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THE MOTION OF A FLOATING SPHERE IN SURFACE WAVES

BY

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Introduction:

In a recent report[(1)*], the author has considered some approximate solutions to the problem of a freely floating circular cylinder in surface gravity waves. The object of the present report is the treatment of the analogous problem for a semi-immersed sphere.

The general problem of floating obstacles has been treated in detail by Fritz John[(2)]. The results, however, while establishing the uniqueness and existence of solutions, are cast in terms of integral equations which in the three dimensional case possess infinite kernels. This being the case, attempts to obtain numerical solutions along these lines would seem to present formidable problems.

An explicit solution in the case of a semi-immersed sphere is obtained in this report by a generalization of the method of separation of variables. The method is based on the procedure used by Ursell[(3)] in the case of the circular cylinder. In order to satisfy all boundary conditions and obtain the correct behavior at infinity, the velocity potential is taken as a superposition of that due to a point source and an infinite series of non-orthogonal harmonic functions. To obtain the motion of a freely floating sphere we first solve the problem of determining the motion produced by forced oscillation. A correction term is then added to account for the incident wave, and a solution is obtained which contains, as unknowns, the amplitude and phase of the motion of the sphere. These quantities may be fixed by means of the equations of motion.

Some remarks on the convergence of the series are made and it is established that the series in the forced oscillation problem converges, provided the frequency of motion is sufficiently small, the proof yielding at the same time an alternative method for the calculation of the coefficients in the expansion.

Some numerical work is underway, but due to the limited computing facilities available, is progressing slowly. In view of the success of the methods in the report on the cylinder, it would be expected that the technique should lead to fairly accurate results.

1. Formulation of the Problem:

We suppose an incompressible, non-viscous fluid, of infinite depth, to fill the region $y > o$ in its undisturbed state. The x-z plane is then to coincide with the free surface. We suppose further that in the absence of any obstacles the motion of the fluid is that of a two-dimensional, sinusoidal,

* Numbers in parentheses refer to References at end of paper.

gravity wave of frequency σ. If now a rigid sphere, of radius, a, is introduced into the free surface, it will move under the action of the incoming waves. In its equilibrium position the sphere is to be semi-immersed with its center at the origin. The motions are assumed to be small and periodic with frequency σ.

A complete determination of the motion is possible if one succeeds in finding the velocity potential, $\phi(x,y,z,t)$, which is a function, harmonic in the domain of the fluid and satisfying certain boundary conditions. The surface elevation above $y = 0$, $\eta(x,z,t)$, and the pressure $p(x,y,z,t)$ are obtained from ϕ by means of the equations,

$$\eta = \frac{1}{g} \frac{\partial \phi}{\partial t}\Big|_{y=0} \tag{1.1}$$

$$p = \rho g \frac{\partial \phi}{\partial t} \tag{1.2}$$

The assumption of sinusoidal time variation means we may take,

$$\phi(x,y,z,t) = \mathrm{Re}\,(W(x,y,z)\, e^{-i\sigma t}) \tag{1.3}$$

where, within the framework of small motions W satisfies,

$$W_{xx} + W_{yy} + W_{zz} = 0 \text{ in fluid} \tag{1.4}$$

$$W_y + KW = 0 \text{ on free surface,} \quad K = \sigma^2/g \tag{1.5}$$

In addition, if v_n is the normal velocity of the sphere,

$$\frac{\partial W}{\partial n} = v_n \text{ on } x^2 + y^2 + z^2 = a^2 \tag{1.6}$$

To deal with the problem of the freely floating sphere, we have first to solve an auxiliary problem in which the sphere moves with a forced vertical oscillation of frequency, σ, in a fluid initially at rest. Introducing spherical co-ordinates (ρ, θ, χ),

$$\rho^2 = x^2 + y^2 + z^2 ; \quad \theta = \tan^{-1} \frac{y}{\sqrt{x^2 + z^2}} \qquad \chi = \tan^{-1} z/x,$$

and letting,

$$Y = Y_0\, e^{-i(\sigma t + \epsilon)}$$

represent the vertical motion of the sphere, condition (1.6) becomes,

$$\frac{\partial W}{\partial \rho}\Big|_{\rho = a} = -i\sigma Y_0\, e^{-i\epsilon} \cos\theta \tag{1.6'}$$

This auxiliary problem is not yet completely formulated. Since we are concerned with an infinite region it is necessary to specify conditions at infinity. One can show by a simple argument involving energy conservation that the wave motion due to a bounded obstacle should decay like $(x^2 + z^2)^{-1/2}$ as $(x^2 + z^2)$ tends to infinity along the free surface. Moreover the work of Fritz John[2] shows that this condition leads to a correctly formulated problem, that is, there is one and only one solution. We thus prescribe the further condition,

$$W = 0(\rho^{-1/2}) \quad \text{as } x^2 + z^2 \to \infty \text{ with bounded } y \tag{1.7}$$

The problem of the forced oscillation of a sphere involves symmetry about the y-axis. It is convenient in numerical work to make use of this symmetry in the introduction of the stream function $V(x,y,z)$. If $r^2 = x^2 + z^2$, V is related to W by

$$\frac{\partial V}{\partial r} = -r\frac{\partial W}{\partial y} \qquad \frac{\partial V}{\partial y} = r\frac{\partial W}{\partial r} \tag{1.8}$$

In terms of V, the condition (1.6') becomes,

$$V = \frac{1}{4} i\sigma Y_0 a^2 e^{i\theta} \cos 2\theta \tag{1.6''}$$

Once the forced oscillation problem has been solved, one may proceed to the case of the freely floating sphere. The velocity potential of the incident plane wave is

$$W^{(i)} = A e^{-Ky} e^{iKx} \tag{1.9}$$

where A is a constant. We assume the motion of the sphere to be of the form,

$Y = Y_0 e^{-i(\sigma t + \alpha)}$, where Y_0 and α are to be determined. We denote by $W^{(F)}$ the solution of the corresponding forced oscillation problem and suppose the total velocity potential to be given by,

$$W = W^{(i)} + W^{(F)} + W^{(n)}$$

where $W^{(n)}$ is chosen so as to satisfy,

$$\frac{\partial W^{(n)}}{\partial \rho} = -\frac{\partial W^{(i)}}{\partial \rho} \quad \text{on} \quad \rho = a \tag{1.10}$$

From the resulting solution Y_0 and α may be determined from the equations of motion.

