter C increases as T_w/T_1 decreases (see Figure 5). Thus the effect of cooling a flat plate at a given Mach number is evidently to make three-dimensional disturbances relatively more unstable, so that with sufficient cooling some of them can actually become more unstable than two-dimensional disturbances. Since C for an insulated plate increases from $C = 0$ at $M_1 = 0$ to a value close to unity when $M_1 \simeq 2$, it appears that the amount of cooling required to make any of the three-dimensional disturbances more unstable than two-dimensional ones is extreme at low subsonic Mach numbers, but gradually becomes less and less extreme as M_1 increases until at a Mach number of the order of two very little is required. In particular, at $M_1 = 0.7$, the wall temperature ratio must be reduced below $T_w/T_1 = 0.58$ (see Figure 5).

It is known that at subsonic Mach numbers cooling of the solid surface is very effective in stabilizing the boundary layer, if only two-dimensional disturbances are considered (see [2]). The preceding remarks indicate that probably this conclusion is still valid even if three-dimensional disturbances are considered, since the latter usually have a higher minimum critical Reynolds number than two-dimensional disturbances (except under conditions of extreme cooling).

Even at the moderate supersonic Mach numbers when three-dimensional disturbances begin to play the leading role, cooling still has an important stabilizing effect, although complete stabilization at supersonic Mach numbers as prediced in [2] is impossible according to the present theory. For example, estimates show that for an insulated flat plate at $M_1 = 1.6$, R_{x_c} is of the order of 10^5, whereas the calculated results show that for a cooled plate at the same Mach number $M_1 = 1.6$, with a temperature ratio $T_w/T_1 = 1.073$ which is just small enough for two-dimensional disturbances to be completely stabilized, R_{x_c} is of the order of 10^6.

11. Conclusions

(11.1) An order of magnitude analysis of the complete linearized equations for a three-dimensional disturbance superimposed on a two-dimensional boundary layer shows that they can be reduced to much simpler equations in the first approximation. These simplified equations are still valid at high free-stream Mach numbers if correct velocity and temperature profiles for the boundary layer in such cases are used. At very high Mach numbers, however, the accuracy of a first approximation may not be very great, so that higher order approximations and therefore more complicated equations with more terms retained may have to be considered.

(11.2) The present equations for a three-dimensional disturbance transform to equations which have the same form mathematically as those for a two-dimensional disturbance, but which do not, however, correspond to a real two dimensional disturbance. Although no direct conclusion regarding the importance of three-dimensional disturbances is now possible, the use of these transformed two-dimensional equations leads to a considerable mathematical simplification of the theory.

(11.3) Corrected viscous solutions have been obtained which lead to a condition for the characteristic values of very much the same form as that in the previous theory of Lees and Lin. Detailed mathematical discussions will be given in another report. The present formulation of the theory permits more accurate numerical calculations at supersonic Mach numbers, when the wave-speed c is often large. It reduces to the Lees-Lin formulation for small values of c, so that the validity of their results in such cases is verified.

(11.4) Consideration of the present analytical and numerical results suggests that probably three-dimensional disturbances are of little importance at subsonic free-stream Mach numbers, since they usually have a higher minimum critical Reynolds number than two-dimensional ones, except possibly under conditions of extreme surface cooling. However, further numerical calculations for oblique waves in this range of Mach numbers would be useful.

(11.5) As the Mach number increases, three-dimensional disturbances enter the picture under conditions which become less and less extreme, until finally for M_1 between one and two they begin to play the leading role in many problems of practical interest.

(11.6) At supersonic free-stream Mach numbers the boundary layer can never be completely stabilized with respect to <u>all</u> three-dimensional disturbances. Although oblique waves whose directions of propagation are at an angle less than arc cos $1/M_1$ with the free-stream direction can be completely stabilized by sufficient cooling when M_1 is not too large, those whose directions of propagation are at a greater angle cannot.

(11.7) The graph of critical temperature ratio (below which strictly two-dimensional disturbances are completely stabilized) versus Mach number (Figure 6) is not significantly different from that found by Van Driest [11], in spite of the corrections to the viscous solutions. Although these corrections are very important for the accurate determination of the curve of neutral stability at supersonic Mach numbers, they are unimportant for the determination of the critical temperature ratios.

(11.8) For Mach numbers up to about two, cooling of the solid surface is very effective in stabilizing the boundary layer. At higher Mach numbers it appears that the amount of cooling necessary to raise the minimum critical Reynolds number significantly becomes practically prohibitive.

(11.9) Very little more can be said about the problem of boundary-layer stability at Mach numbers greater than two, since the present methods of numerical calculation for the curve of neutral stability (and thus the minimum critical Reynolds number) do not appear to be adequate in such cases. The main difficulty is in evaluating the inviscid solutions of the disturbance equations when c and $\tilde{\alpha}$ are both large. This problem is being studied further.

(11.10) A more serious difficulty than the preceding is that at very high Mach numbers the present asymptotic theory, which is essentially a first-order theory, may in itself by inaccurate (as pointed out in (11.1)). More information about this matter will be provided by numerical results (including estimates of certain parameters) for high Mach numbers, which will be obtained as soon as the computational difficulties mentioned in (11.9) are resolved.

(11.11) It is believed that the approximate method of calculation of the neutral curve described in Section 8 is adequate up to a Mach number of the order of two, especially in cases involving cooling, when c and $\tilde{\alpha}$ are both small. Additional neutral curves for the oblique waves in this range of Mach numbers or perhaps further calculations of the type described in Sections 9 and 10, would be of considerable value. It might also be of interest to determine in a few typical cases the effect of using a more accurate approximation than (8.16 a), by keeping terms of higher order in $\tilde{\alpha}$, as Lees has done [2].

(11.12) For practical purposes, the transition Reynolds number is much more important than the minimum critical Reynolds number. Thus, information concerning the rate of amplification of the boundary-layer disturbances after they enter the unstable region would be extremely useful. Further investigations are being directed towards developing the amplification theory for three-dimensional disturbances, which may be somewhat different from that for strictly two-dimensional disturbances.

Appendix. Procedure for the numerical calculations

(a) The transformation to w as independent variable

Most of the stability calculations can be carried out most conveniently with w as independent variable. This formulation has also been found to be very useful for the boundary-layer calculations (see [12] and [13]).

First, introduce new non-dimensional variables as follows:

$$(1) \qquad s = \frac{\tau}{\tau_w}, \quad \eta = \frac{\mu}{\mu_w}, \quad \theta = \frac{T}{T_w} = \frac{\rho_w}{\rho}$$

where $\tau = \mu w'$ and w, T, ρ, μ are as given in the List of Symbols.* Then relations (8.2a) and (8.2b) can be transformed to

$$(2a) \qquad \tilde{\alpha}\tilde{R} = \frac{z^3}{I^2}\left(\frac{R}{p}\,\frac{\tau_w^2\,T_w}{\mu_w}\right)$$

$$(2b) \qquad \lambda(c) = \frac{3}{2c^{3/2}}\,I(c) - 1$$

where

$$(3) \qquad I(c) = \int_0^c \sqrt{(c-w)\frac{\eta}{\theta}}\;\frac{dw}{s}$$

Also, relation (8.16b) for $v_o(c)$ becomes

*The present symbol θ is not to be confused with the same symbol used elsewhere in this report

(4) $$\nu_0(c) = \frac{w_w' c}{T_w} \tilde{K}_{11} = -\frac{w_w' c}{T_w} \pi \left[\frac{T^2}{w'^3} \frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$$

$$= \left[\pi w \frac{\eta\theta}{s}\left(\frac{\eta'}{\eta} + \frac{\theta'}{\theta} - \frac{s'}{s}\right)\right]_{w=c}$$

where θ, η, s are considered as functions of w and θ', η', s' are their derivatives with respect to w.

The non-dimensional viscosity coefficient η is related to θ most accurately by the Sutherland viscosity law (see [13], p.7)

(5) $$\eta = \theta^{3/2} \frac{1 + \frac{0.505}{T_w}}{\theta + \frac{0.505}{T_w}}$$

The non-dimensional temperature θ is given as follows (see [13], p.8) when the specific heats C_p and C_v are constants:

(6) $$\theta = 1 - \left(1 - \frac{1}{T_w}\right)\theta_I(w) + (\gamma - 1)M_1^2 \frac{1}{T_w} \theta_{II}(w)$$

where $\gamma = C_p/C_v = 1.4$, and $\theta_I(w)$ and $\theta_{II}(w)$ are the functions tabulated on p.27 of [13]. Crocco [12] has shown that these functions are practically independent of the particular viscosity-temperature relationship used.

For the case of an insulated wall, the ratio of wall to free-stream temperature is given by

(7) $$T_w = 1 + \sqrt{\sigma} \frac{\gamma - 1}{2} M_1^2$$

When $\rho\mu$ = constant = $\rho_w\mu_w$, it follows from (1) that $\eta = \theta$.

The function $s(w) = g_*(w)/g_*(0)$ is determined from the function $g_*(w)$ which satisfies the following differential equation (see [13], p.6):

$$(8) \qquad g_* g_*'' + 2w\rho\mu = 0$$

with boundary conditions $g_*'(0) = 0$, $g_*(1) = 0$. If $\rho\mu = \rho_w\mu_w$ then

$$(9) \qquad \frac{g_*}{K}\frac{g_*''}{K} + 2w = 0$$

where $K^2 = \rho_w\mu_w$. Therefore, $g_*(w)/K = \bar{g}_*(w)$, the Blasius shear-stress function (i.e. $\bar{g}_*\bar{g}_*'' + 2w = 0$ with $\bar{g}_*'(0) = 0$, $\bar{g}_*(1) = 0$). This function is tabulated in [13], p.25.

For $\rho\mu = \rho_w\mu_w$, therefore, s satisfies the following equation:

$$(10) \qquad s s'' + \frac{2}{\bar{g}_*^2(0)} w = 0$$

with boundary conditions $s'(0) = 0$, $s(1) = 0$, since $s = g_*(w)/g_*(0) = \bar{g}_*(w)/\bar{g}_*(0)$. A comparison of this Blasius distribution for s with the distributions obtained in [13] when the accurate Sutherland viscosity law is used shows that in many cases it is actually quite a close approximation to the correct one.

The function θ satisfies the following differential equation (see [13], p.6):

$$(11) \qquad \left(\theta'' + \sigma(\gamma-1)M_1^2\frac{1}{T_w}\right)s + (1-\sigma)\theta' s' = 0$$

with the boundary conditions $\theta(0) = 1$, $\theta(1) = 1/T_w$.

The functions $s(w)$, $s'(w)$, $\theta_I(w)$, $\theta_I'(w)$, $\theta_{II}(w)$, $\theta_{II}'(w)$ are tabulated in Table 8. The values for θ_I and θ_{II} have been re-calculated by the method described in [13], and are somewhat more accurate than the values tabulated in [13], p.27. The Prandtl number σ is taken to be 0.75.

For small values of c, I(c) and λ(c) may be obtained most conveniently from power series in c. With the aid of equations (2b), (3) and (10) and the fact that $\bar{g}_*(0) = 0.66411$, these series are determined as follows (for $\rho\mu = \rho_w\mu_w$):

$$(12)\qquad I(c) = \frac{2}{3} c^{3/2}\left\{1 + \frac{8}{315}\left[\frac{2}{(0.66411)^2}\right]c^3 + \frac{1216}{675675}\left[\frac{2}{(0.66411)^2}\right]^2 c^6 + O(c^9)\right\}$$

$$(13)\qquad \lambda(c) = \frac{8}{315}\left[\frac{2}{(0.66411)^2}\right]c^3 + \frac{1216}{675675}\left[\frac{2}{(0.66411)^2}\right]^2 c^6 + O(c^9)$$

$$= 0.115167\, c^3 + 0.037008\, c^6 + O(c^9)$$

The functions I(c) and λ(c) are tabulated in Table 8, the values for small c being obtained by use of (12) and (13) and those for large c by numerical integration of (3) and use of (2b).

