BOUNDARY VALUE PROBLEMS FOR $\nabla^2 u + k^2 u = 0$

BY

IVAR STAKGOLD

TECHNICAL REPORT NO. 17

JANUARY 29, 1954

PREPARED UNDER CONTRACT Nonr-225(11)
(NR-041-086)
FOR
OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

A BOUNDARY VALUE PROBLEM FOR $\nabla^2 u + k^2 u = 0$

By

Ivar Stakgold

Stanford University[1]

Introduction and summary.

Part I: The solution of $u_{xx} + u_{yy} + k^2 u = 0$, with u (or $\frac{\partial u}{\partial r}$) on a half-line is obtained by a simpler and more direct method than those previously found in the literature. The method applies equally well to a wedge of arbitrary opening. An idea of Magnus (1) is used to simplify the solution when the boundary values can be expanded in a series of Bessel functions.

Part II: The problem of diffraction by an infinite cone with a spherical tip is treated by a method similar to the used in part I. Both point source and plane wave excitations are considered. When the cone opening θ_0 is $\frac{\pi}{2}$, the problem reduces to reflection from a plane with a hemispherical boss. This simpler problem has been solved by Twersky (7); our results agree with his solution.

Part I.

(a) We first determine the Green's function for the problem.

A two-dimensional unit source is located at the point (ρ', θ') in front of a semi-infinite sheet ($\theta = 0$).

We wish to solve the two-dimensional problem

---

[1] On leave from Harvard University.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + k^2 u = -\frac{\delta(\rho-\rho')\,\delta(\theta-\theta')}{\rho'} \tag{1}$$

$$u(\rho, 0+) = 0$$

$$u(\rho, 2\pi-) = 0 \quad .$$

u obeys the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0$$

(r is measured from an arbitrary fixed origin).

It is immediately clear that the angular eigenfunctions of the homogeneous equation are just $\sin\frac{n}{2}\theta$. This is a complete orthogonal set of functions over $0 < \theta < 2\pi$, satisfying the boundary conditions.

We write $u(\rho,\theta) = \sum_{n=1}^{\infty} a_n(\rho)\sin\frac{n}{2}\theta$ where the $a_n(\rho)$ are to be determined.

We multiply equation (1) by $\sin\frac{n}{2}\theta$ and integrate from 0 to 2π, obtaining:

$$\pi a_n''(\rho) + \frac{\pi}{\rho}a_n'(\rho) + k^2 a_n(\rho) + \frac{1}{\rho}\cdot\int_0^{2\pi}\sin\frac{n\theta}{2}\frac{\partial^2 u}{\partial\theta^2}\,d\theta$$

$$= \frac{-\sin\frac{n}{2}\theta'}{\rho'}\,\delta(\rho-\rho')$$

(A priori it is possible that the coefficients $a_n(\rho)$ are no better than $O(\frac{1}{n^2})$ for large n, therefore we may not be able to differentiate twice termwise with respect to θ).

Integrating twice by parts and using the boundary conditions on u and sin $\frac{n\theta}{2}$, we readily obtain

$$a_n''(\rho) + \frac{a_n'(\rho)}{\rho} + (k^2 - \frac{n^2}{4})\, a_n(\rho) = \frac{-\sin\frac{n}{2}\theta'}{\pi\rho'}\,\delta(\rho-\rho') \ .$$

Hence

$$a_n(\rho) = \begin{cases} A\, H^{(1)}_{\frac{n}{2}}(k\rho) & \rho > \rho' \\ B\, J_{\frac{n}{2}}(k\rho) & \rho < \rho' \end{cases} \ .$$

This satisfies the radiation condition at infinity and the finiteness condition at the origin.

We now require

$$a_n(\rho'+) = a_n(\rho'-), \qquad \frac{da_n}{d\rho}(\rho'+) - \frac{da_n}{d\rho}(\rho'-) = \frac{-\sin\frac{n}{2}\theta'}{\pi\rho'} \ .$$

Recalling the Wronskian relation:

$$\rho'[H'^{(1)}_{\frac{n}{2}}(k\rho')\, J_{\frac{n}{2}}(k\rho') - H^{(1)}_{\frac{n}{2}}(k\rho')\, J'_{\frac{n}{2}}(k\rho')] = \frac{2i}{\pi k} \ .$$

We have:

$$a_n(\rho) = \frac{i}{2}\sin\frac{n}{2}\theta' \begin{cases} H^{(1)}_{\frac{n}{2}}(k\rho)\, J_{\frac{n}{2}}(k\rho') & \rho > \rho' \\ H^{(1)}_{\frac{n}{2}}(k\rho')\, J_{\frac{n}{2}}(k\rho) & \rho < \rho' \end{cases}$$

We use the standard abbreviation:

$$a_n(\rho) = \frac{i}{2} \sin \frac{n}{2} \theta' \; H^{(1)}_{\frac{n}{2}}(k\rho>) \; J_{\frac{n}{2}}(k\rho<). \quad \text{The symbol } \rho > \text{ means } \begin{cases} \rho & \text{when } \rho > \rho' \\ \rho' & \text{when } \rho < \rho' \end{cases}.$$

Therefore we have the Green's function:

$$(2) \qquad u(\rho,\theta) = G(\rho,\theta;\rho',\theta') = \frac{i}{2}\sum_{n=1}^{\infty} \sin \frac{n}{2}\theta \sin \frac{n}{2}\theta' \; H^{(1)}_{\frac{n}{2}}(k\rho>) \; J_{\frac{n}{2}}(k\rho<)$$

(Formulas of this general nature, derived by the use of the idea of sources on a Riemann surface, are to be found in A. Sommerfeld (2)).

