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1. Introduction and Summary

In this paper we describe sequential life test procedures. As in a recent paper [3] devoted to non-sequential methods we consider the special case in which the underlying distribution of the length of life is given by the exponential density

$$(1) \qquad f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \qquad x > 0$$

where the unknown parameter $\theta > 0$ can be thought of physically as the mean life. Our primary aim is to test the simple hypothesis H_0: $\theta = \theta_0$ against the simple alternative H_1: $\theta = \theta_1$ ($\theta_0 > \theta_1$) with type I and II errors equal to preassigned values α and β respectively. The test is carried out by drawing n items at random from the population and placing them all on a life test. We consider both the replacement case, in which failed items are immediately replaced by new items, and the non-replacement case.

An interesting feature of the tests is that they can be terminated either at failure times with rejection of H_0 or at any time between failures with acceptance of H_0. Since abnormally long intervals between failures furnish "information" in favor of H_0 and abnormally short intervals furnish "information" in favor of H_1, the above features are not only reasonable but actually desirable. These features were pointed out in

[4]. Similar problems involving a continuous time parameter have recently appeared in [2] and [5].

In this paper we obtain likelihood ratio tests, give formulae for the O.C. curve, for the expected number of failures $E_\theta(r)$ and the expected waiting time $E_\theta(t)$ before a decision is reached. In the replacement case where the number of items on test throughout the experiment is the same, namely n, it is shown that $E_\theta(t) = \frac{\theta}{n} E_\theta(r)$. A table giving values of $L(\theta)$, $E_\theta(r)$ and $E_\theta(t)$ for certain choices of $\frac{\theta_o}{\theta_1}$, α, and β is given for the replacement case.

2. Basic Formulae

Wald's work on sequential analysis [7] can be used virtually without modification in a situation where decisions are made continuously. In fact, in a truly continuous situation, Wald's formulae become exact since there is then no excess over the boundary. It will become clear as we proceed, that in the problem at hand, the situation can be termed semi-continuous (not to be confused with the concept of the same name in real variable theory) since there is no excess over the boundary used for accepting H_o, but there will in general be some excess over the boundary used in accepting H_1.

Let us assume that the underlying p.d.f. is (1). n items are drawn at random from (1) and placed on life test. We wish to test H_o: $\theta = \theta_o$ against H_1: $\theta = \theta_1$ with type I error = α and type II = β. Information is available continuously and it is readily verified that a continuous analogue of the sequential probability ratio test of Wald can be used. The decision as time unfolds depends on

$$(2) \qquad B < \left(\frac{\theta_o}{\theta_1}\right)^r e^{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right)V(t)} < A,$$

where B and A are constants, depending on α and β, such that $B < 1 < A$. The decision to continue experimentation is made as long as the inequality (2) holds. If, at the time the experiment is stopped, the first inequality in (2) is violated we accept H_0; if the second inequality is violated, we accept H_1. As in Wald's case the test obtained by setting $B = \frac{\beta}{1-\alpha}$, $A = \frac{1-\beta}{\alpha}$ is a satisfactory solution of the problem(1) from a practical point of view.

$V(t)$ in (2) is a statistic which can be interpreted as the total life observed up to time t. In the replacement case

(3) $$V(t) = nt.$$

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(1) It has been pointed out by Wald that in order to have a test of exactly strength (α,β), A and B in (2) should be replaced by A^* and B^*, where $A^* \leq \frac{1-\beta}{\alpha}$ and $B^* \geq \frac{\beta}{1-\alpha}$. In the present case, due to the fact that information is available continuously in time, we know that $B^* = B = \frac{\beta}{1-\alpha}$, since acceptance of H_0 involves no excess over the boundary. However, acceptance of H_1 does in general entail a positive excess over the boundary, and all we can say initially about A^* is that it should lie between $A\theta_1/\theta_0$ and A. Thus using $A = \frac{1-\beta}{\alpha}$ instead of A^* is an approximation. The approximate test based on using A and B is completely suitable since its strength (α',β') is such that $\alpha' \leq \alpha$, $\beta \leq \beta' \leq \frac{\beta}{1-\alpha}$, and $\alpha' + \beta' \leq \alpha + \beta$ [7, pp. 45-6]. Since α and β are generally small ($\leq .10$ say), a procedure based on using A and B provided essentially the same protection against errors of the first and second kind as does a test based on A^* and B^*. While one can, using formulae in [2], compute A^* in such a way as to give exactly strength (α,β), there seems to be little reason in most practical problems for expending the time and effort involved.

