PROBABILITY DISTRIBUTIONS RELATED TO RANDOM TRANSFORMATIONS OF A FINITE SET

BY

H. RUBIN AND R. SITGREAVES

TECHNICAL REPORT NO. 19A (final)

JANUARY 22, 1954

This work was sponsored by the Army, Navy, and Air Force through the Joint Services Advisory Committee for Research Groups in Applied Mathematics and Statistics by Contract No. N6onr 25140 (NR-342-022)

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

PROBABILITY DISTRIBUTIONS RELATED TO RANDOM TRANSFORMATIONS OF A FINITE SET

by

H. Rubin and R. Sitgreaves

1. Introduction. Let X be a set of n elements, and let $\mathcal{J}$ be the set of all transformations of X into X. In the definitions, the notation T^k, where $T \in \mathcal{J}$, has its usual meaning, with k a positive integer or zero. For $k = 0$, $T^0 x = x$.

For a given $x \in X$ and $T \in \mathcal{J}$, the set of all elements $y \in X$ such that

$$T^j x = T^k y$$

for at least one pair of numbers j, k, is called the structure in T containing x, and is denoted by $S_T(x)$. Trivially, $x \in S_T(x)$ since $T^j x = T^j x$ for all j. The number of elements in $S_T(x)$ is the size of the structure containing x.

An element $y \in X$ is called a cyclical element in T if $T^m y = y$ for some $m > 0$. A cyclical element y belongs to a cycle of length k, if

$$T^j y \neq y \qquad 0 < j < k$$

and

$$T^k y = y \quad .$$

It follows that each element

$$z = T^j y \qquad 0 < j < k$$

is also a cyclical element in T, belonging to the same cycle as y, since for any m

$$T^m z = T^m(T^j y) = T^{m+j} y = T^j(T^m y)$$

so that

$$T^m z = T^j(T^m y) \neq T^j y = z \quad \text{for} \quad 0 < m < k$$

and

$$T^k z = T^j(T^k y) = T^j y = z \quad .$$

Every structure $S_T(x)$ contains a non-empty subset of cyclical elements, say $K_T(x)$. Clearly, the elements of $K_T(x)$ represent a single cycle, since for any cyclical elements y, z, with $y \in S_T(x)$ and $z \in S_T(x)$, there exist numbers j, k, such that

$$T^j y = T^k z \quad .$$

An element $y \in S_T(x)$ is said to be a predecessor of x if there exists a $k \geq 0$ such that $T^k y = x$. An element y is said to be a successor of x if there exists a number $k \geq 0$ such that $T^k x = y$. The set of all successors of x is called the six-length of x . It will be noted that x is both a successor and a predecessor of itself. If $x \in K_t(x)$, every element in $S_T(x)$ is a predecessor of x , while the set $K_T(x)$ is the six-length of x .

Consider the following functions defined on $[x]$. Let

m = number of structures in T

c = size of the structure containing x

s = number of elements in the six-length of x

p = number of predecessors of x .

We assume that elements are selected at random from $X \times \mathcal{J}$, each pair (x, T) having probability $1/n^{n+1}$ of being chosen. Exact probability distributions are derived for the functions m, c, s, and p, and asymptotic approximations are given for the distributions of c, s, and p as n becomes large.

2. <u>Graphic representation of the set $\mathcal{J}$</u>. Let $\mathcal{J}^*$ be the subset of $\mathcal{J}$ consisting of all transformations of X onto X, i.e., $\mathcal{J}^*$ is the set of all permutations of the elements of X. Then, two elements of $\mathcal{J}$, say, T_i and T_j are said to belong to the same structure class, if for at least one $T^* \varepsilon \mathcal{J}^*$

$$(2.1) \qquad T_i = T_j \cdot T^* \quad .$$

It is clear that the maximum number of elements of $\mathcal{J}$ which can belong to the same structure class is $n!$.

A graphical presentation of the various structure classes when $n = 5$ is given in Figure 1. The number of elements of $\mathcal{J}$ belonging to each class is also given. The classes are grouped according to the value of m, the number of structures in the transformation, and $N_\emptyset$, the total number of cyclical elements.

3. <u>Auxiliary distributions</u>. Let D be some subset of X containing d elements, and let $\mathcal{J}_D$ be the subset of $\mathcal{J}$ containing all transformations T for which $D \supset TD$. For any pair (x, T) let ℓ denote the length of the cycle in $S_T(x)$. Then

$$(3.1) \qquad P\{K_T(x) \subset X - D,\ s = k,\ \ell = j\ (j \leq k) \mid D,\ T \varepsilon \mathcal{J}_D\}$$

$$= P\{x \notin D;\ Tx \neq x \text{ and } Tx \notin D;\ T^2x \neq x \text{ or } Tx,$$

and $T^2x \notin D$; ... ; $T^{k-1}x \neq x$, Tx, ... , or $T^{k-2}x$,

and $T^{k-1}x \notin D$; and $T^k x = T^{k-1}x \,|\, D \}$

$$= \frac{n-d}{n} \cdot \frac{n-d-1}{n} \cdots \frac{n-d-k+1}{n} \cdot \frac{1}{n}$$

$$= \frac{(n-d)!}{(n-d-k)!\; n^{k+1}} \quad .$$

Or

$$(3.2) \qquad P\left\{x \notin K_T(x)\ ,\ K_T(x) \subset X - D\ ,\ \ell = j \,\middle|\, D\ ,\ T \in \mathcal{J}_D\right\}$$

$$= \sum_{k=j+1}^{n-d} \frac{(n-d)!}{(n-d-k)!\; n^{k+1}}$$

for $j = 1, 2, \ldots, n-d$.

For any $T \in \mathcal{J}$, let N_D be the number of cyclical elements in $X - D$. Now, for any $T \in \mathcal{J}_n$ and any $x \notin D \cup K_T(x)$, either $K_T(x) \subset D$ or $K_T(x) \subset X - D$. Hence

$$(3.3) \qquad P\left\{x \notin K_T(x)\ ,\ K_T(x) \subset X - D\ ,\ \ell = j \,\middle|\, D\ ,\ T \in \mathcal{J}_D\ ,\ N_D = q\right\}$$

$$= P\left\{x \notin D \cup K_T(x) \,\middle|\, D\ ,\ N_D = q\right\}$$

$$\cdot P\left\{K_T(x) \subset X - D \,\middle|\, x \notin D \cup K_T(x)\ ,\ D\ ,\ N_D = q\right\}$$

$$\cdot P\left\{\ell = j \,\middle|\, x \notin D \cup K_T(x)\ ,\ K_T(x) \subset X - D\ ,\ D\ ,\ N_D = q\right\}$$

$$= \frac{n-d-q}{n} \cdot \frac{q}{d+q} \cdot \frac{1}{q}$$

$$= \frac{n-d-q}{n(d+q)} \quad .$$

Or

$$(3.4) \qquad P\{x \notin K_T(x),\ K_T(x) \subset X - D,\ \ell = j \mid D,\ T \in \mathcal{J}_D\}$$

$$= \sum_{q=j}^{n-d} \frac{n-d-q}{n(d+q)} P\{N_D = q \mid D,\ T \in \mathcal{J}_D\}$$

for $j = 1, 2, \ldots, n-d$.

Comparing (3.2) and (3.4) for $\ell = j$ and $\ell = j+1$, we obtain

$$(3.5) \qquad P\{x \notin K_T(x),\ K_T(x) \subset X - D,\ \ell = j \mid D,\ T \in \mathcal{J}_D\}$$

$$- P\{x \notin K_T(x),\ K_T(x) \subset X - D,\ \ell = j+1 \mid D,\ T \in \mathcal{J}_D\}$$

$$= \frac{n-d-j}{n(d+j)} P\{N_D = j \mid D,\ T \in \mathcal{J}_D\}$$

$$= \frac{(n-d)!}{(n-d-j-1)!\, n^{j+2}}.$$

It follows that

$$(3.6) \qquad P\{N_D = j \mid D,\ T \in \mathcal{J}_D\} = \frac{(n-d)!\,(d+j)}{(n-d-j)!\, n^{j+1}}$$

for $j = 1, 2, \ldots, n-d$.

It remains to evaluate $P\{N_D = 0 \mid D,\ T \in \mathcal{J}_D\}$. For this purpose, consider the function g of a real variable t, where

$$g(t) = \sum_{j=0}^{n-d} \frac{(n-d)!\,(d+j)\, t^{d+j}}{(n-d-j)!\, n^{j+1}}.$$

Let

$$f(t) = \sum_{j=0}^{n-d} \frac{(n-d)!\, t^j}{(n-d-j)!\, n^{j+1}}.$$

Then

$$g(t) = t^{d+1} f'(t) + d\, t^d f(t).$$

But

$$f'(t) = \sum_{j=0}^{n-d} \frac{(n-d)!\, j\, t^{j-1}}{(n-d-j)!\, n^{j+1}}$$

$$= (n-d) \sum_{j=0}^{n-d} \frac{(n-d)!\, t^{j-1}}{(n-d-j)!\, n^{j+1}} - \sum_{j=0}^{n-d-1} \frac{(n-d)!\, t^{j-1}}{(n-d-1-j)!\, n^{j+1}}$$

$$= \frac{(n-d)}{t} f(t) - \frac{n}{t^2} f(t) + \frac{1}{t^2}$$

so that

$$g(t) = n(t^d - t^{d-1})\, f(t) + \frac{1}{t^2}$$

and $g(1) = 1$. But

$$g(1) = \sum_{j=0}^{n-d} \frac{(n-d)!\,(d+j)}{(n-d-j)!\, n^{j+1}}$$

and

$$P\left\{N_D = 0 \middle| d\,,\ T \in \mathcal{J}_D\right\} = 1 - \sum_{j=0}^{n-d} \frac{(n-d)!\,(d+j)}{(n-d-j)!\, n^{j+1}} + \frac{d}{n}$$

so that

(3.7) $$P\left\{N_D = 0 \middle| D\,,\ T \in \mathcal{J}_D\right\} = \frac{d}{n} \quad .$$

If $D = \emptyset$ where $\emptyset$ is the empty set, $d = 0$ and $\mathcal{J}_\emptyset = \mathcal{J}$. For any $T \in \mathcal{J}$, $N_\emptyset$ is thus the total number of cyclical elements in X . From (3.6) and (3.7) we obtain

(3.8) $$P\left\{N_\emptyset = j\right\} = \frac{(n-1)!\, j}{(n-j)!\, n^j} \qquad j = 1, 2, \ldots, n \quad .$$

4. **Distribution of the number of structures.** We can write

$$(4.1) \qquad P\{m=i\} = \sum_{j=i}^{n} P\{m=i,\ N_{\phi} = j\}$$

$$= \sum_{j=i}^{n} P\{m=i \mid N_{\phi}=j\} \quad P\{N_{\phi}=j\}$$

But

$P\{m=i \mid N_{\phi}=j\}$ = Prob. of i cycles in a permutation of j elements

$= \alpha(i,j)$, say

we have

$$\alpha(1,1) = 1$$

$$\alpha(1,2) = \frac{1}{2} \qquad \alpha(2,2) = \frac{1}{2}$$

$$\alpha(1,3) = \frac{1}{3} \qquad \alpha(2,3) = \frac{1}{2} \qquad \alpha(3,3) = \frac{1}{6}$$

In general

$$\alpha(i,j) = \frac{j-1}{j}\alpha(i,j-1) + \frac{1}{j}\alpha(i-1,j-1)$$

for $i \leq j$. Clearly $\alpha(i,j) = 0$ for $i > j$. It will be noted that for each j, $\alpha(1,j) = \frac{1}{j}$ and $\alpha(j,j) = \frac{1}{j!}$. Also, we have

$$\alpha(i,j) = \text{coeff. of } t^i \text{ in } \frac{\Gamma(t+j)}{\Gamma(t)j!}$$

The distribution of cycles in permutations of a finite number of elements has also been considered by Gontcharoff (1).

It follows that

$$(4.2) \qquad P\{m=i\} = \sum_{j=i}^{n} \frac{(n-1)!\ j}{(n-j)!\ n^j}\ \alpha(i,j) \quad .$$

Values of $\alpha(i,j)$ for $i,j = 1,2,\ldots,25$, $i \leq j$, are given in Table 1.

5. <u>Distribution of structure size.</u> Let X_j be a subset of X containing j elements. Also, let $\mathcal{J}^{(j)}$ be the set of all transformations of X_j into itself, and for any $T \varepsilon \mathcal{J}^{(j)}$, let $m^{(j)}$ denote the number of structures in T.