The results of our analysis will be a determination of a wave function, W, satisfying the correct boundary conditions and behaving properly at infinity. The question of uniqueness, however, remains partially open, since the work of Fritz John assures a unique solution to the problem of the freely floating sphere only under the condition $Ka < 3/2$

2. Formal Solution:

A discussion of the functions involved in the solution to our problem is given in Appendix I. It is shown that in order to find a potential satisfying equations (1.4), (1.5), (1.6'') and (1.7) one is led to a superposition of characteristic functions

$$W_{2k} = \frac{P_{2k}(\cos\theta)}{\rho^{2k+1}} - \frac{K}{2k}\frac{P_{2k-1}(\cos\theta)}{\rho^{2k}} \qquad k = 1, 2, \ldots \tag{2.1}$$

where the P_n's are Legendre polynomials, and the potential $S\,W_0^*$, due to a point source of strength, S, at the origin,

$$W_0^* = \frac{2}{\rho} + 2\pi k i e^{-ky} H_0^{(1)}(kr) - \frac{4k}{\pi}\int_0^\infty \frac{k\cos\tau y + \tau\sin\tau y}{\tau^2 + k^2} K_0(\tau r)\,d\tau \qquad (2.2)$$

where

$$H_0^{(1)}(kr) = J_0(kr) + i Y_0(kr) \qquad K_0(\tau r) = \frac{\pi i}{2} H_0^{(1)}(kr)$$

The corresponding stream functions are,

$$V_{2k} = \frac{P_{2k}(\cos\theta) - \cos\theta\, P_{2k}(\cos\theta)}{\rho^{2k}} - \frac{k}{2k-1}\,\frac{P_{2k}(\cos\theta) - \cos\theta\, P_{2k-1}(\cos\theta)}{\rho^{2k-1}} \qquad (2.3)$$

$$V_0^* = -\frac{2y}{\rho} + 2ki\pi r H_1^{(1)}(kr) + \frac{4k}{\pi}\int_0^\infty \frac{k\sin\tau y - \tau\cos\tau y}{\tau^2 + k^2}\, r K_1(\tau r)\,d\tau \qquad (2.4)$$

with $K_1(\tau r) = -\frac{\pi}{2} H_1^{(1)}(i\tau r)$. For the stream function, V, of the forced oscillation problem we take,

$$V/S = V_0^* + \sum_1^\infty A_{2m} V_{2m}(\rho,\theta)\, a^{2m} \qquad (2.5)$$

The constant, S, can be related to η_∞, the wave amplitude at infinity. From equation (1.1) together with the asymptotic expression for the Hankel functions, we find for the waves produced by the source, $S\,W_0^*$,

$$\eta_\infty = \frac{2S\sigma}{g}\sqrt{2\pi k/r}$$

or writing $b = \sqrt{r}\,\eta_\infty$,

$$S = \frac{g b}{2\sigma}\sqrt{\frac{1}{2\pi k}} \qquad (2.6)$$

Substituting the series (2.5) into equation (1.6'') gives,

$$F\cos 2\theta = V_0^*(a,\theta) + \sum_1^\infty A_{2m} V_{2m}(a,\theta)\, a^{2m} \qquad (2.7)$$

where F is independent of θ. F is determined by setting $\theta = \pi/2$,

$$F = -V_0^*(a,\pi/2) + \sum_1^\infty A_{2m}\frac{ka}{2m-1} P_{2m}(0) \qquad (2.8)$$

and may be eliminated between (2.7) and (2.8) to give,

$$V_0^*(a,\theta) + V_0^*(a,\pi/2)\cos 2\theta = \sum_1^\infty A_{2m} f_{2m}(a,\theta).$$

$$f_{2m}(a,\theta) = -\left\{\cos\theta\, P_{2m}(\cos\theta) - P_{2m-1}(\cos\theta) - \frac{ka}{2m-1}\left[\cos\theta\, P_{2m-1}(\cos\theta) - P_{2m}(\cos\theta) - P_{2m}(0)\cos 2\theta\right]\right\} \qquad (2.9)$$

The procedure now is to determine any desired number of the A_{2m} by a numerical process. Such a process involving solution of linear equations by an interation process is contained in the construction used in the next section to verify the convergence of the solution. However, a more convenient process is that of least squares since it leads to symmetric matrices with a corresponding decrease in the numerical work. It might be remarked that the rather cumbersome nature of the f_{2m}'s is a consequence of the fact that we do not know, a priori, the value of S and it must be eliminated in the computations.

Once the A_{2m} 's are determined, we can find F from Equation (2.7), and comparison with equation (1.6'') yields,

$$\eta_\infty = \zeta_0 \sqrt{\frac{\pi}{2}} (\kappa a)^{3/2} \frac{1}{|F|} \sqrt{\frac{a}{r}} \qquad \epsilon = \arg F \tag{2.10}$$

For the motion of the sphere, we have

$$Y = \frac{4i S}{\sigma a^2} F e^{-i\sigma t} \tag{2.11}$$

Making use of Equation (.2) to determine the pressure we can also compute the vertical force on the sphere in the form,

$$F_v(t) = 2a^2 \int_0^{\pi/2} p(a,\theta,t) \cos\theta \sin\theta \, d\theta = -2i\rho g \sigma e^{-i\sigma t} a^2 \int_0^{\pi/2} W(a,\theta) \sin\theta \cos\theta \, d\theta$$

$$= -2i\rho g \sigma S a^2 e^{-i\sigma t} \int_0^{\pi/2} \left\{ W_0^*(a,\theta) + \sum_1^\infty A_{2m} W_{2m}(a,\theta) a^{2m} \right\} \cos\theta \sin\theta \, d\theta$$

$$= -2i\rho g \sigma S a e^{-i\sigma t} \left\{ \int_0^{\pi/2} a W_0^*(a,\theta) \cos\theta \sin\theta \, d\theta + \sum_1^\infty (-1)^{m+1} \frac{2m!}{2^{2m}(2m-1)(2m+2)(m!)^2} A_{2m} \right\}$$

$$= -2i\rho g \sigma S a \quad M(\kappa a) e^{-i\sigma t} \tag{2.12}$$

Comparing equations (2.11) and (2.12) we may rewrite the force as,

$$F_v(t) = \frac{1}{2} \rho g a^3 \left\{ \frac{Re(M\bar{F})}{|F|^2} \right\} \frac{d^2 Y}{dt^2} + \frac{1}{2} \rho g \sigma a^3 \left\{ \frac{Im(M\bar{F})}{|F|^2} \right\} \frac{dY}{dt} \tag{2.13}$$

with bars denoting complex conjugates. This latter form is the familiar decomposition of the force into a term involving a virtual mass times acceleration, and a damping factor times velocity.

Having considered the forced oscillation problem, we may turn to the motion of a freely floating sphere. We note that in the above solution was fixed by a knowledge of Ka (since we prescribed the phase of the wave motion at infinity). In order to obtain the potential for an arbitrary phase, α , which is needed in order to proceed as outlined in Section 1, we replace t by $t + \frac{\epsilon - \alpha}{\sigma}$ in the previous result. The corresponding W will be denoted by W_α^F and,

$$W_\alpha^F = W^F e^{-i(\epsilon-\alpha)}.$$

where W is the solution obtained above.

Our task, then, is to find a function, $W^{(n)}$, satisfying equations (1.4) and (1.5) as well as equation (1.6) with $v_n = -\frac{\partial W^{(n)}}{\partial n}$. The motion is no longer symmetric with respect to χ and requires more general functions,

$$W_{2k}^{2m} = \left\{ \frac{P_{2k}^{2m}(\cos\theta)}{\rho^{2k+1}} - \frac{K}{2k-2m} \frac{P_{2k-1}^{2m}(\cos\theta)}{\rho^{2k}} \right\} e^{2imx} \qquad k = 1, 2, \cdots \; ; \; m = 0, 1, \cdots k-1$$

$$W_{2k}^{2m-1} = \left\{ \frac{P_{2k+1}^{2m-1}(\cos\theta)}{\rho^{2k+2}} - \frac{K}{2k-2m+2} \frac{P_{2k}^{2m-1}(\cos\theta)}{\rho^{2k+1}} \right\} e^{(2m-1)\chi} \qquad k = 1, 2, \cdots \; ; \; m = 1, 2, \cdots k \qquad (2.14)$$

the $P_n^m(\cos\theta)$ being associated Legendre functions.