In the non-replacement case(2) $V(t)$ is given by

$$(4)\quad V(t) = \sum_{i=1}^{r} (n-i+1)(x_i - x_{i-1}) + (n-r)(t-x_r) \equiv \sum_{i=1}^{r} x_i + (n-r)(t-x_r),$$

where x_i denotes the time of the ith failure ($x_0 = 0$).

It is convenient, if one wishes to graph the data continuously in time, to write (2) in the form

$$(5)\qquad -h_1 + rs < V(t) < h_0 + rs$$

where h_0, h_1, and s are positive constants given by

$$(6)\qquad h_0 = \frac{-\log B}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}, \quad h_1 = \frac{\log A}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}, \quad s = \frac{\log \frac{\theta_0}{\theta_1}}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}.$$

Further it can be shown as in [7, pp. 48-50] that the O.C. curve, i.e. the probability of accepting H_0 when θ is the true parameter value, is approximately given by a pair of parametric equations

$$(7)\qquad L(\theta) = \frac{A^h - 1}{A^h - B^h}, \quad \theta = \frac{\left(\frac{\theta_0}{\theta_1}\right)^h - 1}{h\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}$$

by letting the parameter h run through all real values.

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(2) In the non-replacement case it may happen that no decision has been reached by the time $t = x_n$, when all n items have failed. This will then require that we either put more items on test and wait until (2) is violated or else have a rule which will tell us how to terminate the experiment and with what decision at $t = x_n$. Fortunately n is often at our disposal and so can be chosen sufficiently large so that the probability of reaching no decision by time x_n is negligible. For large enough n, it really makes very little difference how we truncate experimentation. We could, e.g., adopt the rule that H_1 is accepted if (2) is satisfied for all $t \leq x_n$.

The value of $L(\theta)$ at the five points $\theta = 0$, θ_1, s, θ_o and ∞ enables one to sketch the entire curve. These values are respectively 0, β, β, $\log A/(\log A - \log B)$, $1-\alpha$, and 1.

We now give, in terms of $L(\theta)$, a formula for $E_\theta(r)$, the expected number of observations required to reach a decision when θ is the true parameter value. Since the logarithm of the middle expression in (4) is either $\log B$ or $\log A$ at the time experimentation stops, we have, neglecting only the excess over $\log A$,

(8) $$E_\theta(r) \log \frac{\theta_o}{\theta_1} - E_\theta(V(t)) \left[\frac{1}{\theta_1} - \frac{1}{\theta_2}\right] E_\theta(V(t)) \sim L(\theta) \log B + \left[1 - L(\theta)\right] \log A.$$

It is proved in the next section that

(9) $$E_\theta(V(t)) = \theta\, E_\theta(r).$$

Hence we have from (9) and (10)

(10) $$E_\theta(r) \sim \begin{cases} \dfrac{L(\theta) \log B + \left[1 - L(\theta)\right] \log A}{\log \dfrac{\theta_o}{\theta_1} - \theta\left(\dfrac{1}{\theta_1} - \dfrac{1}{\theta_o}\right)} = \dfrac{h_1 - L(\theta)(h_o + h_1)}{s - \theta} & \text{for } \theta \neq s \\[2ex] \dfrac{-\log A \log B}{\log\left(\dfrac{\theta_o}{\theta_1}\right)^2} = \dfrac{h_o\, h_1}{s^2} & \text{for } \theta = s. \end{cases}$$

If we let $k = \theta_o/\theta_1$, $E_\theta(r)$ becomes particularly simple when $\theta = \theta_1$, s, or θ_o. The result is

(11) $$E_{\theta_1}(r) \sim \left[\beta \log B + (1 - \beta) \log A\right] \Big/ \left[\log k + (k - 1)\right]$$
$$E_s(r) \sim - \log A \log B/(\log k)^2$$
$$E_{\theta_o}(r) \sim \left[(1 - \alpha) \log B + \alpha \log A\right] \Big/ \left[\log k - (k - 1)\right].$$

In table 1 we give $E_\theta(r)$ for the five values $\theta = 0, \theta_1, s, \theta_o, \infty$, for four values of $k(3/2, 2, 5/2, 3)$, and the four number pairs (α,β) which can be made with the numbers .01 and .05.

3. A basic identity

In this section (9) is derived. While this result can be obtained as a consequence of a theorem of Doob on continuous parameter martingales [1, p. 376], it seems desirable to give a simpler proof. We shall consider the replacement case, where $V(t) = nt$, although the proof can be trivially modified so as to hold in non-replacement and truncated situations.

