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THE FREEZING OF MOLTEN MATERIAL IN FINITE REGIONS

Ordnance Corps Project No. TA3-5006: Optimum Utilization of Explosives

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ABSTRACT

The analytical treatment of the problem of solidification in a slab of thickness 2a is based on Lightfoot's solution for the case of a semi-infinite mass, initially at constant temperature and bounded by the plane surface $x = 0$ kept at constant temperature. The rate of advance of the solid walls moving from the two boundary planes $x = 0$ and $x = 2a$, kept at constant temperature, is given in closed form. Solidification in a sphere, in an infinite or finite cylinder, or in a parallelepiped, is obtained from the solution for the slab by correlation of temperatures due to the initial supply of heat. Numerical solutions can be calculated rapidly.

1. INTRODUCTION:

An analytical solution of the problem of freezing in a finite region has been attempted by Lightfoot* who studied the rate of solidification of a medium, originally at constant temperature, and bounded by the plane surfaces $x = 0$ and $x = 2a$ kept at constant temperature. The problem was treated by the method of images but essential factors were neglected and the solution obtained by Lightfoot is neither complete nor exact.

An exact solution has been obtained by the same author for the case of a semi-infinite mass of molten material, originally at constant temperature, and with its bounding plane surface $x = 0$ maintained at constant (zero) temperature. In the given solution, the diffusivity k, the specific heat c, and the density ρ of the medium, are assumed constant for all temperatures, and the same for the liquid as for the solid medium.

The solution of the problem of solidification in finite regions, given below, has been derived from Lightfoot's solution for a semi-infinite medium. The following method will give the position of the surface of separation of solid and liquid phases as a function of time. Results of this study may be of help in the investigation of a complete solution of the problem of solidification.

2. Solidification in a Semi-infinite Medium*

The solution of the problem of solidification of a mass of molten material requires, in addition to known solutions, consideration of the effect on the temperature caused by the evolution of latent heat of fusion. In Lightfoot's analysis of the problem, the surface of separation of solid and liquid phases has been regarded as a moving source of heat. The speed of of the plane source was determined from the condition that the temperature at the moving plane surface, at any position, is equal to the melting point of the material.

The position x of the moving plane, at time t, in a medium initially at temperature ϕ, and with its bounding plane $x = 0$ kept at zero temperature, is given by

$$x = 2\lambda\sqrt{kt} \tag{1}$$

where k is the thermal diffusivity of the medium, and the constant λ is obtained from

$$V = \phi\,\Theta(\lambda) + \frac{L\sqrt{\pi}}{c}\,\lambda e^{\lambda^2}\,\Theta(\lambda)\left\{1 - \Theta(\lambda)\right\} \tag{2}$$

* Proc. London Math. Soc. (2), 31, (1930), 97

The terms used in (2) are as follows:

v, the melting point of the medium

ϕ, the initial temperature

L, the latent heat of fusion

c, the specific heat

and

$$\theta(k) = \frac{2}{\sqrt{\pi}} \int_0^k e^{-\xi^2} d\xi \tag{3}$$

With this solution, the temperature v_1 at a point x_1, at time t, for $x_1 < 2k\sqrt{\kappa t}$ will be:

$$v_1 = \phi\,\theta\left(\frac{x_1}{2\sqrt{\kappa t}}\right) + \frac{L\sqrt{\pi}}{c} k e^{k^2} \theta\left(\frac{x_1}{2\sqrt{\kappa t}}\right)\left\{1 - \theta(k)\right\} \tag{4}$$

and for $x_2 > 2k\sqrt{\kappa t}$

$$v_2 = \phi\,\theta\left(\frac{x_2}{2\sqrt{\kappa t}}\right) + \frac{L\sqrt{\pi}}{c} k e^{k^2} \theta(k)\left\{1 - \theta\left(\frac{x_2}{2\sqrt{\kappa t}}\right)\right\} \tag{5}$$

According to (4) and (5), we notice that, while solidification proceeds from the plane $x = 0$, the temperatures due to the original supply of heat, and to the latent heat of fusion, can be evaluated separately.

For $x = 2k\sqrt{\kappa t}$, at the surface of separation of solid and liquid phases, (4) and (5) coincide with (2).

The terms in (2) are constant for any position of the moving plane surface source, and can be written as:

$$V_\phi = \phi\,\theta(k) \tag{6}$$

$$V_L = \frac{L\sqrt{\pi}}{c} k e^{k^2} \theta(k)\left\{1 - \theta(k)\right\} \tag{7}$$

$$V = V_\phi + V_L \tag{8}$$

3. Solidification in a Slab of Thickness $2a$

An exact solution of the problem of solidification of a mass of molten material, initially at constant temperature, can be derived from the results given in the preceding section.