Expression (2) may be rewritten

$$G(\rho,\theta;\rho',\theta') = \frac{i}{2}\sum_{n=1}^{\infty} \sin n\theta \sin n\theta' \; H^{(1)}_{n}(k\rho>) \; J_{n}(k\rho<)$$

$$+ \frac{i}{2}\sum_{n=0}^{\infty} \sin \frac{2n+1}{2}\theta \sin \frac{2n+1}{2}\theta' \; H^{(1)}_{\frac{2n+1}{2}}(k\rho>) \; J_{\frac{2n+1}{2}}(k\rho<) .$$

It is easy to recognize the first series. Had we carried out a similar analysis for a completely infinite sheet, the Green's function would turn out to be $i \sum_{n=1}^{\infty} \sin n\theta \sin n\theta' \; H^{(1)}_{n}(k\rho>) \; J_{n}(k\rho<)$.

We have therefore

$$\frac{i}{2}\sum_{n=1}^{\infty} \sin n\theta \sin n\theta' \; H^{(1)}_{n}(k\rho>) \; J_{n}(k\rho<) = \frac{i}{8}\left\{H^{(1)}_{o}(kR) - H^{(1)}_{o}(kR^{*})\right\}$$

where R is the distance between the source and the observation point,

R^* is the distance between the reflected source about $\theta = 0$ and the observation point. This result could have been obtained from the familiar addition theorem for cylinder functions. We may write

$$(3)\quad G(\rho,\theta;\rho',\theta') = \frac{i}{8}\left\{H_0^{(1)}(kR) - H_0^{(1)}(kR^*)\right\}$$

$$+ \frac{i}{2}\sum_{n=0}^{\infty} \sin\frac{2n+1}{2}\,\theta\,\sin\frac{2n+1}{2}\,\theta'\; H^{(1)}_{\frac{2n+1}{2}}(k\rho_>)\; J_{\frac{2n+1}{2}}(k\rho_<)\ .$$

We are now in a position to solve the problem of diffraction from a half-plane. Let the incoming field be e^{ikx} propagating towards increasing x. The half-plane is located in the plane $x = 0$, with $y \geq 0$. We use the coordinate system indicated in Fig. 1. The total field is to vanish for $x = 0$, $y \geq 0$.

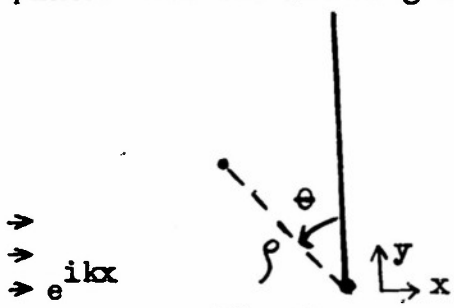

Fig. 1

We can think of e^{ikx} as being the limiting case of a point source removed to $x = -\infty$.

Indeed suppose there is a two-dimensional unit source at $\theta' = \frac{\pi}{2}$, $\rho = \rho'$.

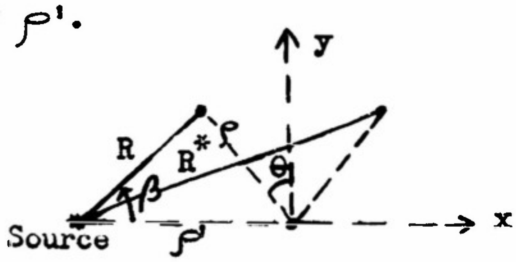

Fig. 2

The field of this point source is just $\frac{i}{4} H_0^{(1)}(kR)$.

We have:

$$R = \frac{(\rho' - \rho\sin\theta)}{\cos} = \frac{\rho' + x}{\cos}\ .$$

When ρ' is taken very large, the asymptotic expansion for $\frac{i}{4} H_o^{(1)}(kR)$ yields

$$\frac{i}{4}\sqrt{\frac{2}{\pi kR}}\; e^{-\frac{i\pi}{4}}\; e^{ikR} \quad .$$

To obtain the limit e^{ikx} as $\rho' \to \infty$, consider

$$\frac{4}{i}\sqrt{\frac{\pi k \rho'}{2}}\; e^{\frac{i\pi}{4}}\; e^{-ik\rho'} \left\{\frac{i}{4} H_o^{(1)}(kR)\right\} \quad .$$

As $\rho' \to \infty$, $\cos\beta \to 1$, $R - \rho' \to 0$, and it is easy to see that the limit of the above expression is e^{ikx}.

It is reasonable to suppose that performing the same limiting process on (3) will yield the diffraction a plane wave normally incident on a half plane. The result is derived rigorously in (b).

Thus, the total field is given by:

$$u(\rho,\theta) = \lim_{\rho'\to\infty} \frac{4}{i}\sqrt{\frac{\pi k\rho'}{2}}\; e^{\frac{i\pi}{4}}\; e^{-ik\rho'}\; \Big[\frac{i}{8}\left\{H_o^{(1)}(kR) - H_o^{(1)}(kR^*)\right\}$$

$$+ \frac{i}{2}\sum_{n=0}^{\infty} \sin\frac{2n+1}{2}\theta \,\sin\frac{2n+1}{2}\frac{\pi}{2}\, H^{(1)}_{\frac{2n+1}{2}}(k\rho_>)\, J_{\frac{2n+1}{2}}(k\rho_<)\Big] \quad .$$

The terms outside the summation yield $\dfrac{e^{ikx} - e^{-ikx}}{2}$.