In the replacement case (9) becomes

$$E_\theta(t) = E_\theta(r)\,\frac{\theta}{n}\,. \tag{12}$$

Thus we are relating expected waiting time to reach a decision to the expected number of failures.

To prove (12) we introduce a "large" integer N and let x_N denote the time of the Nth failure. Let t denote the (first) time at which the inequality (2) is violated or x_N, whichever comes sooner. Then we can write

$$x_N = t + (x_{r+1} - t) + (x_{r+2} - x_{r+1}) + \dots + (x_N - x_{N-1}). \tag{13}$$

We either accept H_1 before x_N, in which case $t = x_r$, or accept H_o before x_N, in which case $x_r < t < x_{r+1}$, or take no action before x_N, in which case $t = x_N$ and $r = N$. Since N is fixed in advance

$$E(x_N) = N\,\frac{\theta}{n}\,. \tag{14}$$

Further it is easily verified that the (N-r) random variables

$(x_{r+1} - t)$, $(x_{r+2} - x_{r+1})$, ..., $(x_N - x_{N-1})$ are independently and identically distributed with the exponential density $\frac{n}{\theta} e^{-nx/\theta}$, $x > 0$. Hence if we take the expectation of both sides of (14), first holding r fixed and then taking the expectation with respect to r, we obtain

(15) $$N \frac{\theta}{n} = E_\theta(t;N) + \left[N - E_\theta(r;N)\right] \frac{\theta}{n}$$

or

(16) $$E_\theta(t;N) = E_\theta(r;N) \frac{\theta}{n} .$$

Formula (16) holds for all N. For $N \rightarrow \infty$ the probability of coming to a decision before x_N tends to unity. Moreover as $N \rightarrow \infty$

(17) $$E_\theta(r;N) \uparrow E_\theta(r) \quad \text{and} \quad E_\theta(t;N) \uparrow E_\theta(t),$$

where $E_\theta(r)$ and $E_\theta(t)$ are respectively the expected number of failures and expected waiting time to reach a decision if $N = \infty$. Thus it follows, letting $N \rightarrow \infty$, that (16) becomes

(12) $$E_\theta(t) = E_\theta(r) \frac{\theta}{n} .$$

The non-replacement case can be treated in exactly the same way. This is because

(18) $$V(x_n) = V(t)+(n-r)(x_{r+1}-t)+(n-r-1)(x_{r+2}-x_{r+1}) + \cdots + (x_n-x_{n-1}).$$

As before, $t = x_r$, if H_1 is accepted; $x_r < t < x_{r+1}$, if H_0 is accepted; and $t = x_n$, if no decision is reached by the time all n items have failed. The last (n-r) components on the right-hand side of (18) are mutually independent random variables, each distributed with the p.d.f. (1). Thus it follows as in the replacement case that

(19) $$E_\theta(V(t);n) = \theta\, E_\theta(r;n).$$

As n increases, $E_\theta(r;n) \uparrow E_\theta(r)$, the expected number of failures in reaching a decision in the replacement case. Thus no matter how we decide to terminate experimentation $E_\theta(V(t);n)$ can be replaced by $E_\theta(V(t)) = \theta\, E_\theta(r)$, when n is large. In practice, for "large" n one could take $n > 3 \max_\theta E_\theta(r)$.

<u>Remark</u>: It should be noted that while we can relate expected waiting time to the expected number of failures in the replacement case by (12), (19) relates expected total life (not waiting time) to the expected number of failures in the non-replacement case. Actually one has to know the probability distribution of r in order to compute $E_\theta(t)$ exactly in the non-replacement case. It can be shown, in the non-replacement case that the formula for $E_\theta(t)$ is given by

$$(20) \qquad E_\theta(t) = \sum_{k=1}^{n} \Pr(r = k|\theta)\, E_\theta(X_{k,n}),$$

where $E_\theta(X_{k,n}) = \theta \sum_{i=1}^{k} 1/n-i+1$. In the replacement case one has (analogous to (20)) the formula

$$(20') \qquad E_\theta(t) = \sum_{k=1}^{\infty} \Pr(r = k|\theta)\, E_\theta(X_{k,n}),$$

where n is the sample size maintained throughout the experiment and $E_\theta(X_{k,n}) = k\theta/n$. Thus, in the replacement case (20') clearly becomes (12). (20) is valid for all(4) life test procedures which involve non-replacement. Similarly (20') holds for all(4) life test procedures, where items which fail are replaced.

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(4) i.e., for truncated or untruncated, sequential, or any similar procedure. Of course the probability distribution of r does depend on the procedure which is followed. In [6] explicit formulae for $\Pr(r = k|\theta)$ are worked out for three procedures.