In the calculations the quantities $(\delta/x_1)\sqrt{R_x}$ and $\tau_w/\mu_w\tau_w$ are required. In the Crocco method the velocity distribution is given by

$$(14)\qquad \frac{x_2}{x_1}\sqrt{R_x} = 2\int_0^w \frac{\mu}{g_*}\, dw$$

(see [13], p.20) so that

$$(15)\qquad \frac{\delta}{x_1}\sqrt{R_x} = 2\int_0^{w_1} \frac{\mu}{g_*}\, dw$$

where w_1 is the value chosen to define the edge of the boundary layer. In the present calculations the value of w_1 is taken to be 0.999. For $\rho\mu = \rho_w\mu_w$ equation (15) becomes

$$(16)\qquad \frac{\delta}{x_1}\sqrt{R_x} = 2(\mu_w T_w)^{1/2}\left\{\int_0^{0.999}\frac{dw}{\bar{g}_*} - \left(1-\frac{1}{T_w}\right)\int_0^{0.999}\frac{\theta_I}{\bar{g}_*}\,dw + (\gamma-1)M_1^2\frac{1}{T_w}\int_0^{0.999}\frac{\theta_{II}}{\bar{g}_*}\,dw\right\}$$

$$= 2(\mu_w T_w)^{1/2}\left\{3.00 - 2.0580\left(1-\frac{1}{T_w}\right) + 0.1460(\gamma-1)M_1^2\frac{1}{T_w}\right\}$$

From equation (40) p. 17 of [13] we see that

$$(17)\qquad \tau_w = (\mu w')_w = \frac{\delta}{\bar{\mu}_1^* U_1}\left(\mu^*\frac{\partial \bar{u}_1}{\partial x_2}\right)_w$$

$$= \frac{\delta}{\bar{\mu}_1^* U_1}\,\frac{1}{2}\,\bar{\rho}_1^* U_1^2\,\frac{g_*(0)}{\sqrt{R_x}} = \frac{1}{2}\,\frac{\delta}{x_1}\sqrt{R_x}\; g_*(0)$$

$$= \frac{1}{2}\,\frac{\delta}{x_1}\sqrt{R_x}\;\sqrt{\rho_w\mu_w}\;\bar{g}_*(0)$$

Therefore

$$(18)\qquad \frac{\tau_w}{\mu_w T_w} = \frac{0.66411}{2}\;\frac{\frac{\delta}{x_1}\sqrt{R_x}}{\mu_w^{1/2}\,T_w^{3/2}}$$

The value of μ_w is determined from the Sutherland viscosity law (see [13] p. 7):

$$(19) \qquad \mu_w = T_w^{3/2} \frac{1.505}{T_w + 0.505}$$

(b) <u>The transformation to ζ as independent variable in the outer part of the boundary layer.</u>

In the evaluation of certain integrals that are required in the calculations it is more convenient to use some independent variable other than w in the outer part of the boundary layer. In the present case when $\rho\mu = \rho_w\mu_w$, the normal-distance parameter ζ of the standard Blasius solution for the non-dimensional stream function $f(\zeta)$ is suitable, where

$$(20) \qquad w = \frac{1}{2} f'(\zeta) \;, \quad \text{and} \quad f''' + f f'' = 0 \quad \text{with}$$

boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 2$ (see [14], p. 135).

For the Blasius case, $\bar{\mu}^* =$ constant $= \bar{\mu}_1^*$, $\bar{\rho}^* =$ constant $= \bar{\rho}_1^*$, $\zeta = \frac{1}{2}(U_1/\bar{\nu}_1^* x_1)^{\frac{1}{2}} x_2$ and

$$(21) \qquad \tau = \mu w' = \frac{1}{4} f''(\zeta)\left(U_1/\bar{\nu}_1^* x_1\right)^{1/2}\delta = \frac{1}{4}\frac{\delta}{x_1}\sqrt{R_x}\, f''(\zeta)$$

But from (17)

$$(22) \qquad \tau = \frac{1}{2}\frac{\delta}{x_1}\sqrt{R_x}\;\bar{g}_*(w)$$

since $\rho_w/\mu_w = \bar{\rho}_w^* \bar{\mu}_w^* / \bar{\rho}_1^* \bar{\mu}_1^* = 1$ in this case. Therefore

(23) $$\frac{1}{2} f''(\zeta) = \bar{g}_*(w)$$

The functions $\theta_I(\zeta)$, $\theta_{II}(\zeta)$, $\theta'_I(\zeta)$, $\theta'_{II}(\zeta)$ for large values of ζ may be obtained with the aid of the following asymptotic formula for $\frac{1}{2} f'(\zeta)$:

(24) $$\frac{1}{2} f'(\zeta) = 1 + 2n \int_{\infty}^{\zeta} e^{-(\zeta - \frac{m}{2})^2} d\zeta$$

where m = 1.73, n = 0.231 (see [15], p. 37).

The functions $\theta_I(w)$ and $\theta_{II}(w)$ are defined on pp. 7-8 of [13] as follows:

(25) $$\theta_I(w) = \frac{I(w)}{I(1)} \quad ; \quad \theta_{II}(w) = \sigma\left[\theta_I(w) J(1) - J(w)\right]$$

where

(26) $$I(w) = \int_0^w \left[\frac{\bar{g}_*(w)}{\bar{g}_*(0)}\right]^{\sigma-1} dw$$

(27) $$J(w) = \int_0^w \left[\frac{\bar{g}_*(w_1)}{\bar{g}_*(0)}\right]^{\sigma-1} dw_1 \int_0^{w_1} \left[\frac{\bar{g}_*(w_2)}{\bar{g}_*(0)}\right]^{1-\sigma} dw_2$$

When the transformation to ζ as independent variable is made, the following asymptotic formulas may be obtained with the use of (24):

(28) $$\theta_I(\zeta) = 1 - \frac{[f''(0)]^{1-\sigma} (4n)^{\sigma}}{2\, I(1)\sqrt{\sigma}} \int_{\sqrt{\sigma}(\zeta - \frac{m}{2})}^{\infty} e^{-\zeta^2} d\zeta$$

$$(29) \qquad \theta_{I}'(\zeta) = \frac{[f''(0)]^{1-\sigma}(4m)^{\sigma}}{2\, I(1)}\, e^{-\sigma(\zeta-\frac{m}{2})^2}$$

$$(30) \qquad J(\zeta) = J(1) - \frac{[f''(0)]^{1-\sigma} I_{2-\sigma}(1)(4m)^{\sigma}}{2\sqrt{\sigma}} \int_{\sqrt{\sigma}(\zeta-\frac{m}{2})}^{\infty} e^{-\zeta^2} d\zeta \; +$$

$$+ \frac{4m^2}{\sqrt{2(2-\sigma)}} \left\{ \frac{1}{2}\left(\frac{2}{2-\sigma}\right)^{1/2} \int_{\sqrt{2}(\zeta-\frac{m}{2})}^{\infty} \frac{e^{-\zeta^2}}{\zeta} d\zeta - \frac{1}{4}\left(\frac{2}{2-\sigma}\right)^{3/2} \left[\frac{1}{4}\frac{e^{-2(\zeta-\frac{m}{2})^2}}{(\zeta-\frac{m}{2})^2} - \int_{\sqrt{2}(\zeta-\frac{m}{2})}^{\infty} \frac{e^{-\zeta^2}}{\zeta} d\zeta \right] \right\}$$

$$(31) \qquad J'(\zeta) = \frac{1}{2}[f''(0)]^{1-\sigma} I_{2-\sigma}(1)(4m)^{\sigma}\, e^{-\sigma(\zeta-\frac{m}{2})^2} -$$

$$- \frac{4m^2}{\sqrt{2(2-\sigma)}} \left\{ \frac{1}{2}\left(\frac{2}{2-\sigma}\right)^{1/2} \frac{e^{-2(\zeta-\frac{m}{2})^2}}{\zeta-\frac{m}{2}} - \frac{1}{8}\left(\frac{2}{2-\sigma}\right)^{3/2} \frac{e^{-2(\zeta-\frac{m}{2})^2}}{(\zeta-\frac{m}{2})^3} \right\}$$

where $\sigma = 0.75$, $I(1) = 1.1070$, $f''(0) = 1.32822$, $J(1) = 0.5768$ and $I_{2-\sigma}(1) = 0.9252$. Then $\theta_{II}(\zeta)$ and $\theta_{II}'(\zeta)$ follow from (25). From the preceding formulas $\theta_{I}(\zeta)$, $\theta_{I}'(\zeta)$, $\theta_{II}(\zeta)$, $\theta_{II}'(\zeta)$ may now be calculated as functions of ζ, with the aid of tables of the exponential function, error function, and exponential integral. The results of such calculations are given in Table 8.

The value of the integral $I(c)$ as defined by equation (3) when $c = 1$ is obtained by changing to ζ as variable of integration in the outer part of the boundary layer as follows:

(32) $$I(1) = \int_0^1 \sqrt{1-w}\,\frac{dw}{s} = \int_0^{0.99} \sqrt{1-w}\,\frac{dw}{s} + 0.66411\int_{\zeta(0.99)}^{\infty} \sqrt{1-w(\zeta)}\,d\zeta$$

$$= \int_0^{0.99} \sqrt{1-w}\,\frac{dw}{s} + 0.66411\int_{\zeta(0.99)}^{3.0} \sqrt{1-w(\zeta)}\,d\zeta + 0.66411\sqrt{2m}\int_{(3.0-\frac{m}{2})}^{\infty} \sqrt{\int_{\zeta}^{\infty} e^{-s^2} ds}\,d\zeta$$

$$= 0.850 + 0.022 + 0.008 = 0.880$$

The evaluation of the integrals occuring in (16) is also carried out most easily by changing to ζ near the outer edge of the boundary layer:

(33) $$\int_0^{0.999} \frac{dw}{\bar{g}_*} = \int_0^{\zeta(0.999)} d\zeta = \zeta(0.999) = 3.00$$

(34) $$\int_0^{0.999} \frac{\theta_I(w)}{\bar{g}_*(w)}\,dw = \int_0^{0.99} \frac{\theta_I(w)}{\bar{g}_*(w)}\,dw + \int_{\zeta(0.99)}^{3.0} \theta_I(\zeta)\,d\zeta$$

$$= 1.5243 + 0.5337 = 2.0580$$

(35) $$\int_0^{0.999} \frac{\theta_{II}(w)}{\bar{g}_*(w)}\,dw = \int_0^{0.99} \frac{\theta_{II}(w)}{\bar{g}_*(w)}\,dw + \int_{\zeta(0.99)}^{3.0} \theta_{II}(\zeta)\,d\zeta$$

$$= 0.1439 + 0.0021 = 0.1460$$

These integrals are evaluated by numerical integration with the use of the functions tabulated in Table 8.

(c) Calculation of the integrals $\tilde{H}_1(c)$ and $\tilde{K}_{1_r}(c)$

The integral $\tilde{H}_1(c)$ defined in (8.7a) may be expressed, after several transformations, as follows:

$$(36) \quad \tilde{H}_1(c) = \int_0^1 \frac{(w-c)^2}{T}\,dy = \frac{\mu_w}{T_w T_w} \int_0^{0.999} \frac{(w-c)^2}{s}\,dw = \frac{\mu_w}{T_w T_w}\,\mathfrak{z}_*(0) \int_0^{3.0} (w-c)^2\,d\zeta$$

$$= \left(\frac{\mu_w}{T_w}\right)^{1/2} \left(\frac{\delta}{x_1}\sqrt{R_x}\right)^{-1} \left(3.60330 - 8.540\,c + 6.00\,c^2\right)$$

where the integrals $\int_0^3 w(\zeta)\,d\zeta$ and $\int_0^3 w^2(\zeta)\,d\zeta$ are evaluated by numerical integration.

For the integral $\tilde{K}'_{1_r}$ defined in (8.7b) by

$$(37) \quad \tilde{K}_{1_r} = \mathcal{R}_e \int_0^1 \frac{T - \tilde{M}_1^2 (w-c)^2}{(w-c)^2}\,dy = \mathcal{R}_e \int_0^1 \frac{T}{(w-c)^2}\,dy - \tilde{M}_1^2$$

the path of integration goes below the singular point y_c in the complex y-plane. Much of the integration can be carried out most conveniently in the w-plane, so that

$$(38) \quad \mathcal{R}_e \int_0^1 \frac{T}{(w-c)^2}\,dy = \frac{\mu_w T_w}{T_w}\,\mathcal{R}_e \int_0^{w_1} \frac{\theta^2}{s(w-c)^2}\,dw$$

$$= \frac{\mu_w T_w}{T_w} \left\{ \mathcal{R}_e \int_0^{w_a} \frac{\theta^2}{s(w-c)^2}\,dw + \int_{w_a}^{w_1} \frac{\theta^2}{s(w-c)^2}\,dw \right\}$$

$$= \frac{\mu_w T_w}{T_w} \left\{ \mathcal{R}_e \int_0^{w_a} \frac{\theta^2}{s(w-c)^2}\,dw + K_{12} \right\}$$

where $0 < c < w_a < w_1$, and $w_1 = 0.999$.