In the summation, $\rho < \rho'$ and the asymptotic expansion for $H^{(1)}_{\frac{2n+1}{2}}(k\rho')$ is used:

$$H^{(1)}_{\frac{2n+1}{2}}(k\rho') \sim \sqrt{\frac{2}{\pi k\rho'}}\; e^{ik\rho'}\; e^{-in\pi} \quad .$$

We obtain

$$(4) \quad u(\rho,\theta) = i \sin kx + 2e^{-\frac{i\pi}{4}} \sum_{n=0}^{\infty} (i)^{-n} \sin \frac{2n+1}{2} \theta \sin \frac{2n+1}{2} \frac{\pi}{2} J_{\frac{2n+1}{2}}(k\rho) .$$

This formula can easily be identified with that of Magnus (1) who uses the time dependence e^{iwt} and a rotated coordinate system.

The results for the case where the normal derivative vanishes on the half-line are easily found:

$$(3a) \quad G(\rho,\theta;\rho',\theta') = \frac{i}{8}\left\{H_o^{(1)}(kR) + H_o^{(1)}(kR^*)\right\}$$

$$+ \frac{i}{2} \sum_{n=0}^{\infty} \cos \frac{2n+1}{2} \theta \cos \frac{2n+1}{2} \theta' \, H^{(1)}_{\frac{2n+1}{2}}(k\rho>) \, J_{\frac{2n+1}{2}}(k\rho<)$$

$$(4a) \quad u(\rho,\theta) = \cos kx + 2e^{-\frac{i\pi}{4}} \sum_{n=0}^{\infty} (i)^{-n} \cos \frac{2n+1}{2} \theta \cos \frac{2n+1}{2} \frac{\pi}{2} J_{\frac{2n+1}{2}}(k\rho) .$$

It is clear that the case of oblique incidence is easily handled by the same method.

(b) Consider the following two-dimensional boundary value problem:

$$\nabla^2 u + k^2 u = 0 \quad u(\rho,\theta+) = u(\rho,2\pi-) = \text{given function } f(\rho);$$

u is continuous throughout the plane; grad u is continuous everywhere except possibly on the half-line $\theta = 0$; u obeys a suitable condition at infinity.

Remark on the condition at infinity.

The condition at infinity is not the Sommerfeld radiation condition

$$\lim_{\rho \to \infty} \sqrt{\rho} \left(\frac{\partial u}{\partial \rho} - iku\right) = 0 .$$

In the case of diffraction by an obstacle having some infinite dimension, neither the total field nor the scattered field can be expected to satisfy the radiation condition.

As a simple example, consider total reflection of a plane wave by an infinite sheet.

The incident field is e^{ikx}.

The total field is $\begin{cases} e^{ikx} - e^{-ikx} & x < 0 \\ 0 & x > 0 \end{cases}$.

The "scattered" field is $- e^{ik|x|}$. None of these fields obeys the Sommerfeld condition. The same is true for diffraction by a semi-infinite sheet.

This is not fatal, however, since all we really need is

$$\lim_{\rho \to \infty} \int_S \left\{ H_0^{(1)}(kR) \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} H_0^{(1)}(kR) \right\} d\sigma = 0 . \tag{5}$$

The "scattered" field in the problem of total reflection obeys condition (5).

In general, the geometrical optics field (i.e., no diffraction) will satisfy (5) (see D. S. Jones (3)). Since the scattered field is expected to differ from the geometrical optics field only by a field satisfying the Sommerfeld condition, (5) will hold for the scattered field of the half-plane problem. In any event we adopt (5) as the imposed condition at infinity. It is clear that if (5) holds with the free space Green's function $H_0^{(1)}$, it would also hold with the Green's function G of part I.

Let us apply Green's theorem to u and the Green's function G given by (2). As the boundary of the region of integration is removed to infinity, we obtain (in virtue of (5)):

$$u(\rho',\theta') = \int_0^\infty d\rho \frac{f(\rho)}{\rho} \left\{ \frac{\partial u}{\partial \theta}\bigg|_{\theta=0+} - \frac{\partial u}{\partial \theta}\bigg|_{\theta=2\pi-} \right\}$$

$$= \int_0^\infty \frac{d\rho}{\rho} f(\rho) \left\{ \sum_{n=0}^{\infty} \frac{i}{2} (2n+1) \sin \frac{2n+1}{2} \theta' \, H^{(1)}_{\frac{2n+1}{2}}(k\rho >) \right. .$$

Under suitable conditions, this may be written as

$$(6) \quad u(\rho',\theta') = \frac{i}{2} \sum_{n=0}^{\infty} (2n+1) \sin \frac{2n+1}{2} \theta' \int_0^\infty d\rho \frac{f(\rho)}{\rho} H^{(1)}_{\frac{2n+1}{2}}(k\rho >) \, J_{\frac{2n+1}{2}}(k\rho <) .$$

At first, it might appear that $\lim\limits_{\theta' \to 0+} u(\rho',\theta') = 0$; this is not true. We will show that the integral in (6) is no better than $O(\frac{1}{n^2})$, hence the coefficient of $\sin \frac{2n+1}{2} \theta'$ is no better than $O(\frac{1}{n})$; the series will exhibit discontinuities at $\theta' = 0, 2\pi$, and $\lim\limits_{\theta' \to \left\{ \begin{smallmatrix} 0+ \\ 2\pi- \end{smallmatrix} \right.} u(\rho',\theta') = f(\rho')$.