4. Numerical examples

Problem 1: Find a sequential replacement procedure for testing $H_o: \theta_o$ 7500 hours against $H_1: \theta_1 = 2500$ hours with $\alpha = \beta = .05$. The constant number of items under test is $n = 100$.

Solution: In this case (2) becomes

$$\frac{1}{19} < 3^r e^{-V(t)/3750} < 19,$$

where $V(t) = 100t$ hours.

Problem 2: Compute $E_\theta(r)$ and $E_\theta(t)$ for $\theta = 0, \theta_1, s, \theta_o$, and ∞.

Solution: From table 1 we can read $E_\theta(r)$ for this case, since $k = \theta_o/\theta_1 = 3$ and $\alpha = \beta = .05$. For $\theta = 0, \theta_1$ (= 2500), s (= 4115), θ_o (= 7500), ∞ respectively we have $E_\theta(r) = 3, 6.1, 7.2, 2.9, 0$. $E_\theta(t)$ is found most easily in the replacement case for all values of $\theta(\neq \infty)$ by using (12), $E_\theta(t) = E_\theta(r)$. Thus we get for $\theta = 0, \theta_1, s, \theta_o$ respectively, $E_\theta(t) = 0$, 150, 29 220. For $\theta = \infty$, the expected waiting time to reach a decision is given by t_∞, where $e^{-100t_\infty/3750} = \frac{1}{19}$. This gives $t_\infty = E_\infty(t) = 110$.

Remark: More generally, in terms of B, n, θ_o, and k, $t_\infty = \dfrac{-\theta_o \log B}{n(k-1)}$.

This means that if no items fail by time t_∞ stop experimentation at t_∞ with acceptance of H_o.

Problem 3. Assume that we are testing the hypothesis in Problem 1. A sample of size 100 is placed on test. Items which fail are replaced by new items drawn from the same lot. The experiment is started at time $t = 0$. The first failure occurs at $x_1 = 20.1$ hours, the second failure at $x_2 = 10$ 5 hours, the third failure at $x_3 = 121.7$ hours, the fourth failure at x_4 = 167.4 hours, the fifth failure at x_5 = 179.2 hours. (All times are measured from t = 0.)

(a) Verify that no decision has been reached by time x_5.

(b) We keep waiting for the sixth failure and note that it has not yet occurred at 287.5 hours (time measured from t = 0). Verify that we can stop experimentation at time t = 287.5 with the acceptance of H_0.

Solution: It can be readily verified that in this case (5) becomes $-100 + 37.5r < t < 100 + 37.5r$. This region is drawn in Figure 1. The life test data are plotted on the figure by moving vertically so long as we are waiting for the next failure to occur and moving horizontally by one unit (in r) at each failure time. Clearly the path crosses into the region of acceptance, when r = 5, at time t = 100 + (37.5)5 = 287.5. Since the sixth failure has not yet occurred we can stop experimentation at t = 287.5 with the acceptance of H_0.

Remark: As a matter of fact we happen to know in this case that the sixth failure occurs at x_6 = 346.7 hours. Thus we saved 346.7 - 287.5 = 59.2 hours by observing the life test continuously in time.

Problem 4: The first seven failure times in a sample of 100 (with replacement) are x_1 = 19.3, x_2 = 45.8, x_3 = 49.9, x_4 = 96.7, x_5 = 115.2, x_6 = 127.7, x_7 = 131.2. Verify that if the hypotheses being tested are those in Problem 1, then H_0 is rejected at time x_7 = 131.2 hours.

Solution: See Figure 2.

Remark: Note that while the decision in Problem 3 is made between x_5 and x_6, the decision in Problem 4 is made at the failure time x_7. Also there is an excess over the boundary.

Problem 5: Find a truncated (non-sequential) replacement procedure for testing the hypothesis in Problem 1. Use constant sample size $n = 100$.

Solution: From results in [6], it can be verified that the truncated replacement procedure meeting the requirements is as follows:

If $\min[x_{10}, 407.5] = 407.5$, truncate the experiment at 407.5 with acceptance of H_0. If $\min[x_{10}, 407.5] = x_{10}$, truncate experimentation at x_{10} with acceptance of H_1. The O.C. curves of this test procedure and the one in Problem 1 are essentially the same.

Problem 6: Compute $E_\theta(r)$ and $E_\theta(t)$ for the plan in Problem 5 for $\theta = 0, \theta_1, s, \theta_0, \infty$.