For any pair (x, T) selected at random from $X \times \mathcal{J}$, the size of the structure containing x is the number of elements in $S_T(x)$. Then the probability that a picked structure has size j is given by

$$(5.1) \qquad P\{c = j\} = (\text{Number of ways in which } X_j \text{ can be chosen})\cdot$$

$$P\{X_j \text{ forms the picked structure}\}$$

$$= \binom{n}{j} P\{x \varepsilon X_j\} \cdot P\{TX_j \subset X_j\} \cdot P \cdot \{T(X-X_j) \subset X-X_j\}$$

$$\cdot P\{m^{(j)} = 1\}$$

$$= \binom{n}{j} \frac{j}{n} \cdot \left(\frac{j}{n}\right)^j \left(\frac{n-j}{n}\right)^{n-j} P\{m^{(j)} = 1\} .$$

From (4.2) we have

$$P\{m^{(j)} = 1\} = \sum_{\ell=1}^{j} \frac{(j-1)!\,\ell}{(j-\ell)!\, j^{\ell}} \alpha(1, \ell)$$

$$= \sum_{\ell=1}^{j} \frac{(j-1)!}{(j-\ell)!\, j^{\ell}} .$$

It follows that

$$(5.2) \qquad P\{c=j\} = \frac{(n-1)!}{(n-j)!\,(j-1)!} \frac{j^j (n-j)^{n-j}}{n^n} \sum_{\ell=1}^{j} \frac{(j-1)!}{(j-\ell)!\, j^{\ell}} .$$

6. <u>Distribution of six-lengths.</u> We obtain the joint distribution of six-length and cycle length directly from (3.1) by letting $D = \emptyset$. Thus

$$(6.1) \qquad P\left\{s=k,\ \hat{\lambda}=j\ (j\leq k)\right\} = \frac{(n-1)!}{(n-k)!\, n^k}$$

It follows that

$$(6.2) \qquad P\left\{s=k\right\} = \frac{(n-1)!\, k}{(n-k)!\, n^k}$$

and

$$(6.3) \qquad P\left\{\hat{\lambda}=j\right\} = \sum_{k=j}^{n} \frac{(n-1)!}{(n-k)!\, n^k}$$

The expected six-length is given by

$$(6.4) \qquad E(s) = \sum_{k=1}^{n} \frac{(n-1)!\, k^2}{(n-k)!\, n^k}$$

Writing $k^2 = n^2 - n(n-k) - (k+1)(n-k) + (n-k)$, this becomes

$$(6.5) \qquad E(s) = n\left\{\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^{k-1}} - \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k-1)!\, n^{k}}\right.$$

$$\left. - \sum_{k=1}^{n-1} \frac{(n-1)!\,(k+1)}{(n-k-1)!\, n^{k+1}} + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k-1)!\, n^{k+1}}\right\}$$

$$= n\left\{\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^{k-1}} - \sum_{k=2}^{n} \frac{(n-1)!}{(n-k)!\, n^{k-1}}\right.$$

$$\left. - \sum_{k=1}^{n} \frac{(n-1)!\, k}{(n-k)!\, n^{k}} + \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^{k}}\right\}$$

$$= n\left\{1 - 1 + \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^{k}}\right\}$$

$$= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^{k-1}} .$$

It is of interest to note that

$$E\left(\frac{1}{s}\right) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\, n^k} = \frac{1}{n} E(s) \tag{6.7}$$

7. Distribution of predecessors. Let x be a given element in X, and now let $X_j = x \cup X^*_{j-1}$ where X^*_{j-1} is a subset of $X-x$ containing $(j-1)$ elements. As before, let $\mathcal{J}^{(j)}$ be the set of all transformations of X_j into itself, and let $\mathcal{J}^{(j)}_x$ be the subset of $\mathcal{J}^{(j)}$ consisting of all transformations T for which $Tx = x$. For any $T \varepsilon \mathcal{J}^{(j)}$, let $N^{(j)}_x$ be the number of cyclical elements in $X_j - x$, i.e., in X^*_{j-1}. Then the probability that x has j predecessors, where x is counted as a predecessor of itself, is given by

$$\begin{aligned} P\{p = j|x\} &= \text{Number of ways in which } X^*_{j-1} \text{ can be chosen.} \\ &\quad P\{T X^*_{j-1} \subset X_j\} \cdot P\{T(X - X_j) \subset X - X_j\} \\ &\quad \cdot P\{N^{(j)}_x = 0 | x,\ T \varepsilon \mathcal{J}^{(j)}_x\} \\ &= \binom{n-1}{j-1}\left(\frac{j}{n}\right)^{j-1}\left(\frac{n-j}{n}\right)^{n-j} \cdot \frac{1}{j} \\ &= \frac{(n-1)!\, j^{j-2}\, (n-j)^{n-j}}{(n-j)!\, (j-1)!\, n^{n-1}} \end{aligned} \tag{7.1}$$

It follows that

$$\begin{aligned} P\{p=j\} &= \sum_{k=1}^{n} \frac{1}{n} \left(\frac{(n-1)!\, j^{j-2}\, (n-j)^{n-j}}{(n-j)!\, (j-1)!\, n^{n-1}}\right) \\ &= \frac{(n-1)!\, j^{j-2}\, (n-j)^{n-j}}{(n-j)!\, (j-1)!\, n^{n-1}} \ . \end{aligned} \tag{7.2}$$

8. Distribution of m, c, s, and p when n = 5. The probability distributions of m, c, s, and p when n = 5 are as follows:

k	$P\{m = k\}$	$P\{c = k\}$	$P\{s = k\}$	$P\{p = k\}$
1	$\frac{1569}{3125}$	$\frac{256}{3125}$	$\frac{125}{625}$	$\frac{256}{625}$
2	$\frac{1220}{3125}$	$\frac{324}{3125}$	$\frac{200}{625}$	$\frac{108}{625}$
3	$\frac{305}{3125}$	$\frac{408}{3125}$	$\frac{180}{625}$	$\frac{72}{625}$
4	$\frac{30}{3125}$	$\frac{568}{3125}$	$\frac{96}{625}$	$\frac{64}{625}$
5	$\frac{1}{3125}$	$\frac{1569}{3125}$	$\frac{24}{625}$	$\frac{125}{625}$

These distributions are shown graphically in Figure 2.

9. Asymptotic expansion of $\sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!\, k^j}$. An asymptotic expansion for the quantity $\sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!\, k^j}$ which appears in the probability distribution of structure size as well as the expression for the expected six-length, can be obtained as follows: We have

$$(9.1) \qquad \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!\, k^j} = \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-1-j)!\, k^{j+1}}$$

$$= \frac{1}{k} \int_0^{\infty} e^{-x} \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-1-j)!\, j!} \left(\frac{x}{k}\right)^j dx$$

$$= \frac{1}{k} \int_0^{\infty} e^{-x} \left(1 + \frac{x}{k}\right)^{k-1} dx$$

Let $x = k(e^y - 1)$, $dx = ke^y\,dy$. It follows that

$$\frac{1}{k}\int_0^\infty e^{-x}\left(1+\frac{x}{k}\right)^{k-1} dx = \int_0^\infty e^{-k(e^y-1-y)}\,dy$$

$$= \int_0^\infty e^{-k\sum_{j=2}^{\infty} y^j/j!}\,dy .$$

Now let $y = (2u/k)^{\frac{1}{2}}$, $dy = (1/(2ku))^{\frac{1}{2}}\,du$

so that

$$(9.2)\qquad \int_0^\infty e^{-k\sum_{j=2}^{\infty} y^j/j!}\,dy = \int_0^\infty \frac{1}{(2ku)^{\frac{1}{2}}}\, e^{-u-\frac{\sqrt{2}u^{3/2}}{3k^{1/2}}-\frac{u^2}{6k}-\frac{\sqrt{2}u^{5/2}}{30k^{3/2}}\cdots}\,du$$

$$= \frac{1}{(2k)^{1/2}}\int_0^\infty e^{-u}u^{-1/2}\sum_{j=0}^{\infty}\frac{a_j(u)}{k^j}\,du$$

$$= \frac{1}{3k}\int_0^\infty e^{-u}u\sum_{j=0}^{\infty}\frac{b_j(u)}{k^j}\,du$$

where $a_0(u) = 1$

$a_1(u) = u^3/9 - u^2/6$

$a_2(u) = u^6/486 - u^5/54 + 13u^4/360 - u^3/90$

$a_3(u) = u^9/65.610 - u^8/2.916 + 23u^7/9.720 - 37u^6/6.480$

$+ 76u^5/9.450 - u^4/2.520$

. . . .

$b_0(u) = 1$

$b_1(u) = u^3/27 - u^2/6 + u/10$

$b_2(u) = u^6/2.430 - u^5/162 + u^4/40 - u^3/36 + u^2/210$

. . . .

Consider the integral

$$\int_0^\infty e^{-k(e^{-y}-1+y)}\,dy = \int_0^\infty e^{-k\sum_{j=2}^{\infty}(-1)^j y^j/j!}\,dy$$

$$= \int_0^\infty \frac{1}{(2ku)^{1/2}}\, e^{-u + \frac{\sqrt{2}u^{3/2}}{3k^{1/2}} - \frac{u^2}{6k} + \frac{\sqrt{2}u^{5/2}}{30k^{3/2}} \cdots}\,du$$

$$= \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u}\, u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j}\,du$$

$$+ \frac{1}{3k} \int_0^\infty e^{-u}\, u \sum_{j=0}^{\infty} \frac{b_j(u)}{k^j}\,du \quad .$$

Or

$$\frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u}\, u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j}\,du$$

$$= \frac{1}{2} \int_0^\infty \left(e^{-k(e^{y}-1-y)} + e^{-k(e^{-y}-1+y)} \right) dy$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{-k(e^{y}-1-y)}\,dy$$

Let $y = \log\ t \qquad dy = \frac{dt}{t}$

Or

$$\frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u}\, u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j}\,du = \frac{1}{2} e^k \int_0^\infty e^{-kt}\, t^{k-1}\,dt$$

$$= \frac{1}{2}\,(e/k)^k\, \Gamma(k) \quad .$$

Thus we obtain

$$(9.3)\qquad \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!\, k^j} \sim \frac{e^k\,\Gamma(k)}{2k^k} - \frac{1}{3k} - \frac{4}{135k^2} + \frac{8}{2.835k^3} \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

10. <u>Asymptotic approximation for the distribution of structure size.</u> We have (cf. 5.2)

$$(10.1)\qquad P\{c = k\} = \frac{(n-1)!}{(n-k)!\,(k-1)!}\;\frac{k^k\,(n-k)^{n-k}}{n^n}\;\sum_{j=1}^{k}\frac{(k-1)!}{(k-j)!\,k^j}\ .$$

The mode of the distribution is at $k = n$. We have

$$(10.2)\qquad P\{c = n\} = \sum_{j=1}^{n}\frac{(n-1)!}{(n-j)!\,n^j}\ .$$

An approximation for this expression is given in the preceding section.

Except at the tails, i.e., if neither k nor $n-k$ is small, we have

$$(10.3)\qquad P\{c = k\} \sim \frac{e^{-n}\,n^{n-1/2}\,(2\pi)^{1/2}\,k^k\,(n-k)^{n-k}}{e^{-(n-k)-k}\,(n-k)^{n-k+1/2}\,k^{k-1/2}\,(2\pi)\,n^n}\sqrt{\frac{\pi}{2k}}$$

$$= \frac{1}{2n^{1/2}\,(n-k)^{1/2}}\ .$$

The asymptotic density of $x = \frac{c}{n}$ is given by

$$p(x) = \frac{1}{2(1-x)^{1/2}} \qquad\qquad 0 \le x \le 1\ .$$

11. <u>Asymptotic approximation for the distribution of six-lengths</u>, we have (cf. 6.2)

$$(11.1)\qquad P\{s = k\} = \frac{(n-1)!\,k}{(n-k)!\,n^k}\ .$$

The mode of the distribution is at $k \sim \sqrt{n}$. Let $k = \sqrt{n}\, x$. Then, as n becomes large

$$(11.2) \qquad P\left\{s = \sqrt{n}\, x\right\} \sim \frac{e^{-n}\, n^{n - 1/2}\, (2\pi)^{1/2}\, \sqrt{n}\, x}{e^{-(n - \sqrt{n}\, x)} (n - \sqrt{n}\, x)^{n - \sqrt{n}\, x + 1/2} (2\pi)^{1/2}\, n^{\sqrt{n}\, x}}$$

$$= \frac{x}{e^{\sqrt{n}\, x}\, n^{1/2} \left(1 - \frac{x}{\sqrt{n}}\right)^{n - \sqrt{n}\, x + 1/2}}$$

$$\sim \frac{x}{e^{\sqrt{n}\, x}\, n^{1/2} \left(1 - \frac{x}{\sqrt{n}}\right)^{n} \left(1 - \frac{x}{\sqrt{n}}\right)^{-\sqrt{n}\, x}}$$

$$\sim \frac{x}{e^{\sqrt{n}\, x}\, n^{1/2}\, e^{-\sqrt{n}\, x - 1/2\, x^2} \cdot e^{x^2}}$$

$$= \frac{x\, e^{-(1/2)x^2}}{\sqrt{n}}$$

The asymptotic density of $\frac{s}{\sqrt{n}}$ is thus

$$(11.3) \qquad P(x) = x\, e^{-(1/2)x^2} \quad .$$

12. Asymptotic approximation for the distribution of predecessors. The distribution of the number of predecessors (cf. 7.2) is

$$(12.1) \qquad P\left\{p = k\right\} = \frac{(n-1)!\, k^{k-2}\, (n-k)^{n-k}}{(n-k)!\, (k-1)!\, n^{n-1}} \quad .$$

This is a U-shaped distribution with an antimode at $(3/4)n$. The probability that $p = n$ is given by

$$P\left\{p = n\right\} = \frac{1}{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad .$$

As n becomes large and k remains small relative to n, we have, using Stirlings approximations for $(n-1)!$ and $(n-k)!$

$$(12.2) \qquad \lim_{n\to\infty} P\left\{ \; = k\right\} = \frac{k^{k-2}\, e^{-k}}{(k-1)!}$$

As k increases, but is still small relative to n, we have

$$(12.3) \qquad P\left\{ p = k\right\} \sim \frac{1}{\sqrt{2\pi}\; k^{3/2}}$$

Values of the limiting probability that $p > k$ have been computed for some selected values of k and n. These values follow

$P\left\{p > k\right\}$
(in percent)

k	n = 5	n = ∞
1	59.04	63.21
2	41.76	49.68
3	30.24	42.21
4	20.00	37.33
5	0	33.82

	n = ∞
10	24.55
25	15.782
100	7.957
250	5.041
1,000	2.522
2,500	1.5956
10,000	.7979
25,000	.5046

$P\left\{\frac{h}{p} > k\right\}$
(in percent)

k	$n = 10^8$	$n = 10^9$	$n = 10^{10}$	$n = \infty$
10^5	.2522	.2523	.2523	.2523
2.5×10^5	.15938	.15956	.159578	.15958
10^6	.07939	.07975	.079786	.07979
2.5×10^6	.04983	.05040	.050454	.05046
10^7	.02394	.02510	.025217	.02523

13. Case of each element having 0 or k predecessors. Let $\mathcal{J}$ be the set of transformations of X into X such that each element $x \in X$ has either no immediate predecessors or exactly k immediate predecessors, where k is a given number. For a given $T \in \mathcal{J}$, let A_0 be the set of elements with no immediate predecessors, and $A_k = X - A_0$ the set with k immediate predecessors. Also, let a_0 and $a_k = n - a_0$ be the number of elements in A_0 and A_k, respectively. Since $TX = A_k$, we must have $n = ka_k$; that is, for the set $\mathcal{J}$ to be non-empty, n must be a multiple of k, say, rk, and $a_k = r$ for all $T \in \mathcal{J}$.

