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The Solution by Iteration of Nonlinear Integral Equations

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DEPARTMENT OF THE ARMY PROJECT No. 503-06-002
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0007K

BALLISTIC RESEARCH LABORATORIES

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ABSTRACT

In the mathematical formulation of many physical problems there are frequently encountered integral equations of the linear as well as non-linear type. The adequate solution of such equations then becomes a practical necessity.

It turns out that for large classes of non-linear integral equations describing certain wave phenomena the well known method of successive approximations provides a feasible means of constructing a solution, securing its uniqueness, and, in the process, furnishing practical estimates of relevant magnitudes and errors. Furthermore, the method also lends itself quite easily to treatment on high speed computing machinery.

These principal conclusions are illustrated - and borne out - by means of an appropriately chosen example.

1. INTRODUCTION

It is a well known fact that many physical problems lead to nonlinear integral equations. The following examples serve to illustrate this point.

i. Nystrom has shown [1] that the general first boundary value problem in the theory of ordinary differential equations may be reduced to an ordinary differential equation of the type

$$y'' = F(x,y) \qquad -1/2 \leq x \leq 1/2$$

with the boundary conditions $y(-1/2) = y(1/2) = 0$. This equation, in turn, is equivalent to the integral equation

$$y(x) = \int_{-1/2}^{1/2} K(x,s)\, F(s,y(s))\, ds,$$

$K(x,s)$ representing the Green's function of the boundary value problem.

ii. Boundary value problems of the form

$$\Delta u = u_{xx} + u_{\xi\xi} = \exp u$$

with $u = f(x,\xi)$ on the boundary C of the given region B, are reducible to integral equations, as follows [2]: Let $w(x,\xi)$ be the solution of the boundary value problem

$$\Delta w = 0, \quad w = f \text{ on } C.$$

Then the function $y = u-w$ clearly satisfies

$$\Delta y - y \exp w = [\exp(y) - y] \exp w$$

with $y = 0$ on C. If now $K(x,\xi;\, s,\sigma) \geq 0$ denotes the Green's function of the problem

$$\Delta y - y \exp w = 0, \; y = 0 \text{ on } C,$$

then it may be shown that $y(x, \xi)$ is a solution of the nonlinear integral equation

$$y(x, \xi) + \iint_B K(x, \xi ; s, \sigma) \left[\exp (y) - y\right] \exp (w) \, ds d\sigma = 0.$$

iii. In the discussion of certain wave phenomena L. Lichtenstein [3] was led to the study of integral equations of the form

$$L\, y\, (x) = \sum_{i=1}^{\infty} \int_a^b \int_a^b K_i (s, s_1 \ldots s_i) y(s_1) \cdots y(s_i) \, ds_1 \ldots ds_i = 0,$$

with L denoting a linear operator, and the K_i symmetric and continuous functions in the (n+1) dimensional unit cube.

Physical situations leading to similar nonlinear integral equations have also been described by Schmeidler [4].

The solution of such integral equations may be accomplished in general by a variety of methods, with the choice of the most appropriate one naturally depending upon the form and type of the integral equations of interest. Such methods may emply closed analytical expressions, power series expansions, the replacement of integrations by summations, successive approximations, variational methods, etc. [5].

The approach to be used here will be the iteative one, to be applied to integral equations of a generalized Lichtenstein type:

$$y(x) - \sum_{i=1}^{r} \int_a^b \cdots \int_a^b K_i (x, s_1, \ldots s_i) F_i (s_1, \ldots s_i) \, y(s_1) \ldots y(s_i) \, ds_1 \ldots ds_i = 0. \qquad (1)$$

Integral equations containing an "inhomogeneous" term, e.g., of the form

$$y^* (x) - K_o (x) - \int K_1 F_1^* (y^*) - \iint K_2 F_2^* (y^*) = 0$$

can obviously be reduced to the form (1) by the substitution

$$y = y^* - K_o.$$

For kernels K_i whose squares are integrable, and functions F_i that have bounded derivatives the existence and uniqueness of a solution will be proved, and an upper bound for max $|y|$ will be given. The proof is based on the method of successive iterations, a method which also easily provides an estimate of the error inherent in the nth approximation. The results will then be generalized to systems of equations of type (1), and, finally, an example will be given.