Solution: From results in [6], $E_\theta(r) = 10, 9.93, 8.75, 5.39, 0$, $E_\theta(t) = \frac{\theta}{n} E_\theta(r)$. For $\theta = 0, \theta_1, s, \theta_0, \infty$ respectively, $E_\theta(t) = 0, 248, 360, 404.5, 407.5$ respectively.

Remark: In Figures 3 and 4 we compare the $E_\theta(r)$ and $E_\theta(t)$ curves for Problems 2 and 6. This will give some idea of the savings in the expected number of failures and time to reach a decision.

Problem 7: Find t_∞ in Problem 1 if $\alpha = \beta = .01$.

Solution: $t_\infty = \dfrac{-\theta_0 \log B}{n(k-1)} = 230$. This is about twice the value of t_∞ when $\alpha = \beta = .05$.

Table 1

Values of $E_\theta(r)$ for sequential tests for various values of $k = \theta_o/\theta_1$ and α, β.

$k = \theta_o/\theta_1 = 3/2$

	$\alpha = .01$		$\alpha = .05$	
θ \ β	.01	.05	.01	.05
0	11	11	7	7
θ_1	62	60	40	37
s	128	83	83	53
θ_o	48	31	44	28
∞	0	0	0	0

$k = \theta_o/\theta_1 = 2$

	$\alpha = .01$		$\alpha = .05$	
θ \ β	.01	.05	.01	.05
0	7	7	4	4
θ_1	23	23	15	14
s	44	28	28	18
θ_o	15	9.5	14	8.6
∞	0	0	0	0

$k = \theta_o/\theta_1 = 5/2$

	$\alpha = .01$		$\alpha = .05$	
θ \ β	.01	.05	.01	.05
0	5	5	3	3
θ_1	14	14	9.2	8.4
s	25	16	16	10
θ_o	7.7	5.0	7.2	4.5
∞	0	0	0	0

$k = \theta_o/\theta_1 = 3$

	$\alpha = .01$		$\alpha = .05$	
θ \ β	.01	.05	.01	.05
0	4	4	3	3
θ_1	10	10	6.7	6.1
s	18	11	11	7.2
θ_o	5.0	3.2	4.6	2.9
∞	0	0	0	9