Formula (6) gives the solution of our problem in a rather untractable form. Magnus (1) noted that integrals of the type (6) are considerably simplified when $f(\rho)$ admits a series development in Bessel functions of integral order. Let

$$I_{m,n} = \int_0^\infty \frac{J_m(k\rho)}{\rho} H^{(1)}_{\frac{2n+1}{2}}(k\rho >) \, J_{\frac{2n+1}{2}}(k\rho <) \, d\rho .$$

We split the integral into integrals from 0 to ρ' and ρ' to ∞. In each integral we let $t = k\rho$

$$I_{m,n} = \left. \frac{t\left\{ J'_m(t)\, J_{\frac{2n+1}{2}}(t) - J'_{\frac{2n+1}{2}}(t)\, J_m(t) \right\}}{m^2 - \left(\frac{2n+1}{2}\right)^2} \right]_0^{k\rho'} H^{(1)}_{\frac{2n+1}{2}}(k\rho')$$

$$+ \left. \frac{t\left\{ J'_m(t)\, H^{(1)}_{\frac{2n+1}{2}}(t) - H'^{(1)}_{\frac{2n+1}{2}}(t)\, J_m(t) \right\}}{m^2 - \left(\frac{2n+1}{2}\right)^2} \right]_{k\rho'}^{\infty} J_{\frac{2n+1}{2}}(k\rho') .$$

The contribution from 0 vanishes, the contribution at infinity is

$$\frac{2i}{\pi} \frac{e^{\frac{i\pi}{2}\left(m - \frac{2n+1}{2}\right)}}{m^2 - \left(\frac{2n+1}{2}\right)^2} .$$

If we use in addition the formula for the Wronskian of part I, we obtain

$$I_{m,n} = -\frac{2i}{\pi} \frac{e^{-\frac{i\pi}{4}}\,(i)^{m-n}}{m^2 - \left(\frac{2n+1}{2}\right)^2} J_{\frac{2n+1}{2}}(k\rho') + \frac{2i}{\pi} \frac{J_m(k\rho')}{m^2 - \left(\frac{2n+1}{2}\right)^2} .$$

For the case $f(\rho) = J_m(\rho)$ (6) reduces to:

$$u_m(\rho',\theta') = \frac{(i)^m e^{-\frac{i\pi}{4}}}{\pi} \sum_{n=0}^{\infty} (i)^{-n} \frac{(2n+1)\sin\frac{2n+1}{2}\theta'}{m^2 - \left(\frac{2n+1}{2}\right)^2} J_{\frac{2n+1}{2}}(k\rho')$$

$$- \frac{J_m(k\rho')}{\pi} \sum_{n=0}^{\infty} \frac{2n+1}{m^2 - \left(\frac{2n+1}{2}\right)^2} \sin\frac{2n+1}{2}\theta' .$$

The first of these Fourier series in θ' has coefficients $O(\frac{1}{n^2})$ due to the rapid decrease with n of $J_{\frac{2n+1}{2}}(k\rho')$. This series represents a continuous function which vanishes at $\theta' = 0, 2\pi$. Hence the limiting values as $\theta' \to 0+, 2\pi-$ are also zero.

The second series has coefficients $O(\frac{1}{n})$ and may exhibit a discontinuity at $\theta' = 0$. As a matter of fact it can be shown by an elementary computation that

$$-\pi \cos m\theta = \sum_{n=0}^{\infty} \frac{2n+1}{m^2 - (\frac{2n+1}{2})^2} \sin \frac{2n+1}{2} \theta' \quad .$$

Thus:

$$\lim_{\theta' \to \left\{ \begin{smallmatrix} 0+ \\ 2\pi - \end{smallmatrix} \right.} u_m(\rho',\theta') = J_m(k\rho')$$

as required. We can write:

$$u_m(\rho',\theta') = \frac{(i)^m e^{-\frac{i\pi}{4}}}{\pi} \sum_{n=0}^{\infty} (i)^{-n} \frac{(2n+1) \sin \frac{2n+1}{2} \theta'}{m^2 - (\frac{2n+1}{2})^2} J_{\frac{2n+1}{2}}(k\rho') + J_m(k\rho') \cos m\theta'.$$

In general if $f(\rho)$ admits an expansion $f(\rho) = \sum_{m=0}^{\infty} \beta_m J_m(k\rho)$ we will obtain

$$(7) \quad u(\rho',\theta') = \sum_{m=0}^{\infty} \beta_m J_m(k\rho') \cos m\theta'$$

$$+ \frac{e^{-\frac{i\pi}{4}}}{\pi} \sum_{m,n=0}^{\infty} \beta_m (i)^{m-n} (2n+1) \frac{\sin \frac{2n+1}{2} \theta'}{m^2 - (\frac{2n+1}{2})^2} J_{\frac{2n+1}{2}}(k\rho') \; .$$

It is now easy to solve the problem of diffraction by a half-plane. For simplicity we take normal incidence. The total field $W(\rho,\theta)$ may be split up in an incident field e^{ikx} and a scattered field $u(\rho,\theta)$ satisfying (5). The values of $u(\rho,\theta)$ on the half-plane are just $f(\rho) = -1$. Since $-1 = -\sum_{p=1}^{\infty} \epsilon_p J_{2p}(k\rho)$, we can use (7), with $\beta_m = -\epsilon_m (-i)^m \cos m\frac{\pi}{2}$ $(\epsilon_o = 1,\ \epsilon_m = 2,\ m > 1)$, thus obtaining

$$u(\rho',\theta') = \sum_{m=0}^{\infty} \epsilon_m J_{2m}(k\rho') \cos 2m\theta'$$

$$+ \sum_{n=0}^{\infty} \frac{e^{-\frac{i\pi}{4}}}{\pi} (i)^{-n} (2n+1) \sin \frac{2n+1}{2}\theta' \, J_{\frac{2n+1}{2}}(k\rho') \sum_{m=0}^{\infty} -\epsilon_m \frac{\cos m\frac{\pi}{2}}{m^2 - (\frac{2n+1}{2})^2} .$$

The first sum is recognized as $-\cos(k\rho' \sin\theta') = -\cos kx'$. The last sum can be evaluated by an elementary computation (see Magnus (1), p. 175)

$$\sum_{m=0}^{\infty} -\epsilon_m \frac{\cos m\frac{\pi}{2}}{m^2 - (\frac{2n+1}{2})^2} = \frac{2}{2n+1} \sin\left(\frac{2n+1}{2}\frac{\pi}{2}\right) .$$

Thus we have a single summation for $u(\rho',\theta')$ and we obtain

$$(8)\quad W(\rho,\theta) = i \sin kx + 2e^{-i\frac{\pi}{4}} \sum_{n=0}^{\infty} (i)^{-n} \sin \frac{2n+1}{2}\theta \sin \frac{2n+1}{2}\frac{\pi}{2} \cdot J_{\frac{2n+1}{2}}(k\rho)$$

which agrees with formula (4) of (a).