The total number of transformations in $\mathcal{J}$ is $\varphi(r,k)$, say, where

(13.1) $\varphi(r,k)$ = (number of ways in which the sets A_0 and A_k can be chosen) $\times$ (number of transformations mapping X onto A_k so that each element of A_k has exactly k predecessors)

$$= \frac{(rk)!\,(rk)!}{r!\,(r(k-1))!\,k!^r} .$$

Clearly, the probability that any pair (x,T) will be chosen is

(13.2) $$\frac{r!\,(r(k-1))!\,k!^r}{(rk)!\,(rk)!\,rk} .$$

A procedure similar to that used in the general case is again useful in deriving exact distributions for the functions m, c, s, and p. Let D be a given subset of X containing d elements, and let $\mathcal{J}_D$ be the subset of $\mathcal{J}$ containing all transformations T for which $D \supset TD$. Let $\mathcal{J}_D^{(TD)}$ be the subset of $\mathcal{J}_D$ for which TD is the specified transformation on D. For any $T \in \mathcal{J}_D^{(TD)}$, let $A_k^{(TD)}$ be the subset of D specified by TD as belonging to A_k; i.e., $TD = A_k^{(TD)}$ for all $T \in \mathcal{J}_D^{(TD)}$. Suppose that for a given TD, $A_k^{(TD)}$ contains ν elements, and suppose we know D and TD only. Clearly, $\nu \leq \min(d,r)$ and $d \leq k\nu$, so that $k\nu - d$ predecessors for the elements of $A_k^{(TD)}$ remain to be chosen from X-D. Moreover, if $d \leq r$, any of the $(d-\nu)$ elements of $D-A_k^{(TD)}$, for which no predecessor is specified by TD, may also be in A_k, so that there are altogether

$$k\nu - d + k(d-\nu) = (k-1)d \tag{13.3}$$

possible immediate predecessors for elements of D in X-D. In case $d > r$, as many as $(r-\nu)$ of the $(d-\nu)$ elements of $D-A_k^{(TD)}$ may also be in A_k, so that there are altogether

$$k\nu - d + k(r-\nu) = rk-d \tag{13.4}$$

possible immediate predecessors for elements of D in X-D. In either case, the number of possible immediate predecessors for elements of D in X-D depends only on d, and not on the particular TD selected,. We can write

$$d = k(r-a)-b \quad \text{or} \quad kr-d = ka+b \tag{13.5}$$

where b is given by (13.3) if $d \leq r$ and by (13.4) if $d > r$.

Now

$$\begin{aligned}
&P\{x\in A_o, K_T(x)\subset X\text{-}D,\ s=j,\ \ell=q,\ q<j \mid D, T\in\mathcal{T}_D\} \\
&= P\{x\in A_o, x\notin D; Tx\neq x \text{ and } Tx\notin D; T^2x\neq x \text{ or } Tx, \text{ and} \\
&\quad T^2x\notin D;\ldots;T^{j-1}x\neq x, Tx,\ldots, \text{or } T^{j-1}x, \text{ and } T^{j-1}x\notin D; \\
&\quad \text{and } T^jx = T^{j-q}x \mid D, T\in\mathcal{T}_D\} \\
&= \frac{(k-1)a+b}{ka+b}\cdot\frac{ka}{ka+b-1}\cdot\frac{k(a-1)}{ka+b-2}\cdots\frac{k(a-j+2)}{ka+b-j+1}\cdot\frac{(k-1)}{ka+b} \\
&= \frac{((k-1)a+b)(k-1)k^{j-1}a!(ka+b-j)!}{(ka+b)(ka+b)!(a-j+1)!}
\end{aligned} \tag{13.6}$$

Also

$$\begin{aligned}
&P\{x\in A_k, K_T(x)\subset X\text{-}D, s=j,\ \ell=q,\ q<j \mid D, T\in\mathcal{T}_D\} \\
&= \frac{(k-1)k^{j-1}a!(ka+b-j)!}{(ka+b)(ka+b)!(a-j)!}
\end{aligned} \tag{13.7}$$

and

(13.8) $$P\{x \in A_k, K_T(x) \subset X-D, s = j, \ell = j \mid D, T \in \mathcal{J}_D\}$$

$$= \frac{k^j a!(ka+b-j)!}{(ka+b)(ka+b)!(a-j)!} \cdot P\{K_T(x) \subset X-D, s = j, \ell = j \mid D, T \in \mathcal{J}_D\}$$

$$1 \leq j \leq a$$

so that

(13.9) $$P\{K_T(x) \subset X-D, s = j, \ell = q, q < j \mid D, T \in \mathcal{J}_D\}$$

$$= \frac{(k-1)k^{j-1} a!(ka+b-j+1)!}{(ka+b)(ka+b)!(a-j+1)!} \qquad 2 \leq j \leq a+1$$

and

(13.10) $$P\{x \notin K_T(x), K_T(x) \subset X-D, \ell = q \mid D, T \in \mathcal{J}_D\}$$

$$= \sum_{j=q+1}^{a+1} \frac{(k-1)k^{j-1} a!(ka+b-j+1)!}{(ka+b)(ka+b)!(a-j+1)!} \quad .$$

As before, for any $T \in \mathcal{J}_D$, let N_D be the number of cyclical elements in X-D. Then, we have

(13.11) $$P\{x \notin K_T(x), K_T(x) \subset X-D, \ell = q \mid D, T \in \mathcal{J}_D, N_D = j\}$$

$$= P\{x \notin D \cup K_T(x) \mid D, T \in \mathcal{J}_D, N_D = j\}$$

$$\cdot P\{K_T(x) \subset X-D \mid x \notin D \cup K_T(x), D, T \in \mathcal{J}_D, N_D = j\}$$

$$\cdot P\{\ell = j \mid x \notin D \cup K_T(x), K_T(x) \subset X-D, D, T \in \mathcal{J}_D, N_D = j\}$$

$$= \frac{(ka+b-j)}{ka+b} \cdot \frac{(k-1)j}{(k-1)j+b} \cdot \frac{1}{j} = \frac{(k-1)(ka+b-j)}{(ka+b)((k-1)j+b)} \quad .$$

Or

(13.12) $$P\{x \notin K_T(x), K_T(x) \subset X-D, \ell = q \mid D, T \in \mathcal{J}_D\}$$

$$= \sum_{j=q}^{a} \frac{(k-1)(ka+b-j)}{(ka+b)((k-1)j+b)} P\{N_D = j \mid D, T \in \mathcal{J}_D\} \quad .$$

Comparing (13.10) and (13.12) for $\ell = q$ and $\ell = q+1$, we obtain

$$(13.13)\quad P\left\{x \notin K_T(x), K_T(x) \subset X-D,\ \ell = q \mid D, T \in \mathcal{T}_D\right\} - P\left\{x \notin K_T(x), K_T(x) \subset X-D,\ \ell = q+1 \mid D, T \in \mathcal{T}_D\right\}$$

$$= \frac{(k-1)(ka+b-q)}{(ka+b)((k-1)q+b)}\ P\left\{N_D = q \mid D, T \in \mathcal{T}_D\right\}$$

$$= \frac{(k-1)k^q a!(ka+b-q)!}{(ka+b)(ka+b)!(a-q)!}\ .$$

Or

$$(13.14)\quad P\left\{N_D = q \mid D, T \in \mathcal{T}_D\right\} = \frac{((k-1)q+b)k^q a!(ka+b-q-1)!}{(ka+b)!(a-q)!}$$

for $1 \leq q \leq a$. Since

$$(13.15)\quad \sum_{q=0}^{a} \frac{((k-1)q+b)k^q a!(ka+b-q-1)!}{(ka+b)!(a-q)!}$$

$$= \sum_{q=0}^{a} \frac{k^q a!(ka+b-q-1)!}{(ka+b)!(a-q)!}(ka+b-q-k(a-q))$$

$$= \sum_{q=0}^{a} \frac{k^q a!(ka+b-q)!}{(ka+b)!(a-q)!} - \sum_{q=0}^{a-1} \frac{k^{q+1} a!(ka+b-q-1)!}{(ka+b)!(a-q-1)!}$$

$$= \sum_{q=0}^{a} \frac{k^q a!(ka+b-q)!}{(ka+b)!(a-q)!} - \sum_{q=1}^{a} \frac{k^q a!(ka+b-q)!}{(ka+b)!(a-q)!} = 1$$

we have

$$(13.16)\quad P\left\{N_D = 0 \mid D, T \in \mathcal{T}_D\right\} = \frac{b}{ka+b}\ .$$

If $D = \phi$, where ϕ is the empty set, $\mathcal{T}_\phi = \mathcal{T}$, and $b = 0$, $a = r$. For any $T \in \mathcal{T}$, N_ϕ is the total number of cyclical elements in X. Hence

$$(13.17)\quad P\left\{N_\varphi = j\right\} = \frac{k^{j-1} j(k-1)(r-1)!(kr-j-1)!}{(kr-1)!(r-j)!} \qquad 1 \leq j \leq r\ .$$

The distribution of the number of structures is given by

$$(13.18)\quad P\{m = i\} = \sum_{j=1}^{r} P\left\{m = i, N_\varphi = j\right\}$$

$$= \sum_{j=1}^{r} P\left\{m = i \mid N_\varphi = j\right\} P\left\{N_\varphi = j\right\}$$

$$= \sum_{j=1}^{r} \frac{k^{j-1} j(k-1)(r-1)!(kr-j-1)!}{(kr-1)!(r-j)!}\ \alpha(i,j) \qquad 1 \leq i \leq r$$

where $\alpha(i,j)$ is the probability of i cycles in a permutation of j elements. Since $\alpha(1,j) = 1/j$

$$(13.19) \qquad P\{m = 1\} = \sum_{j=1}^{r} \frac{k^{j-1}(k-1)(r-1)!(kr-j-1)!}{(kr-1)!(r-j)!}$$

For any pair (x,T), $S_T(x)$ has been defined as the structure in T containing x, and c, the number of elements in $S_T(x)$, as the size of the structure containing x. The possible values of c are $k,2k,\ldots,rk$, for suppose $S_T(x)$ contains ν elements from A_k $(1 \leq \nu \leq r)$. Then, since $S_T(x)$ is the set of immediate predecessors of these ν elements, $c = \nu k$.

Let X_{jk} be a subset of X containing jk elements. We have

$$(13.20) \qquad P\{c = jk\} = (\text{number of ways in which } X_{jk} \text{ can be chosen})$$
$$\times P\{X_{jk} \text{ forms the picked structure}\}$$
$$= \binom{kr}{kj} \cdot P\{x \in X_{jk}\} \cdot P\{TX_{jk} \subset X_{jk}\}$$
$$\cdot P\{T(X-X_{jk}) \subset X-X_{jk}\} \cdot P\{m = 1 | n = jk\}$$
$$= \frac{(kr)!}{(kj)!(k(r-j))!} \cdot \frac{kj}{kr} \cdot \frac{(kj)!(kj)!(k(r-j))!(k(r-j))!}{j!(j(k-1))!(r-j)!((r-j)(k-1))!}$$
$$\cdot \frac{r!(r(k-1))!}{(rk)!(rk)!} \sum_{t=1}^{j} \frac{k^{t-1}(k-1)(j-1)!(kj-t-1)!}{(kj-1)!(j-t)!}$$
$$= \frac{(k(r-j))!(r-1)!(r(k-1))!}{(kr)!(j(k-1)-1)!(r-j)!((r-j)(k-1))!} \sum_{t=1}^{j} \frac{(kj-t-1)!k^{t-1}}{(j-t)!} .$$

We obtain the joint distribution of sir-length and cycle length directly from (13.8) and (13.9) by letting $D = \phi$. Thus, we obtain

$$(13.21) \qquad P\{s = j, \ell = q, q < j\} = \frac{(k-1)k^{j-2}(r-1)!(kr-j+1)!}{(kr)!(r-j+1)!} \qquad 2 \leq j \leq r+1$$

and

$$(13.22) \qquad P\{s = j, \ell = j\} = \frac{k^{j-1}(r-1)!(kr-j)!}{(kr)!(r-j)!} \qquad 1 \leq j \leq r .$$

From these we obtain the marginal distributions

$$(13.23)\qquad P\{\ell = q\} = \frac{k^{q-1}(r-1)!(kr-q)!}{(kr)!(r-q)!} + \sum_{j=q+1}^{r+1} \frac{(k-1)k^{j-2}(r-1)!(kr-j+1)!}{(kr)!(r-j+1)!}$$

$$= \frac{k^{q}(r-1)!(kr-q)!}{(kr)!(r-q)!} + \sum_{j=q+1}^{r} \frac{(k-1)k^{j-1}(r-1)!(kr-j)!}{(kr)!(r-j)!}$$

and

$$(13.24)\qquad P\{s = 1\} = \frac{1}{kr}$$

$$P\{s = j\} = \frac{(k-1)(j-1)k^{j-2}(r-1)!(kr-j+1)!}{(kr)!(r-j+1)!} \qquad 2 \leq j \leq r+1 \quad .$$