In order to calculate the first integral expand θ^2/s in a power series about $w = c$:

$$(39) \quad \frac{\theta^2}{s(w-c)^2} = \frac{A}{(w-c)^2} + \frac{B}{w-c} + C + D(w-c) + \cdots$$

where $A = (\theta^2/S)_c$, $B = (\theta^2/S)'_c$, $C = \frac{1}{2}(\theta^2/S)''_c$, $D = \frac{1}{6}(\theta^2/S)'''_c, \ldots$

Then

$$(40) \quad \Re \int_0^{w_a} \frac{\theta^2}{s(w-c)^2}\,dw = \Re \int_0^{w_a} \left[\frac{A}{(w-c)^2} + \frac{B}{w-c}\right] dw + \int_0^{w_a} \left[C + D(w-c) + \cdots\right] dw$$

$$= K_{10} + K_{11}$$

The value of K_{10} is evidently given by

$$(41) \quad K_{10} = -A\,\frac{w_a}{c(w_a-c)} + B \ln \frac{w_a-c}{c}$$

The other integral

$$(42) \quad K_{11} = \int_0^{w_a} \left[C + D(w-c) + \cdots\right] dw = \int_0^{w_a} \left[\frac{\theta^2}{s(w-c)^2} - \frac{A}{(w-c)^2} - \frac{B}{w-c}\right] dw$$

has a regular integrand for all values of w. The upper limit w_a is as close to w_1 as the integration with respect to w can be easily carried. The rapid decrease of s towards zero for large w therefore limits this value. In the present calculations a value

$w_a = 0.90$ is taken.

The coefficients A,B,C,... are given by

$$(43)\quad \begin{cases} A = \left(\frac{\theta^2}{s}\right)_c , \quad B = \left(\frac{\theta^2}{s}\right)_c \left(2\frac{\theta'}{\theta} - \frac{s'}{s}\right)_c , \\ C = \left(\frac{\theta^2}{s}\right)_c \left(\frac{\theta''}{\theta} + \frac{\theta'^2}{\theta^2} - 2\frac{\theta' s'}{\theta s} + \frac{s'^2}{s^2} - \frac{s''}{2s}\right)_c , \ldots \end{cases}$$

The higher derivatives of s and θ at w = c may all be obtained in terms of s,s',θ,θ' at w = c by successive differentiation of the differential equations (10) and (11), as follows:

$$(44)\quad \begin{cases} s'' = -\frac{2}{\mathfrak{z}_*^2(0)}\frac{w}{s} , \quad s''' = -\frac{s's''}{s} - \frac{2}{\mathfrak{z}_*^2(0)\, s} , \ldots \\ \theta'' = -\sigma(\gamma-1)M_1^2\frac{1}{T_w} - (1-\sigma)\frac{\theta' s'}{s} , \\ \theta''' = -\frac{1}{s}\left[s'\left(\theta''+\sigma(\gamma-1)M_1^2\frac{1}{T_w}\right)+(1-\sigma)(\theta'' s' + \theta' s'')\right] , \ldots \end{cases}$$

After the quantities A,B,C,... at w = c are determined, the integral K_{11} is evaluated by numerical integration. The value of the integrand at w = c is given by C, and its values at other points are obtained by direct substitution in

$$(45)\quad \frac{\theta^2}{s(w-c)^2} - \frac{A}{(w-c)^2} - \frac{B}{w-c}$$

In general, it is found to be possible to evaluate the integrand in this way at points which are close enough to w = c to permit the integration to be carried out with sufficient accuracy.

The final integral

(46) $$K_{12} = \int_{w_0}^{w_1} \frac{\theta^2}{s(w-c)^2}\,dw = \bar{q}_*(0)\int_{\varsigma(w_0)}^{3.0} \frac{\theta^2(\varsigma)}{[w(\varsigma)-c]^2}\,d\varsigma = 0.66411\int_{\varsigma(w_0)}^{3.0} \frac{\theta^2(\varsigma)}{[w(\varsigma)-c]^2}\,d\varsigma$$

is easily determined by numerical integration

The functions $K_{10}(c)$, $K_{11}(c)$, $K_{12}(c)$ for the case $M_1 = 1.6$, $T_w = 1.073$ are listed in Table 8.

(d) <u>The determination of the curve of neutral stability</u>

The relation (8.1) for the characteristic values is equivalent to the two real relations

(47a) $$\Phi_i(z) = \frac{(1+\lambda)v}{(1+\lambda u)^2 + \lambda^2 v^2}$$

(47b) $$\Phi_r(z) = \frac{(1+\lambda)[u(1+\lambda u)+\lambda v^2]}{(1+\lambda u)^2 + \lambda^2 v^2}$$

With $v = v_0(c)$ these may be solved for $z(c)$ and $u(c)$ by a successive approximations procedure, with the aid of the values of $\Phi_r(z)$ and $\Phi_i(z)$ in Table 1, as follows:

(48a) $$\Phi_i\left(z^{(m+1)}\right) = \frac{(1+\lambda)v_0}{(1+\lambda u^{(m)})^2 + \lambda^2 v_0^2}$$

(48b) $$u^{(m+1)} = \Phi_r\left(z^{(m+1)}\right)\left[\frac{(1+\lambda u^{(m)})^2 + \lambda^2 v_0^2}{(1+\lambda)(1+\lambda u^{(m)})}\right] - \frac{\lambda v_0^2}{1+\lambda u^{(m)}}$$

Since λ is usually quite small, good initial approximations are obtained by solving

$$(49) \qquad \Phi_i(z) = \nu_0(c) \,, \qquad \mu = \Phi_r(z)$$

From the discussion of Section 8, it follows that there are usually two solutions (z,u) if $v_0(c) > 0$ and c is not too large, and that there is a maximum value of c for which these two solutions coincide and beyond which no solutions exist. The maximum value of c and the corresponding value of z must satisfy

$$(50) \qquad \nu_0(c) = \psi_i(z,c) = \frac{(1+\lambda)\Phi_i}{[1+\lambda(1-\Phi_r)]^2 + \lambda^2\Phi_i^2}$$

and

$$(51) \qquad \frac{\partial \psi_i}{\partial z} = 0$$

which may be written as

$$(52) \qquad \Phi_i' = -2\lambda \Phi_i \Phi_r' \frac{1+\lambda(1-\Phi_r)}{[1+\lambda(1-\Phi_r)]^2 - \lambda^2\Phi_i^2}$$

The solution (z,c) of (50) and (52) may be obtained by the successive approximations scheme

$$(53a) \qquad \Phi_i'\left(z^{(m+1)}\right) = \left[-2\lambda \Phi_i \Phi_r' \frac{1+\lambda(1-\Phi_r)}{[1+\lambda(1-\Phi_r)]^2 - \lambda^2\Phi_i^2}\right]_{\substack{c = c^{(m)} \\ z = z^{(m)}}}$$

$$\nu_0\left(c^{(m+1)}\right) = \left[\frac{(1+\lambda)\Phi_i}{[1+\lambda(1-\Phi_r)]^2 + \lambda^2\Phi_i^2}\right]_{\substack{c = c^{(m)} \\ z = z^{(m+1)}}}$$

with the aid of the functions $\Phi_r, \Phi_i, \Phi_r', \Phi_i'$ in Table 1 and the tabulated values of $v_o(c)$ and $\lambda(c)$. Initial approximations for z and c are given by

$$(54) \quad \Phi_i' = 0 \quad \text{and} \quad v_0 = \Phi_i \;, \text{ i.e. by} \quad z = 3.22 \text{ and } v_0(c) = 0.580.$$

With z and c determined, u is given directly by (8.4a).

When z and u have been calculated for each value of c, $\tilde{\alpha}$ is then obtained by solving (8.16a). The quantity $u - \tilde{L}$ appearing in this equation may be expressed in terms of known quantities as follows

$$(55) \quad u - \tilde{L} = u - \frac{\mu_w' c}{T_w}\left\{\tilde{K}_{1_p} + \frac{T_w}{\mu_w' c}\right\} = u(c) + \frac{T_w}{\mu_w T_w}\tilde{M}_1^2 c - \left[1 + c(K_{10} + K_{11} + K_{12})\right]$$

The value of $\tilde{R}$ is then given by

$$(56) \quad \tilde{\alpha}\tilde{R} = \frac{(0.66411)^2}{9}\frac{z^3}{I^2}\left(\frac{\delta}{x_1}\sqrt{R_x}\right)^2 = 0.0490047\, z^3\left(\frac{\frac{\delta}{x_1}\sqrt{R_x}}{I}\right)^2$$

which follows from (2a) and (18).

The quantities α and R are now determined by means of the transformations (6.8), and then $\alpha_x, R_x, \alpha_\theta, R_\theta$ are calculated by means of

(57a) $$\alpha_x = \alpha \frac{x_1}{\delta} = \alpha R \left(\frac{\delta}{x_1} \sqrt{R_x} \right)^{-2}$$

(57b) $$R_x = R \frac{x_1}{\delta} = R^2 \left(\frac{\delta}{x_1} \sqrt{R_x} \right)^{-2}$$

(57c) $$\alpha_\theta = \alpha \frac{\theta^*}{\delta}$$

(57d) $$R_\theta = R \frac{\theta^*}{\delta}$$

where

(58) $$\frac{\theta^*}{\delta} = \int_0^\infty \rho w(1-w)\,dy = \frac{\mu_w}{T_w T_w} \int_0^1 \frac{w(1-w)}{s}\,dw$$

$$= \frac{\mu_w}{T_w T_w} \bar{g}_*(0) \int_0^\infty w(1-w)\,d\zeta = \left(\frac{\mu_w}{T_w}\right)^{1/2} \frac{0.6667}{\frac{\delta}{x_1}\sqrt{R_x}}$$

For the case $M_1 = 1.6$, $T_w = 1.073$, the parameters μ_w, $\frac{\delta}{x_1}\sqrt{R_x}$, $T_w/\mu_w T_w$ which appear in the calculations are given by

(59) $$\mu_w = 1.0595, \quad \frac{\delta}{x_1}\sqrt{R_x} = 6.396, \quad \frac{T_w}{\mu_w T_w} = 1.357$$

(e) <u>Calculation of the parameter</u> $C = M_1\sqrt{2(1-c)^2 + \frac{B}{A}}$

The values of the parameter C given in Table 5 have been calculated for $\sigma = 1$ and $\rho\mu = 1$, for the sake of convenience, since then many of the quantities required (such as u-L and c) can be obtained to a sufficient degree of approximation from the data given by Lees [2], for Mach numbers up to $M_1 = 1.3$. In addition, the boundary-layer thickness parameter $\frac{\delta}{x_1}\sqrt{R_{x}}$ is taken to be 6.00 for the plate with heat transfer, and 5.60 for the insulated plate, as in [2]. The general trend of the results would be expected to be quite similar, however, for values of σ and $\rho\mu$ close to unity and for slightly different definitions of the boundary-layer thickness, since the various calculated quantities are not very sensitive to variations in σ, $\rho\mu$ and δ. It might be mentioned that at $M_1 = 2.472$ corresponding to the last calculated value of C for the insulated plate, the quantity B/A is zero since $\bar{H}_1 = (1-c)^2$ in this case, so that apparently B/A, which is positive for smaller values of M_1, becomes negative for larger values.

(f) Determination of the critical temperature ratios for stabilization of the boundary layer

From the considerations of Section 10 it follows that the critical temperature ratio in any particular case can be obtained by solving

$$(60) \qquad v_0(c) = \psi_{i_m}(c) \quad , \quad c = 1 - \frac{1}{M_1}$$

for the temperature ratio T_w.

For given c, (53a) with $c^{(n)} = c$ is solved for z and then $\psi_{i_m}(c)$ is given by $\psi_{i_m}(c) = \psi_i(z,c)$, where ψ_i is defined by (8.4b). The function $\psi_{i_m}(c)$ calculated in this manner is tabulated in Table 8.

When $\rho\mu = \rho_w\mu_w$ equation (60) becomes

$$(61)\quad \left[\pi w \frac{\theta^2}{s}\left(2\frac{\theta'}{\theta} - \frac{s'}{s}\right)\right]_{w=c} = \psi_{im}(c) \;, \quad c = 1 - \frac{1}{\tilde{M}_1}$$

with the use of relation (4) for v_o. Now, according to equation (6), θ is a linear function of $1/T_w$ so that (61) can be reduced to a quadratic equation for $1/T_w$ as follows:

$$(62)\quad \frac{1}{T_w^2} + 2\frac{spg' - s\theta_I' g - s'pg}{(2sg' - s'g)g}\frac{1}{T_w} - \frac{2s\theta_I' p + s'p^2 + \frac{s^2\psi_{im}(c)}{\pi c}}{(2sg' - s'g)g} = 0$$

where

$$(63)\quad p = 1 - \theta_I \;, \quad g = \theta_I + (\gamma-1)M_1^2\theta_{II} \;, \quad g' = \theta_I' + (\gamma-1)M_1^2\theta_{II}'$$

The quantities $\theta_I(w)$, $\theta_I'(w)$, $\theta_{II}(w)$, $\theta_{II}'(w)$, $s(w)$, $s'(w)$ are all evaluated at $w = c = 1 - 1/\tilde{M}_1$. The coefficients in (62) are calculated with the assistance of Table 8 and the equation is then easily solved for T_w.