It has been shown that (8) can be transformed into the Sommerfeld solution (see Magnus (1), p. 179) and into the Kontorowich-Lebedev solution ((4), p. 241).

Part II.

We consider now the diffraction by a cone with a spherical tip. The problem of the cone without a tip has been treated by Sollfrey (5).

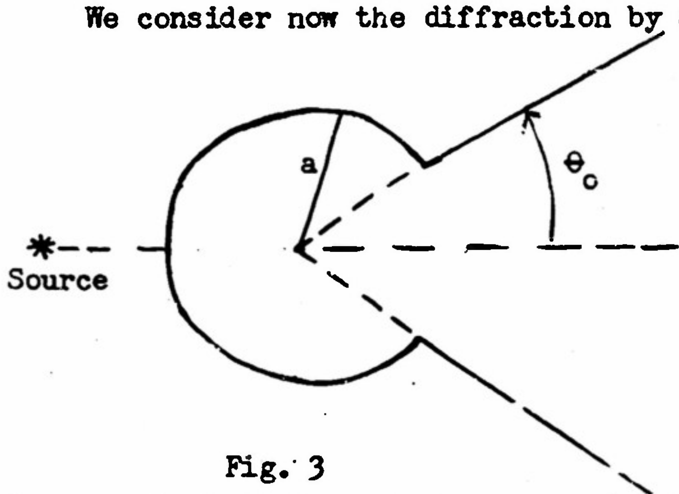

Fig. 3

The cone is infinite and the source is taken at $\theta = \pi$, $r = r'$. The problem can be solved with only slight additional difficulty for other locations of the source.

The total field satisfies

$$\nabla^2 u + k^2 u = \frac{-\delta(r - r')\,\delta(\theta)}{2\pi r^2 \sin\theta} \qquad \text{(the field is independent of } \varphi)$$

The boundary condition is taken as $u = 0$ on both the sphere and cone.

$$(9) \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + k^2 u = \frac{-\delta(r-r')\,\delta(\theta)}{2\pi r^2 \sin\theta} .$$

When $r \neq r'$, the homogeneous equation can readily be separated yielding the equation for the θ part of the separable solutions

$$\frac{1}{\sin_\theta} \frac{\partial}{\partial\theta} (\sin \theta w') + \lambda w = 0 \qquad \begin{array}{l} w(\theta_0) = 0 \\ w(\pi) \quad \text{finite} \end{array}$$

λ is a separation constant. If we let $\lambda = \nu(\nu+1)$, $x = \cos \theta$,

$$\frac{d}{dx} [(1-x^2) \frac{dw}{dx}] + \nu(\nu+1)w = 0 \qquad \begin{array}{l} w(-1) = \text{finite} \\ w(\cos \theta_0) = 0 \quad . \end{array}$$

The only solution which is finite at -1 is $P_\nu(-x)$, the Legendre function of the first kind.

The eigenvalues ν are determined from the equation $P_\nu(- \cos \theta_0) = 0$. The functions $P_\nu(- \cos \theta)$ from a complete orthogonal set in the interval $\theta_0 < \theta < \pi$, with weight $\sin \theta$. The normalization factor

$$\int_{\theta_0}^{\pi} P_\nu^2(- \cos \theta) \sin \theta \, d\theta = A_\nu$$

will be evaluated later.

We return to equation (9) and consider for u an expansion of the form

$$u = \sum_\nu a_\nu(r) \, P_\nu(- \cos \theta) \quad .$$

We multiply (9) by $r^2 \, P_\mu(- \cos \theta) \sin \theta$ and integrate from θ_0 to π . We obtain the differential equation for $a_\mu(r)$:

$$\text{(10)} \qquad \frac{d}{dr} (r^2 \, a_\mu') + k^2r^2a_\mu - \mu(\mu+1)a_\mu = \frac{- \delta(r-r')}{2\pi A_\nu}$$

The solution of (10) may be written as:

$$a_\mu(r) = C \frac{H_{\mu+1/2}(kr>)}{\sqrt{r>}} \frac{Z_{\mu+1/2}(kr<)}{\sqrt{r<}}$$

where $Z_{\mu+1/2}$ is a cylinder function which vanishes at $r = a$. We have taken care of the boundary conditions at $r = a$, $r = \infty$, and of continuity at $r = r'$. We determined C from the jump condition

$$a'_\mu(r'+) - a'_\mu(r'-) = -\frac{1}{2\pi r'^2}$$

$$a'_\mu(r'+) = C \frac{Z_{\mu+1/2}(kr')}{\sqrt{r'}} \cdot \left\{ \frac{k}{\sqrt{r'}} H'^{(1)}_{\mu+1/2}(kr') - H^{(1)}_{\mu+1/2}(kr') \frac{1}{2r'^{3/2}} \right\}$$

$$a'_\mu(r'-) = C \frac{H^{(1)}_{\mu+1/2}(kr')}{\sqrt{r'}} \cdot \left\{ \frac{k}{\sqrt{r'}} Z_{\mu+1/2}(kr') - Z_{\mu+1/2}(kr') \cdot \frac{1}{2r'^{3/2}} \right\} .$$