With $y = \rho\cos\theta$, $x = \rho\sin\theta\cos\chi$, we have from equation (1.9)

$$\left.\frac{\partial W^{(n)}}{\partial\rho}\right|_{\rho=a} = -KAe^{-ika\cos\theta}\{[\cos\theta\cos(ka\sin\theta\cos\chi) + \sin\theta\cos\chi\sin(ka\sin\theta\cos\chi)]$$

$$+ i[\cos\theta\sin(ka\sin\theta\cos\chi) - \sin\theta\cos\chi\cos(ka\sin\theta\cos\chi)]\}$$

$$= -KAe^{-ka\cos\theta}\{R(ka,\theta,\chi) + iI(ka,\theta,\chi)\} \qquad (2.15)$$

Now R and I are even functions of χ with respect to $\chi = 0$, hence each will have an expansion in $\{W_{2k}^{\lambda}\}$ with $e^{i\lambda\chi}$ replaced by $\cos\lambda\chi$. Moreover, R and I are even and odd respectively with respect to $\chi = \pi/2$, hence we take,

$$W^{(n)} = KA\sum_{n=1}^{\infty}\sum_{m=0}^{n-1} A_{2n}^{2m} a^{2n+2}\left\{\frac{P_{2n}^{2m}(\cos\theta)}{\rho^{2n+1}} - \frac{K}{2n-2m}\frac{P_{2n-1}^{2m}(\cos\theta)}{\rho^{2n}}\right\}\cos 2m\chi$$

$$+ iKA\sum_{n=1}^{\infty}\sum_{m=1}^{n} A_{2n}^{2m-1} a^{2n+3}\left\{\frac{P_{2n+1}^{2m-1}(\cos\theta)}{\rho^{2n+2}} - \frac{K}{2n-2m+2}\frac{P_{2n}^{2m-1}(\cos\theta)}{\rho^{2n+1}}\right\}\cos(2m-1)\chi$$

$$(2.16)$$

with

$$e^{-ka\cos\theta}R(ka,\theta,\chi) = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} A_{2n}^{2m}\left\{(2n+1)P_{2n}^{2m}(\cos\theta) - \frac{2n\,ka}{2n-2m}P_{2n-1}^{2m}(\cos\theta)\right\}\cos 2m\chi \qquad (2.17)$$

$$e^{-ka\cos\theta}I(ka,\theta,\chi) = \sum_{n=1}^{\infty}\sum_{m=1}^{n} A_{2n}^{2m-1}\left\{(2n+2)P_{2n+1}^{2m-1}(\cos\theta) - \frac{(2n+1)ka}{2n-2m+2}P_{2n}^{2m-1}(\cos\theta)\right\}\cos(2m-1)\chi \qquad (2.18)$$

The dependence on χ in equations (2.17) and (2.18) may be eliminated by making use of the orthogonality of the $\cos m\chi$ to write,

$$\sum_{n=m+1}^{\infty} A_{2n}^{2m}\left\{(2n+1)P_{2n}^{2m}(\cos\theta) - \frac{2n\,ka}{2n-2m}P_{2n-1}^{2m}(\cos\theta)\right\}$$

$$= \frac{4}{\pi}e^{-ka\cos\theta}\int_0^{\pi/2} R(ka,\theta,\chi)\cos 2m\chi\,d\chi = A_{2m}(ka,\theta)$$

$$(2.19)$$

$$\sum_{n=m}^{\infty} A_{2n}^{2m-1}\left\{(2n+2)P_{2n+1}^{2m-1}(\cos\theta) - \frac{(2n+1)\kappa a}{2n-2m+2} P_{2n}^{2m-1}(\cos\theta)\right\}$$

$$= \frac{4}{\pi} e^{-\kappa a\cos\theta}\int_0^{\pi/2} I(\kappa a,\theta,\chi)\cos(2m-1)\chi \, d\chi = A_{2m-1}(\kappa a,\theta) \tag{2.20}$$

for m = 0, 1, 2, ... in (2.19), 1, 2, 3, ... in (2.20) Finally we may make use of the identities,

$$\int_0^{\pi/2} \cos(z\cos\chi)\cos 2m\chi \, d\chi = \frac{\pi}{2}(-1)^m J_{2m}(z)$$

$$\int_0^{\pi/2} \sin(z\cos\chi)\cos(2m-1)\chi \, d\chi = \frac{\pi}{2}(-1)^{m-1} J_{2m-1}(z)$$

to carry out the integrations for the functions $A_k(\kappa a,\theta)$.

$$A_{2m} = (-1)^m e^{-\kappa a\cos\theta}\left\{2\cos\theta J_{2m}(\kappa a\sin\theta) + \sin\theta\left[J_{2m+1}(\kappa a\sin\theta) - J_{2m-1}(\kappa a\sin\theta)\right]\right\} \tag{2.21}$$

$$A_{2m-1} = (-1)^m e^{-\kappa a\cos\theta}\left\{2\cos\theta J_{2m-1}(\kappa a\sin\theta) + \sin\theta\left[J_{2m-2}(\kappa a\sin\theta) - J_{2m}(\kappa a\sin\theta)\right]\right\} \tag{2.22}$$

The determination of $w^{(n)}$ is thus reduced to a numerical process which is analogous to that used in finding $w^{(F)}$. Once $w^{(n)}$ is computed the vertical force on the sphere may be found from,

$$F_v(t) = -4\sigma\rho g a^2 \int_0^{\pi/2}\int_0^{2\pi} (w^{(i)} + w_\infty^{(F)} + w^{(n)}) \sin\theta\cos\theta \, d\chi \, d\theta \tag{2.23}$$

The dynamic equation of motion,

$$\frac{2}{3}\pi\rho a^3 \frac{d^2 y}{dt^2} + \pi\rho g a^2 y = F_v(t) \tag{2.24}$$

will then yield two equations, by equating real and imaginary parts, for the determination of y_0 and α .

3. Convergence of the Formal Solutions:

The aim of this section is to obtain some results concerning the convergence of the infinite series used to construct solutions in Section 2. The analysis will be made ultimately to rest on well known results in the theory of expansions in series of Legendre functions, however certain manipulations must be performed before one is in a position to invoke these results. We will deal only with the forced oscillation problem. Although similar procedures should carry through for the freely floating sphere.