(g) Concluding remarks

It is to be noticed that the value of the constant C in the viscosity-temperature relation $\bar{\mu}^*/\bar{\mu}^*_1 = C\,\bar{T}^*/\bar{T}_1^*$ does not enter into the formulas (3) and (4) for I(c) and $v_o(c)$, since for any C, $\eta = \theta$ for this linear relation. Therefore the critical temperature ratios do not depend on C, which in the present investigation, it may be recalled, has been chosen so as to give the correct value for the viscosity at the wall. However, in the calculations for the neutral curve the value of C has a small influence.

It should also be mentioned that with this linear viscosity relation, the functions $I(c)$, $\lambda(c)$ and $\psi_{1_m}(c)$ become <u>universal functions of c</u> (for a zero pressure gradient), so that the values of these functions listed in Table 8 can be used for all values of M_1, T_w/T_1, and σ.

For more complicated situations (involving the accurate Sutherland viscosity law, pressure gradient, etc), these functions are different for each case. However, even in these cases some simplification is possible. In particular, from equation (8.4b) we see that

$$\psi_i(z, c) = \chi_i(z, \lambda) \quad , \quad \text{with} \quad \lambda = \lambda(c) \quad , \tag{64}$$

so that

$$\psi_{i_m}(c) = \chi_{i_m}(\lambda) \quad , \quad \text{with} \quad \lambda = \lambda(c) \quad , \tag{65}$$

where $\chi_{i_m}(\lambda)$ is the maximum of $\chi_1(z, \lambda)$ for a given λ. Clearly $\chi_{1_m}(\lambda)$ is a <u>universal function of λ</u>, which may be calculated once and for all. Thus in any given case only the function $\lambda(c)$ has to be calculated, and the value of $\psi_{1_m}(c)$ for a particular c is then equal to $\chi_{1_m}(\lambda)$ for the value of λ corresponding to this value of c. A table of the function $\chi_{i_m}(\lambda)$ could evidently be obtained by just re-tabulating the function $\psi_{1_m}(c)$ in Table 8 against λ.

List of Symbols

Dimensional quantities	Non-dimensional quantities	Reference quantities
Positional co-ordinates		
(1) x_1	x	δ
(2) x_2	y	δ
(3) x_3	z	δ
Time		
(4) t^*	t	$\frac{\delta}{U_1}$
Velocity components		
(5) $u_1 = \bar{u}_1 + u_1'$	$w(y)+f(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	U_1
(6) $u_2 = \bar{u}_2 + u_2'$	$v(y)+\alpha\phi(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	U_1
(7) $u_3 = u_3'$	$h(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	U_1
Density		
(8) $\rho^* = \bar{\rho}^* + \rho'^*$	$\rho(y)+r(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	$\bar{\rho}_1^*$
Pressure		
(9) $p^* = \bar{p}^* + p'^*$	$p(y)+\pi(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	$\bar{p}_1^*$
Temperature		
(10) $T^* = \bar{T}^* + T'^*$	$T(y)+\theta(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	T_1^*
Viscosity coefficients		
(11) $\mu^* = \bar{\mu}^* + \mu'^*$	$\mu(y)+\frac{d\bar{\mu}}{dT}\theta(y)\exp[i(\alpha x+\beta z-\alpha ct)]$,	$\bar{\mu}_1$
(12) $\lambda^* = \bar{\lambda}^* + \lambda'^*$	$\lambda(y)+\frac{d\lambda}{dT}\theta(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	$\bar{\mu}_1^*$

Dimensional quantities	Non-dimensional quantities	Reference quantities
Thermal conductivity		
(13) $k^* = \bar{k}^* + k'^*$	$\frac{\mu(y)}{\sigma(y)} + \overline{\frac{d}{dT}(\frac{\mu}{\sigma})}\theta(y)\exp[i(\alpha x+\beta z-\alpha ct)]$	$C_p/\bar{\mu}_1^*$
Wave-numbers of the disturbance		
(14) α_1	$\begin{cases} \alpha \\ \alpha_\theta \\ \alpha_x \end{cases}$	δ^{-1} θ^{*-1} x_1^{-1}
(15) α_3	β	δ^{-1}
Wave-velocity of the disturbance		
(16) c^*	c	U_1
Specific heat at constant volume		
(17) C_v	1	C_v
Specific heat at constant pressure		
(18) C_p	γ	C_v
Gas constant per gram		
(19) R^*	$\gamma - 1$	C_v
Acceleration due to gravity		
(20) g	$\frac{1}{F_r}$	$\frac{U_1^{*}}{\delta}$
Momentum thickness of boundary layer		
(21) $\theta^* = \int_0^\infty \rho w(1-w)dx_2$		

Dimensional quantities	Non-dimensional quantities	Reference quantities

Reynolds number

(22) Based on boundary layer thickness δ $$R = \frac{\bar{\rho}_1^* U_1 \delta}{\bar{\mu}_1^*}$$

(23) Based on momentum thickness θ^* $$R_\theta = \frac{\bar{\rho}_1^* U_1 \theta^*}{\bar{\mu}_1^*}$$

(24) Based on distance x_1 from leading edge $$R_x = \frac{\bar{\rho}_1^* U_1 x_1}{\bar{\mu}_1^*}$$

Mach number

(25) $$M_1 = \frac{U_1}{\sqrt{\gamma_1 R^* \bar{T}_1^*}}$$

Froude number

(26) $$F = \frac{U_1}{\sqrt{g\delta}}$$

Prandtl number

(27) $$\sigma = \frac{C_p \mu^*}{k^*}$$

Remarks

(a) A bar $(\overline{\quad})$ denotes mean value, a dash $(\quad)'$ denotes fluctuation. The subscript $(\quad)_1$ refers to the value at the edge of the boundary layer at the particular x-location considered, and the subscript $(\quad)_w$ refers to wall value. The subscripts $(\quad)_r$ and $(\quad)_i$ denote the real and imaginary parts of a quantity, respectively.

(b) The non-dimensional mean quantities $\bar{u}_1(x_1,x_2)/U_1$, $\bar{u}_2(x_1,x_2)/U_1$, $\bar{\rho}(x,y)$, $\bar{p}(x,y)$,...are represented as functions of y alone, that is, as w(y), v(y), ρ(y), p(y),...respectively. This corresponds to an approximation depending on the boundary-layer nature of the mean flow (see Section 4).

(c) The quantities $C_p, C_v, \mu, \lambda, k, \sigma, \gamma$ are all considered to be functions of T only. The fluctuation Q' of any such quantity Q is given by $Q' = \overline{dQ/dT}\ T'$.

(d) For moderate Mach numbers and rates of heat transfer C_p, C_v, γ, σ vary only slightly in the boundary layer. They are therefore regarded as constants equal to their free stream values in the numerical calculations.

(e) The reference length δ is the boundary-layer thickness at the particular value of x considered. Its definition is somewhat arbitrary, and may be taken in any convenient manner. In the present numerical computations it is taken to be the value of x_2 at which $u_1/U_1 = 0.999$ (see Appendix).

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Table 1

z	$\Phi_r(z)$	$\Phi_i(z)$	$\Phi_r'(z)$	$\Phi_i'(z)$
1.0	0.80630	-2.60557		
1.1				
1.2	1.77012	-2.29854	3.71	2.781
1.3			2.54	
1.4	2.26836	-1.71669	1.505	2.937
1.5			.850	2.633
1.6	2.44985	-1.18600	.4245	2.384
1.7			.1390	2.127
1.8	2.48104	- .75892	- .06428	1.907
1.9			- .2140	1.726
2.0	2.43927	- .41253	- .3337	1.573
2.1			- .4377	1.443
2.2	2.35196	- .12348	- .5333	1.325
2.3			- .6242	1.213
2.4	2.22724	+ .11916	- .7108	1.1007
2.5			- .7912	.9833
2.6	2.06929	.31558	- .8625	.8584
2.7			- .9209	.7254
2.8	1.88566	.46043	- .9627	.5853
2.9			- .9850	.4414
3.0	1.68938	.54872	- .9853	.2978
3.1			- .9640	.1590
3.2	1.49726	.58082	- .9223	.0296
3.3			- .8629	= .0867
3.4	1.32516	.56401	- .7893	- .1862
3.5			- .7054	- .2695
3.6	1.18429	.52074	- .6151	- .3332
3.7			- .5222	- .3786
3.8	1.07982	.43560	- .4302	- .4067
3.9			- .3422	- .4193
4.0	1.01118	.35220	- .2609	- .4189
4.1			- .1865	- .4060
4.2	.97361	.27133	- .1172	- .3793
4.3			- .0625	- .3538
4.4	.96056	.20038	- .01470	- .3225
4.5			- .0061	
4.6	.95989	.13601	- .0034	
4.7			+ .0476	
4.8	.97659	.09503	+ .1794	
4.9				
5.0	.99582	.07266		

Table 2(a)

The function $v_o(c)$; $M_1 = 0.7$, $\rho\mu = 1$, $\sigma = 1$

$T_w/T_1 = 0.70$ c	v_o	$T_w/T_1 = 0.80$ c	v_o	$T_w/T_1 = 0.90$ c	v_o	$T_w/T_1 = 1.098$ c	v_o	$T_w/T_1 = 1.25$ c	v_o
.0262	.0825	.0237	.0486	.0433	.0517	.0353	-.0009	.0346	-.0237
.0521	.1645	.0472	.0965	.0863	.1028	.0705	-.0024	.0692	-.0476
.0777	.2466	.0705	.1443	.1291	.1568	.1058	-.0025	.1040	-.0698
.1030	.3297	.0937	.1925	.1714	.2173	.1410	.0006	.1389	-.0886
.1281	.4146	.1168	.2417	.2135	.2885	.1762	.0090	.1738	-.1021
.1529	.5023	.1397	.2926	.2551	.3746	.2114	.0248	.2088	-.1085
.1701	.5661	.1625	.3457	.2963	.4805	.2464	.0501	.2439	-.1057
.1726	.5754	.1851	.4017	.3166	.5426	.2813	.0872	.2789	-.0917
		.2075	.4614	.3268	.5762	.3161	.1389	.3138	-.0641
		.2298	.5253			.3505	.2082	.3485	-.0203
		.2409	.5592			.3847	.2985	.3831	.0427
		.2475	.5801			.4185	.4137	.4174	.1286
						.4352	.4821	.4512	.2414
						.4452	.5270	.4846	.3859
						.4559	.5790	.5092	.5184
								.5190	.5779

Table 2(b)

The function $v_o(c)$; $M_1 = 4$, $\rho\mu = \rho_w\mu_w$, $\sigma = 0.75$

c \ T_w/T_1	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.1	2.275	1.572	1.144	.858	.654	.502	.385
0.2	4.367	2.856	1.976	1.409	1.019	.735	.521
0.3	5.830	3.632	2.388	1.605	1.083	.710	.433
0.4	6.571	3.888	2.406	1.498	.899	.480	.177
0.5	6.207	3.645	2.164	1.192	.563	.133	-.172
0.6	6.828	3.741	1.969	.925	.264	-.179	-.487
0.7	9.095	4.809	2.557	1.245	.422	-.135	-.496
0.8	18.98	10.94	6.676	4.172	2.587	1.365	.793
0.9	84.3	52.6	35.5	25.3	18.71	14.23	11.00
0.95	343	218	150.8	110.0	83.6	65.6	52.6

Table 3

(a) $\tilde{M}_1 = 0.6$, $M_1 = 1.6$, $T_w/T_1 = 1.073$

c	α	$R\times10^{-4}$	α_θ	$R_\theta\times10^{-3}$	α_x	$R_x\times10^{-6}$
0	0	∞	0	∞	∞	∞
.15	.0515	40.7	.00534	42.1	512	4240
.175	.0642	21.3	.00665	22.1	334	1110
.2	.0779	12.2	.00807	12.7	233	365
.25	.110	4.82	.0114	5.00	129	56.7
.275	.129	3.23	.0134	3.35	102	25.5
.325	.178	1.63	.0184	1.69	70.8	6.52
.3625	.231	1.09	.0239	1.13	61.7	2.92
.375	.268	1.05	.0278	1.09	68.8	2.70
.3625	.278	1.40	.0288	1.45	94.7	4.76
.325	.253	2.63	.0262	2.73	162	16.9
.275	.208	6.23	.0216	6.45	317	94.8
.25	.186	9.89	.0193	10.3	450	239
.2	.144	28.2	.0149	29.2	988	1938
0	0	∞	0	∞	∞	∞