In determining the distribution of the number of predecessors, we recall that the selected element x is counted as a predecessor of itself. The possible values of p, therefore, are $1, k+1, \ldots, (r-1)(k+1)$, if x is not a cyclical element in T, and $k, 2k, \ldots, rk$, if x is cyclical. It will be noted that if $p = 1$, the selected element x belongs to A_o, while if $p > 1$, $x \in A_k$ in the chosen transformation. Hence,

$$(13.25)\qquad P\{p = 1\} = P\{x \in A_o\} = \frac{k-1}{k} \quad .$$

As before, let X_{jk} be a subset of X containing jk elements. Then

$$(13.26)\qquad P\{p = jk\} = (\text{number of ways in which } X_{jk} \text{ can be chosen})$$

$$\cdot P\{TX_{jk} \subset X_{jk}\} \cdot P\{T(X - X_{jk}) \subset X - X_{jk}\}$$

$$\cdot P\{m = 1, x \in K_T(x) \mid n = jk\}$$

$$= \frac{(kr)!}{(kj)!(k(r-j))!} \cdot \frac{(kj)!(kj)!(k(r-j))!(k(r-j))!}{j!(j(k-1))!(r-j)!((r-j)(k-1))!}$$

$$\cdot \frac{r!(r(k-1))!}{(rk)!(rk)!} \sum_{t=1}^{j} \frac{1}{j} \cdot \frac{1}{rk} \cdot P\{N_\varphi = j\}$$

$$= \frac{(r-1)!(r(k-1))!(kj-1)!(k(r-j))!}{(j-1)!(j(k-1))!(r-j)!((r-j)(k-1))!(rk)!} \quad .$$

To obtain $P\{p=jk+1\}$, we consider the following: Let X^* be a set of $jk+1$ elements, and let x^* be a given element of X^*. Consider the set of all transformations of X^*, say $\mathcal{J}^*$, such that x^* is mapped into itself and in addition has k immediate predecessors in X^*-x^* while each element in X^*-x^* has either zero of k immediate predecessors. The total number of such transformations is

$$\frac{(jk)!(jk)!}{(j-1)!(j(k-1)+1)!k!^j} .$$

Let N_{x^*} be the total number of cyclical elements in X^*-x^*. Then, in a manner similar to that used for the sets D and $\mathcal{J}_D$, we find that

$$P\{N_{x^*}=q|x^*,T\in\mathcal{J}^*\}=\frac{((k-1)q+k)k^q(j-1)!(jk-q-1)!}{(jk)!(j-1-q)!} \qquad 0\leq q\leq j-1$$

so that

$$P\{N_{x^*}=0|x^*,T\in\mathcal{J}^*\}=\frac{1}{j} .$$

Let us now return to the sets X and $\mathcal{J}$, and for any pair (x,T) let $P_T(x)$ denote the set of predecessors of x. Then

$$\begin{aligned}
(13.27)\qquad P\{p=jk+1\} &= (\text{number of ways in which sets } x,\ X_{jk},\text{ and } X_{k(r-j)-1}\text{ can be chosen})\cdot P\{x\cup X_{jk}=P_T(x)\}\times P\{x\text{ is selected from }X\} \\
&= \frac{(rk)!}{(jk)!(k(r-j)-1)!}\,P\{T(x\cup X_{k(r-j)-1})\subset X_{k(r-j)-1}\} \\
&\quad\cdot P\{T(x\cup X_{jk})\in\mathcal{J}^*|x=x^*,x\cup X_{jk}=X^*\} \\
&\quad\cdot P\{N_{x^*}=0|x=x^*,x\cup X_{jk}=X^*,T\in\mathcal{J}^*\}\cdot\frac{1}{rk} \\
&= \frac{(rk-1)!}{(jk)!(k(r-j)-1)!}\cdot\frac{(k(r-j)-1)!}{(r-j)!((k-1)(r-j)-1)!}\;\frac{(k(r-j))!}{k!^{r-j}} \\
&\quad\cdot\frac{(jk)!(jk)!}{(j-1)!(j(k-1)+1)!k!^j}\cdot\frac{r!(r(k-1))!k!^r}{(rk)!(rk)!}\cdot\frac{1}{j} \\
&= \frac{(k(r-j))!(jk)!r!(r(k-1))!}{(r-j)!((k-1)(r-j)-1)!j!(j(k-1)+1)!(rk)!rk}
\end{aligned}$$

for $j=0,1,\ldots,r-1$.

Asymptotic approximations for the various distributions can be found as follows. For fixed k, as r becomes large, we have

$$(13.28)\qquad P\{N_\varphi = j\} \sim \frac{k^{j-1} j(k-1) r^{r-\frac{1}{2}} (kr-j)^{kr-j-\frac{1}{2}}}{(kr)^{kr-\frac{1}{2}} (r-j)^{r-j+\frac{1}{2}}}$$

$$= \frac{j(k-1) k^{kr-\frac{3}{2}} r^{kr+r-j-1} (1-\frac{j}{kr})^{kr-j-\frac{1}{2}}}{k^{kr-\frac{1}{2}} r^{kr+r-j} (1-\frac{j}{r})^{r-j+\frac{1}{2}}}$$

$$\sim \frac{j(k-1)}{kr} e^{\frac{j^2}{2kr} - \frac{j^2}{2r}}$$

$$= \frac{j(k-1)}{kr} e^{-\frac{1}{2}\frac{(k-1)}{kr} j^2}$$

or the asymptotic density of $(\frac{k-1}{kr})^{1/2} N_\varphi$ is

$$(13.29)\qquad P(x) = xe^{-\frac{1}{2}x^2} .$$

Given that $N_\varphi = j$, the conditional distribution of m, the number of structures, is asymptotically normal as j becomes large with

$$(13.30)\qquad E(m|j) \sim \log j + \gamma + \frac{1}{2j} ,$$

where γ is Euler's constant and

$$(13.31)\qquad Var(m|j) \sim \log j + \gamma - \frac{\pi^2}{6} + \frac{3}{2j} - \frac{1}{2j^2} .$$

The marginal distribution of m is asymptotically normal with expected value and variance each approximately equal to $\frac{1}{2}\log(\frac{kr}{k-1})$.

The asymptotic distribution of structure size is given by

$$(13.32)\quad P\{c = jk\} \sim \frac{e^{-k(r-j)}(k(r-j))^{k(r-j)+\frac{1}{2}}\, e^{-r}\, r^{r-\frac{1}{2}}\, e^{-r(k-1)}(r(k-1))^{r(k-1)+\frac{1}{2}}}{e^{-kr}(kr)^{kr+\frac{1}{2}}\, e^{-j(k-1)}(j(k-1))^{j(k-1)-\frac{1}{2}}\, e^{-(r-j)}(r-j)^{r-j+\frac{1}{2}}}$$

$$\cdot \frac{1}{\sqrt{2\pi}\, e^{-(r-j)(k-1)}\{(r-j)(k-1)\}^{(r-j)(k-1)+\frac{1}{2}}}$$

$$\cdot \sum_{t=1}^{j} \frac{e^{-(kj-t)}(kj-t)^{kj-t-\frac{1}{2}}\, k^{t-1}}{e^{-(j-t)}(j-t)^{j-t+\frac{1}{2}}}$$

$$= \frac{(k-1)^{1/2}}{\sqrt{2\pi}\, k^{3/2} r^{1/2} (r-j)^{1/2} j^{1/2}} \sum_{t=1}^{j} \frac{(1-\frac{t}{kj})^{kj-t-\frac{1}{2}}}{(1-\frac{t}{j})^{j-t+\frac{1}{2}}}$$

$$\sim \frac{(k-1)^{1/2}}{\sqrt{2\pi}\, k^{3/2} r^{1/2} (r-j)^{1/2} j^{1/2}} \int_0^{\infty} e^{-\frac{(k-1)}{2kj}t^2}\, dt$$

$$= \frac{1}{2kr^{1/2}(r-j)^{1/2}}$$

the asymptotic density of $x = \frac{c}{kr}$ is

$$P(x) = \frac{1}{2(1-x)^{1/2}} \qquad 0 \leq x \leq 1 \quad .$$

From (13.21) we obtain, as r becomes large

$$(13.33)\quad P\{s = j,\ \ell = q, q < j\} \sim \frac{(k-1)k^{j-2} e^{-r-kr+j-1} r^{r-\frac{1}{2}} (kr-j+1)^{kr-j+\frac{3}{2}}}{e^{-kr-r+j-1}(kr)^{kr+\frac{1}{2}}(r-j+1)^{r-j+\frac{3}{2}}}$$

$$= \frac{(k-1)}{kr} \frac{(1-\frac{j-1}{kr})^{kr-j+\frac{3}{2}}}{(1-\frac{j-1}{r})^{r-j+\frac{3}{2}}}$$

$$\sim \frac{(k-1)}{kr}\, e^{-\frac{1}{2}\frac{(k-1)}{kr}(j-1)^2} \quad .$$

From (13.22), we obtain

$$(13.34) \qquad P\{s=j,\ \ell=j\} \sim \frac{k^{j-1} e^{-r-kr+j} r^{r-\frac{1}{2}} (kr-j)^{kr-j+\frac{1}{2}}}{e^{-kr-r+j} (kr)^{kr+\frac{1}{2}} (r-j)^{r-j+\frac{1}{2}}}$$

$$\sim \frac{1}{kr} e^{-\frac{1}{2}\frac{(k-1)}{kr} j^2}$$

From these, we have

$$(13.35) \qquad P\{s=j\} \sim \frac{(k-1)(j-1)}{kr} e^{-\frac{1}{2}\frac{(k-1)}{kr}(j-1)^2}$$

or the asymptotic density of $t=\frac{(s-1)}{\sqrt{kr}}$ is $p(t)=(k-1)te^{-\frac{1}{2}(k-1)t^2}$. Also

$$(13.36) \qquad P\{\ell=q\} \sim \int_q^\infty \frac{(k-1)}{kr} e^{-\frac{1}{2}\frac{(k-1)}{kr} j^2} dj$$

$$= \left(\frac{2\pi(k-1)}{kr}\right)^{1/2} \Phi\left(\left(\frac{k-1}{kr}\right)^{1/2} q\right)$$

where $\Phi(a)=\int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$.

The asymptotic distribution of predecessors can be found directly from (13.26) and (13.27) with the use of Stirling's approximations, In this manner we obtain

$$(13.37) \qquad P\{p=jk\} \sim \frac{(kj)!(k-1)^{j(k-1)}}{j!(j(k-1))!k^{jk+1}r^{1/2}(r-j)^{1/2}} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty$$

and

$$(13.38) \qquad P\{p=jk+1\} \sim \frac{(kj)!(k-1)^{j(k-1)+1}}{j!(j(k-1)+1)!k^{jk+1}}$$

These results can also be obtained in the following manner. Since $\sum_{j=1}^{r} P\{p=jk\}$ = probability that an element selected at random is cyclic, and this probability approaches zero as r becomes large, it is clear that $P\{p=jk\} \longrightarrow 0$ as $r \longrightarrow \infty$. Writing for simplicity $P\{p=jk+1\} = \psi_{jk+1}$, we have

(13.39)
$$
\begin{aligned}
\psi_1 &= \frac{k-1}{k} \\
\psi_{k+1} &= \frac{1}{k}\,\psi_1^k \\
&\ \vdots \\
\psi_{jk+1} &= \frac{1}{k} \sum_{\substack{r_1 \ldots r_k \\ \sum_{t=1}^{k} r_t = j-1}} \psi_{r_1 k+1}\,\psi_{r_2 k+1} \cdots \psi_{r_k k+1} \\
&\ \vdots \qquad\qquad\qquad \vdots
\end{aligned}
$$

Consider the function $y = h(x) = \sum_{t=0}^{\infty} x^{tk+1}\,\psi_{tk+1}$. Then

(13.40)
$$y = h(x) = \left(x\left(\frac{k-1}{k} + \frac{1}{k}(h(x))^k\right)\right) = xg(y) \quad \text{, say.}$$

It follows that

(13.41)
$$
\begin{aligned}
\psi_{jk+1} &= \frac{1}{(jk+1)!}\left(\frac{d^{(jk)}}{dy}\,(g(y))^{jk+1}\Big|_{y=0}\right) \\
&= \frac{(jk)!}{j!\,(j(k-1)+1)!}\;\frac{(k-1)^{j(k-1)+1}}{k^{jk+1}}
\end{aligned}
$$

the result obtained in (13.38). When j is large but still small relative to r, we have

(13.42)
$$
\begin{aligned}
P\{p = jk+1\} &\sim \frac{e^{-jk}(jk)^{jk+\frac{1}{2}}(k-1)^{j(k-1)+1}}{\sqrt{2\pi}\,e^{-j-jk+j-1}\,j^{j+\frac{1}{2}}(j(k-1)+1)^{j(k-1)+\frac{3}{2}}\,k^{jk+1}} \\
&\sim \frac{1}{\sqrt{2\pi}\,j^{3/2}(k-1)^{1/2}k^{1/2}}
\end{aligned}
$$

14. Case of each element having at most k predecessors. Let $\mathcal{J}$ be the set of transformations of X into X such that each element $x \in X$ has at most k immediate predecessors where k is a given number. For a given $T \in \mathcal{J}$, let A_i be the set of elements with exactly i immediate predecessors $(i = 0,1,2,\ldots,k)$, and let B_j be the set of elements of X such that $TB_j = A_j$ $(j = 1,2,\ldots,k)$. Clearly

(14.1) $$X = \bigcup_{i=0}^{k} A_i = \bigcup_{i=1}^{k} B_i \quad .$$

Also, if a_i is the number of elements in A_i, then ia_i is the number of elements in B_i, and

(14.2) $$\sum_{i=0}^{k} a_i = \sum_{i=1}^{k} ia_i = n \quad .$$

Let $N^* = (n_0, n_1, \ldots, n_k)$ be a set of $(k+1)$ integers satisfying (14.2). Then

(14.3) $$\mathcal{J} = \bigcup_{N^*} \mathcal{J}_{N^*}$$

where $\mathcal{J}_{N^*}$ is the set of all transformations $T \in \mathcal{J}$ for which $a_0 = n_0, a_1 = n_1, \ldots, a_k = n_k$. The number of transformations in $\mathcal{J}_{N^*}$ is

(14.4) $\varphi(N^*)$ = (number of ways in which the sets $A_0, A_1, \ldots, A_k$ can be chosen)

× (number of ways in which the sets $B_1, B_2, \ldots, B_k$ can be chosen)

× $\sum_{i=1}^{k}$ (number of transformations mapping B_i onto A_i such that each element of A_i has exactly i predecessors)