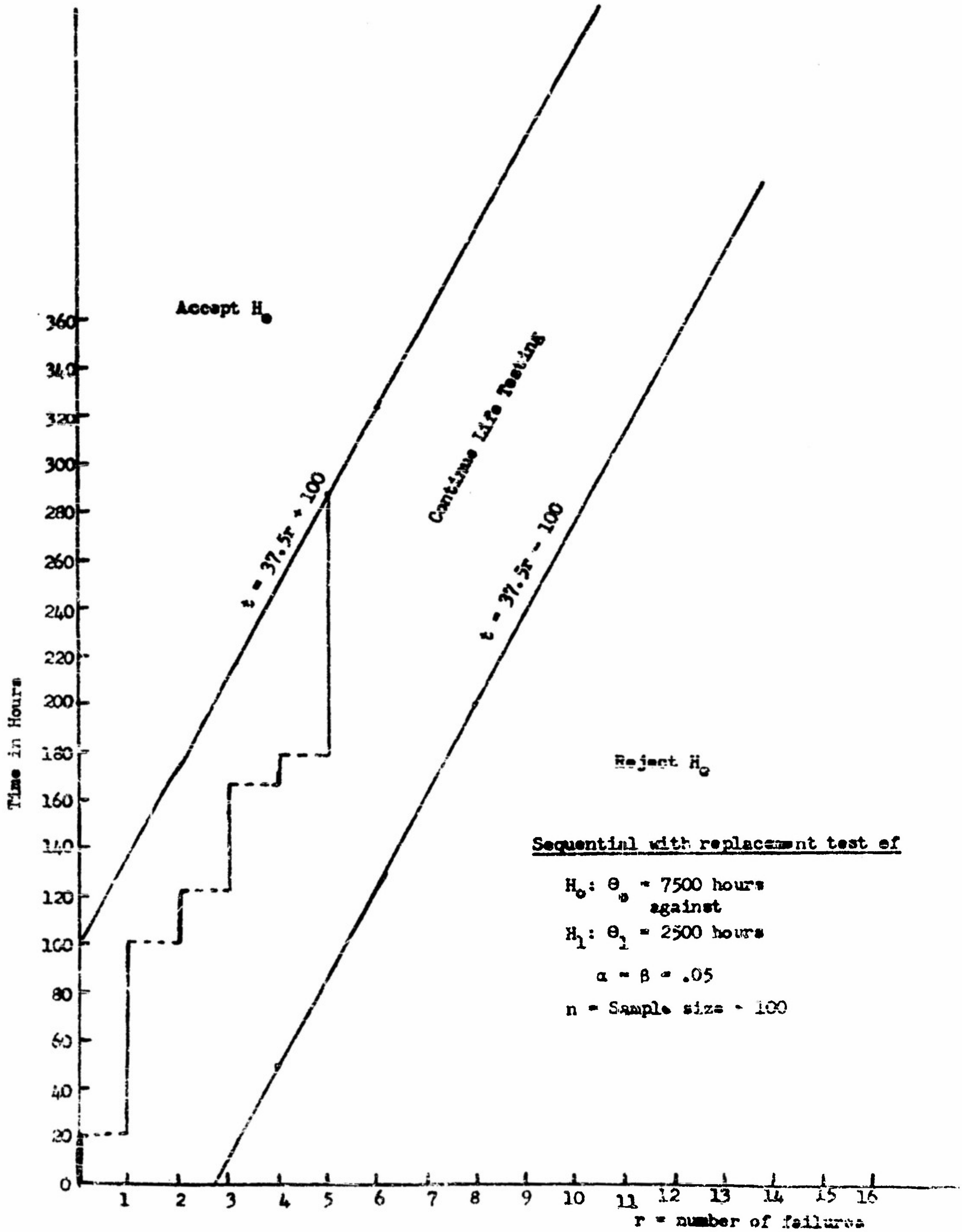

Figure 1: Graphical treatment of data in Problem 3.