Table 3

(b) $\widetilde{M}_1 = 1.0$, $M_1 = 1.6$, $T_w/T_1 = 1.073$

c	α	$R \times 10^{-4}$	α_θ	$R_\theta \times 10^{-3}$	α_x	$R_x \times 10^{-6}$
0	0	∞	0	∞	∞	∞
.05	.00869	4930	.000900	5160	10600	606x10^5
.1	.0264	249	.00273	258	1600	151000
.15	.0515	40.7	.00534	42.2	513	4050
.175	.0671	20.4	.00695	21.1	335	1017
.2	.0849	11.2	.00880	11.6	233	307
.25	.128	4.14	.01330	4.28	130	41.8
.275	.155	2.70	.0161	2.79	103	17.8
.3	.186	1.84	.0193	1.90	83.6	8.25
.325	.222	1.31	.0230	1.35	70.9	4.17
.35	.267	.969	.0277	1.00	63.3	2.30
.3625	.295	.856	.0306	.887	61.7	1.79
.375	.343	.822	.0355	.852	68.8	1.65
.3625	.349	1.11	.0362	1.15	95.0	3.02
.35	.337	1.40	.0350	1.45	115	4.76
.325	.307	2.17	.0318	2.25	163	11.5
.3	.274	3.38	.0284	3.50	227	27.9
.275	.241	5.39	.0250	5.58	318	71.0
.25	.209	8.81	.0217	9.13	449	189
.2	.151	26.8	.0156	27.8	990	1760
.15	.0993	109	.0103	113	2640	28900
.1	.0553	748	.00573	775	10120	137x10^4
.05	.0193	19180	.00200	19900	90300	9x10^8
0	0	∞	0	∞	∞	∞

Table 3

(c) $\tilde{M}_1 = 1.4$, $M_1 = 1.6$, $T_w/T_1 = 1.073$

c	α	$R \times 10^{-4}$	α_θ	$R_\theta \times 10^{-3}$	α_x	$R_x \times 10^{-6}$
.2857	0	∞	0	∞		∞
.2875	.0219	17.18	.00227	17.80	92.0	721
.292	.0421	8.63	.00436	8.94	88.8	182
.3	.0662	5.17	.00686	5.36	83.7	65.3
.325	.125	2.33	.0129	2.41	70.9	13.2
.35	.183	1.41	.0189	1.46	63.3	4.89
.3625	.216	1.17	.0224	1.21	61.8	3.35
.375	.263	1.07	.0272	1.11	68.8	2.81
.3625	.250	1.55	.0259	1.61	94.9	5.87
.35	.225	2.09	.0233	2.17	115	10.7
.325	.166	4.01	.0172	4.15	163	39.3
.3	.0936	9.92	.00969	10.28	227	241
.292	.0605	17.05	.00627	17.67	252	711
.2875	.0318	34.5	.00329	35.7	268	2904
.2857	0	∞	0	∞		∞

Table 4

$M_1 = 1.6$, $T_w/T_1 = 1.073$

Direction angle $= \cos^{-1}\tilde{M}_1/M_1$

$\tilde{M}_1$	$\cos^{-1}\tilde{M}_1/M_1$	$R_{x_o} \times 10^{-6}$
0	90°	∞
.1	86.4	75.1
.2	82.8	19.2
.4	75.5	5.23
.6	68.0	2.70
.8	60.0	1.89
1.0	51.3	1.65
1.030	49.9	1.64
1.2	41.4	1.80
1.4	29.0	2.81
1.5	20.4	5.08
1.576	10	19.6
1.594	5	76.7
	0	∞

Table 5(a)

The function $C = M_1 \sqrt{2(1-c)^2 + B/A}$ for zero heat transfer; $\sigma = 1$, $\rho\mu = 1$.

M_1	0	0.5	0.9	1.3	1.7	2.472
C	0	0.455	0.759	0.941	1.006	0.885

Table 5(b)

The function $C = M_1 \sqrt{2(1-c)^2 + B/A}$ for various temperature ratios at $M_1 = 0.7$; $\sigma = 1$, $\rho\mu = 1$

T_w/T_1	0.70	0.80	0.90	1.25
C	0.871	0.797	0.722	0.544

Table 6

$\sigma = 0.75$; $\rho\mu = \rho_w\mu_w$

M_1	T_w/T_1
1	0
1.1	.582
1.2	.766
1.25	.827
1.5	1.018
2	1.262
2.5	1.453
3	1.590
3.5	1.678
4	1.700
4.5	1.641
5	1.548
6	1.146
6.5	.842
7	.453
7.25	.240
7.48	0

Table 7

$M_1 = 4$, $\cos\widetilde{M}_1/M_1$ = direction angle, $\rho\mu = \rho_w\mu_w$, $\sigma = 0.75$

$\widetilde{M}_1$	$\cos\widetilde{M}_1/M_1$	T_w/T_1	$\widetilde{M}_1$	$\cos\widetilde{M}_1/M_1$	T_w/T_1
4.	0°	1.700	1.429	69.1	1.885
3.61	25.4°	1.600	1.25	71.8	1.935
3.33	33.7	1.540	1.143	73.5	1.800
2.67	48.1	1.474	1.096	74.1	1.600
2.5	51.3	1.483	1.068	74.5	1.400
2.0	60.0	1.585	1.051	74.8	1.200
1.968	60.5	1.600	1.036	75.0	1.000
1.667	65.4	1.740	1.026	75.2	0.800
1.567	66.9	1.800	1.000	75.5	0

Table 8

(a) The functions $I(c), \lambda(c), \Psi 1_m(c)$ for the case $\rho\mu = \rho_w\mu_w$, $\sigma = 0.75$

c	I(c)	λ(c)	$\Psi 1_m(c)$	c	I(c)	λ(c)	$\Psi 1_m(c)$
0	0	0	.58114	.6	.3183	.0273	.61307
.1	.02108	$.11520\times10^{-3}$	.58216	.7	.4085	.0462	.63625
:2	.05968	$.9237\times10^{-3}$	.58251	.8	.5137	.0769	.67622
.3	.10989	3.136×10^{-3}	.58529	.9	.6431	.1298	.75297
.4	.1698	.0068	.58891	.95	.7269	.1775	.83165
.5	.2393	.0152	.59868	1.0	.880	.320	1.15

(b) The functions $\theta_I(w), \theta_I'(w), \theta_{II}(w), \theta_{II}'(w), \sigma(w), \sigma'(w)$

for the case $\rho\mu = \rho_w \mu_w$

w	$\theta_I(w)$	$\theta_I'(w)$	$\theta_{II}(w)$	$\theta_{II}'(w)$	$\sigma(w)$	$\sigma'(w)$
0	0	.90334	0	.3908	1	0
.05	.045168	.90336	.01860	.3533	.99991	-.0054
.1	.090339	.90351	.03534	.3164	.99925	-.0226
.15	.13552	.90392	.05022	.2785	.99746	-.0511
.2	.18074	.90472	.06325	.2412	.99395	-.0911
.25	.22600	.90603	.07436	.2040	.98816	-.1424
.275	.[illegible]4866	.90693	.07916	.18548	.98423	-.1725
.3	.[illegible]7135	.90803	.08356	.16694	.97951	-.2057
.325	.29407	.90933	.08762	.14842	.97392	-.2419
.35	.31682	.91086	.09104	.12990	.96738	-.2813
.3625	.32821	.91172	.09259	.12064	.96373	-.3022
.375	.33962	.91265	.09406	.11138	.95982	-.3240
.4	.36246	.91472	.09665	.09286	.95115	-.3701
.45	.40831	.91984	.10032	.05578	.93015	-.4724
.5	.45447	.92651	.10210	.01854	.90366	-.5899
.55	.50099	.93511	.10217	-.01897	.87088	-.7241
.6	.54802	.94615	.10027	-.05692	.83093	-.8773
.65	.59566	.96038	.09647	-.09557	.78278	-1.0527
.7	.64413	.97890	.09062	-.13529	.72520	-1.2556
.75	.69367	1.00351	.0829	-.17668	.65662	-1.493
.8	.74463	1.03736	.0728	-.22080	.57504	-1.778
.85	.79766	1.08667	.0606	-.26973	.47756	-2.132
.875		1.12110			.42153	-2.361
.9	.85381	1.16654	.0457	-.3285	.35959	-2.610
.92	.8776		.0390			
.925	.8837	1.23045	.0368		.29051	-2.94
.94	.9026					
.95	.91563	1.33085	.0274	-.4152	.21227	-3.347
.96	.9292	1.39146	.0212		.17764	-3.61
.97	.94354	1.47535	.0187	-.4762	.14055	-3.851
.98	.9589	1.60531			.10027	-4.194
.99	.97607	1.86175	.0080	-.5890	.05543	-4.65
1.00	1.0		0		0	

(c) The function $w = 1/2f'(\zeta)$; see p. 44 for definition of $f(\zeta)$.

ζ	$w(\zeta)$	ζ	$w(\zeta)$	ζ	$w(\zeta)$	ζ	$w(\zeta)$
0	0	.8	.5168	1.6	.8761	2.4	.9878
.1	.0664	.9	.5748	1.7	.9018	2.5	.9915
.2	.1328	1.0	.6298	1.8	.9233	2.6	.9942
.3	.1989	1.1	.6813	1.9	.9411	2.7	.9962
.4	.2647	1.2	.7290	2.0	.9555	2.8	.9975
.5	.3298	1.3	.7725	2.1	.9670	2.9	.9984
.6	.3938	1.4	.8115	2.2	.9759	3.0	.9990
.7	.4563	1.5	.8460	2.3	.9827		

(d) The functions $\Theta_I(\zeta), \Theta_{II}(\zeta), \Theta_I'(\zeta), \Theta_{II}'(\zeta)$

ζ	$\Theta_I(\zeta)$	$\Theta_{II}(\zeta)$	$\Theta_I'(\zeta)$	$\Theta_{II}'(\zeta)$
1.5	.79575	.0650	.33770	-.11296
1.6	.82787	.0548	.30470	-.09461
1.7	.85666	.0459	.27087	-.08276
1.8	.88205	.0381	.23719	-.07312
1.9	.90414	.0313	.20462	-.06411
2.0	.92304	.0253	.17388	-.05535
2.1	.93900	.0202	.14557	-.04705
2.2	.95225	.0159	.12004	-.03923
2.3	.96312	.0123	.097529	-.03214
2.4	.97187	.0094	.078050	-.02588
2.5	.97883	.0071	.061536	-.02049
2.6	.98427	.0053	.047792	-.01596
2.7	.98848	.0039	.03657	-.01224
2.8	.99167	.0028	.02757	-.00923
2.9	.99406	.0020	.02046	-.00686
3.0	.99582	.0014	.01497	-.00502

(e) Functions for calculation of curve of neutral stability

for $M_1 = 1.6$, $T_w/T_1 = 1.073$, $\rho\mu = \rho_w\mu_w$, $\sigma = 0.75$, $\gamma = 1.4$.

c	I(c)	$\lambda(c)\times 10^3$	$\tilde{H}_1(c)$	$v_0(c)$	z(c)	u(c)
0	0	0	.5599	0	2.2971	2.2956
.05	.007869	.01440	.4959	.08892	2.3730	2.2461
.10	.02108	.11520	.4365	.16316	2.4408	2.1978
.15	.03874	.3891	.3818	.22701	2.5038	2.1500
.175	.04884		.3562		2.5349	
.20	.05968	.9237	.3318	.2877	2.5674	2.0990
.25	.08348	1.8085	.2864	.3531	2.6428	2.0354
.275	.09637	2.411	.2655	.3875	2.6864	1.9969
.2875	.10305		.2555		2.7122	
.292	.10550		.2519		2.7224	
.3	.10989	3.136	.2157	.4276	2.7416	1.9464
.325	.12401	3.997	.2271	.4734	2.8136	1.8782
.35	.13873	5.006	.2097	.5258	2.9172	1.7767
.3625	.14631	5.570	.2014	.5559	2.9996	1.6946
.375	.15404	6.176	.1934	.5882	3.2188	1.48130
.3625	.14631	5.570	.2014	.5559	3.4604	1.2792
.35	.13873	5.006	.2097	.5258	3.5648	1.2064
.325	.12401	3.997	.2271	.4734	3.7105	1.1217
.3	.10989	3.136	.2157	.4276	3.8234	1.0697
.292	.10550		.2519		3.8556	
.2875	.10305		.2555		3.8729	
.275	.09637	2.411	.2655	.3875	3.9184	1.0348
.25	.08348	1.8085	.2864	.3531	3.9994	1.0111
.20	.05968	.9237	.3318	.2877	4.1584	.9785
.15	.03874	.3891	.3818	.22701	4.3238	.9638
.10	.02108	.11520	.4365	.16316	4.5112	.9587
.05	.007869	.01440	.4959	.08892	4.85	.98
0	0	0	.5599	0	∞	1.00