For a freezing medium in the region $2a$, initially at temperature ϕ, with the bounding plane surfaces $x = 0$ and $x = 2a$ kept at zero, we evaluate the sum of the temperatures due to the original supply of heat and to latent heat of fusion released on solidification by the two solid sections of the medium $(0, x)$ and $(2a, 2a - x)$ as in Fig. 1.

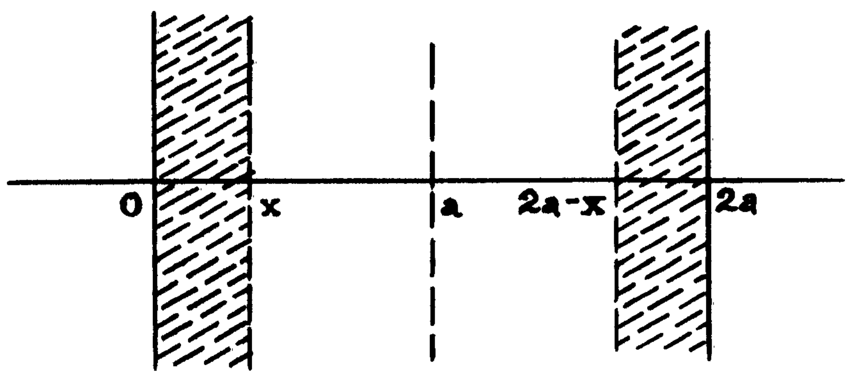


Fig. 1

The surfaces of separation of solid and liquid phases are taken as two moving plane sources of heat, and the speed of these two sources is determined from the condition that the temperature at the moving plane sources is equal to the melting point of the medium.

For the temperature at the moving sources we write the condition

$$V = V_\phi + (V_L)_1 + (V_L)_2 \tag{9}$$

where V_ϕ refers to the original supply of heat, and $(V_L)_1$, $(V_L)_2$ are temperatures due to heat of fusion from solid section $(0, x)$ and $(2a, 2a - x)$ respectively.

It is evident that solidification will proceed from both $x = 0$ and $x = 2a$, and for t small, the positions of the moving planes will be

$$x = 2k_0\sqrt{Kt}, \quad 2a - x = 2a - 2k_0\sqrt{Kt} \tag{10}$$

The constant k_0 of (10) is the same as for a semi-infinite medium and is given by (2). In terms of the dimensionless variable

$$\tau = \frac{K}{a^2}t \tag{11}$$

forms (10) will become

$$\frac{x}{a} = 2k_0\sqrt{\tau} \qquad \frac{2a-x}{a} = 2 - 2k_0\sqrt{\tau} \tag{12}$$

In the region $2a$, the temperature V_ϕ, at any point x, at a given τ, is the known solution

$$V_\phi = \frac{4\phi}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2a}e^{-(2n+1)^2\pi^2\tau/4} \tag{13}$$

During an initial period of time, $(V_L)_1$ and $(V_L)_2$ will be obtained from (4) and (5) as follows:

$$\text{For } \frac{x_1}{a} < 2k_0\sqrt{\tau} \quad (V_L)_1 = \frac{L\sqrt{\pi}}{c}k_0 e^{k_0^2}\,\theta\left(\frac{x_1}{2a\sqrt{\tau}}\right)\left\{1-\theta(k_0)\right\} \tag{14}$$

$$\text{and} \quad (V_L)_2 = \frac{L\sqrt{\pi}}{c}k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta\left(\frac{2a-x_1}{2a\sqrt{\tau}}\right)\right\} \tag{15}$$

$$\text{For } \frac{x_1}{a} > 2k_0\sqrt{\tau} \quad (V_L)_1 = \frac{L\sqrt{\pi}}{c}k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta\left(\frac{x_1}{2a\sqrt{\tau}}\right)\right\} \tag{16}$$

$(V_L)_2 =$ same as (15)

For the moving plane source $\frac{x}{a} = 2k_0\sqrt{\tau}$ the terms in (9) will be evaluated from:

$$V_\phi = \frac{4\phi}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2a}\, e^{-(2n+1)^2\pi^2\tau/4} \tag{17}$$

$$(V_L)_1 = \frac{L\sqrt{\pi}}{c}\, k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta(k_0)\right\} \tag{18}$$

$$(V_L)_2 = \frac{L\sqrt{\pi}}{c}\, k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta\left(\frac{2a-x}{2a\sqrt{\tau}}\right)\right\} \tag{19}$$