2. ASSUMPTIONS AND NOTATIONS.

For the sake of simplicity, we shall take $r = 2$ in eq. (1), i.e. consider in the interval I: $a \leq x,s,t \leq b$ the equation

$$y(x) - \int K_1(x,s)\, F_1(s,y(s))\,ds - \iint K_2(x,s,t) F_2(s,t,y(s), y(t))\,ds\,dt = 0. \tag{2}$$

The kernels K_1 and K_2 will be assumed to have integrable squares, so that $\int K_1^2(x,s)\,ds$, $\iint K_2^2(x,s,t)\,ds\,dt$ exist and are continuous functions of $x \in I$. For the maximum of a continuous function $f(s)$ defined in I we shall employ the notation

$$\|f\| = \max_{x \in I} |f(x)| .$$

Then

$$K^2 = \left\| \int K_1^2\,ds + \iint K_2^2\,ds\,dt \right\| \text{ certainly exists.}$$

The continuous functions $F_1(s,u)$, $F_2(s,t,u,v)$, defined for $s, t \in I$, $u,v \in J$, where J denotes a certain interval $\langle -c,c \rangle$, are assumed to satisfy conditions of the form

$$|F_1(s,u_1) - F_1(s,u_2)| \leq L_1 |u_1 - u_2| \tag{3}$$

$$|F_2(s,t,u_1,v_1) - F_2(s,t,u_2,v_2)| \leq L_2(|u_1-u_2| + |v_1 - v_2|) ,$$

where L_1, L_2 are positive numbers independent of s, t, u, v. The quantities

$$M_1^2 = \| F_1^2(s,0)\| , \quad M_2^2 = \|F_2^2(s, t, 0, 0)\|$$

then also exist.

Relations (3) certainly hold if, for example, F_1 and F_2 have continuous partial derivatives $\partial F_1/\partial u$, $\partial F_2/\partial u$, $\partial F_2/\partial v$ for arguments restricted, respectively, to the intervals I, J.

Further, we put

$$A^2 = b - a, \ \alpha = 1 \text{ or } 2 \text{ for } A \leq 1 \text{ or } A > 1 .$$

Other quantities to be employed later on are

$$L^2 = L_1{}^2 + L_2{}^2, \ M^2 = M_1{}^2 + M_2{}^2,$$

$$A^\alpha KM = E, \ A^\alpha KL = D.$$

In the following extensive use will be made of Schwarz's inequality

$$\left|\int f(x)g(x)dx\right| \leq \left(\int f^2 \, dx\right)^{1/2} \left(\int g^2 dx\right)^{1/2}. \tag{4}$$

This relation is applicable also to multiple integrals and sums. If we put

$$\left|\int f\right| = I^2(f^{1/2}),$$

then eq. (4) may be expressed in the form

$$I^2(f^{1/2}g^{1/2}) \leq I(f)I(g). \tag{5}$$

3. LEMMA.

We are now in a position to state the following Lemma:

Let $y \in J$ be a solution of Eq. (2), and let the F_i satisfy conditions of type (3). Then $\|y\|(1 - 2D) \leq E.$

Proof. It follows from eq. (2) that

$$|y| \leq \left|\int K_1F_1\right| + \left|\iint K_2F_2\right|.$$

In this inequality the right side is, by (5), identically in x,

$$\leq I(K_1) \ I \ (F_1) + I \ (K_2) \ I \ (F_2).$$

From Schwarz's inequality for sums it may now be deduced that

$$|y| \leq \left[I^2(K_1) + I^2(K_2)\right]^{1/2} \left[I^2(F_1) + I^2(F_2)\right]^{1/2}$$

$$\leq K\left[I^2(F_1) + I^2(F_2)\right]^{1/2}.$$

Since, by assumption, $y \in J$, we may make use of relationships (3), getting

$$I^2(F_2) \leq \iiint \left[|F_2(s,t,o,0)| + L_2(|y(s)| + |y \ \ (t)| \ )\right]^2 dsdt$$

$$\leq A^{2\alpha}(M_2 + 2 \, L_2 \, \|y\| \,)^2.$$

Similarly,

$$I^2(F_1) \leq A^{2\alpha}(M_1 + 2\,L_1\,\| y \|)^2.$$

By addition, then,

$$I^2(F_1) + I^2(F_2) \leq A^{2\alpha}(M + 2L\,\| y \|)^2.$$

Thus

$$\| y \| \leq K\,A^{\alpha}(M + 2L\| y \|) = E + 2D\| y \|,$$

which proves the lemma.

4. EXISTENCE THEOREM. For functions F_1, F_2 satisfying conditions (3) with

$$0 \leq E/(1 - 2D) \leq C \tag{6}$$

eq. (2) has exactly one solution.