Before proceeding we wish to make plausible the type of expansion that was adopted in Section 2. For this purpose we consider the two simplified problems which arise when one lets σ become infinitely large or small. The free surface condition (1.5) may then be replaced by the simpler conditions $W = 0$, and $W_y = 0$, respectively. In addition the conditions (1.7) may be deleted and it suffices to demand that W be regular (i.e., vanishing like $1/\rho$) at infinity. Consideration of formula (I.6) of Appendix I shows that $W_0^* \to 0$ as $\sigma \to \infty$, and $W_0^* \to \frac{3}{\rho}$ as $\sigma \to 0$. Moreover, we have,

$$\frac{1}{k} W_2 \longrightarrow - \frac{P_1(\cos\theta)}{\rho} \qquad as \qquad \sigma \to \infty \tag{3.1}$$

$$W_{2k} \longrightarrow \frac{P_{2k}(\cos\theta)}{\rho^{2k+1}} \qquad as \qquad \sigma \to 0 \tag{3.2}$$

Now it is possible to solve the two limiting problems by standard methods of separation of variables. For the case $W = 0$ on the free surface, the solution involves a constant times the dipole potential, $P_1(\cos\theta)/\rho$, which will fit the limiting form of our expansion. Similarly for the case $W_y = 0$, we can fit any given function over the immersed surface of the sphere with an infinite series of even indexed Legendre polynomials. Here again the leading term is provided by W_0^* and the further terms by W_{2k} 's.

These remarks indicate also how the speed of convergence should depend on σ. For the case of large σ , the leading term above should be nearly adequate. If, on the other hand the calculations for the case $\sigma = 0$ are carried out, it is found that the coefficients of the $P_{2m}(\cos\theta)$ are of order $n^{-5/2}$ so that the convergence is slow. Our method will yield a proof of convergence only under the assumption of small σ which, however, from the above discussion will be the case likely to give difficulty.

A precise statement of the main goal of this section is the following:

<u>Theorem:</u> If $f(\theta) = k\cos\theta$ where k is a constant, there exist constants S, A_2, A_4,.. such that the series,

$$\Sigma = S\left\{ W_0^* + \sum_1^\infty A_{2m} W_{2m} a^{2m+2} \right\}$$

has the property

$$\frac{\partial \Sigma}{\partial \rho}\Big|_{\rho=a} = S\left\{ \frac{\partial W_0^*}{\partial \rho}\Big|_{\rho=a} - \sum_1^\infty A_{2m}\left[(2m+1)P_{2m}(\cos\theta) - ka\, P_{2m-1}(\cos\theta)\right] \right\} = f(\theta) \tag{3.3}$$

the convergence of the infinite series being uniform and absolute in $0 \le \theta \le \pi/2$.

Suppose first that equation (3.3) holds. We obtain, then, by setting $\theta = 0$,

$$\frac{k}{S} - \frac{\partial W_0^*(a,0)}{\partial \rho} = - \sum_1^\infty A_{2m}\left[(2m+1)P_{2m}(1) - ka\, P_{2m-1}(1)\right] \tag{3.4}$$

Since $P_n(1) = 1$ for all n we obtain by subtracting equations (3.3) and (3.4).

$$- \frac{\partial W_0^*(a,\theta)}{\partial \rho} + \frac{\partial W_0^*(a,0)}{\partial \rho}\cos\theta = - \sum_1^\infty A_{2m}\left\{(2m+1)(P_{2m}(\cos\theta) - \cos\theta) - ka(P_{2m-1}(\cos\theta) - \cos\theta)\right\} \tag{3.5}$$

We should remark at this point that the series (2.7) is obtained from (3.3) by integration so that the uniform convergence of the latter implies convergence of the former.

Now the left hand member of equation (3.5) is a function, continuous in $0 \le \theta \le \pi/2$, and vanishing on $\theta \equiv 0$. As such it can be extended across $\theta = \pi/2$ as an even function of θ, to define a new function, continuous with a piece-wise continuous first derivative in $0 \le \theta \le \pi$. It will have, then, an expansion of the form,

$$\sum_{m=0}^{\infty} a_{2m} P_{2m}(\cos\theta)(4m+1) \qquad a_{2m} = O(m^{-5/2})$$

and,

$$a_0 = -\sum_{m=1}^{\infty} a_{2m} \tag{3.6}$$

If the series on the right converges uniformly, it defines a function with the same properties, and hence will possess a similar expansion,

$$\sum_{m=0}^{\infty} b_{2m} P_{2m}(\cos\theta)(4m+1)$$

where the b_{2m}'s satisfy (3.6). If equality holds we have necessarily $a_{2m} = b_{2m}$ for $m = 1, 2, \ldots$ and therefore also $a_0 = b_0$.

The proof of our theorem consists in giving a method of construction of a sequence $\{A_{2m}\}$ for which the series does converge, and $b_{2m} = a_{2m}$ for $m = 1, 2..$ We make use of the formula,

$$\int_0^{\pi/2} P_{2m}(\cos\theta) P_{m'}(\cos\theta)\sin\theta \, d\theta \quad \text{if } m = m' ; \qquad = 0 \text{ if } n \ne n' \text{ and } n-n' \text{ is even}$$

$$= (-1)^{(m+m'+1)/2} \frac{m!\,m'!}{2^{m+m'-1}(m-m')(m+m'+1)\left[\frac{m}{2}!\,\frac{m'-1}{2}!\right]^2} \qquad \text{if } n - n' \text{ is odd.}$$

n even, n' odd.

Then if equation (3.5) holds,

$$a_{2m} = -\frac{2m+1}{4m+1} A_{2m} + \frac{2m!\,(-1)^{m+1}}{2^{2m}(2m-1)(2m+2)(m!)^2} \sum_{m=1}^{\infty} A_{2m}(2m+1-ka)$$

$$+ ka \sum_{m=1}^{\infty} (-1)^{m+m} \frac{2m!\,(2m-1)!\,A_{2m}}{2^{2m+2m-2}(2m-2m+1)(2m+2m)(m!)^2(m-1)!^2} \qquad m = 1, 2, 3, \cdots \tag{3.7}$$

and our problem is to establish the existence of a solution of these equations.

In order to proceed we interpret equations (3.7) as a single vector equation. Denote by $\bar{x}^0$ and $\bar{x}$ the infinite dimensional vectors having a_{2m}, and $A_{2m}(2m+1)/(4m+1)$ as components, i.e., $\bar{x}^0 = (x_1^0, x_2^0, \cdots)$, $x_m^0 = a_{2m}$; $\bar{x} = (x_1, x_2 \cdots)$, $x_m = A_{2m}\frac{2m+1}{4m+1}$ Further, by $F(\bar{x})$ denote the matrix,

$$F_{mm} = \frac{2m!\,(4m+1)(4m+1-ka)\,(-1)^{m+1}}{2^{2m}(2m-1)(2m+2)(m!)^2(2m+1)}$$

$$+ \frac{(4m+1)ka\,2m!\cdot(2m-1)!\,(-1)^{m+m}}{2^{2m+2m-2}(2m+1)(2m-2m+1)(2m+2m)(m!)^2(m-1)!^2}$$

and then the equations (3.7) become,

$$\bar{x} = \bar{x}^0 + F(\bar{x}) \tag{3.7'}$$

Consider now the set of all infinite dimensional vectors, $\bar{x}$, satisfying the condition,

$$\|\bar{x}\| = \sum_{m=1}^{\infty} (4m+1)\,|x_m| < \infty \tag{3.8}$$

This set forms a linear, normed vector space X, which, with respect to the norm defined in (3.8), is complete, i.e., is a Banach space. By this we mean that if $\{\bar{x}^n\}$ is a sequence of vectors of X, satisfying $\|x^n - x^m\| \to 0$ as $m, n \to \infty$ there exists a vector $\bar{x}$, also in X, such that $\lim_{n\to\infty} \|\bar{x} - \bar{x}^n\| = 0$. From equation (3.5) the vector $\bar{x}^0$ defined by the a_{2m}'s is contained in X, and we proceed to show that equation (3.7') has a solution.