Or

(14.5) $$\varphi(N^*) = \frac{n!}{n_0! n_1! \ldots n_k!} \cdot \frac{n!}{n_1!(2n_2)! \ldots (kn_k)!} \cdot n_1! \frac{(2n_2)!}{(2!)^{n_2}} \cdots \frac{(kn_k)!}{(k!)^{n_k}}$$

$$= \frac{n!\, n!}{n_0! n_1! \ldots n_k! (2!)^{n_2} \ldots (k!)^{n_k}} \quad .$$

The total number of transformations in $\mathcal{J}$ is given by $\sum_{N^*} \varphi(N^*)$. To obtain the asymptotic value of this sum as n becomes large, we proceed as follows: If we approximate $n_i!$ by $\sqrt{2\pi}\, e^{-n_i} n_i^{n_i + \frac{1}{2}}$ and take the logarithm of $\varphi(N^*)$, we obtain

(14.6) $$\log \varphi(N^*) = 2 \log n! - \left(\frac{k+1}{2}\right) \log 2\pi - \sum_{i=0}^{k} \left(n_i + \frac{1}{2}\right) \log n_i$$

$$- \sum_{i=2}^{k} n_i \log(i!) + \sum_{i=0}^{k} n_i \quad .$$

Since N^* satisfies (14.2), i.e.,

$$\sum_{i=0}^{k} n_i = \sum_{i=1}^{k} i n_i = n$$

we can write

$$n_o = \sum_{i=2}^{k} (i-1) n_i \quad , \quad n_1 = n - \sum_{i=2}^{k} i n_i$$

and

(14.7) $$\log \varphi(N^*) = \Psi(n_2, n_3, \ldots, n_k) \quad , \text{ say.}$$

Neglecting the terms $-\frac{1}{2}\log n_i$ $(i = 0,1,\ldots,k)$ and differentiating Ψ with respect to n_j $(j = 2,\ldots,k)$, we obtain

(14.8) $$\frac{\partial \Psi}{\partial n_j} = -(j-1)\log n_o - (j-1) + j \log n_1 + j - \log n_j - \log j! - 1$$

$$= \log \left(\frac{n_1^j}{n_o^{j-1} n_j j!} \right) \quad .$$

Also

(14.9) $$\frac{\partial^2 \Psi}{\partial n_i \partial n_j} = - \frac{(i-1)(j-1)}{n_o} - \frac{ij}{n_1} - \frac{\delta_{ij}}{n_i}$$

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$.

Let $n_o^*, n_1^*, \ldots, n_k^*$ be values of $n_o, n_1, \ldots, n_k$, respectively, which maximize Ψ, i.e., $n_o^*, n_1^*, \ldots, n_k^*$ are a set of values satisfying the $(k+1)$ equations

$$\frac{\partial \Psi}{\partial n_j} = 0 \qquad j = 2,\ldots,k$$

$$\sum_{i=0}^{k} n_i = n$$

$$\sum_{i=1}^{k} i n_i = n \quad .$$

From (14.8) we obtain

(14.10) $$n_j^* = \frac{n_o^*}{j!} \left(\frac{n_1^*}{n_o^*} \right)^j \qquad (j = 2,\ldots,k) \quad .$$

It will be observed that (14.10) holds trivially for $j = 0$ and 1.

Let $n_o^* = \alpha n$ and $n_1^* = \alpha \beta n$. Then

(14.11) $$n_j^* = \frac{\alpha \beta^j n}{j!} \qquad (j = 0,1,\ldots,k) \quad .$$

Since

$$\alpha n \sum_{j=0}^{k} \frac{\beta^j}{j!} = \alpha n \sum_{j=1}^{k} \frac{j\beta^j}{j!} = n \quad ,$$

we have

(14.12) $$\sum_{j=2}^{k} \frac{(j-1)\beta^j}{j!} = 1$$

and

(14.13) $$\alpha = \frac{(\beta-1)k!}{\beta^{k+1}} \quad .$$

It is easy to see that the desired value of β is the single positive real root of (14.12), say, β^*. The value of β^* decreases as k increases, with $\beta^* = \sqrt{2}$ when $k = 2$, and $\beta^* = 1$ when $k = \infty$. From (14.11) we obtain that

(14.14) $$n_j^* = \frac{n(\beta^*-1)k!}{\beta^{*k+1-j} j!} \quad .$$

Let

(14.15) $$Q = (q_{ij}) = \left(- \frac{\partial^2 \Psi}{\partial n_{i+1} \partial n_{j+1}} \bigg|_{n_0 = n_0^*, n_1 = n_1^*, \ldots, n_k = n_k^*} \right)$$

$$= \left(\frac{ij}{n_0^*} + \frac{(i+1)(j+1)}{n_1^*} + \frac{\delta_{ij}}{n_{i+1}^*} \right) \qquad i,j = 1,2,\ldots,k-1 \ .$$

Then

(14.16) $$\varphi(N^*) \sim \frac{n! n^{n+\frac{1}{2}} e^{-\frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} q_{ij}(n_{i+1}-n_{i+1}^*)(n_{j+1}-n_{j+1}^*)}}{(2\pi)^{k/2} \prod_{i=0}^{k} n_i^{*n_i^*+1/2} \, i!^{n_i^*}}$$

and

(14.17) $$\sum_{N^*} \varphi(N^*) \sim \frac{n! n^{n+\frac{1}{2}}}{(2\pi)^{k/2} \prod_{i=0}^{k} n_i^{*n_i^*+\frac{1}{2}} \, i!^{n_i^*}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} q_{ij}(n_{i+1}-n_{i+1}^*)(n_{j+1}-n_{j+1}^*)} dn_2 \ldots dn_k$$

$$= \frac{n! n^{n+\frac{1}{2}} |Q|^{-\frac{1}{2}}}{(2\pi)^{1/2} \prod_{i=0}^{k} n_i^{*n_i^*+\frac{1}{2}} \, i!^{n_i^*}} \quad .$$

Let γ' and ρ' denote the row vectors $(1,2,\ldots,k-1)$ and $(2,3,\ldots,k)$, respectively, and let D be the $(k-1)\times(k-1)$ diagonal matrix

$$D = \begin{pmatrix} \frac{1}{n_2^*} & 0 & \ldots & 0 \\ 0 & \frac{1}{n_3^*} & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & \ldots & \frac{1}{n_k^*} \end{pmatrix} .$$

Then

$$|Q| = \begin{vmatrix} 1 & 0 & \frac{1}{\sqrt{n_0^*}}\gamma' \\ 0 & 1 & \frac{1}{\sqrt{n_0^*}}\rho' \\ -\frac{1}{\sqrt{n_0^*}}\gamma & -\frac{1}{\sqrt{n_1^*}}\rho & D \end{vmatrix} = |D|\cdot \begin{vmatrix} 1+\frac{1}{n_0^*}\gamma' D^{-1}\gamma & \frac{1}{\sqrt{n_0^* n_1^*}}\gamma' D^{-1}\rho \\ \frac{1}{\sqrt{n_0^* n_1^*}}\gamma' D^{-1}\rho & 1+\frac{1}{n_1^*}\rho' D^{-1}\rho \end{vmatrix}$$

$$= \prod_{i=0}^{k} \frac{1}{n_i^*} \begin{vmatrix} n_0^* + \sum_{j=2}^{k}(j-1)^2 n_j^* & \sum_{j=2}^{k} j(j-1)n_j^* \\ \sum_{j=2}^{k} j(j-1)n_j^* & n_1^* + \sum_{j=2}^{k} j^2 n_j^* \end{vmatrix}$$

$$= n\left(\sum_{j=2}^{k} j(j-1)n_j^*\right)\prod_{i=0}^{k}\frac{1}{n_i^*} .$$

Hence

$$(14.18) \qquad \sum_{N^*} \varphi(N^*) \sim \frac{n!n^n}{\left(2\pi \sum_{j=2}^{k} j(j-1)n_j^*\right)^{1/2} \prod_{i=0}^{k} (n_i^*!)^{n_i^*}} .$$

From (14.11), (14.12), and (14.13), we have

$$(14.19) \qquad \sum_{j=2}^{k} j(j-1)n_j^* = \sum_{j=2}^{k} j(j-1)\frac{\alpha\beta^{*j}n}{j!} = \beta^*\alpha n\sum_{j=1}^{k-1}\frac{1\beta^{*j}}{j!} = \beta^* n\left(1-\frac{\alpha\beta^{*k}}{(k-1)!}\right)$$

$$= n(k-(k-1)\beta^*) .$$

Since (14.19) is necessarily positive, this yields $1+\frac{1}{k-1}$ as an upper bound for β^*. Also

$$(14.20)\qquad \prod_{i=0}^{k}(n_i^*\, i!)^{n_i^*} = \prod_{i=0}^{k}(\alpha\beta^{*i}n)^{\frac{\alpha\beta^{*i}n}{i!}} = (\alpha\beta^* n)^n = \frac{n^n(\beta^*-1)^n k!^n}{\beta^{*kn}}$$

so that

$$(14.21)\qquad \sum_{N^*}\varphi(N^*) = \frac{n!\,\beta^{*kn}}{(2\pi n)^{1/2}(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^n k!^n} = \frac{e^{-n}n^n\beta^{*kn}}{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^n k!^n}$$

correct to terms of within order $1/n$.

It is easy to see that the variables $\frac{n_0}{\sqrt{n}}, \frac{n_1}{\sqrt{n}}, \ldots, \frac{n_k}{\sqrt{n}}$ have a limiting singular multivariate normal distribution with expected values $\frac{n_0^*}{\sqrt{n}}, \frac{n_1^*}{\sqrt{n}}, \ldots, \frac{n_k^*}{\sqrt{n}}$. The non-singular covariance matrix of the limiting distribution of $\frac{n_2}{\sqrt{n}}, \frac{n_3}{\sqrt{n}}, \ldots, \frac{n_k}{\sqrt{n}}$ is

$$(14.22)\qquad \Sigma = Q^{-1} = \left(\delta_{ij}\frac{n_i^*}{n} - \frac{n_i^*}{n}\frac{n_j^*}{n} - \frac{(i-1)(j-1)}{\sum_{t=2}^{k} t(t-1)\frac{n_t^*}{n}}\frac{n_i^*}{n}\frac{n_j^*}{n}\right)$$

$$= \left(\delta_{ij}\frac{\alpha\beta^{*i}}{i!} - \frac{\alpha\beta^{*i}\alpha\beta^{*j}}{i!\,j!} - \frac{(i-1)(j-1)\alpha\beta^{*i}\alpha\beta^{*j}}{(k-(k-1)\beta^*)\,i!\,j!}\right)$$

$$= \left(\frac{\delta_{ij}(\beta^*-1)k!}{\beta^{*k+1-i}\,i!} - \frac{(\beta^*-1)^2 k!^2}{\beta^{*2k+2-i-j}\,i!\,j!}\left(1+\frac{(i-1)(j-1)}{k-(k-1)\beta^*}\right)\right)$$

It is clear that the variables $\frac{n_2}{n}, \frac{n_3}{n}, \ldots, \frac{n_k}{n}$ converge stochastically to their expected values.

When $k=2$, we have $n_0 = n_2$, $n_1 = n-2n_2$, $\beta^* = \sqrt{2}$, and

$$(14.23)\qquad \begin{aligned} n_0^* &= \tfrac{1}{2}(2-\sqrt{2})n \\ n_1^* &= (\sqrt{2}-1)n \\ n_2^* &= \tfrac{1}{2}(2-\sqrt{2})n \end{aligned}$$

with

$$(14.24) \qquad \varphi(N^*) = \frac{n!(2+\sqrt{2})^{n+\frac{1}{2}}}{(\pi n)^{\frac{1}{2}}\, 2^{\frac{n+2}{2}}}$$

correct to within terms of order $1/n$. The asymptotic distribution of $\frac{n_2}{\sqrt{n}}$ is normal with expected values $\frac{1}{2}(2-\sqrt{2})\sqrt{n}$ and variance $\frac{1}{\sqrt{2}\,(2+\sqrt{2})^2}$.