Temperature V_ϕ, $(V_L)_1$, and $(V_L)_2$, at the moving plane source $\frac{2a-x}{2a} = 2k_0\sqrt{\tau}$ will be obtained with (17), (19), and (18), respectively. The sum of (17), (18), and (19), is equal to the melting point of the medium for a range of values $0 \leq \tau \leq \tau_0$. The upper limit τ_0 varies for each numerical case and depends upon the initial temperature ϕ and the ratio $\frac{L}{c}$. For τ small, the temperature at the center of the slab is given by

$$v_{x=a} = \frac{4\phi}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi}{2}\, e^{-(2n+1)^2\pi^2\tau/4} + 2\,\frac{L\sqrt{\pi}}{c}\, k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta\left(\frac{1}{2\sqrt{\tau}}\right)\right\} \tag{20}$$

and we can verify that, when

$\tau \leq \tau_0 \qquad v_{(x=a)} = \phi$

$\tau > \tau_0 \qquad v_{(x=a)} < \phi$

In each numerical case, therefore, the upper limit τ_0 can be obtained with (20). We see also that, during the entire period of time τ_0, the boundary conditions of the problem will be satisfied.

In the case of a semi-infinite medium and $x > 2k\sqrt{K t}$, or $\frac{x}{a} > 2k_0\sqrt{\tau}$, the temperature given by (5) is the sum of the two terms

$$V_\phi = \phi\,\theta\left(\frac{x}{2a\sqrt{\tau}}\right) \tag{21}$$

$$V_L = \frac{L\sqrt{\pi}}{c}\, k_0 e^{k_0^2}\,\theta(k_0)\left\{1-\theta\left(\frac{x}{2a\sqrt{\tau}}\right)\right\} \tag{22}$$

Solution (21) is correct for large values of τ but, for small values, V_ϕ must be equal to ϕ at a finite distance from the boundary $\underline{x} = 0$. Where $V_\phi = \phi$, we will obtain $(V_L) = 0$. In general, the boundary condition is satisfied when the temperature at the center of a slab of thickness $\underline{4a}$, initially at ϕ and with bounding surfaces kept at zero is

$$(V_\phi)_{\substack{x=2a\\ \tau=\tau_0}} = \phi = \frac{4\phi}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi}{2}\, e^{-(2n+1)^2\pi^2\tau/16} \tag{23}$$

This condition can be verified for any τ_0, including the limiting case where

$$\frac{x_0}{a} = 2k_0\sqrt{\tau_0} = 1$$

and we can write

$$(V_L)_{\substack{x=2a\\ \tau=\tau_0}} = 0$$

During the period of time τ_0, the two plane sources move from the boundaries toward the center of the slab as in semi-infinite medium. The temperature distributions V_ϕ, $(V_L)_1$ and $(V_L)_2$ and their sum, computed for a numerical example, with $\tau = \tau_0$, are shown on Fig. 2.

In the region $\underline{2a}$, initially and till solidification is complete, the correct mathematical forms of $(V_L)_1$ and $(V_L)_2$, at the plane source moving from $\underline{x} = 0$ to $\underline{x} = \underline{a}$ are:

$$(V_L)_1 = \frac{L}{ac}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{2a}\int_0^{\tau}\sin\frac{n\pi x'}{2a}\, e^{-n^2\pi^2\left(\frac{\tau-\tau'}{4}\right)}\frac{dx'}{d\tau'}d\tau' \tag{24}$$

$$(V_L)_2 = \frac{L}{ac}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{2a}\int_0^{\tau}\sin\frac{n\pi(2a-x')}{2a}\, e^{-n^2\pi^2\left(\frac{\tau-\tau'}{4}\right)}\frac{dx'}{d\tau'}d\tau' \tag{25}$$

During the interval $0 \leq \tau \leq \tau_0$, $(V_L)_1$ is a constant obtained from (18) while $(V_L)_2$, evaluated from (19), is initially zero but may assume increasing values as τ approaches τ_0. However, during the entire interval $0 \leq \tau \leq \tau_0$

$$(V_L)_2 = V - \left\{V_\phi + (V_L)_1\right\} \tag{26}$$

The rate of solidification in the range $0 \leq x \leq x_0$ might be obtained directly from (24) and (25) if the series of integrals in the above forms could be evaluated exactly. So far, available graphical or numerical techniques are unsatisfactory.

Lightfoot shows that a rate of solidification can be obtained by trial method, and assumes a position of the moving planes given by

$$\frac{x}{a} = b - 2K\sqrt{\tau_2 - \tau} \qquad \begin{matrix} \tau \geq \tau_0 \\ x \geq x_0 \end{matrix}$$

where b, K, and τ_2, are chosen for continuity and satisfy the conditions

$$\frac{b}{a} = 2K\sqrt{\tau_0}\left\{1 + \frac{K^2}{k_0^2}\right\} \qquad \tau_2 = \tau_0\left\{1 + \frac{K^2}{k_0^2}\right\}$$

The constant K remains undetermined and its value must be adjusted for correct temperature at prescribed time. According to the author, this method requires "great labour".