To prove this theorem we use successive iteration, starting with arbitrary continuous function $y_0 \in J$, and defining a sequence $\{y_n\}$ of continuous functions by

$$y_{n+1}(x) = \int K_1(x,s)\,F_1(s,y_n(s))\,ds + \iint K_2(x,s,t)F_2(s,t,y_n(s), \tag{7}$$

$$y_n(t)\quad ds\,dt,\ n = 0,1,2\ldots$$

First we show by induction that $y_n \in J$. Let, then, y_{n-1} have this property. By definition (7), following the same reasoning that was used to prove the lemma, there is obtained.

$$\| y_n \| \leq E + 2D\,\| y_{n-1} \|.$$

From the inductive assumption and condition (6) it may now be inferred that

$$\| y_n \| \leq E + 2DC \leq C,$$

i.e. $\quad y_n \in J$.

Next, the sequence is easily shown to be uniformly convergent, as follows:

From

$$| y_{n+1} - y_n | \leq \left| \int K_1(F_1(y_n) - F_1(y_{n-1}))\,ds \right|$$

$$+ \left| \iint K_2(F_2(y_n) - F_2(y_{n-1}))\,ds\,dt \right|$$

it follows, by means of Schwarz's inequality, that

$$|y_{n+1} - y_n| \leq K\left[I^2(F_1(y_n) - F_1(y_{n-1})) + I^2(F_2(y_n) - F_2(y_{n-1}))\right]^{1/2}.$$

Taking advantage of relations (3) the right side [illegible]

$$\leq 2\,K\,A^2 L\,\| y_n - y_{n-1} \|,$$

so that

$$\| y_{n+1} - y_n \| \leq 2D \| y_n - y_{n-1} \|, \tag{8}$$

or, by induction,

$$\| y_{n+1} - y_n \| \leq \| y_1 - y_0 \| \, (2D)^n, \; n \geq 0.$$

Since $1 - 2D > 0$ $\sum_{n=0}^{\infty} (2D)^n$ is convergent, and it follows that the partial sums y_n of $y_0 + \sum_{n=1}^{\infty} (y_n - y_{n-1})$ are absolutely and uniformly convergent.

The continuous limit function y of the $\{y_n\}$ is easily shown to be a solution of eq. (2). Putting

$$\int K_1 F_1(y)\, ds + \iint K_2 F_2(y)\, ds\, dt = y^*(x)$$

we find, namely, that

$$|y_{n+1} - y^*| \leq K\left[I^2(F_1(y_n) - F_1(y)) + I^2(F_2(y_n) - F_2(y))\right]^{1/2}.$$

But the continuity of the F_i in conjunction with the uniform convergence of y_n towards y suffice to make the difference, $F_i(y_n) - F_i(y)$ arbitrarily small by taking n sufficiently large. Thus also $\lim y_n = y^*$ uniformly in x. It follows that $y^* = y$, i.e. that y is a solution of eq. (2).

Finally, to show that y is the only solution of eq. (2) under the conditions stated above let it be assumed that there exists another solution $Y(x)$. Clearly

$$\| Y - y \| \leq \| y_{n+1} - Y \| + \| y_{n+1} - y \|,$$

and $\| y_{n+1} - y \| < \epsilon/2$ for $n > N_1$. The reasoning applied above to establish the relationship (8) now leads to

$$\| y_{n+1} - Y \| \leq (2D)^n \| y_n - Y \|,$$

or

$$\| y_{n+1} - Y \| \leq (2D)^n \| y_o - Y \|,$$

whence

$$\| y_{n+1} - Y \| < \epsilon/2 \text{ for } n > N_2.$$

Consequently

$$\| Y - y \| < \epsilon .$$

Due to the arbitrariness of ϵ and the meaning of $\| Y - y \|$ it is then necessary that $Y(x) - y(x) \equiv 0$.

By the lemma, which is applicable here, and by condition (6),

$$\| y \| \leq E/(1 - 2D) \leq C.$$

For increasing values of K, L, M, therefore, the upper bound $E/(1 - 2D)$ of $\| y \|$ also increases.

An estimate of the error $\| y - y_n \|$ of the n^{th} approximation y_n is easily obtained. From

$$\| y - y_n \| \leq \| y_{n+1} - y_n \| + \| y - y_{n+1} \|$$

it follows that

$$\| y - y_n \| \leq \sum_{i=n}^{n+m} \| y_{i+1} - y_i \| + \| y - y_{n+m+1} \| .$$

For fixed n, $m \rightarrow \infty$, then, since $y_m \rightarrow y$ uniformly,

$$\| y - y_n \| \leq \sum_{i=n}^{\infty} \| y_{i+1} - y_i \|$$

$$= \sum_{i=0}^{\infty} \| y_{i+1} - y_i \| - \sum_{i=0}^{n-1} \| y_{i+1} - y_n \| .$$