As before, for any pair (x,T) let s denote the six-length of x and ℓ the cycle length. Also, let S be the set of elements in the six of Tx, i.e., $S = (Tx, T^2x, \ldots)$, and let s_i be the number of elements in $S \cap A_i$ $(i = 1,2,\ldots,k)$. Clearly, $\sum_{i=1}^{k} s_i = s-1$. Then

$$(14.25) \qquad P\left\{s = r, \ell = q, q < r, s_1 = \nu_1, \ldots, s_k = \nu_k, x \in A_o, T^{r-1-q}x \in A_i \,\middle|\, N^*\right\}$$

= (number of ways in which the sets $A_o, A_1, \ldots, A_k$ can be chosen) $\times$ (number of ways in which x can be chosen) $\times$ (number of ways in which $S \cap A_1, S \cap A_2, \ldots, S \cap A_k$ can be chosen) $\times$ (number of ways in which the remaining elements of the sets $B_1, B_2, \ldots, B_k$ can be chosen) $\times \sum_{t=1}^{k}$ (number of ways in which the remaining predecessors for the elements of A_t can be chosen from these remaining elements of B_t) $\times \frac{1}{n \cdot \varphi(N^*)}$

$$= \frac{n!}{n_o!\,n_1!\cdots n_k!} \cdot n_o \cdot \frac{n_1!\cdots n_k!}{\nu_1!(n_1-\nu_1)!\cdots \nu_k!(n_k-\nu_k)!} \cdot \nu_i (r-2)!$$

$$\cdot \frac{(n-r)!}{(n_1-\nu_1)!\,\nu_2!(2(n_2-\nu_2))!\cdots(i-2)!((i-1)(\nu_i-1))!(i(n_i-\nu_i))!\cdots(k(n_k-\nu_k))!}$$

$$\cdot \frac{(n_1-\nu_1)!\,\nu_2!(2(n_2-\nu_2))!\cdots((i-1)(n_{i-1}-\nu_{i-1}))!((i-1)(\nu_i-1))!\cdots(k(n_k-\nu_k))!}{2!^{n_2-\nu_2}\,2!^{\nu_3}\cdots(i-1)!^{\nu_{i-1}}(i-1)!^{\nu_i-1}\,i!^{n_i-\nu_i}\cdots(k-1)!^{\nu_k}k!^{n_k-\nu_k}}$$

$$\cdot \frac{n_o!\,n_1!\cdots n_k!\,2!^{n_2}\cdots k!^{n_k}}{n\cdot n!\cdot n!}$$

$$= \frac{n_o}{n}\,\frac{(n-r)!(r-2)!}{n!} \qquad (i-1)\nu_i \prod_{t=1}^{k} \frac{n_t!\,t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}$$

It follows that

(14.26) $$P\{s=r,\ \ell=q, q<r,\ a_1=\nu_1,\dots,a_k=\nu_k, x\in A_0 \mid N^*\}$$
$$= \frac{n_0}{n}\,\frac{(n-r)!(r-2)!}{n!}\sum_{i=2}^{k}(i-1)\nu_i\prod_{t=1}^{k}\frac{n_t!t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}\ .$$

Similarly, we find

(14.27) $$P\{s=r,\ \ell=q, q<r,\ a_1=\nu_1,\dots,a_k=\nu_k, x\in A_j, j>0 \mid N^*\}$$
$$= \frac{(n_1-\nu_1)}{n}\,\frac{(n-r)!(r-2)!}{n!}\sum_{i=2}^{k}(i-1)\nu_i\prod_{t=1}^{k}\frac{n_t!t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}$$

(14.28) $$P\{s=r,\ \ell=r, a_1=\nu_1,\dots,a_k=\nu_k, x\in A_j, j>0 \mid N^*\}$$
$$= \frac{j(n_1-\nu_1)}{n}\,\frac{(n-r)!(r-1)!}{n!}\prod_{t=1}^{k}\frac{n_t!t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}$$

so that

(14.29) $$P\{s=r,\ \ell=q, q<r,\ a_1=\nu_1,\dots,a_k=\nu_k \mid N^*\}$$
$$= \frac{(n-r+1)!(r-2)!}{n\cdot n!}\sum_{i=2}^{k}(i-1)\nu_i\prod_{t=1}^{k}\frac{n_t!t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}$$

and

(14.30) $$P\{s=r,\ \ell=r, a_1=\nu_1,\dots,a_k=\nu_k \mid N^*\}$$
$$= \frac{(n-r)!(r-1)!}{n\ n!}\sum_{i=1}^{k}i(n_i-\nu_i)\prod_{t=1}^{k}\frac{n_t!t^{\nu_t}}{\nu_t!(n_t-\nu_t)!}\ .$$

To find the joint marginal probability distribution of s and ℓ, it is necessary to sum (14.29) and (14.30) over all values of the ν_i such that $0 \le \nu_i \le n_i$, $i=1,2,\dots,k$, $\sum_{i=1}^{k}\nu_i = r-1$, and to take the expected value of each sum with respect to N^*. To obtain the asymptotic value of $P\{s=r,\ \ell=q, q<r\}$ we replace each factorial in (14.29) by the first term of its Stirling's approximation and consider the logarithm of the resulting expression as a function of $\nu_2,\dots,\nu_k$, namely

(14.31) $$g(\nu_2,\ldots,\nu_k) = (n-2+\tfrac{3}{2})\log(n-r+1)+(r-\tfrac{3}{2})\log(r-1)-(n+\tfrac{3}{2})\log n$$
$$-(\tfrac{k-1}{2})\log 2\pi+\log(\sum_{i=2}^{k}(i-1)\nu_i)+\sum_{t=1}^{k}(n_t+\tfrac{1}{2})\log n_t$$
$$+\sum_{t=1}^{k}\nu_t\log t-\sum_{t=1}^{k}(\nu_t+\tfrac{1}{2})\log\nu_t-\sum_{t=1}^{k}(n_t-\nu_t+\tfrac{1}{2})\log(n_t-\nu_t)\ .$$

Neglecting the terms $\log(\sum_{i=2}^{k}(i-1)\nu_i)$, $-\frac{1}{2}\log\nu_t$, and $-\frac{1}{2}\log(n_t-\nu_t)$ which are relatively very small as n becomes large, and differentiating g with respect to ν_i, $i=2,\ldots,k$, we obtain

(14.32) $$\frac{\partial g}{\partial\nu_i} = \log i-\log\nu_i+\log\nu_1-\log(n_i-\nu_i)-\log(n_1-\nu_1)$$
$$=\log\left(\frac{i\nu_1(n_i-\nu_i)}{\nu_i(n_1-\nu_1)}\right)\ .$$

Also

(14.33) $$\frac{\partial^2 g}{\partial\nu_i\partial\nu_j} = -\frac{\delta_{ij}}{\nu_i}-\frac{\delta_{ij}}{n_i-\nu_i}-\frac{1}{\nu_1}-\frac{1}{n_1-\nu_1} = -\left(\frac{\delta_{ij}n_i}{\nu_i(n_i-\nu_i)}+\frac{n_1}{\nu_1(n_1-\nu_1)}\right).$$

Let $\nu_1^*,\nu_2^*,\ldots,\nu_k^*$ be the values of $\nu_1,\nu_2,\ldots,\nu_k$, respectively, satisfying the k equations

$$\sum_{i=1}^{k}\nu_i = r-1$$
$$\frac{\partial g}{\partial\nu_i}=0 \qquad i=2,\ldots,k\ .$$

From (14.32) we have

(14.34) $$\nu_i^* = \frac{in_i\nu_1^*}{n_1+(i-1)\nu_1^*}\ .$$

It will be observed that (14.34) is satisfied trivially when $i=1$.

Let

(14.35) $$Q^* = (q_{ij}^*) = \left(\frac{-\partial^2 g}{\partial\nu_{i+1}\partial\nu_{j+1}}\bigg|_{\nu_1=\nu_1^*,\nu_2=\nu_2^*,\ldots,\nu_k=\nu_k^*}\right)$$
$$=\left(\frac{\delta_{ij}n_{i+1}}{\nu_{i+1}^*(n_{i+1}-\nu_{i+1}^*)}+\frac{n_1}{\nu_1^*(n_1-\nu_1^*)}\right) \qquad i,j=1,2,\ldots,k-1.$$

Let $\dfrac{n_i}{\nu_i^*(n_i-\nu_i^*)} = \theta_i$. Then

$$|Q^*| = \begin{vmatrix} \theta_1+\theta_2 & \theta_1 & \cdots & \theta_1 \\ \theta_1 & \theta_1+\theta_3 & \cdots & \theta_1 \\ \cdot & \cdot & \cdots & \cdot \\ \theta_1 & \theta_1 & \cdots & \theta_1+\theta_k \end{vmatrix} = \begin{vmatrix} 1 & \theta_1^{1/2} & \theta_1^{1/2} & \cdots & \theta_1^{1/2} \\ -\theta_1^{1/2} & \theta_2 & 0 & \cdots & 0 \\ -\theta_1^{1/2} & 0 & \theta_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\theta_1^{1/2} & 0 & 0 & \cdots & \theta_k \end{vmatrix}$$

$$= \prod_{t=2}^{k} \theta_t \left(1 + \sum_{i=2}^{k} \frac{\theta_1}{\theta_i}\right) = \sum_{i=1}^{k} \frac{\nu_i^*(n_i-\nu_i^*)}{n_i} \cdot \prod_{t=1}^{k} \frac{n_t}{\nu_t^*(n_t-\nu_t^*)}$$

But

(14.36) $$P\{s=r, \ell=q, q<r \mid N^*\} \sim \frac{(n-r+1)^{n-r+\frac{3}{2}}(r-1)^{r-\frac{3}{2}}\left(\sum_{i=2}^{k}(i-1)\nu_i^*\right)\prod_{t=1}^{k} n_t^{n_t+\frac{1}{2}} t^{\nu_t^*}}{n^{n+\frac{3}{2}}(2\pi)^{k-\frac{1}{2}}\prod_{t=1}^{k}\nu_t^{*\,\nu_t^*+\frac{1}{2}}(n_t-\nu_t^*)^{n_t-\nu_t^*+\frac{1}{2}}}$$

$$\cdot \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{i=1}^{k-1} q_{ij}^*(\nu_{i+1}-\nu_{i+1}^*)(\nu_{j+1}-\nu_{j+1}^*)}\, d\nu_2 \ldots d\nu_k$$

$$= \frac{(n-r+1)^{n-r+\frac{3}{2}}(r-1)^{r-\frac{3}{2}}\left(\sum_{i=2}^{k}(i-1)\nu_i^*\right)\prod_{t=1}^{k} n_t^{n_t}\, t^{\nu_t^*}}{n^{n+\frac{3}{2}}\left(\sum_{i=1}^{k}\frac{\nu_i^*(n_i-\nu_i^*)}{n_i}\right)^{1/2}\prod_{t=1}^{k}\nu_t^{*\,\nu_t^*}(n_t-\nu_t^*)^{n_t-\nu_t^*}}.$$

Substituting the value of ν_i^* $(i=2,\ldots,k)$ from (14.34) we obtain

(14.37) $$P\{s=r, \ell=q, q<r \mid N^*\}$$

$$\sim \frac{(n-r+1)^{n-r+\frac{3}{2}}(r-1)^{r-\frac{3}{2}}\left(\sum_{i=2}^{k} i(i-1)n_i\right)\prod_{t=1}^{k}(n_1+(t-1)\nu_1^*)^{n_t}}{n^{n+2}\nu_1^{*\,r-\frac{3}{2}}(n_1-\nu_1^*)^{n-n_0-r+\frac{3}{2}}}$$

$$= \left(\frac{(r-1)(n_1-\nu_1^*)}{\nu_1^*(n-r+1)}\right)^{r-\frac{3}{2}} \frac{\left(\sum_{i=2}^{k} i(i-1)\frac{n_i}{n}\right)}{n} \frac{\left(1-\frac{r-1}{n}\right)^n \prod_{t=1}^{k}\left(1+\frac{(t-1)\nu_1^*\left(\frac{n}{n_1}\right)}{n}\right)^{n\left(\frac{n_t}{n}\right)}}{\left(1-\frac{\nu_1^*\left(\frac{n}{n_1}\right)}{n}\right)^{n\left(1-\frac{n_0}{n}\right)}}$$

$$= \left(\frac{(r-1)(n_1-\nu_1^*)}{\nu_1^*(n-r+1)}\right)^{r-\frac{3}{2}} \frac{\left(\sum_{i=2}^{k} i(i-1)\frac{n_i}{n}\right)}{n} \cdot \exp\left\{-(r-1)-\frac{(r-1)^2}{2n}+\nu_1^*\left(\frac{n}{n_1}\right)\sum_{t=1}^{k}(t-1)\frac{n_t}{n}\right.$$
$$\left. -\frac{\nu_1^{*2}}{2n}\left(\frac{n}{n_1}\right)^2\sum_{t=1}^{k}(t-1)^2\frac{n_t}{n}+\nu_1^*\left(\frac{n}{n_1}\right)\left(1-\frac{n_0}{n}\right)+\frac{\nu_1^{*2}}{2n}\left(\frac{n}{n_1}\right)^2\left(1-\frac{n_0}{n}\right)\right\}.$$

Or

$$(14.38)\quad P\left\{s=r,\ell=q,q<r|N^*\right\}\sim\left(\frac{(r-1)(n_1-\nu_1^*)}{\nu_1^*(n-r+1)}\right)^{r-\frac{3}{2}}\frac{\left(\sum_{t=2}^{k}t(t-1)\frac{n_t}{n}\right)}{n}$$
$$\cdot e^{-(r-1)-\frac{(r-1)^2}{2n}+\nu_1^*\left(\frac{n}{n_1}\right)+\nu_1^{*2}\left(\frac{n}{n_1}\right)^2-\frac{\nu_1^{*2}}{2n}\left(\frac{n}{n_1}\right)^2\sum_{t=2}^{k}t(t-1)\frac{n_t}{n}}.$$

But

$$(r-1)=\nu_1^*\sum_{i=1}^{k}\frac{in_i}{n_1+(i-1)\nu_1^*}=\nu_1^*\sum_{i=1}^{k}\frac{i\frac{n_i}{n}}{\frac{n_1}{n}+\frac{(i-1)\nu_1^*}{n}}\sim\nu_1^*\left(\frac{n}{n_1}\right)$$ as n becomes large.

Hence

$$(14.39)\quad P\left\{s=r,\ell=q,q<r|N^*\right\}\sim\frac{\left(\sum_{t=2}^{k}t(t-1)\frac{n_t}{n}\right)}{n}e^{-\frac{(r-1)^2}{2n}\sum_{t=2}^{k}t(t-1)\frac{n_t}{n}}.$$

However, as n becomes large, $\sum_{t=2}^{k}t(t-1)\frac{n_t}{n}$ converges in probability to $(k-(k-1)\beta^*)$ so that

$$(14.40)\quad P\left\{s=r,\ell=q,q<r\right\}=\frac{(k-(k-1)\beta^*)}{n}e^{-\frac{(r-1)^2}{2n}(k-(k-1)\beta^*)}$$

correct to terms of within order 1/n.