<u>Lemma</u>: If B is a Banach space, and F is an operator on B satisfying a Lipschitz condition,

$$\|F(\bar{x}) - F(\bar{y})\| \leq M \|\bar{x} - \bar{y}\| \qquad 0 < M < 1 \tag{3.9}$$

then the equation (3.7'), for $\bar{x}^0 \in B$, has a unique solution $\bar{x} \in B$.

<u>Proof</u>: Define the sequence $\{\bar{x}^n\}$ as follows,

$$\bar{x}^0 = \bar{x}^0, \quad \bar{x}^1 = \bar{x}^0 + F(\bar{x}^0), \quad \bar{x}^2 = \bar{x}^0 + F(\bar{x}^1), \quad \cdots \quad \bar{x}^{n+1} = \bar{x}^0 + F(\bar{x}^n), \cdots$$

$$\bar{x}^1 - \bar{x}^0 = F(\bar{x}^0).$$

$$\bar{x}^2 - \bar{x}^1 = F(\bar{x}^1) - F(\bar{x}^0) \qquad \therefore \|x^2 - \bar{x}^1\| \leq M \|F(\bar{x}^0)\|$$

$$\bar{x}^3 - \bar{x}^2 = F(\bar{x}^2) - F(\bar{x}^1) \qquad \therefore \|\bar{x}^3 - \bar{x}^2\| \leq M \|\bar{x}^2 - \bar{x}^1\| \leq M^2 \|F(\bar{x}^0)\|$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\bar{x}^n - \bar{x}^{n-1} = F(\bar{x}^{n-1}) - F(\bar{x}^{n-2}) \qquad \therefore \|x^n - x^{n-1}\| \leq M^{n-1} \|F(\bar{x}^0)\|$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

For $n > m$

$$\bar{x}^n - \bar{x}^m = F(\bar{x}^{n-1}) - F(\bar{x}^{m-1}) = [F(\bar{x}^{n-1}) - F(\bar{x}^{n-2})] + [F(\bar{x}^{n-2}) - F(\bar{x}^{n-3})]$$

$$+ \cdots + [F(\bar{x}^m) - F(\bar{x}^{m-1})]$$

$$\therefore \|\bar{x}^n - \bar{x}^m\| \leq M \|\bar{x}^{n-1} - \bar{x}^{n-2}\| + M \|x^{n-2} - \bar{x}^{n-3}\| + \cdots M \|\bar{x}^m - \bar{x}^{m-1}\|$$

$$\leq F(\bar{x}^\circ)\left\{M^{n-1}+M^{n-2}+\cdots+M^{m}\right\} = M^{m} F(\bar{x}^\circ)\left\{1+M+\cdots+M^{n-m-1}\right\}$$

$$\leq M^{m}\; F(\bar{x}^\circ)\;\frac{1}{1-M}$$

$$\therefore \;\|\bar{x}^{n}-\bar{x}^{m}\| \longrightarrow 0 \quad \text{as} \quad m,n \longrightarrow \infty$$

Hence there exists $\bar{x}$ such that $\|\bar{x}^{m}-\bar{x}\| \longrightarrow 0$ as $m \longrightarrow \infty$. Moreover from the Lipschitz condition we immediately infer $F(\bar{x}) + \bar{x}^\circ = \bar{x}$, as well as the uniqueness of the solution.

In order to enter the form of the lemma we must verify that the matrix F_{mn} satisfies the Lipschitz condition. We have,

$$\|F(\bar{x})-F(\bar{y})\| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2m!}{2^{2m}(2m-1)(2m+2)\,m!^{2}}\left(1+\tfrac{1}{3}Ka\right)(4n+1)\,|x_n-y_n|$$

$$+\,Ka\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2m!\;2n!}{2^{2m+2n}(2n+1)(2m-2n+1)(n+m)(m!)^{2}(n!)(n-1)!}(4n+1)\;|x_n-y_n| \qquad (3.10)$$

We make use of the numerical series,

$$\sum_{1}^{\infty}\frac{2n!}{2^{2n}(n!)^{2}}\;\frac{1}{2n+1} = \frac{\pi}{2}-1$$

and estimate as follows:

$$\sum_{m=1}^{\infty}\frac{2m!}{2^{2m}(2m-1)(2m+2)(m!)^{2}} \leq \frac{3}{4}\sum_{1}^{\infty}\frac{2m!}{2^{2m}(m!)^{2}(2m+1)} = \frac{3}{4}\left(\frac{\pi}{2}-1\right)$$

$$\frac{2n!}{2^{2n}(2n+1)\,n!\,(n-1)!} \leq \frac{1}{6}$$

$$\sum_{m=1}^{\infty}\frac{2m!}{2^{2m}(m+n)(m!)^{2}} \leq \frac{3}{2}\sum_{1}^{\infty}\frac{2m!}{2^{2m}(m!)^{2}(2m+1)} = \frac{3}{2}\left(\frac{\pi}{2}-1\right)$$

From (3.10) we obtain, then,

$$\|F(\bar{x})-F(\bar{y})\| \leq \left(\frac{\pi}{2}-1\right)\left[\frac{3}{4}+\frac{7}{12}Ka\right]\|\bar{x}-\bar{y}\| \qquad (3.11)$$

It follows that the Lipschitz condition (3.9) is satisfied for,

$$Ka < \frac{12}{7}\left[\frac{1}{\frac{\pi}{2}-1}-\frac{3}{4}\right]$$

or approximately, $Ka < 2$

We conclude, therefore, that the equations (3.7) possess a solution, $A_2, A_4, \ldots$ such that

$$\bar{X} = (A_2, A_4, \cdots) \in \mathfrak{X} \quad ; \quad \text{i.e.} \quad A_{2m} = O(m^{-2-\epsilon}) \ , \ \epsilon > 0$$

It follows that the series in the right hand member of equation (3.4) converges and we use that equation to define the constant S. Moreover the A_{2m}'s have been determined so that equation (3.5) holds and by multiplying (3.4) by $\cos\theta$ and adding to (3.5) we deduce that equation (3.3) holds. This completes the proof of the theorem, for sufficiently small ka.

__Summary:__

A solution to the problem of a freely floating semi-immersed sphere is obtained in a form which admits of numerical calculations. The solution consists of fitting an infinite series of non-orthogonal functions to a given boundary condition on a portion of the sphere. Some estimates are made regarding convergence and some affirmative results in this direction are obtained.

Once the numerical calculations of the coefficients are carried out certain quantities of physical interest may be obtained, e.g., the wave motion at large distances from a sphere moving with forced sinusoidal oscillations, and the virtual mass and damping factors for a freely floating sphere.