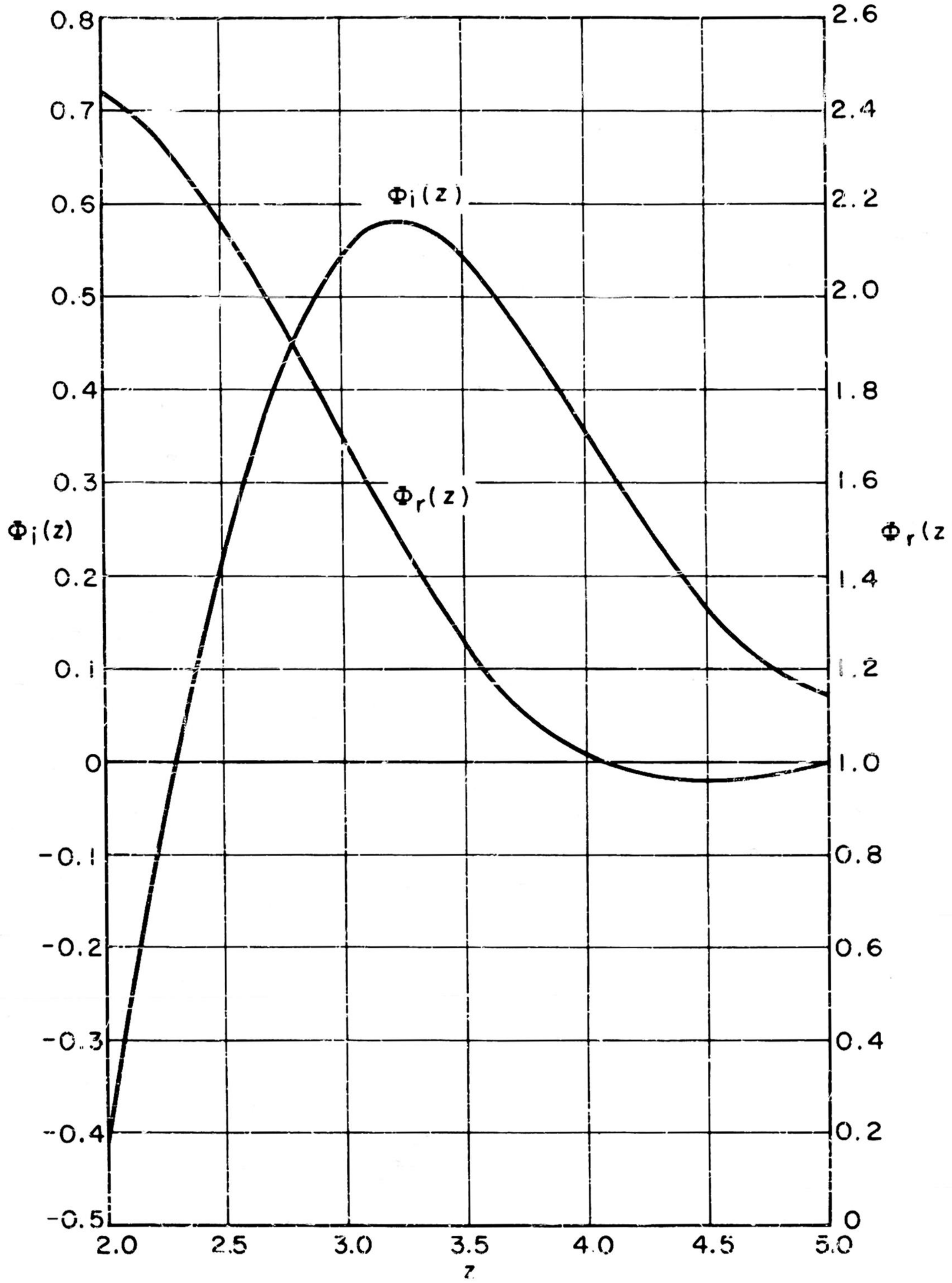
(f) $M_1 = 1.6$, $T_w/T_1 = 1.073$, $\sigma = 0.75$, $\rho\mu = \rho_w\mu_w$

c	$cK_{10}(c)$	$cK_{11}(c)$	$cK_{12}(c)$
0	-1	0	0
.05	-1.010	.002	.046
.1	-1.070	.006	.103
.15	-1.182	.025	.177
.2	-1.306	.057	.264
.25	-1.454	.104	.377
.275	-1.539	.137	.445
.3	-1.631	.174	.525
.325	-1.732	.215	.611
.35	-1.844	.264	.712
.3625	-1.905	.292	.768
.375	-1.970	.321	.828

(g) $M_1 = 1.6$, $T_w/T_1 = 1.073$, $\sigma = 0.75$, $\rho\mu = \rho_w\mu_w$, $\tau_w/\mu_w T_w = 1.857$

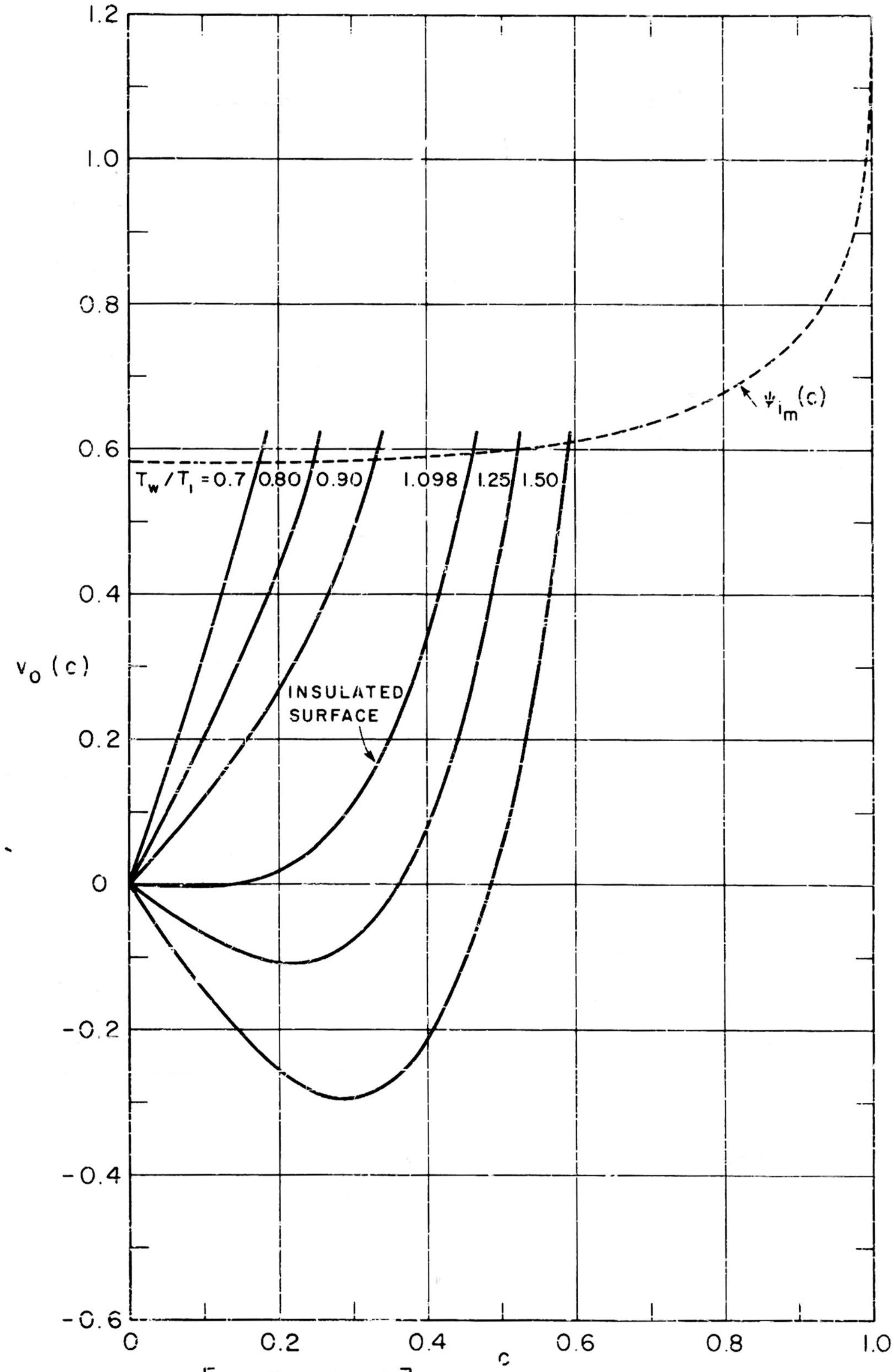
$$u - \tilde{L} = u(c) + (\tau_w/\mu_w T_w)\tilde{M}_1^2 c - [1 + c(K_{10} + K_{11} + K_{12})]$$

c \ $\tilde{M}_1$	0.6	1.0	1.4	c \ $\tilde{M}_1$	0.6	1.0	1.4
0		2.296		.3625	1.367	1.798	2.444
.05		2.305		.35	1.309	1.725	2.349
.1		2.344		.325	1.245	1.632	2.212
.15	2.263	2.408		.3	1.202	1.559	2.094
.175		2.434		.292			2.061
.2	2.218	2.455		.2875			2.042
.25	2.176	2.473	2.919	.275	1.176	1.502	1.993
.275	2.138	2.464	2.955	.25	1.152	1.449	1.894
.2875			2.965	.2	1.098	1.334	
.292			2.967	.175			
.3	2.079	2.435	2.970	.15	1.077	1.222	
.325	2.002	2.388	2.968	.1		1.105	
.35	1.879	2.296	2.920	.05		1.037	
.3625	1.782	2.214	2.860	0		1.0	
.375		1.999	2.667				



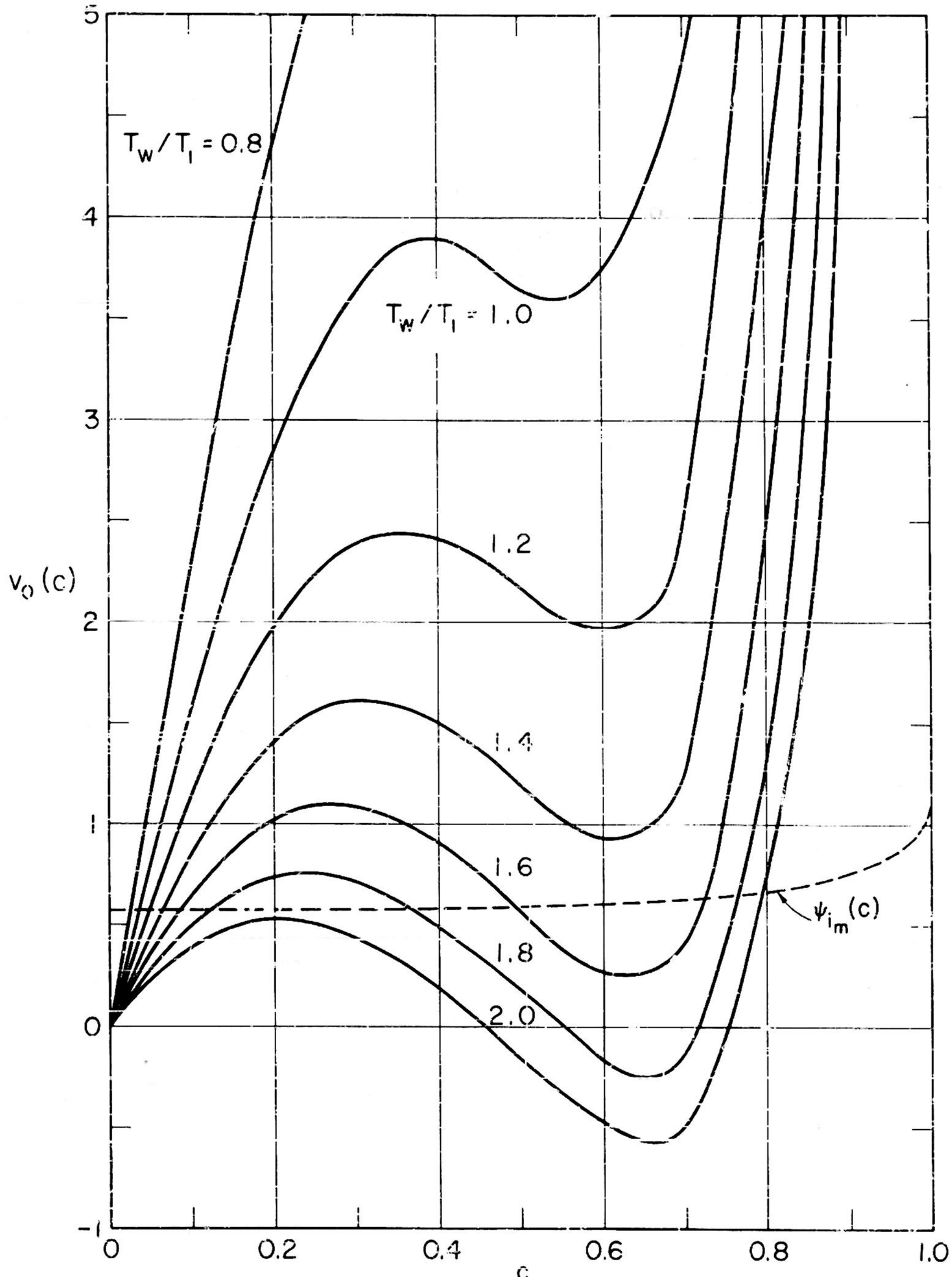
THE FUNCTIONS $\Phi_r(z)$ AND $\Phi_i(z)$