Hence

$$C \frac{k}{r'} \text{ Wronskian } \left\{ Z_{\mu+1/2}, H^{(1)}_{\mu+1/2} \right\} = -\frac{1}{2\pi r'^2 A_\nu} .$$

We write

$$Z_{\mu+1/2}(kr) = H^{(1)}_{\mu+1/2}(kr) - \frac{H^{(1)}_{\mu+1/2}(ka)}{H^{(2)}_{\mu+1/2}(ka)} H^{(2)}_{\mu+1/2}(kr) .$$

It is known that

$$\text{Wronskian } \left\{ H^{(2)}_\nu(kr), H^{(1)}_\nu(kr) \right\} = -\frac{4i}{\pi kr} .$$

Hence:

$$C = \frac{i}{8A_\nu} \frac{H^{(2)}_{\mu+1/2}(ka)}{H^{(1)}_{\mu+1/2}(ka)} .$$

We may now write the total field:

$$u(r,\theta) = \frac{i}{8} \sum_\nu \frac{P_\nu(-\cos\theta)}{A_\nu} \frac{H^{(2)}_{\nu+1/2}(ka)}{H^{(1)}_{\nu+1/2}(ka)} \frac{H^{(1)}_{\nu+1/2}(kr>)}{\sqrt{r>}} \frac{Z_{\nu+1/2}(kr<)}{\sqrt{r<}} .$$

The sum is taken over all ν such that $P_\nu(-\cos\theta_0) = 0$, and $Z_{\nu+1/2}$ is defined by (3).

We now consider the diffraction of a plane wave by our obstacle. This is to be obtained by removing the point source to infinity in a suitable manner. Using the symbols in figure 2 (with $\rho' = r'$), and letting the source go to $-\infty$ along the x axis:

$$\frac{e^{ikR}}{4\pi R} \sim \frac{e^{ikr'}\, e^{ikx}}{4\pi r'} .$$

The total field $v(r,\theta)$ from a plane wave c^{ikx} will be:

$$\lim_{r'\to\infty} \frac{4\pi r'}{e^{ikr'}} u(r,\theta)$$

$$= \lim_{r'\to\infty} \frac{4\pi r'}{e^{ikr'}} \sum_\nu \frac{P_\nu(-\cos\theta)}{A_\nu} \frac{H^{(2)}_{\nu+1/2}(ka)}{H^{(1)}_{\nu+1/2}(ka)} \frac{Z_{\nu+1/2}(kr)}{\sqrt{r}} \frac{H^{(1)}_{\nu+1/2}(kr')}{\sqrt{r'}} .$$

Asymptotically

$$H^{(1)}_{\nu+1/2}(kr') \sim \sqrt{\frac{2}{\pi kr'}}\, e^{i(kr' - \frac{\nu\pi}{2} - \frac{\pi}{2})} .$$

The total field is:

$$v(r,\theta) = \sum_{\nu} \frac{P_\nu(-\cos\theta)}{A_\nu} \frac{H^{(2)}_{\nu+1/2}(ka)}{H^{(1)}_{\nu+1/2}(ka)} \frac{Z_{\nu+1/2}(kr)}{\sqrt{r}} \cdot \sqrt{\frac{\pi}{2k}}\, e^{-\frac{i\nu\pi}{2}} . \tag{12}$$

A_ν may be obtained from (see Sollfrey (5)):

$$A_\nu = \int_{\theta_0}^{\pi} P_\nu^2(-\cos\theta) \sin\theta\, d\theta$$

$$= -\frac{1}{2\nu+1} \sin^2\theta_0 \left\{\frac{\partial}{\partial(-\cos\theta)} P_\nu(-\cos\theta)\right\}_{\theta=\theta_0} \left\{\frac{\partial}{\partial\nu} P_\nu(-\cos\theta)\right\}_{\theta=\theta_0} .$$

In the case $\theta_0 = \frac{\pi}{2}$, ν is determined from $P_\nu(0) = 0$. Hence ν is an arbitrary odd integer, $2n+1$

$$A_{2n+1} = \int_{\frac{\pi}{2}}^{\pi} P^2_{2n+1}(-\cos\theta) \sin\theta\, d\theta = \int_0^1 P^2_{2n+1}(x)\, dx = \frac{1}{4n+3} .$$

We have

$$v(r,\theta) = \sum_{n=0}^{\infty} (4n+3)\, P_{2n+1}(-\cos\theta) \frac{H^{(2)}_{2n+3/2}(ka)}{H^{(1)}_{2n+3/2}(ka)} \frac{Z_{2n+3/2}(ka)}{\sqrt{r}} \sqrt{\frac{\pi}{2k}}\, (-i)^n .$$

This was obtained by Twersky (7) using a combination of an image method and superposition.

REFERENCES

(1) W. Magnus, "Über die Beugung elektromagnetischer Wellen an einer Halbebene", Zeitschrift für Physik, 117, 1941, pp. 168-179.

(2) A. Sommerfeld, Math. Annalen, 47, 1896, pp. 317-374.

(3) D. S. Jones, "Removal of an Inconsistency in the Theory of Diffraction", Proceedings of the Cambridge Philosophical Society, 48, 1952, pp. 733-741.

(4) M. J. Kontorowich and N. N. Lebedex, "On a Method of Solution of Some Problems of the Diffraction Theory", Journal of Physics, U.S.S.R., vol. I, No. 3, 1939, pp. 229-241.

(5) W. Sollfrey, "Diffraction of Pulses by Conducting Wedges and Cones", N. Y. U., report EM-45, Math. Res Group.

(6) S. Karp, "Diffraction by a Tipped Wedge", N. Y. U., report EM-52, Math. Res Group.