Similarly, we find that

$$(14.41)\quad P\left\{s=r,\ell=r\right\}=\frac{1}{n}e^{-\frac{(r-1)^2}{2n}(k-(k-1)\beta^*)}.$$

It follows that

(14.42) $$P\{s=r\} = \left(\frac{(r-1)}{n}(k-(k-1)\beta^*) + \frac{1}{n}\right)e^{-\frac{(r-1)^2}{2n}(k-(k-1)\beta^*)}$$

$$\rightarrow \frac{(r-1)}{n}(k-(k-1)\beta^*)e^{-\frac{(r-1)^2}{2n}(k-(k-1)\beta^*)}$$

as $n \rightarrow \infty$. The asymptotic density of $x = \frac{r-1}{\sqrt{n}}$ is

(14.43) $$p(x) = (k-(k-1)\beta^*)\, x\, e^{-\frac{x^2}{2}(k-(k-1)\beta^*)} \qquad 0 \leq x \leq \infty .$$

When $k=2$, we have

(14.44) $$p(x) = (2-\sqrt{2})\, x\, e^{-\frac{x^2}{2}(2-\sqrt{2})} .$$

It will be observed that the probability that an element selected at random is cyclic is given by

(14.45) $$\sum_{r=1}^{n} P\{s=r, q=r\} = \frac{1}{n}\sum_{0}^{n-1} e^{-\frac{r^2}{2n}(k-(k-1)\beta^*)}$$

$$\sim \frac{1}{n}\int_0^{\infty} e^{-\frac{r^2}{2n}(k-(k-1)\beta^*)}\, dr$$

$$= \left(\frac{\pi}{2n(k-(k-1)\beta^*)}\right)^{1/2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

The asymptotic distribution of cycle length is given by

(14.46) $$P\{\ell=q\} = \frac{1}{n}e^{-\frac{(q-1)^2}{2n}(k-(k-1)\beta^*)} + \sum_{r=q+1}^{n}\frac{k-(k-1)\beta^*}{n}e^{-\frac{(r-1)^2}{2n}(k-(k-1)\beta^*)}$$

$$\sim \frac{(k-(k-1)\beta^*)}{n}\int_q^{\infty} e^{-\frac{r^2(k-(k-1)\beta^*)}{2n}}\, dr$$

$$= \left(\frac{2\pi(k-(k-1)\beta^*)}{n}\right)^{1/2} \Phi\left(q\left(\frac{k-(k-1)\beta^*}{n}\right)^{1/2}\right)$$

where $\Phi(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$. The asymptotic density of $h = \frac{\ell}{\sqrt{n}}$ is

$$(14.47) \quad p(h) = (2\pi(k-(k-1)\beta^*))^{1/2}\Phi(h(k-(k-1)\beta^*)^{1/2}) \quad .$$

Since

$$(14.48) \qquad P\{\ell = j\} = \sum_{r=j}^{n} \frac{1}{r} P\{N_\varphi = r\}$$

where N_φ is the total number of cyclical elements, we have

$$(14.49) \quad \frac{1}{r} P\{N_\varphi = r\} \sim \frac{(2\pi(k-(k-1)\beta^*))^{1/2}}{n^{1/2}} \frac{d}{dr}\Phi\left(\frac{r(k-(k-1)\beta^*)^{1/2}}{n^{1/2}}\right)$$

$$= \frac{k-(k-1)\beta^*}{n} e^{-\frac{r^2}{2n}(k-(k-1)\beta^*)} \quad .$$

Or

$$(14.50) \quad P\{N_\varphi = r\} = \frac{r}{n}(k-(k-1)\beta^*) e^{-\frac{r^2}{2n}(k-(k-1)\beta^*)} \quad .$$

The expected value of r is $\left(\frac{n\pi}{2(k-(k-1)\beta^*)}\right)^{1/2}$ with variance $\frac{n}{(k-(k-1)\beta^*)}(2-\frac{\pi}{2})$.

Given that $N_\varphi = r$, the conditional distribution of m, the number of structures, is asymptotically normal as r becomes large with

$$(14.51) \qquad E(m|r) \sim \log r + \gamma + \frac{1}{2r}$$

where γ is Euler's constant, and

$$(14.52) \qquad Var(m|r) \sim \log r + \gamma - \frac{\pi^2}{6} + \frac{3}{2r} - \frac{1}{2r^2} \quad .$$

It follows that the marginal distribution of m is asymptotically normal with expected value and variance given by

$$(14.53) \qquad E(m) \sim \frac{1}{2}\log n \quad \text{and} \quad Var(m) \sim \frac{1}{2}\log n \quad .$$

The asymptotic distribution of structure size is given by

(14.54) $P\{c=j\}$ = (number of ways in which a set of j elements can be chosen) $\times$ (Probability that this set forms the picked structure)

$$\sim \binom{n}{j} \cdot \frac{j}{n} \; \frac{e^{-j} j^j \beta^{*kj}}{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^j k!^j} \; \frac{e^{-(n-j)}(n-j)^{n-j}\beta^{*k(n-j)}}{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^{n-j}k!^{n-j}}$$

$$\frac{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^n k!^n}{e^{-n} n^n \beta^{*kn}} \int_0^\infty \frac{(k-(k-1)\beta^*)}{j} e^{-\frac{r^2}{2j}(k-(k-1)\beta^*)} dr$$

$$= \frac{(n-1)! j^j (n-j)^{n-j}}{(j-1)!(n-j)! n^n} \; \frac{1}{2} \left(\frac{2\pi}{j}\right)^{1/2}$$

$$= \frac{n^{n-\frac{1}{2}} j^{j-\frac{1}{2}} (n-j)^{n-j}}{2j^{j-\frac{1}{2}}(n-j)^{n-j+\frac{1}{2}} n^n}$$

$$= \frac{1}{2n^{1/2}(n-j)^{1/2}} .$$

The asymptotic density of $x = c/n$ is given by

(14.55) $$p(x) = \frac{1}{2(1-x)^{1/2}} \qquad 0 \leq x \leq 1 .$$

These asymptotic results are the same as the results for the general case of all transformations of X into X.

To obtain the asymptotic distribution of predecessors, we consider the following: For any pair (x,T), the probability that x has exactly i predecessors in T is given by $E_{**}(\frac{n_i}{n}) \to \frac{\alpha \beta^{*i}}{i!}$ as n becomes large. Also, as $n \to \infty$, $P\{\bar{x} = x\} = 1/n \to 0$. Hence, writing for convenience, $\frac{\alpha \beta^{*i}}{i!} = \theta_i$ and $P\{p = j\} = \psi_j$, we have

$$(14.56)\qquad \begin{aligned}
\psi_1 &= \theta_0 \\
\psi_2 &= \theta_1 \psi_1 \\
&\vdots \\
\psi_j &= \sum_{i=1}^{k} \theta_i \cdot \sum_{\substack{r_1, r_2, \ldots, r_i \\ \sum_{t=1}^{i} r_t = j-1}} \psi_{r_1} \psi_{r_2} \cdots \psi_{r_i} \\
&\vdots
\end{aligned}$$

Consider the function $y = h(x) = \sum_{t=1}^{\infty} x^t \psi_t$. Then

$$(14.57)\qquad \begin{aligned}
y = h(x) &= x\Big(\theta_0 + \theta_1 \sum_{r_1=1}^{\infty} x^{r_1} \psi_{r_1} + \theta_2 \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} x^{r_1+r_2} \psi_{r_1} \psi_{r_2} \\
&\qquad + \ldots + \theta_k \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} x^{r_1+r_2+\ldots+r_k} \psi_{r_1} \cdots \psi_{r_k}\Big) \\
&= x(\theta_0 + \theta_1 h(x) + \theta_2 h^2(x) + \ldots + \theta_k h^k(x)) \\
&= x(\theta_0 + \theta_1 y + \theta_2 y^2 + \ldots + \theta_k y^k) \\
&= x g(y) \qquad \text{, say.}
\end{aligned}$$

Then

$$(14.58)\qquad h(x) = \sum_{j=1}^{\infty} \frac{\left(h^{(j)}(x)\big|_{x=0}\right)}{j!} x^j = \sum_{j=1}^{\infty} \frac{\left(\frac{d^{(j-1)}}{dy} g^j(y)\big|_{y=0}\right)}{j!} x^j$$

so that

$$(14.59)\qquad \begin{aligned}
\psi_j &= \frac{1}{j!}\left(\frac{d^{(j-1)}}{dy} g^j(y)\Big|_{y=0}\right) = \frac{1}{j!}\left(\frac{d^{(j-1)}}{dy} (\theta_0 + \theta_1 y + \ldots + \theta_k y^k)^j\Big|_{y=0}\right) \\
&= \frac{1}{j} \sum_{\substack{r_0, r_1, \ldots, r_k \\ \sum_{t=0}^{k} r_t = j \\ \sum_{t=1}^{k} t r_t = j-1}} \frac{j!\, \theta_0^{r_0} \theta_1^{r_1} \cdots \theta_k^{r_k}}{r_0!\, r_1! \cdots r_k!}
\end{aligned}$$

$$= \frac{(\beta^*-1)^j k!^j}{\beta^{*jk+1}\, j} \sum_{\substack{r_0, r_1, \ldots, r_k \\ \sum_{t=0}^{k} \frac{r_t}{j} = 1 \\ \sum_{t=1}^{k} \frac{t r_t}{j} = 1 - \frac{1}{j}}} \frac{j!}{r_0! r_1! \ldots r_k! \, 2!^{r_2} \ldots k!^{r_k}}$$

$$\sim \frac{(\beta^*-1)^j k!^j}{\beta^{*jk+1}\, j} \frac{\beta^{*kj}}{(2\pi j)^{1/2}(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^j k!^j}$$

$$= \frac{1}{(2\pi)^{1/2} j^{3/2} \beta^* (k-(k-1)\beta^*)^{1/2}}$$

for large j.

When $k = 2$, $\theta_0 = \frac{1}{2}(2-\sqrt{2})$, $\theta_1 = \frac{1}{2}(2-\sqrt{2})\sqrt{2}$, $\theta_2 = \frac{1}{2}(2-\sqrt{2})$ and $y = h(x) = \frac{1}{2}(2-\sqrt{2})\, X(1+\sqrt{2}\, y+y^2)$. From this we have

$$\psi_j = \frac{(\sqrt{2}-1)^j}{j\sqrt{2}} \sum_{r_2=0}^{[\frac{j-1}{2}]} \frac{j!}{(r_2+1)!(j-1-2r_2)!\, r_2!\, 2^{r_2}}$$

$$\sim \frac{1}{(2\pi)^{1/2} j^{3/2} \sqrt{2}\, (2-\sqrt{2})^{1/2}}$$

for large j.

REFERENCE

(1) Gontcharoff, V. : On the Field of Combinatory Analysis. Bulletin de l'Academie des Sciences U.R.S.S. Serie Mathematique, Vol. 8, No. 1 (1944) pp 1-48 .

Figure 1

Structure Classes in $\mathcal{J}$ when $n = 5$, $n^n = 3.125$

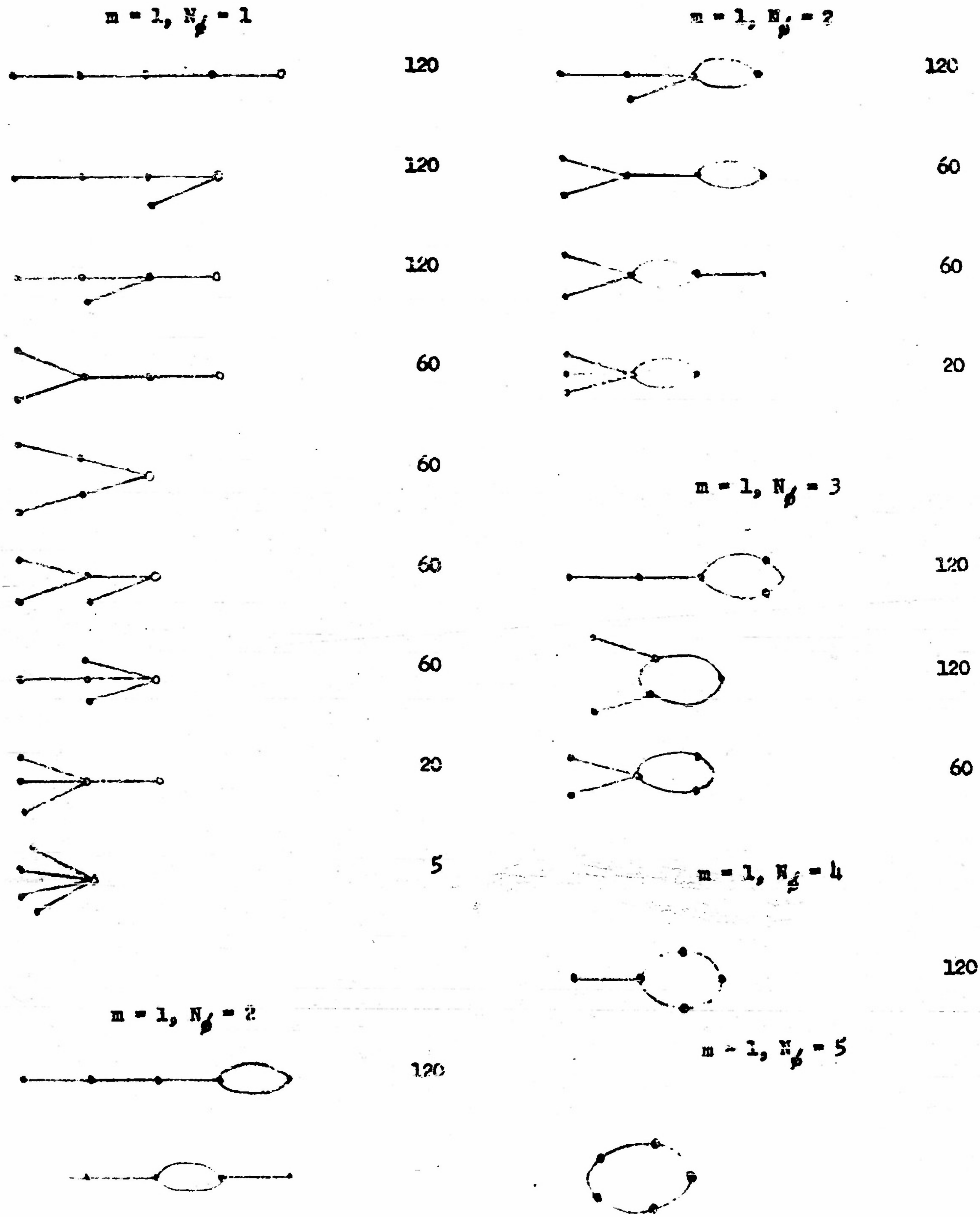


Figure 1 (Continued)