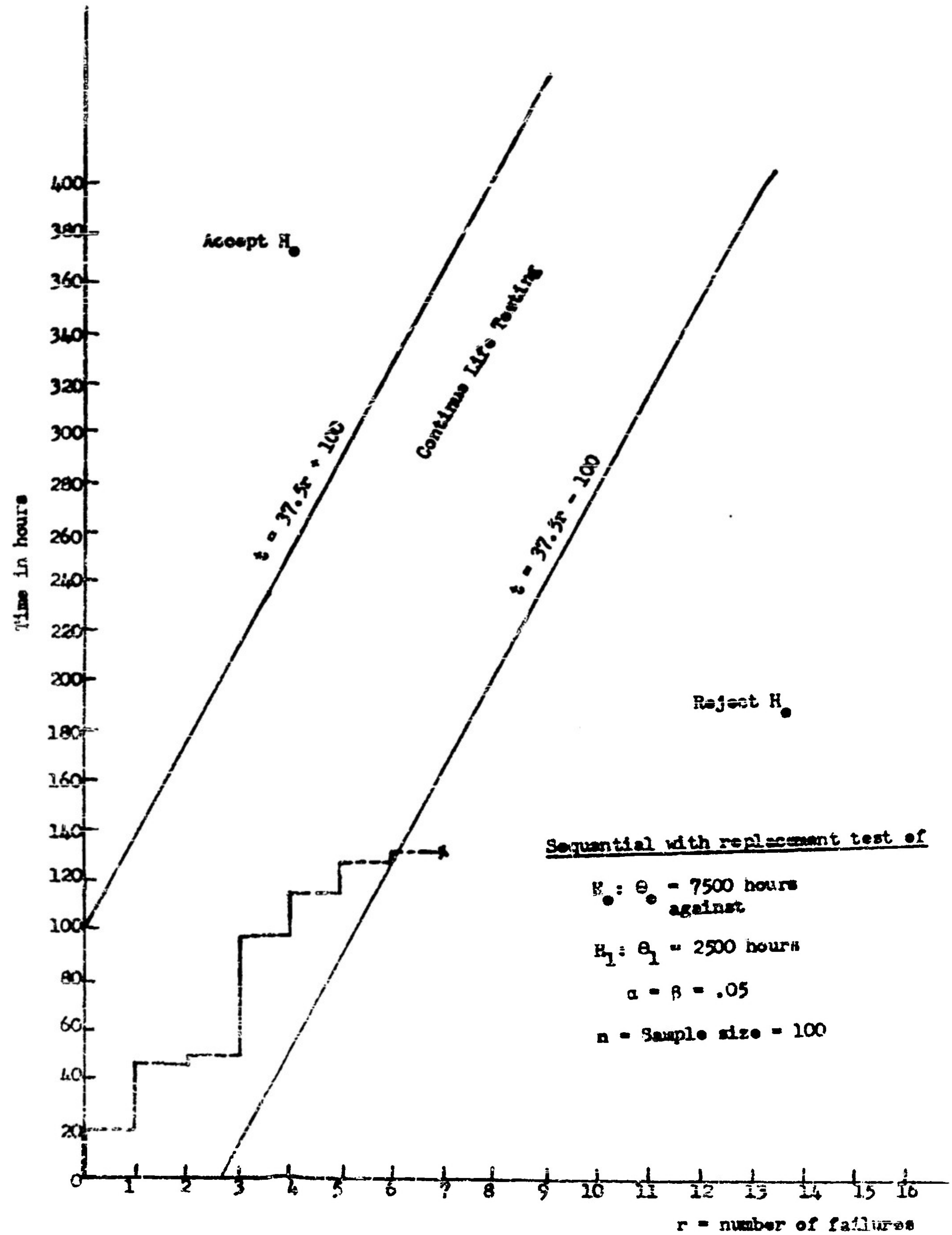

Figure 2: Graphical treatment of data in Problem 4.

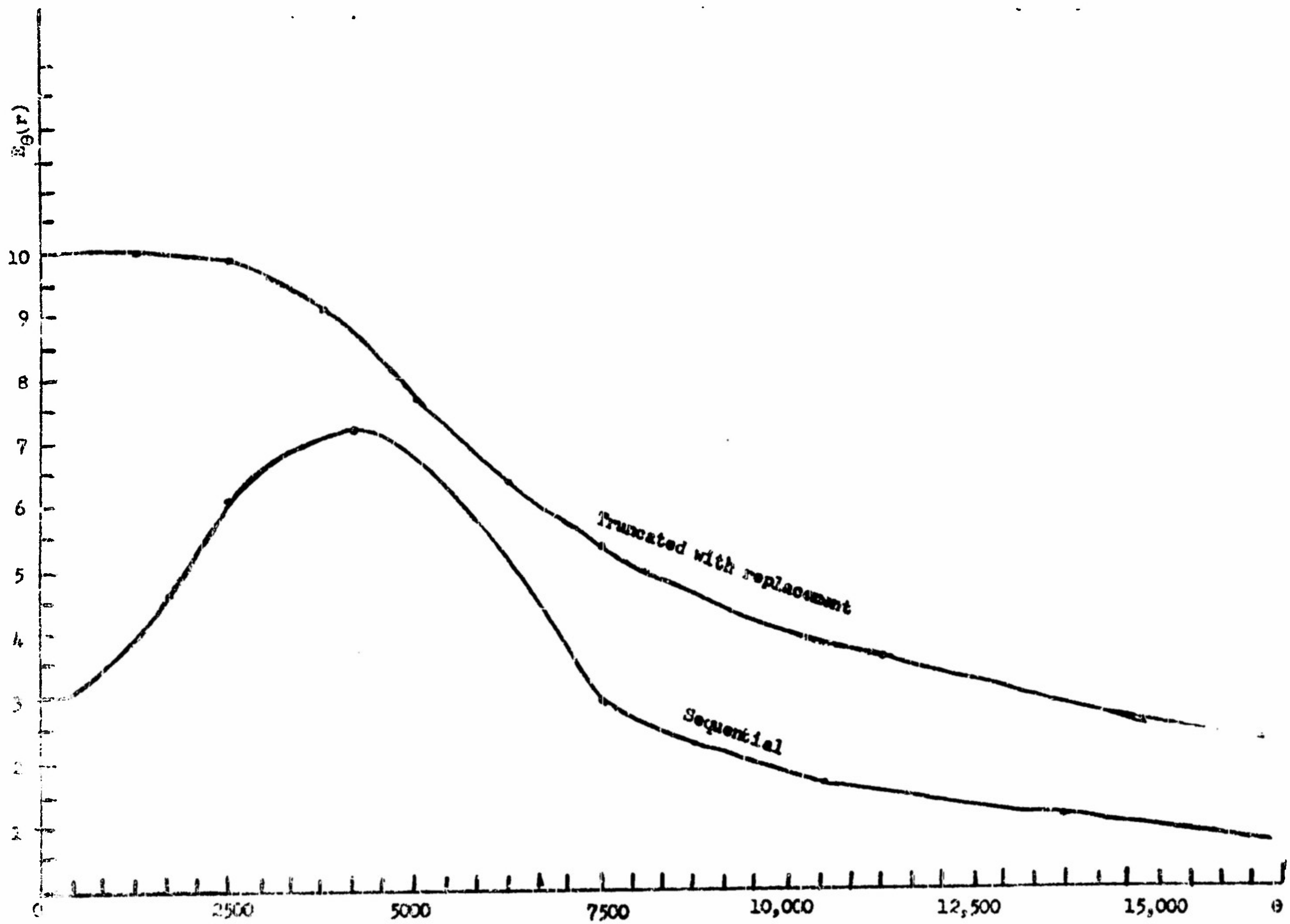

Figure 3: Comparison of $E_\theta(r)$ curves for a sequential and truncated with replacement plan. The O.C. curves for each plan are such that $L(\theta_0)=.95$, $L(\theta_1)=.05$, with $\theta_0 = 7500$, $\theta_1 = 2500$.