(7) V. Twersky, J. Acoust. Soc. Am., 22, 539 (1950).

STANFORD UNIVERSITY

Technical Reports Distribution List
Contract Nonr 225(11)
(NR-041-086)

Chief of Naval Research Office of Naval Research Department of the Navy Washington 25, D. C. Attn: Code 432	5
Scientific Section Office of Naval Research Department of the Navy 1000 Geary Street San Francisco 9, California Attn: Dr. J. D. Wilkes	2 1
Director, Naval Research Lab. Washington 25, D. C. Attn: Technical Information Officer	9
Dr. Eugene Lukacs Head, Statistics Branch Office of Naval Research Department of the Navy Washington 25, D. C. Attn: Code 433	1
Planning Research Division Deputy Chief of Staff Comptroller, U. S. A. F. The Pentagon Washington 25, D. C.	1
Headquarters, U. S. A. F. Director of Research and Development Washington 25, D. C.	1
Chairman Research and Development Board The Pentagon Washington 25, D. C.	1
Asst. Chief of Staff, G-4 for Research and Development U. S. Army Washington 25, D. C.	1
Chief of Naval Operations Operations Evaluation Group-OP342E The Pentagon Washington 25, D. C.	1
Office of Naval Research Department of the Navy Washington 25, D. C. Attn: Code 438	2
Chief, Bureau of Ordnance Department of the Navy Washington 25, D. C. Attn: Re3d Re6a	1 1
Chief, Bureau of Aeronautics Department of the Navy Washington 25, D. C.	1
Chief, Bureau of Ships Asst. Chief for Research & Development Department of the Navy Washington 25, D. C.	1
Director, ONR Branch Office 495 Summer Street Boston 10, Mass.	1
Director, ONR Branch Office 844 N. Rush Street Chicago 11, Illinois	1
Director, ONR Branch Office 346 Broadway New York 13, N. Y.	1
Director, ONR Branch Office 1030 E. Green St. Pasadena 1, California	1
Office of the Asst. Naval Attaché for Research Naval Attaché American Embassy Navy No. 100 Fleet Post Office New York, N. Y.	10

Commander, USNOTS
Pasadena Annex
3202 E. Foothill Boulevard
Pasadena 8, California
Attn: Technical Library 1

Office of Ordnance Research
Duke University
2127 Myrtle Drive
Durham, North Carolina 1

Commanding Officer
Ballistic Research Laboratory
Aberdeen Proving Grounds
Aberdeen, Maryland
Attn: Mr. R. H. Kent 1

Director, David Taylor Model Basin
Washington 25, D. C.
Attn: L. Landweber 1
P. Eisenberg 1
Hydromechanics Lab. 1
Technical Library 1

Director, National Bureau of Standards
Washington, D. C.
Attn: Natl. Hydraulics Lab. 1

Commanding General
U. S. Proving Grounds
Aberdeen, Maryland 1

Commander
U. S. Naval Ordnance Test Station
Inyokern, China Lake, Calif. 1

N. A. C. A.
1724 F St., N.W.
Washington 25, D. C.
Attn: Chief, Office of Aeronautical Intelligence 1

Hydrodynamics Laboratory
National Research Laboratory
Ottawa, Canada 1

Director
Penn. State School of Engineering
Ordnance Research Laboratory
State College, Pa. 1

Statistical Laboratory
Dept. of Mathematics
University of California
Berkeley 4, California 1

Hydrodynamics Laboratory
California Inst. of Technology
1201 East California St.
Pasadena 4, California
Attn: Executive Committee 1

Commanding Officer
Naval Ordnance Laboratory
White Oak, Silver Spring 19, Md.
Attn: Technical Library 1

Ames Aeronautical Laboratory
Moffett Field
Mountain View, California
Attn: Technical Librarian 1

Mr. Samuel I. Plotnick
Asst. to the Director of Research
The George Washington University Research Laboratory
Area B, Camp Detrick
Frederick, Maryland 1

Mathematics Library
Syracuse University
Syracuse 10, N. Y. 1

University of Southern California
University Library
3518 University Avenue
Los Angeles 7, California 1

Library
California Institute of Technology
Pasadena 4, California 1

Engineering Societies Library
29 W. 39th Street
New York, N. Y. 1

John Crerar Library
Chicago 1, Illinois 1

National Bureau of Standards Library
3rd Floor North West Building
Washington 25, D. C. 1

Library
Mass. Inst. of Technology
Cambridge 39, Mass. 1

Louisiana State University Library
University Station
Baton Rouge 3, La. 1

Library
Fisk University
Nashville, Tennessee 1

Mrs. J. Henley Crosland
Director of Libraries
Georgia Institute of Technology
Atlanta, Georgia 1

Technical Report Collection
Rm. 303A, Pierce Hall
Harvard University
Cambridge 38, Mass. 1

Prof. L. V. Ahlfors
Dept. of Mathematics
Harvard University
Cambridge 38, Mass. 1

Prof. Peter G. Bergmann
Dept. of Physics
Syracuse University
Syracuse 10, N. Y. 1

Prof. G. Birkhoff
Dept. of Mathematics
Harvard University
Cambridge 38, Mass. 1

Prof. H. Busemann
Dept. of Mathematics
University of So. California
Los Angeles 7, California 1

Dr. F. H. Clauser
Dept. of Aeronautical Engineering
Johns Hopkins University
Baltimore 18, Maryland 1

Dr. Milton Clauser
Dept. of Aeronautical Engineering
Purdue University
Lafayette, Indiana 1

Dr. E. P. Cooper
U. S. Naval Shipyard
U. S. Navy Radiological Defense Laboratory
San Francisco, California 1