Thus

$$\| y - y_n \| \leq \| y_1 - y_0 \| \; (2D)^n/((1 - 2D) \text{ for } n \geq 0. \tag{9}$$

5. GENERALIZATIONS.

Equation (2) may be considered to be a special case of a more general system

$$y(x) - \sum_{j=1}^{q} \int \cdots \int K_{ij}(x, s_1, \ldots, s_j) F_{ij}(s_1, \ldots s_j, y_1(s_1), \ldots y_1(s_j), \ldots y_p(s_1), \ldots,$$

$$y_p(s_j) ds_1 \ldots ds_j = 0,$$

where $i = 1, 2, \ldots, p$, $p \geq 1$, $q \geq 1$. To such systems the results of the previous sections may be transferred without difficulty; it is necessary only to replace the quantities α, E, D $\| y \|$, etc. by

$$\alpha = 1 \text{ or } q \text{ for } A \leq 1 \text{ or } A > 1,$$

$$\| \sum_j I^2(K_{ij}) \| = K_i^2,$$

$$\sum_j M_{ij}^2 = M_i^2, \quad A^\alpha \sum_i K_i M_i = E, \quad A^\alpha \sum_i K_i L_i = D,$$

$$\| y \| = \| \sum_i y_i(x) \| .$$

Finally, the important subsidiary condition $1 - 2D > 0$ must be replaced by $1 - qD > 0$.

For $p = q = 1$, i.e., for the equation

$$y(x) - \int K(x,s) F(s,y(s))\, ds = 0, \tag{10}$$

this condition $1 - qD > 0$ becomes $0 < AKL < 1$, or

$$0 < A^2K^2L^2 < 1. \tag{11}$$

Let us ascertain its meaning for the case of the linear integral equation of the second kind

$$y(x) - \lambda \int_a^b K(x,s)\, y(s)\, ds = f(x). \tag{12}$$

This type of equation is obtained from eq. (10) by putting

$$F(s,u) = g(s) + \lambda u$$

$$f(x) = \int K(x,s)\, g(s)\, ds.$$

Since in this case $L = \| \partial F/\partial u \| = |\lambda|$, inequality (11) indicates that

$$\lambda^2 \iint K^2(s,t)\, ds dt \leq L^2 A^2 K^2 < 1.$$

Now for the characteristic value of smallest absolute amount of the symmetric kernel there exists the relation [6]

$$\lambda_1^2 \geq \left(\iint K^2 ds dt\right)^{-1} .$$

Consequently, condition (11) becomes

$$|\lambda| < |\lambda_1| ,$$

which is the well-known sufficient condition for the existence of a unique solution of eq. (12) for arbitrary $f(x)$.

6. APPLICATIONS.

There are many practical cases where the kernels may be expressed in bilinear forms

$$K_1(x,s) = \sum_i \phi_{1i}(x)\,\psi_{1i}(s) \tag{13}$$

$$K_2(x,s,t) = \sum_j \phi_{2j}(x)\,\psi_{2j}(s,t),$$

the ϕ_{1i} being linearly independent of the ϕ_{2j}, and the summation to be extended over a finite or infinite number of values i, j. In such cases the iterative procedure (7) clearly becomes

$$y_{n+1}(x) = \sum_i \phi_{1i}(x)\,p_{n+1,i} + \sum_j \phi_{2j}(x)\,q_{n+1,j} \tag{14}$$

$$p_{n+1,i} = \int \psi_{1i}(s)\,F_1\Big(s, \sum_i \phi_{1i}(s)\,p_{ni} + \sum_j \phi_{2j}(s)\,q_{nj}\Big)ds$$

$$q_{n+1,j} = \iint \psi_{2j}\,F_2\Big(s,t, \sum_i (s) + \sum_j (s), \sum_i (t) + \sum_j (t)\Big)ds\,dt.$$

The iterative procedure for the functions $y_n(x)$ is thus reduced to a method of successive approximation for the constants p_{ni}, q_{nj}:

$$p_{n+1,i} = \Phi_1(p_{ni}, q_{nj}) \tag{15}$$

$$q_{n+1,j} = \Phi_2(p_{ni}, q_{nj}).$$

In the limit, then,

$$y(x) = \sum_i \phi_{1i}(x)\,p_i + \sum_j \phi_{2j}(x)\,q_j, \tag{16}$$

where the p_i, q_j satisfy the functional equations

$$p_i = \Phi_1(p_i, q_j)$$

for all i, j.

$$q_j = \Phi_2(p_i, q_j) \tag{17}$$

Whenever the functional equations (17) are directly solvable the solution (16) may be obtained immediately. Otherwise use must be made of the successive approximations (15).