As in the previous report on the cylinder, certain approximate results could be obtained for the sphere. One could, for example, solve the simpler problems in which the free surface condition is replaced by $W = 0$, $W_y = 0$ corresponding to large and small frequencies. Also a combination technique could be used in which one neglects the diffraction effect for small frequencies using only the potential of the forced oscillation problem, and the simplified condition $W = 0$ for large frequencies.

The simplified problems can be solved by means of Green's functions which can be expressed as source systems. We have not included these results for two reasons. First, the solution obtained in this report seems to be complete in itself without the necessity of appealing to approximations. Second, these problems lend themselves to variational techniques which enable one to determine the changes in the Green's functions with small changes in the shape of the obstacle. It is the author's hope to discuss this approach to the problem elsewhere.

APPENDIX I.
WAVE FUNCTIONS.

1. Characteristic Functions:

We are concerned with the determination of certain functions, which we call wave functions, which will permit the construction of a velocity potential satisfying equations (1.4), (1.5), (1.6) and (1.7). As a first step we indtoduce characteristic functions which satisfy the first two of these equations. These arise in a natural way if one attempts a solution by separation of variables. Consider the functions,

$$W_{2k}^{2m}(\rho,\theta,\chi) = \left\{ \frac{P_{2k}^{2m}(\cos\theta)}{\rho^{2k+1}} - \frac{\kappa}{2k-2m}\frac{P_{2k-1}^{2m}(\cos\theta)}{\rho^{2k}} \right\} e^{2im\chi} \quad k=1,2,\cdots,\ m=0,1,\cdots k-1$$

$$\qquad\qquad (I.1)$$

$$W_{2k}^{2m-1}(\rho,\theta,\chi) = \left\{ \frac{P_{2k+1}^{2m-1}(\cos\theta)}{\rho^{2k+2}} - \frac{k}{2k-2m+2}\frac{P_{2k}^{2m-1}(\cos\theta)}{\rho^{2k+1}} \right\} e^{(2m-1)i\chi} \quad k=1,2,\cdots\ m=1,2,\cdots k$$

where $P_n^m(\cos\theta)$ are associated Legendre functions. These functions are harmonic in $\rho > 0$, and we proceed to show that they satisfy the free surface condition (1.5)

$$P_n^m(0) = \frac{\sqrt{\pi}\qquad 2^m}{\Gamma(\frac{n-m}{2}+1)(-\frac{n-m+1}{2})}$$

so that $P_n^m(0)$ is zero if n m is odd. Similarly from the recursion formula

$$\left.\frac{dP_n^m}{dx}\right|_{x=0} = -(n-m+1)\,P_{n+1}^m(0).$$

we infer that $\frac{dP_n^m(0)}{dx}$ is 0 if m - n is even. Using these results

$$\pm\frac{1}{\rho}\left.\frac{\partial W_{2k}^{2m}}{\partial\theta}\right|_{\theta=\pm\pi/2} + \kappa \left.W_{2k}^{2m}\right|_{\theta=\pm\pi/2} = \left\{-\frac{2k-2m}{2k-2m}\kappa + \kappa\right\}\frac{P_{2k}^{2m}(0)}{\rho^{2k+1}}e^{2im\chi} = 0$$

$$\pm\frac{1}{\rho}\left.\frac{\partial W_{2k}^{2m-1}}{\partial\theta}\right|_{\theta=\pm\pi/2} + \kappa \left.W_{2k}^{2m-1}\right|_{\theta=\pm\pi/2} = \left\{-\frac{2k-2m+2}{2k-2m+2}\kappa + \kappa\right\}\frac{P_{2k-1}^{2m-1}(0)}{\rho^{2k+1}}e^{(2m-1)i\chi} = 0$$

It is apparent that a series of the W_{2k}^{m} will never be adequate for our problem since such a series would vanish at least like ρ^{-2} for large ρ, contradicting the condition (1.7). To eliminate this difficulty, we introduce a "source" solution which will satisfy equations (1.4) and (1.5) and also behave properly at infinity.

2. Source Solution:

To obtain the desired solution we solve equation (1.4) when a point source is admitted at (x', y', z'), that is, we seek W* (x, y, z, x', y', z') satisfying,

$$W^*_{xx} + W^*_{yy} + W^*_{zz} = -\delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{I.2}$$

where the right hand side represents a product of delta functions. We assume initially that the source point is (o, y', o) so that W^* will be independent of θ. Equation (I.2) becomes,

$$\frac{1}{r}\left(r\frac{\partial W^*}{\partial r}\right) + \frac{\partial^2 W^*}{\partial y^2} = -\delta(o)\,\delta(y-y')$$

Applying the Hankel transform,

$$w(p, y) = \int_0^\infty r\,J_0(rp)\,W^*(r, y)\,dr$$

yields,

$$\frac{d^2 w}{dy^2} - p^2 w = -\delta(y-y')\,J_0(o) = -\delta(y-y') \tag{I.3}$$

Equation (I.3) is an ordinary differential equation for $w(p,y)$ which may be solved by application of Fourier transforms. The resulting solution is

$$w(p, y) = A e^{-py} + B e^{py} + \frac{1}{p} e^{-p|y-y'|}$$

The boundedness of w as $y \rightarrow \infty$ and the boundary condition (1.5), yield,

$$w(p, y) = \frac{1}{p}\left\{\frac{p+K}{p-K} e^{-p(y+y')} + e^{-p|y-y'|}\right\} \tag{I.4}$$

The transform $w(p, y)$ is regular in $-\pi/2 < \arg p < \pi/2$ except for a simple pole at $p = K$. The choice of a contour for the inversion integral is dictated by the behavior we want at infinity. We choose a contour, c, which consists of the positive real axis except for a small semi-circle in the lower half-plane, about the point $p = K$. Then,

$$W^*(r, y) = \int_{0\,c}^{\infty} J_0(pr)\left\{\frac{p+K}{p-K} e^{-p(y+y')} + e^{-p|y-y'|}\right\} dp \tag{I.5}$$

Using the identity,

$$\int_0^\infty J_0(bt)\,e^{-at}\,dt = \frac{1}{\sqrt{a^2+b^2}}$$

and replacing r^2 by $(x-x')^2 + (z-z')^2$, we obtain finally,

$$W^*(x, y, z, x', y', z') = \frac{1}{R} + \int_{0\,c}^{\infty} \frac{p+K}{p-K} e^{-p(y+y')} J_0(pr)\,dp \tag{I.6}$$

where $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$.

The function W^* is thus seen to be identical with the Green's function obtained by Fritz John(2). It is interesting to note, in passing, that it may be found by a method used by Kennard(4) in the case of two-dimensional motion. In this procedure one first considers the simplified boundary condition, $W = 0$ on free surface, $y = 0$, and treats the resulting problem as an initial elevation.