Prof. R. Courant
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. 1

Dr. A. Craya
Dept. of Aeronautical Engineering
Columbia University
New York 27, N. Y. 1

Dr. K. S. M. Davidson
Experimental Towing Tank
Stevens Inst. of Technology
711 Hudson Street
Hoboken, N. J. 1

Prof. R. J. Duffin
Dept. of Mathematics
Carnegie Inst. of Technology
Pittsburgh 13, Pa. 1

Dr. Carl Eckart
Scripps Inst. of Oceanography
La Jolla, California 1

Prof. A. Erdélyi
Dept. of Mathematics
California Institute of Technology
Pasadena 4, California 1

Prof. K. O. Friedrichs
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. 1

Prof. D. Gilbarg
Grad. Inst. for Applied Mathematics
Indiana University
Bloomington, Indiana 1

Prof. A. T. Ippen
Dept. of Civil and Sanitary Engineering
Mass. Inst. of Technology
Cambridge 39, Mass. 1

Prof. Samuel Karlin
Dept. of Mathematics
Calif. Inst. of Technology
Pasadena 4, California 1

Prof. J. R. Kline
Dept. of Mathematics
University of Pennsylvania
Philadelphia 4, Pa. 1

Dr. R. T. Knapp
Hydrodynamics Laboratory
California Inst. of Technology
Pasadena 4, California 1

Prof. P. A. Lagerstrom
Aeronautics Dept.
California Inst. of Technology
Pasadena 4, California 1

Prof. Benjamin Lepson
Mathematics Department
Catholic University of America
Washington 17, D. C. 1

Dr. Martin Lessen
Aeronautical Engineering Dept.
Penn. State College
State College, Pa. 1

Prof. H. G. Lew
Dept. of Aeronautical Engineering
Penn. State College
State College, Pa. 1

Prof. Hans Lewy
Dept. of Mathematics
University of California
Berkeley, California 1

Prof. C. C. Lin
Dept. of Mathematics
Mass. Inst. of Technology
Cambridge 39, Mass. 1

Prof. W. T. Martin
Dept. of Mathematics
Mass. Inst. of Technology
Cambridge 39, Mass. 1

Dean Paul E. Mohn
School of Engineering
The University of Buffalo
Buffalo, N. Y. 1

Prof. C. B. Morrey
Dept. of Mathematics
University of California
Berkeley 4, California 1

Prof. Z. Nehari
Dept. of Mathematics
Washington University
St. Louis, Mo. 1

Dr. M. S. Plesset
California Inst. of Technology
Pasadena 4, California 1

Prof. W. Prager
Dept. of Mathematics
Brown University
Providence 12, R. I. 1

Prof. P. C. Rosenbloom
Dept. of Mathematics
University of Minnesota
Minneapolis 14, Minn. 1

Dr. H. Rouse
State Inst. of Hydraulic Research
University of Iowa
Iowa City, Iowa 1

Prof. Charles Saltzer
Case Institute of Technology
Cleveland, Ohio 1

Prof. A. C. Schaeffer
Dept. of Mathematics
University of Wisconsin
Madison 6, Wisconsin 1

Prof. L. I. Schiff
Dept. of Physics
Stanford University
Stanford, California 1

Prof. W. Sears
Grad. School of Aeronautical Engineering
Cornell University
Ithaca, N. Y. 1

Prof. Herbert Solomon
Teachers College
Columbia University
New York 27, N. Y. 1

Prof. D. Spencer
Fine Hall, Box 708
Princeton, N. J. 1

Prof. J. J. Stoker
Inst. for Mathematics & Mechanics
New York University
New York 3, N. Y. 1

Dr. V. L. Streeter
Fundamental Mechanics Research
Illinois Inst. of Technology
Chicago 16, Illinois 1

Prof. C. Truesdell
Grad. Inst. for Applied Mathematics
Indiana University
Bloomington, Indiana 1

Prof. Joseph Ullman
Dept. of Mathematics
University of Michigan
Ann Arbor, Michigan 1

Prof. J. K. Vennard
Dept. of Civil Engineering
Stanford University
Stanford, California 1

Prof. Martin Vitousek
University of Hawaii
Honolulu 11, Hawaii 1

Prof. S. E. Warschawski
Dept. of Mathematics
University of Minnesota
Minneapolis 14, Minn. 1

Prof. A. Weinstein
Inst. for Fluid Dynamics &
Applied Mathematics
University of Maryland
College Park, Maryland 1

Prof. A. Zygmund
Dept. of Mathematics
The University of Chicago
Chicago 37, Illinois 1

Mathematics Dept.
University of Colorado
Boulder, Colorado 1

Los Angeles Engineering Field Office
Air Research and Development Command
5504 Hollywood Boulevard
Los Angeles 28, California
Attn: Capt. N. E. Nelson 1

Navy Department
Naval Ordnance Test Station
Underwater Ordnance Dept.
Pasadena, California
Attn: Dr. G. V. Schliestett
Code P8001 1

ASTIA, Western Regional Office
5504 Hollywood Blvd.
Hollywood 28, California 1

Armed Services Technical
Information Agency
Documents Service Center
Knott Building
Dayton 2, Ohio 5

Mr. R. T. Jones
Ames Aeronautical Laboratory
Moffett Field
Mountain View, California 1

Mr. J. D. Pierson
Glenn L. Martin Company
Middle River
Baltimore, Maryland 1

Mr. E. G. Straut
Consolidated-Vultee Aircraft
Corporation
Hydrodynamics Research Lab.
San Diego, California 1

Dr. George E. Forsythe
National Bureau of Standards
Institute for Numerical Analysis
University of California
405 Hilgard Avenue
Los Angeles 24, California 1

Additional copies for project leader and assistants and reserve for future requirements 50