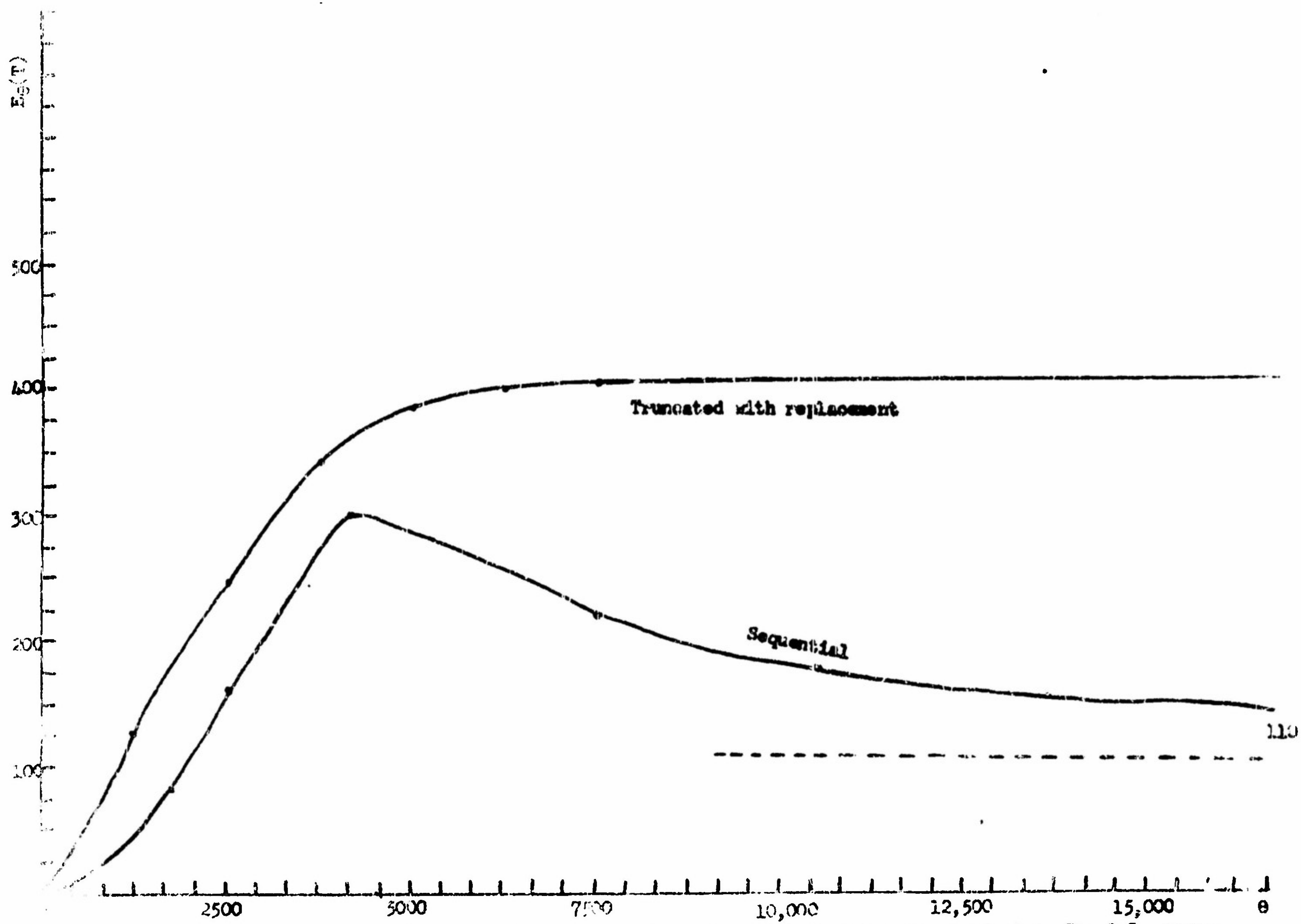

Figure 4: Comparison of $E_\theta(t)$ curves for a sequential and truncated with replacement plan. The O.C. curves for each plan are such that $L(\theta_0)=.95$, $L(\theta_1)=.05$, with $\theta_0 = 7500$, $\theta_1 = 2500$

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