It can be seen by direct differentiation that W^* satisfied equation (1.4) and (1.5). In order to demonstrate that it also satisfied condition (1.7) we make a transformation which at the same time yields a form more amenable to numerical computations. From equation (I.6) we can write,

$$W^* = \frac{1}{R} + 2\pi\kappa i e^{-\kappa(y+y')} J_0(\kappa r) + R_e\, P_r V \int_0^\infty \frac{p+\kappa}{p-\kappa} e^{-p(y+y')} H_0^{(1)}(pr)\,dp$$

where $H_0^{(1)}$ is the Hankel function of the first kind. Again,

$$W^* = \frac{1}{R} + 2\pi\kappa i e^{-\kappa(y+y')} H_0^{(1)}(\kappa r) + R_e \int_{\bar{C}_0}^\infty \frac{p+\kappa}{p-\kappa} e^{-p(y+y')} H_0^{(1)}(pr)\,dp$$

where $\bar{C}$ is the image of C in the real axis. Since $H_0^{(1)}(z) = O\left(\frac{e^{iz}}{\sqrt{z}}\right)$ for large $|z|$, we can move the contour of integration to the positive imaginary axis and obtain,

$$W^* = \frac{1}{R} + 2\pi\kappa i e^{-\kappa(y+y')} H_0^{(1)}(\kappa r) + \frac{2}{\pi}\int_0^\infty \frac{(\tau^2-\kappa^2)\cos\tau y' - 2\kappa\tau\sin\tau(y+y')}{\tau^2+\kappa^2} K_0(\tau r)\,d\tau$$

where $K_0(z) = \frac{\pi}{2} i H_0^{(1)}(iz)$ is a quantity which is real for real z. Finally we use the identity,

$$\int_0^\infty K_0(at)\cos bt\,dt = \frac{\pi}{2}\,\frac{1}{\sqrt{a^2+b^2}} \tag{I.7}$$

to write,

$$W^* = \frac{1}{R} + \frac{1}{R'} + 2\pi\kappa i H_0^{(1)}(\kappa r) e^{-\kappa(y+y')} - \frac{4\kappa}{\pi}\int_0^\infty \frac{\kappa\cos\tau(y+y') + \tau\sin\tau(y+y')}{\tau^2+\kappa^2} K_0(\tau r)\,d\tau \tag{I.8}$$

where $R'^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$. Now $K_0(\tau r) = O(e^{\tau r})$ for large r, and hence we see from (I.8) that,

$$W^* = 2i\sqrt{\frac{2\pi\kappa}{r}}\, e^{-\kappa(y+y')} e^{i(\kappa r + \pi/4)} + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty, \text{ y bounded.}$$

Some further steps toward numerical computations are given in Appendix II where it is also shown that the apparent singularity in equation (I.8) vanishes by subtraction.

3. Axially Symmetric Motion:

Suppose the motion is independent of X. It is then permissible to introduce the stream function, defined by

$$\frac{\partial V}{\partial r} = r\,\frac{\partial W}{\partial y} \qquad \frac{\partial V}{\partial y} = r\,\frac{\partial W}{\partial r} \tag{I.9}$$

The stream function V^* corresponding to W^*, with $x' = z' = 0$ is given by,

$$V^* = -\frac{(y-y')}{\sqrt{r^2+(y-y')^2}} + \int_0^\infty \frac{p+k}{p-k} e^{-p(y+y')} r\, J_1(pr)\, dp \tag{I.10}$$

as may be verified on noting that,

$$\frac{d}{dr}\left(r J_1(pr)\right) = r p\, J_0(pr) \qquad ; \qquad \frac{d}{dr} J_0(pr) = -p\, J_1(pr)$$

The integral term in equation (I.10) may be manipulated as before to yield,

$$V^* = -\frac{(y-y')}{\sqrt{r^2+(y-y')^2}} - \frac{2}{\pi}\int_0^\infty \sin \tau(y+y')\, r\, K_1(\tau r)\, d\tau$$

$$+ 2\pi k\, i\, r\, e^{-k(y+y')} H_1^{(1)}(kr) + \frac{4k}{\pi}\int_0^\infty \frac{k \sin \tau(y+y') - \tau \cos \tau(y+y')}{\tau^2+k^2}\, r\, K_1(\tau r)\, d\tau$$

where $K_1(z) = -\frac{\pi}{2} H_1^{(1)}(iz)$. The first integral may be evaluated by noting that, $r\,K_1(\tau r)$ is $-\frac{d}{d\tau} K_0(\tau r)$, integrating by parts, and using the identity, (I.7), quoted before, we obtain,

$$V^* = -\frac{(y-y')}{\sqrt{r^2+(y-y')^2}} - \frac{(y+y')}{\sqrt{r^2+(y+y')^2}} + 2\pi k\, i\, r\, e^{-k(y+y')} H_1^{(1)}(kr)$$

$$+ \frac{4k}{\pi}\int_0^\infty \frac{k \sin \tau(y+y') - \tau \cos \tau(y+y')}{\tau^2+k^2}\, r\, K_1(\tau r)\, d\tau \tag{I.11}$$

Since the motion is to be independent of χ the only characteristic functions to be considered are,

$$W_{2k}(\rho,\theta) = \left\{ \frac{P_{2k}(\cos\theta)}{\rho^{2k+1}} - \frac{k}{2k}\frac{P_{2k-1}(\cos\theta)}{\rho^{2k}} \right\} \qquad k = 1, 2, \ldots$$

The associated stream functions are,

$$V_{2k} = \frac{P_{2k-1}(\cos\theta) - \cos\theta\, P_{2k}(\cos\theta)}{\rho^{2k}} + \frac{k}{2k-1}\frac{P_{2k}(\cos\theta) - \cos\theta\, P_{2k-1}(\cos\theta)}{\rho^{2k-1}} \tag{I.12}$$

To see this we note that in spherical co-ordinates the equation (I.9) reduces to,

$$\frac{1}{\rho}\frac{\partial V}{\partial \theta} = -\rho \sin\theta \frac{\partial W}{\partial \rho} \qquad\qquad \frac{\partial V}{\partial \rho} = \sin\theta \frac{\partial W}{\partial \theta}.$$

and using the identities,

$$(m+1)P_m(x) = \frac{dP_{m+1}}{dx} - x\frac{dP_m}{dx} \quad ; \quad mP_m = x\frac{dP_m}{dx} - \frac{dP_{m-1}}{dx}$$

it may be verified that V_{2k} and W_{2k} satisfy these equations.

The introduction of the stream function V proves a convenience in solving the forced oscillation problem of Section 2, since it eliminates the necessity of numerically computing derivatives of W^*. The results of the last three sections lead one to choose as the stream function for this problem,

$$V = SV_0^* + \sum_1^\infty A_{2m}V_{2m}$$

where V_0^* is the stream function for a source at the origin. The origin is again chosen as a numerical convenience since it slightly simplifies the computations.