FIGURE 1



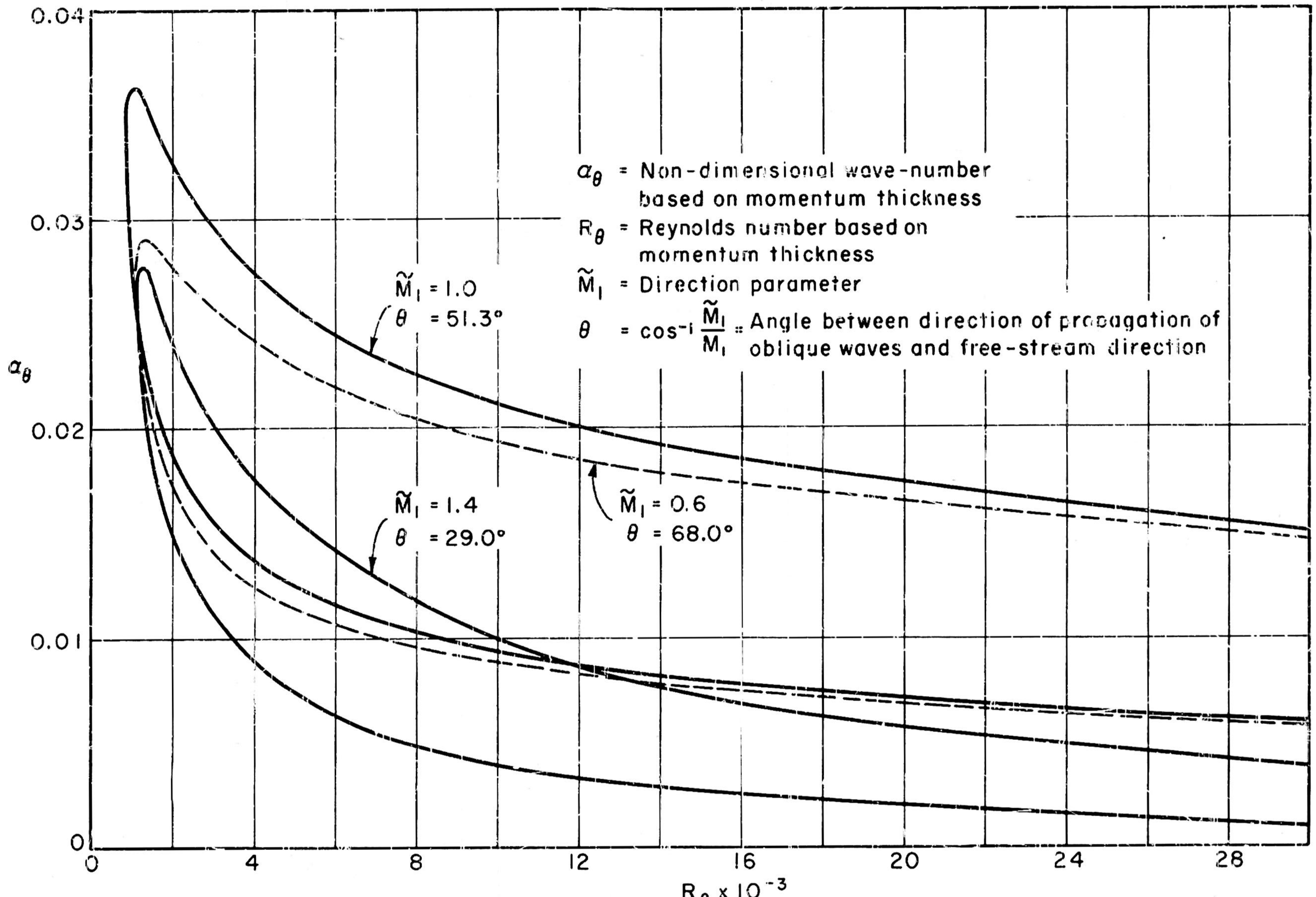
THE FUNCTION $v_0(c) = \left[\pi w \frac{\theta^2}{s}\left(2\frac{\theta'}{\theta} - \frac{s'}{s}\right)\right]_c$ VERSUS c FOR VARIOUS RATIOS T_w/T_1 OF WALL TO FREE-STREAM TEMPERATURE AT A FREE-STREAM MACH NUMBER OF $M_1 = 0.7$

FIGURE 2 (a)



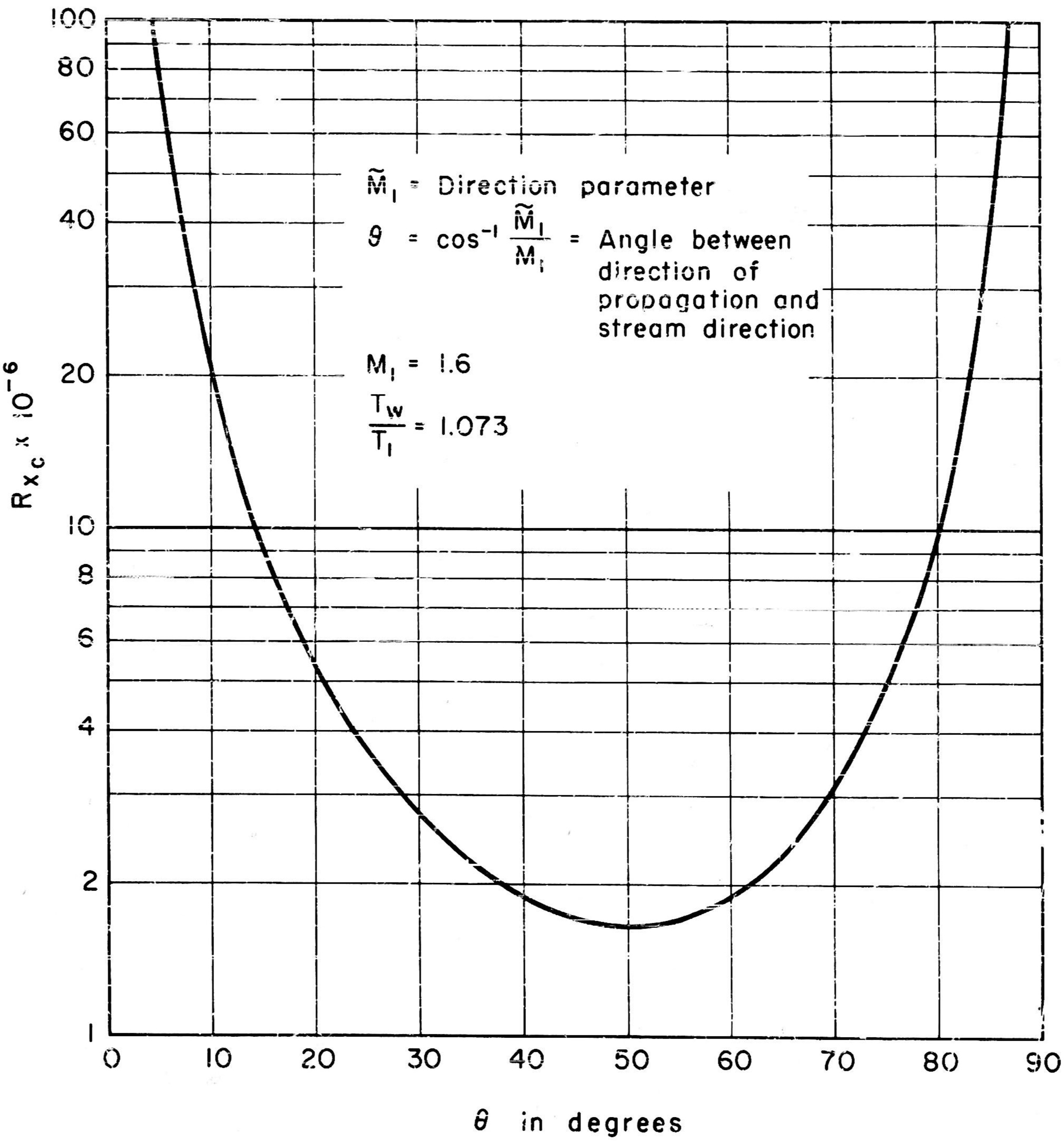
THE FUNCTION $v_0(c) = \left[\pi\, w \frac{\theta^2}{s}\left(2\frac{\theta'}{\theta} - \frac{s'}{s}\right)\right]_c$ VERSUS c FOR VARIOUS RATIOS T_w/T_1 OF WALL TO FREE-STREAM TEMPERATURE AT A FREE-STREAM MACH NUMBER OF $M_1 = 4$

FIGURE 2 (b)



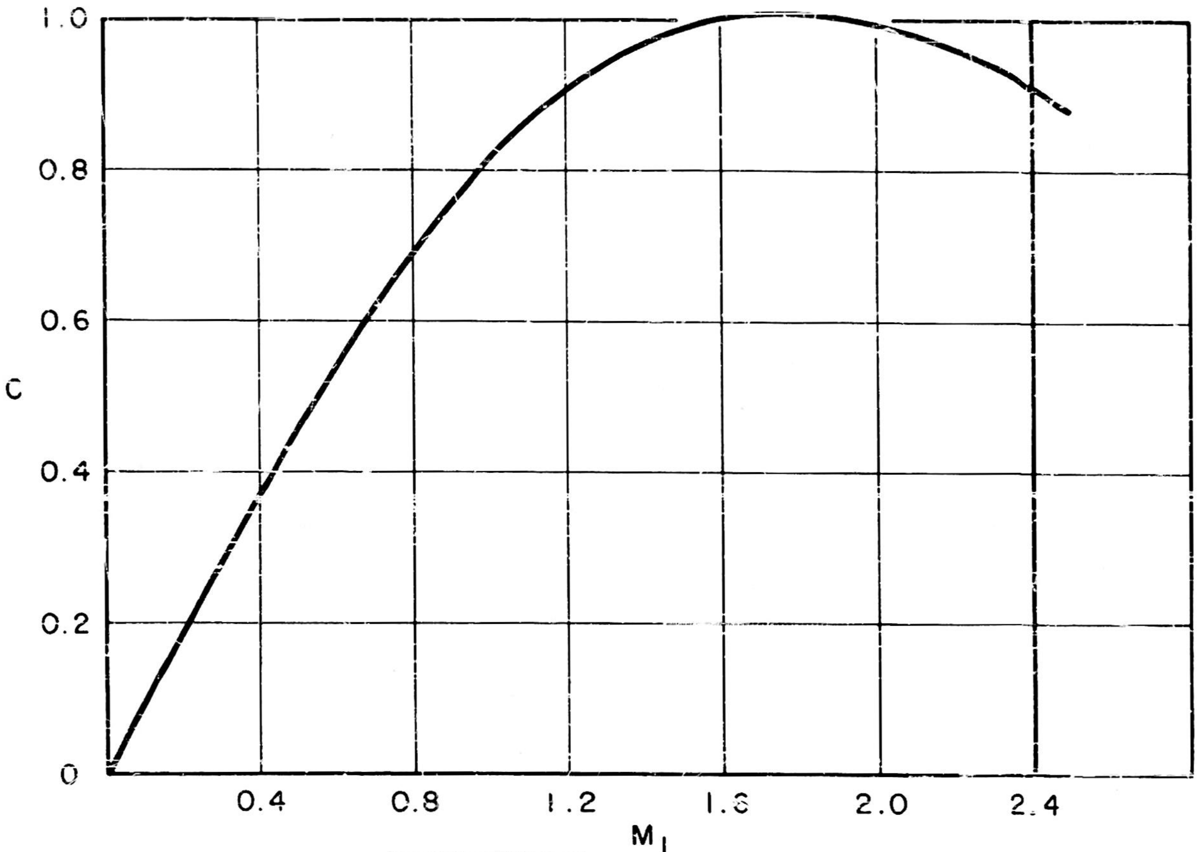
CURVES OF NEUTRAL STABILITY AT A FREE-STREAM MACH NUMBER OF $M_1 = 1.6$ AND A RATIO OF WALL TO FREE-STREAM TEMPERATURE OF $T_w/T_1 = 1.073$