Let us now apply the foregoing deductions to the following case:

$$K_1(x,s) = x(1-s), \qquad I: \langle 0, 1 \rangle$$

$$K_2(x,s,t) = \exp(x-s-t)$$

$$F_1(s,u) = \lambda(u+1) \qquad J: \langle -1, 1 \rangle$$

$$F_2(s,t,u,v) = \lambda\, u\, v.$$

The equation to be solved, then, is

$$y(x) - \lambda \left\{ \int_0^1 x(1-s) \left[y(s)+1 \right] ds + \int_0^1\!\!\int_0^1 \exp(x-s-t)y(s)y(t)\,ds\,dt \right\} = 0. \qquad (18)$$

For the convergence of the iterative procedure discussed above it is sufficient that

$$0 \leq E/(1-2D) \leq 1. \qquad (6)$$

Now

$$A^{\alpha} = 1$$

$$K = \left[(1/3) + (e^2 - 1)^2/4e^2 \right]^{1/2} \approx 1.31$$

$$L = 2^{1/2}\, |\lambda|, \qquad M = |\lambda| .$$

Condition (6) is satisfied, then, for

$$|\lambda| \leq \lambda_0 = \left[K \left(1 + 2\sqrt{2}\right) \right]^{-1} \approx 0.2 \qquad (19)$$

Further, the lemma indicates that for the solution y(x),

$$\| y(x) \| \leq K\, |\lambda| / \left(1 - 2K\, |\lambda|\, \sqrt{2}\right).$$

Since in this example i = j = 1, the iterative process for eq. (18) may be expressed in the form

$$y_{n+1}(x) = \lambda \left(p_{n+1} x + q_{n+1} \exp(x)\right)$$

with

$$p_{n+1} = \int_0^1 (1-s)\left[\lambda(p_n s + q_n \exp(s)) + 1\right] ds$$

$$q_{n+1} = \lambda^2 \left[\int_0^1 (p_n s \exp(-s) + q_n)\, ds\right]^2 .$$

According to the existence and uniqueness theorem $p = \lim p_n$, $q = \lim q_n$ exist. These limits satisfy a system of equations corresponding to equations (17), in this case

$$p(1-\lambda/6) - q\lambda(e-2) = 1/2$$
$$\lambda^2 \left[(1-2/e)p + q\right]^2 = q.$$

Eliminating p now there is obtained for q the equation

$$\lambda^2(\alpha p + \beta)^2 = \gamma q \tag{20}$$

$$\alpha = 6\lambda(e-2)^2 + (6-\lambda)e$$

$$\beta = 3(e-2)$$

$$\gamma = e^2(6-\lambda)^2.$$

Consequently,

$$q = \left\{(\gamma - 2\alpha\beta\lambda^2) \pm \left[\gamma(\gamma - 4\alpha\beta\lambda^2)\right]^{1/2}\right\}/2\alpha^2\lambda^2. \tag{21}$$

Of the two possible solutions indicated here the one corresponding to the positive sign becomes unbounded as λ approaches zero. Since it is desired to keep the solutions for all $|\lambda| \leq \lambda_0$ of norm not exceeding unity the positive sign in eq. (21) must be discarded.

Suppose we wish to determine the solution of eq. (18) for $\lambda = 0.2$. For the initial approximation we take

$$p_o = q_o = 1,$$

i.e

$$y_o(x) = 0.2\,(x + \exp(x)).$$

The resulting approximations

$$p_{n+1} = 0.2\left[p_n/6 + q_n(e-2)\right] + 1/2$$

$$q_{n+1} = 0.04\left[(e-2)p_n/e + q_n\right]^{2}$$

are shown in the table below.

Table I. Successive Approximation

n	p_n	q_n
0	1.0000	1.0000
1	0.6770	0.0639
2	0.5317	0.0024
3	0.5181	0.0008
4	0.5174	0.0008

The exact solution, as determined from eq. (21), is

$$y(x) = 0.2\,(0.51735\,x + 0.00075\,\exp(x)).$$

While there is always convergence to the solution for $y_o \in J$, $|\lambda| < \lambda_o$, the choice of an initial approximation y_o outside of J, or a value of $|\lambda| > \lambda_o$ may lead to divergence. This is the case, for example, for $p_o = q_o = 25$, $\lambda = 0.2$, and $p_o = q_o = 1$, $\lambda = 1$.

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It is worthwhile mentioning that iterative procedures such as those exemplified by eqs. (15) are eminently suited for treatment on high speed computing machinery. The required quadratures may be performed by any suitable standard methods, or by means of the formulas outlined in [7].

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