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A NOTE ON THE COMPATIBILITY OF DISTRIBUTION FUNCTIONS

by

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A class of random variables is generally defined in either of two ways. One, as a class of measurable functions over a space with a probability measure, the other, by defining each element in terms of the properties it shares with the other members of the class - these properties being characterized by distribution functions. The latter development usually takes this form: associated with each finite subset $t_1, \ldots, t_n$ of some set T we have a distribution function $F_{t_1, \ldots, t_n}$. This system of distribution must satisfy the well known consistency relations.

Let the integer k be fixed. Suppose that associated with each finite subset $t_1, \ldots, t_n$ of T, with $n \leq k$, we have a distribution function $F_{t_1, \ldots, t_n}$, and that this system satisfies the consistency relations. Call such a system a k-fold system of distribution functions.

Can a k-fold system of distribution functions always be extended to define a class of random variables?

This question merits some consideration. Most of the convergence criteria for sequences of random variables are determined by the 2-fold distribution functions, as is the covariance function of a stochastic process. One may seek an example of a stochastic process with a particular property determined by the k-fold system of distribution functions, and having specified a satisfactory k-fold system, may ask whether a stochastic process exists with this k-fold system.

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The question also appears to have some significance in the axiomatic development of probability in terms of random experiments. We refer particularly to Cramer, <u>Mathematical Methods of Statistics</u>, paragraph 14.2 An affirmative answer to the above questions would seem necessary for the appropriateness of Axiom 3 of that paragraph.

The following example shows that, at least for 2-fold systems an extension is not always possible.

Let x_1, x_2, x_3 be any three elements from $\{x_t\}$, $t \in T$. Let $a \neq b$ be two real numbers. Then the relations

$$P[x_i = a,\ x_j = b] = P[x_i = b,\ x_j = a] = \tfrac{1}{2},\quad i = 1, 2, 3;$$

$j = 1, 2, 3$; $i \neq j$, uniquely determines a consistent 2-fold system of distribution functions. But no distribution function for x_1, x_2, x_3 exists which is consistent with this system. For any such distribution function will be completely specified by the eight values

$$P[x_1 = c,\ x_2 = d,\ x_3 = e]$$

where c, d, and e are each equal to a or b. Since at least two of them, say c and d, are equal, we have, by the consistency requirement,

$$P[x_1 = c,\ x_2 = c,\ x_3 = e] \leq P[x_1 = c,\ x_2 = c] = 0$$

The question of the compatibility of distribution functions must thus be confined to specific k-fold systems and specific values of k.