Structure Classes in $\mathcal{J}$ when $n = 5$, $n^n = 3.125$

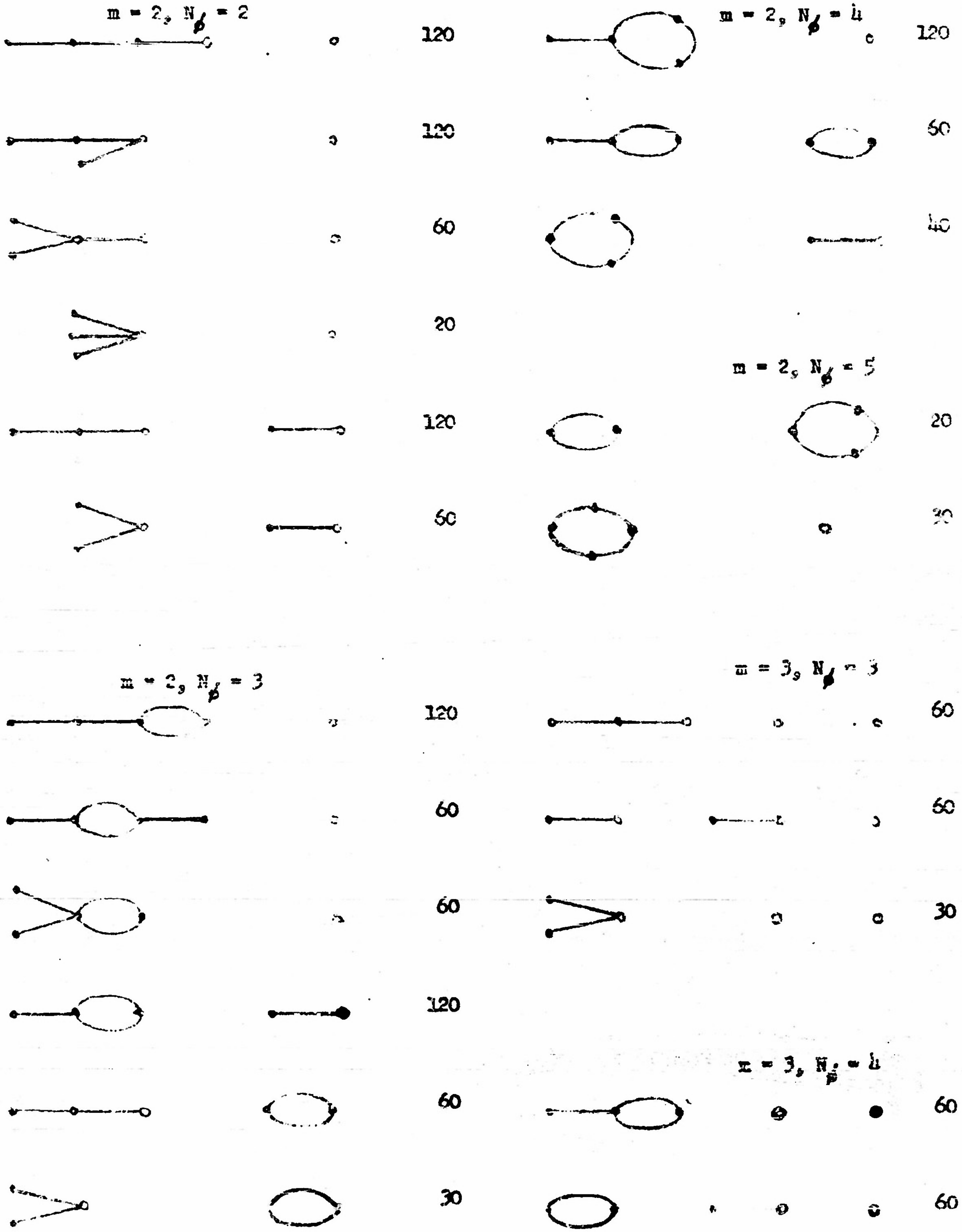


Figure 1 (continued)

Structure Classes in $\mathcal{J}$ when $n = 5$, $n^n = 3.125$

$m = 3$, $N_\phi = 5$

20

15

$m = 4$, $N_\phi = 4$

20

$m = 4$, $N_\phi = 5$

10

$m = 5$, $N_\phi = 5$

1

TABLE 1

Probability of i Cycles in a Permutation of j Elements

$\alpha(i, j)$

i \ j		2	3	4	5
1	1.00000 00000	.50000 00000	.33333 33333	.25000 00000	.20000 00000
2		.50000 00000	.50000 00000	.45833 33333	.41666 66667
3			.16666 66667	.25000 00000	.29166 66667
4				.04166 66667	.08333 33333
5					.00833 33333

i \ j	6	7	8	9	10
1	.16666 66667	.14285 71429	.12500 00000	.11111 11111	.10000 00000
2	.38055 55555	.35000 00000	.32410 71429	.30198 41270	.28289 68254
3	.31250 00000	.32222 22222	.32569 44444	.32551 80776	.32316 46825
4	.11805 55556	.14583 33333	.16788 19445	.18541 66667	.19942 68078
5	.02083 33333	.03472 22222	.04861 11111	.06136 34259	.07421 87500
6	.00138 88889	.00416 66667	.00798 61111	.01250 00000	.01743 63426
7		.00019 84127	.00069 44444	.00150 46296	.00260 41666
8			.00002 [illegible]	.00009 92064	.00023 97487
9				.00000 27557	.00001 24008
10					.00000 02756

i \ j	11	12	13	14	15
1	.09090 90909	.08333 33333	.07692 30769	.07142 85714	.06666 66667
2	.26626 98413	.25165 64454	.23870 85137	.22715 24111	.21677 08218
3	.31950 39682	.31506 77910	.31018 99952	.30508 41751	.29983 87[illegible]
4	.21067 57055	.21974 47273	.22707 72707	.23301 38939	.23781 85793
5	.08560 13007	.09602 41678	.10554 11339	.11422 22866	.12214 17271
6	.02259 83796	.02784 06231	.03309 28957	.03826 77699	.04333 14043
7	.00395 25463	.00550 63657	.00722 50009	.00907 27077	.01101 90452
8	.00045 46958	.00074 61833	.00111 23512	.00154 89690	.00205 05516
9	.00003 30688	.00006 82044	.00012 03566	.00019 12133	.00028 17303
10	.00000 13779	.00000 40188	.00000 89562	.00001 69133	.00002 85333
11	.00000 00250	.00000 01378	.00000 04363	.00000 10449	.00000 21028
12		.00000 00022	.00000 00125	.00000 00428	.00000 01096
13			.00000 00002	.00000 00010	.00000 00038
14				.00000 00000	.00000 00001
15					.00000 00000

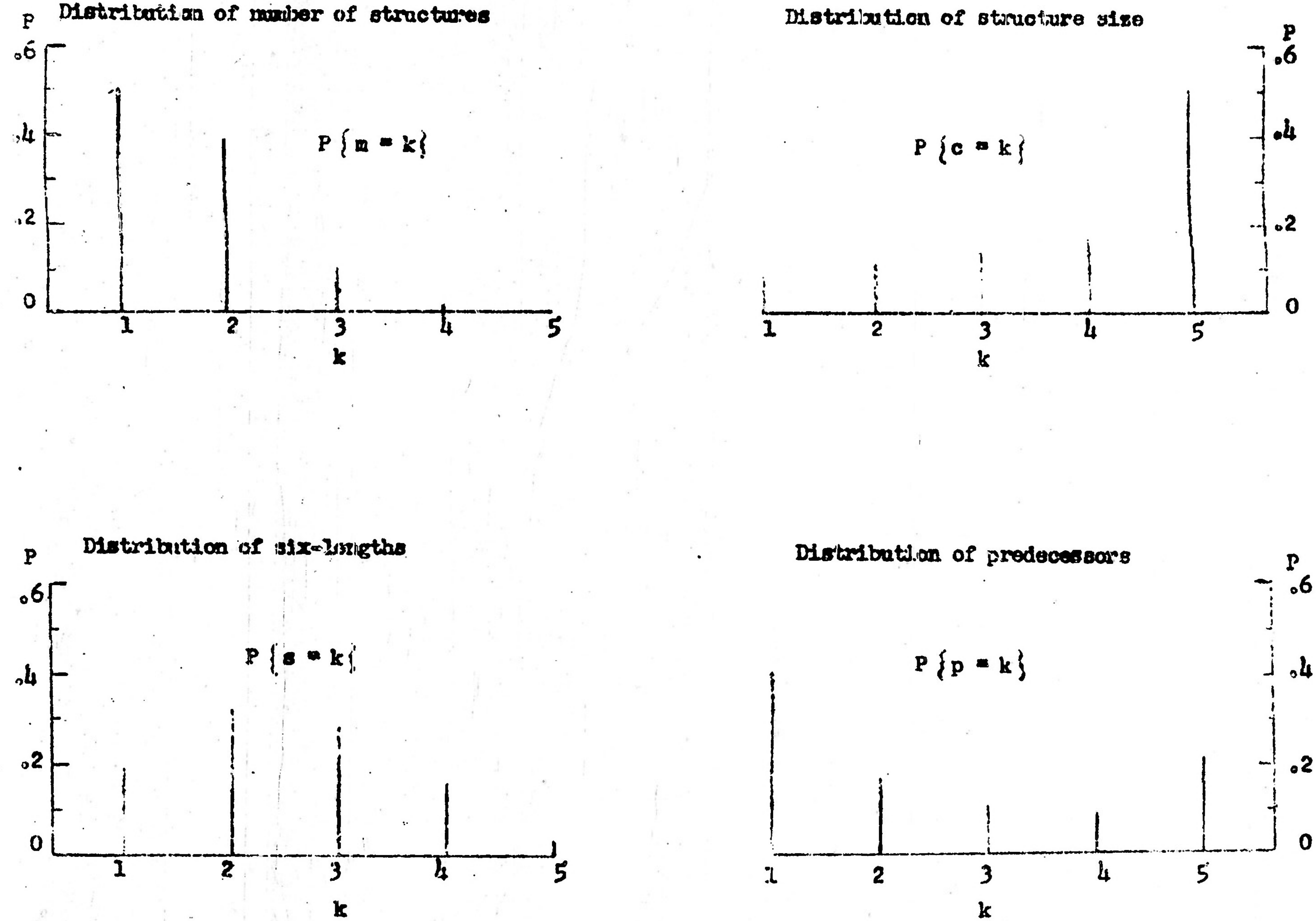
Probability of i Cycles in a Permutation of j Elements

$\alpha(i, j)$

i \ j	16	17	18	19	20
1	.06250 00000	.05882 35294	.05555 55555	.05263 15790	.05000 00000
2	.20738 93121	.19886 64114	.19108 62513	.18395 30567	.17738 69828
3	.29469 38553	.28955 82939	.28451 98560	.27960 22979	.27481 98359
4	.24169 79633	.24481 53687	.24730 10868	.24925 99694	.25077 70858
5	.12937 15303	.13597 89677	.14202 54344	.14756 62582	.15265 09431
6	.04825 70495	.05302 84895	.05763 68494	.06207 83539	.06635 27492
7	.01303 85676	.01511 02430	.01721 68122	.01934 41826	.02148 08912
8	.00261 10825	.00322 44639	.00388 47850	.00458 64706	.00532 43562
9	.00037 22817	.00052 27994	.00067 26919	.00084 19389	.00102 91655
10	.00004 43581	.00006 48242	.00009 02673	.00012 09317	.00015 69821
11	.00000 37547	.00000 61431	.00000 94031	.00001 36592	.00001 90228
12	.00000 02342	.00000 04413	.00000 07580	.00000 12130	.00000 18354
13	.00000 00104	.00000 00236	.00000 00468	.00000 00842	.00000 01407
14	.00000 00003	.00000 00009	.00000 00022	.00000 00045	.00000 00085
15	.00000 00000	.00000 00000	.00000 00001	.00000 00002	.00000 00004
16	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00001
17		.00000 00000	.00000 00000	.00000 00000	.00000 00000
18			.00000 00000	.00000 00000	.00000 00000
19				.00000 00000	.00000 00000
20					.00000 00000

i \ j	21	22	23	24	25
1	.04761 90476	.04545 45454	.04347 82609	.04166 66667	.04000 00000
2	.17132 09360	.16569 81229	.16047 01413	.15559 54796	.15103 83271
3	.27018 01761	.26568 65743	.26133 92503	.25713 63709	.25307 47353
4	.25192 19787	.25275 18968	.25331 42740	.25364 86480	.25378 81569
5	.15732 36170	.16162 35425	.16558 56449	.16924 10045	.17261 73104
6	.07046 21870	.07441 04338	.07820 23081	.08184 32805	.08533 91895
7	.02361 76464	.02574 69436	.02786 27476	.02996 02292	.03203 55513
8	.00609 37150	.00689 02574	.00771 01133	.00854 98064	.00940 62237
9	.00123 36984	.00145 46082	.00169 09408	.00194 17396	.00220 60623
10	.00019 85146	.00024 55685	.00029 81354	.00035 61690	.00041 95918
11	.00002 55923	.00003 34524	.00004 26748	.00005 33190	.00006 54330
12	.00000 26538	.00000 36965	.00000 49902	.00000 65604	.00000 84307
13	.00000 02214	.00000 03320	.00000 04782	.00000 06662	.00000 09020
14	.00000 00148	.00000 00242	.00000 00376	.00000 00559	.00000 00803
15	.00000 00008	.00000 00014	.00000 00024	.00000 00039	.00000 00060
16	.00000 00001	.00000 00001	.00000 00002	.00000 00002	.00000 00005
17	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000
18	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000
19	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000
20	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000
21		.00000 00000	.00000 00000	.00000 00000	.00000 00000
22			.00000 00000	.00000 00000	.00000 00000
23				.00000 00000	.00000 00000
24					.00000 00000
25					

Figure 2


Distribution of number of structures
P {m = k}
Distribution of structure size
P {c = k}
Distribution of six-lengths
P {s = k}
Distribution of predecessors
P {p = k}
p
.6
.4
.2
0
1
2
3
4
5
k