FIGURE 3



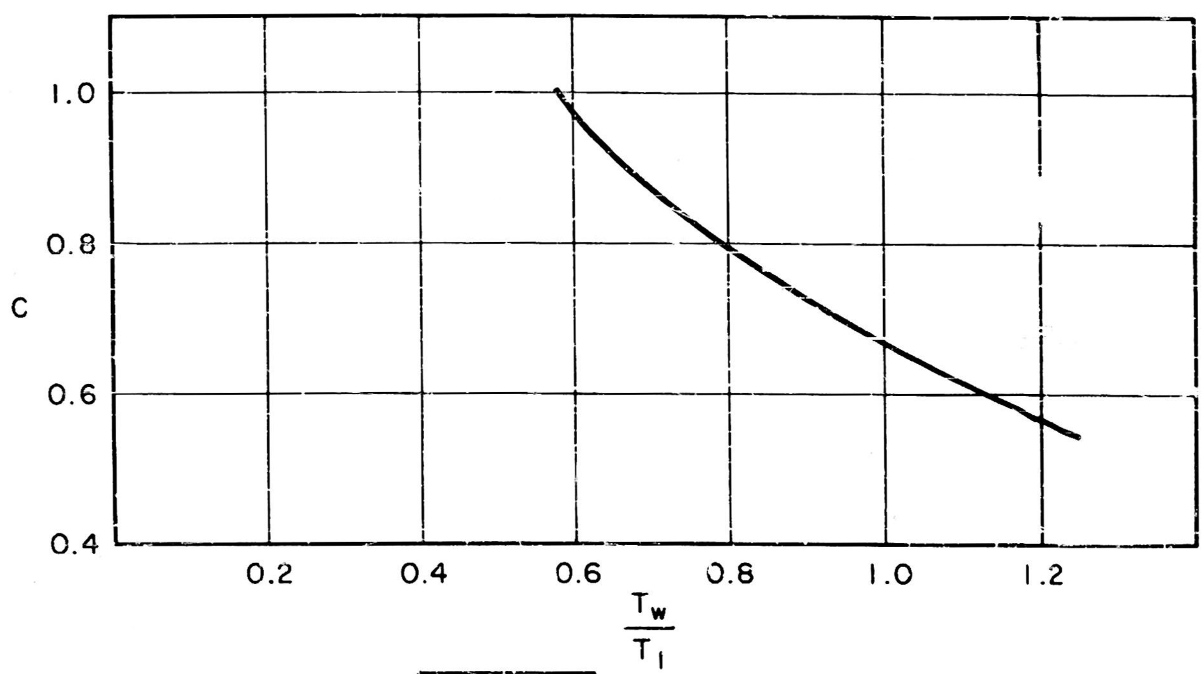
MINIMUM CRITICAL REYNOLDS NUMBER VERSUS DIRECTION OF PROPAGATION OF OBLIQUE WAVES

FIGURE 4



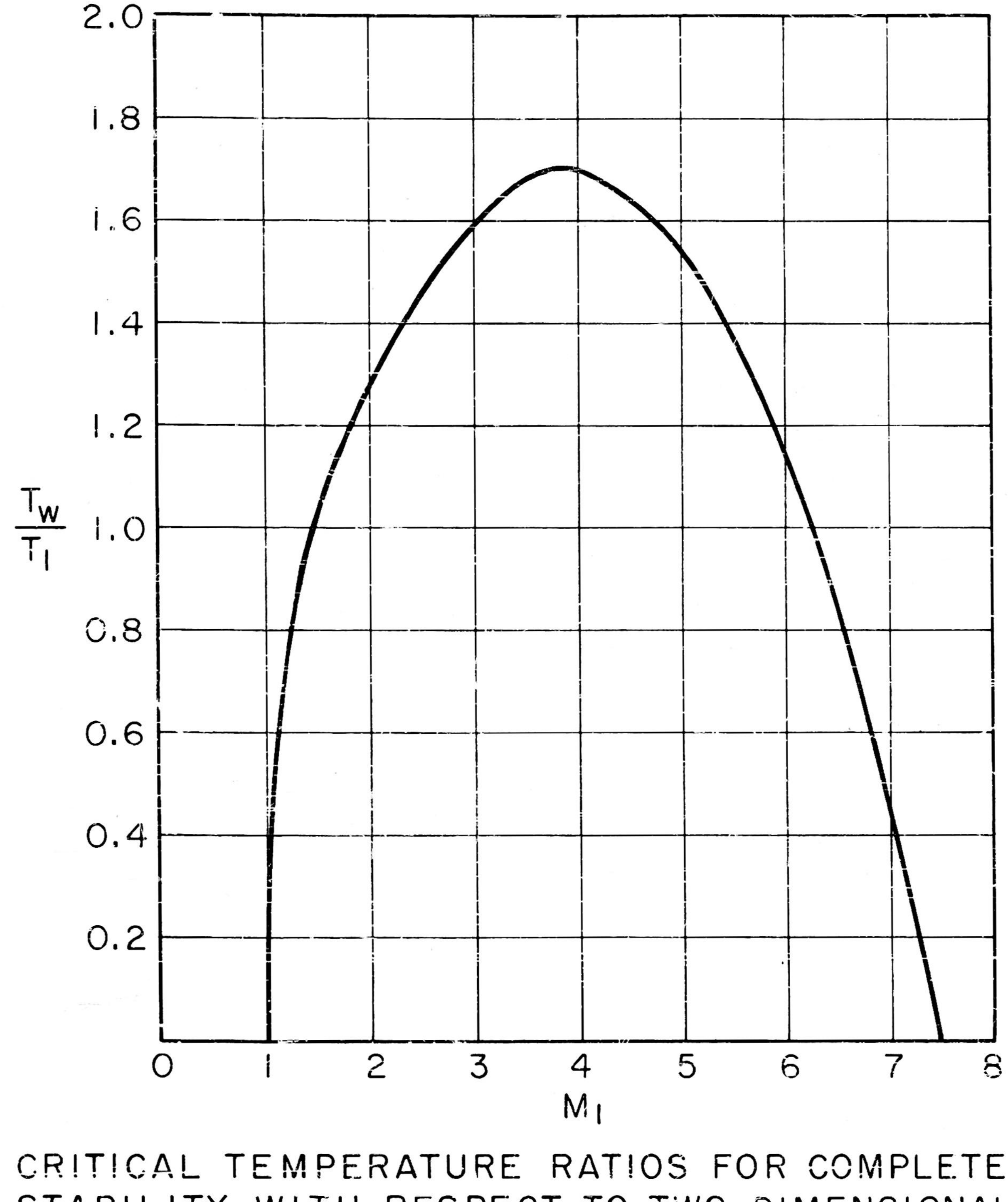
THE FUNCTION $C = M_1\sqrt{2(1-c)^2 + \frac{B}{A}}$ FOR ZERO HEAT TRANSFER AT VARIOUS MACH NUMBERS

FIGURE 5(a)



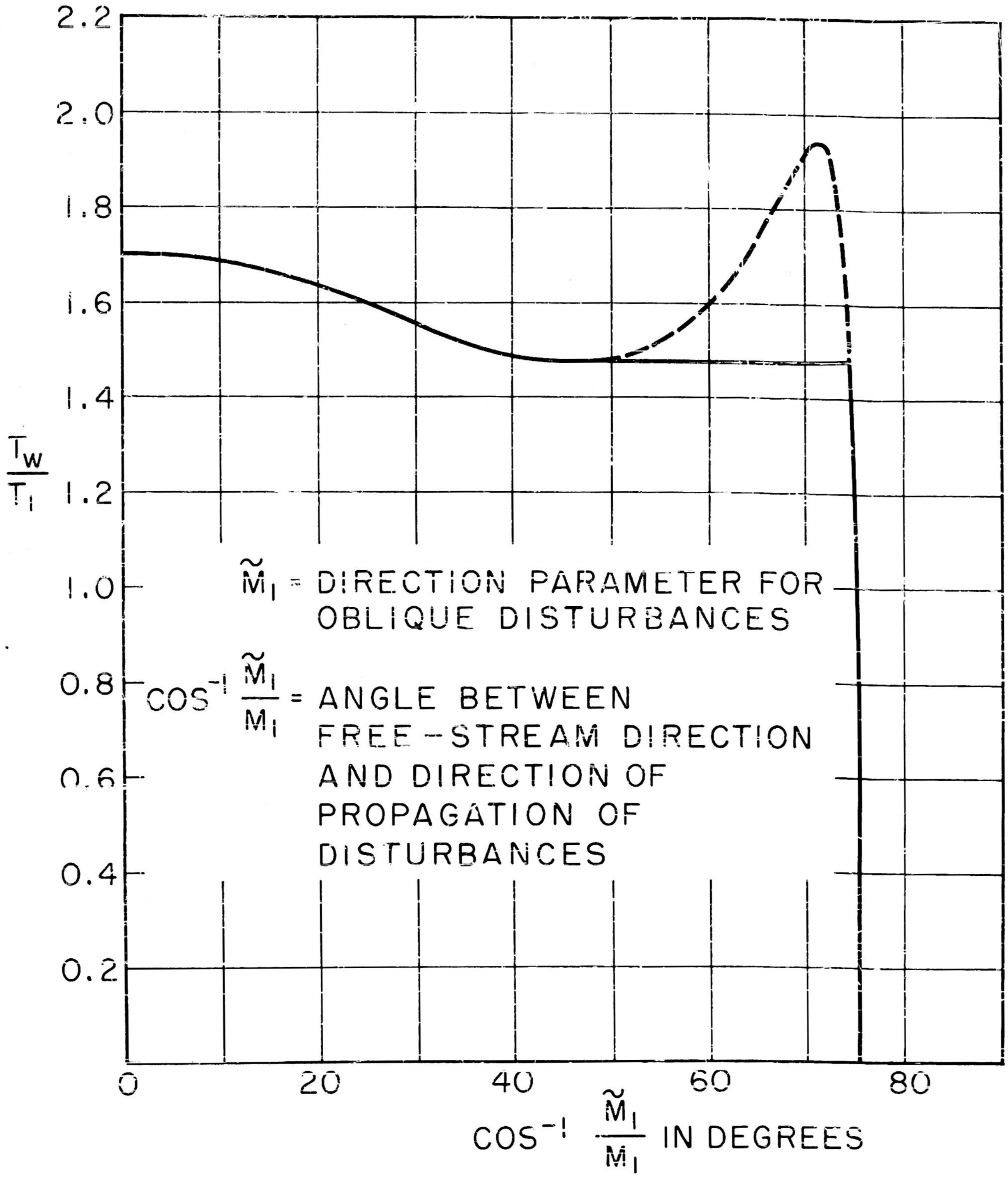
THE FUNCTION $C = M_1\sqrt{2(1-c)^2 + \frac{B}{A}}$ FOR VARIOUS TEMPERATURE RATIOS AT A FREE-STREAM MACH NUMBER OF $M_1 = 0.7$

FIGURE 5(b)



CRITICAL TEMPERATURE RATIOS FOR COMPLETE STABILITY WITH RESPECT TO TWO-DIMENSIONAL DISTURBANCES

FIGURE 6



CRITICAL TEMPERATURE RATIOS FOR COMPLETE STABILITY WITH RESPECT TO THREE - DIMENSIONAL DISTURBANCES AT A FREE-STREAM MACH NUMBER OF $M_1 = 4$

FIGURE 7