APPENDIX II

AIDS TO NUMERICAL COMPUTATIONS

Numerical work based on this report requires computations with the two functions,

$$W_0^*|_{\rho=a} = \frac{2}{\rho} + 2\pi k i e^{-ky}H_0^{(1)}(kr) - \frac{4k}{\pi}\int_0^\infty \frac{k\cos\tau y + \tau\sin\tau y}{\tau^2+k^2}K_0(\tau r)\,d\tau \qquad \text{(II.1)}$$

$$V_0^*|_{\rho=a} = -\frac{2y}{\rho} + 2\pi k i r e^{-ky}H_1^{(1)}(kr) + \frac{4k}{\pi}\int_0^\infty \frac{k\sin\tau y - \tau\cos\tau y}{\tau^2+k^2}\,rK_1(\tau r)\,d\tau \qquad \text{(II.2)}$$

In this form the calculations are lengthy because of the occurrence of the trigonometric functions in the integrand and the infinite range of integration. By suitable manipulations it is possible to reduce these expressions so that they involve only finite intergals with monotonic integrands.

For $W_0^*|_{\rho=a}$ we need the two integrals,

$$R = k\int_0^\infty \frac{\cos\tau y}{\tau^2+k^2}K_0(\tau r)\,d\tau \quad ; \quad S = \int_0^\infty \frac{\tau\sin\tau y}{\tau^2+k^2}K_0(\tau r)\,d\tau$$

where $y = a\cos\theta$, $r = a\sin\theta$. Letting $z = ka$, $\alpha = \cos\theta$, $\beta = \sin\theta$

$$R = z\int_0^\infty \frac{\cos\alpha u}{u^2+z^2}K_0(\beta u)\,du$$

$$S = \int_0^\infty \frac{u\sin\alpha u}{u^2+z^2}K_0(\beta u)\,du.$$

Now

$$K_0(\beta\mu) = \int_0^\infty \frac{\cos \beta\mu t}{\sqrt{t^2+1}}\, dt$$

and we can write,

$$R = \frac{2}{2}\int_0^\infty \frac{dt}{\sqrt{t^2+1}} \int_0^\infty \frac{\cos\mu(\alpha+\beta t) \begin{array}{l} +\cos\mu(\alpha-\beta t) \\ +\cos\mu(\beta t-\alpha) \end{array}}{\mu^2+z^2}\, d\mu \qquad \begin{array}{l} \beta t < \alpha \\ \beta t > \alpha \end{array}$$

$$S = \frac{2}{2}\int_0^\infty \frac{dt}{\sqrt{t^2+1}} \int_0^\infty \frac{\sin\mu(\alpha+\beta t) \begin{array}{l} +\sin\mu(\alpha-\beta t) \\ -\sin\mu(\beta t-\alpha) \end{array}}{\mu^2+z^2}\, d\mu \qquad \begin{array}{l} \beta t < \alpha \\ \beta t > \alpha \end{array}$$

From the identities,

$$\int_0^\infty \frac{x \sin \mu x}{x^2+a^2}\, dx = \frac{\pi}{2} e^{-\mu a} \qquad \text{for } \mu > 0,\ a \geq 0$$

$$\int_0^\infty \frac{\cos \mu x}{x^2+a^2}\, dx = \frac{\pi}{2a} e^{-\mu a} \qquad \text{for } \mu > 0,\ a \geq 0$$

it follows that,

$$\frac{4K}{\pi}(R+S) = 2K e^{-\beta z}\int_0^\infty \frac{e^{-\alpha z t}}{\sqrt{t^2+1}}\, dt + 2K e^{-\alpha z}\int_0^{\alpha/\beta} \frac{e^{\beta z t}}{\sqrt{t^2+1}}\, dt$$

Moreover,

$$\int_0^\infty \frac{e^{-\alpha z t}}{\sqrt{t^2+1}}\, dt = \frac{\pi}{2}\left[S_0(\alpha z) - Y_0(\alpha z)\right]$$

where $S_0(z)$ is the Struve function of order 0. Hence we obtain,

$$W_0^*\Big|_{\rho=a} = \frac{2}{\rho} - \pi K e^{-Ka\cos\theta}\left[S_0(Ka\sin\theta) + Y_0(Ka\sin\theta)\right]$$

$$- 2K e^{-Ka\cos\theta}\int_0^{\operatorname{ctn}\theta} \frac{e^{Kat\sin\theta}}{\sqrt{t^2+1}}\, dt + 2\pi K e^{-Ka\cos\theta} J_0(Ka\sin\theta) \tag{II.3}$$

For small x,

$Y_0(x) \sim -\frac{2}{\pi} \log \frac{2}{\gamma x}$ where γ is Euler's constant. In addition the integral in (II.3) ceases to converge as $\theta \longrightarrow 0$. Hence the formula ceases to be of use at $\theta = 0$. It is most convenient to study W_0^* for small θ, by means of the original equation (II.1). If we replace $K_0(\tau r)$ by its asymptotic estimate, $\log \frac{2}{\gamma \tau r}$ in (II.1) and make use of the formulae,

$$\int_0^\infty \frac{\log x}{x^2+a^2} \cos \mu x \, dx = \frac{\pi}{4a}\left[2e^{-\mu a}\log a + e^{\mu a}\mathcal{E}_i(-\mu a) - e^{-\mu a}\bar{\mathcal{E}}_i(\mu a)\right]$$

$$\int_0^\infty \frac{x \log x}{x^2+a^2} \sin \mu x \, dx = \frac{\pi}{4}\left[2e^{-\mu a}\log a - e^{\mu a}\mathcal{E}_i(-\mu a) - e^{-\mu a}\bar{\mathcal{E}}_i(\mu a)\right]$$

where $\mathcal{E}_i(\varphi)$, $\bar{\mathcal{E}}_i(\varphi)$ are exponential integral functions (see Jahnke and Emde[5]), we obtain,

$$-\frac{4\kappa}{\pi}\int_0^\infty \frac{\kappa \cos \tau y + \tau \sin \tau y}{\tau^2+\kappa^2} K_0(\tau r)\, d\tau \sim 4\kappa e^{-\kappa y}\log \kappa r$$

$$-2\kappa\, \bar{\mathcal{E}}_i(\kappa y) - 2\kappa \log\frac{2}{\gamma}\, e^{-\kappa y} \qquad \text{(II.4)}$$

The limiting value of $W_0^*|_{\rho=a}$ is, therefore,

$$\frac{2}{a} - 2\kappa e^{-\kappa a}\, \mathcal{E}_i^*(\kappa a)$$

Similar calculations can be carried out on the integral in equation (II.2) and lead to,

$$\frac{4\kappa}{\pi}\int_0^\infty \frac{\kappa \sin \tau y - \tau \cos \tau y}{\tau^2+\kappa^2} r K_1(\tau r)\, d\tau = 2\cos\theta - e^{-\kappa a \cos\theta}\int^{\cot\theta} \frac{e^{x \sin\theta t}}{(t^2+1)^{3/2}}\, dt$$

$$+\pi \kappa a \sin\theta\, e^{-\kappa a \sin\theta}\left[Y_1(\kappa a \sin\theta) - S_1(\kappa a \sin\theta) + \frac{2}{\pi}\right] \qquad \text{(II.5)}$$

where S_1 is the Struve function of order 1. and

$$V_0^*(a,\pi/2) = -\pi \kappa a\left[Y_1(\kappa a) + S_1(\kappa a) - \frac{2}{\pi}\right] + 2\pi \kappa a\, e\, J_1(\kappa a) \qquad \text{(II.6)}$$

$$V_0^*(a,0) = 0$$

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