For solidification in the range $x_0 \leq x \leq a$ we assume a position of the moving plane given by

$$\frac{x}{a} = -b + 2k_2\sqrt{\tau - \tau_2} \qquad \begin{matrix} x \geq x_0 \\ \tau \geq \tau_0 \end{matrix} \tag{27}$$

where b, k_2, and τ_2, are to be determined from the condition that the temperature at the moving plane is equal to the melting point of the medium. For continuity of the two functions of τ, (11) and (27) and their first derivatives at $x = x_0$, we make

$$\tau_2 = \tau_0\left(1 - \frac{k_2^2}{k_0^2}\right) \tag{28}$$

We determine b and k_a from the following:

$$\frac{x_o}{a} = -b + 2k_a\sqrt{\tau_o - \tau_2}$$
$$1 = -b + 2k_a\sqrt{\tau_a - \tau_2} \qquad (29)$$

The second equation of (29) refers to complete solidification at a definite τ_a, to be obtained according to the following considerations.

While the source of heat moves from $x = 0$ to $x = x_o$ temperature $(V_L)_1$ is obtained from (18), and is a constant at any point in this range. At the same plane source, $(V_L)_2$ will vary from zero to a known value. Solidification is complete when the source moving from $x = 0$, and the source moving from $x = 2a$, merge at $x = a$. At this position we will obtain $(V_L)_1 = (V_L)_2$ regardless of their value at any other point.

As the plane source moves from $x = x_o$ to $x = a$, temperature $(V_L)_1$ may be:

(a) continuously increasing
(b) continuously decreasing
(c) constant $= \frac{L\sqrt{\pi}}{c} k_o e^{k_o^2} \theta(k_o)\left\{1 - \theta(k_o)\right\}$

The rate of solidification for (c) can be easily obtained, as the temperature, due to the original supply of heat at $x = a$ and $\tau = \tau_a$ will be given by

$$V_\phi = \frac{4\phi}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi}{2}\, e^{-(2n+1)^2\pi^2\tau_a/4}$$
$$= V - \frac{2L\sqrt{\pi}}{c} k_o e^{k_o^2}\theta(k_o)\left\{1 - \theta(k_o)\right\} \qquad (30)$$

The value τ_a of (30) may be found graphically and checked numerically.

The functions of τ, corresponding to (a), (b), and (c), will be obtained by substituting in the second equations of (29) a value τ_a

(a) greater than in (30)
(b) smaller than in (30)
(c) as in (30)

and may be represented as in Fig. 3.

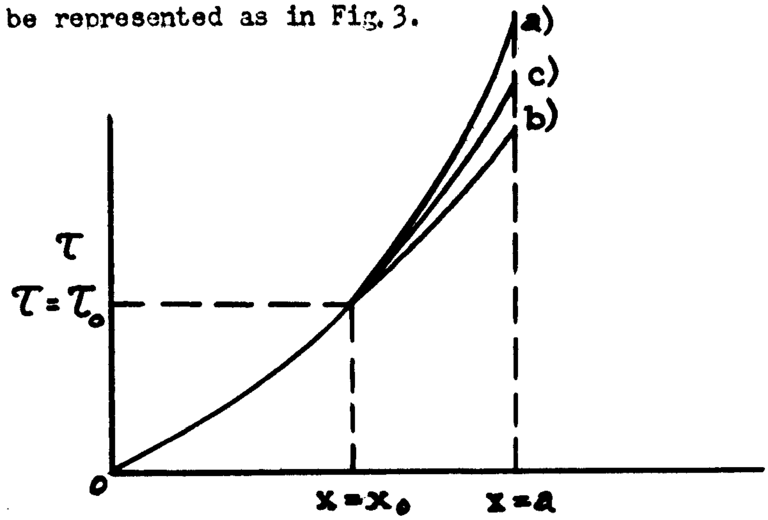


Fig. 3

The speed of the moving planes for (a), (b), and (c), will be given by

$$\frac{dx}{d\tau} = \frac{k_a}{\sqrt{\tau \cdot \tau_a}} \qquad \begin{array}{l} x_0 \leq x \leq a \\ \tau_0 \leq \tau \leq \tau_a \end{array} \tag{31}$$

and for each case will depend on k_a and τ_a .

In any numerical example, the lowest rate of solidification will be obtained for case (a) and the highest rate for case (b).

The rate of solidification is given also by the rate of conduction of heat at the source * , or

$$K\rho c\left\{\frac{d(V_L)_1}{dx}_{solid} - \frac{d(V_L)_1}{dx}_{molten}\right\} \tag{32}$$

* Lightfoot, loc. cit.

Graphically, the rate of conduction for case (a) may be represented as in Fig. 4. A large increase of $(V_L)_1$ implies an increase in the rate of conduction at the source, or rate of solidification. The rate computed with (31) will follow an opposite trend. The difference between graphical and computed rates tends to zero when $(V_L)_{1x=a} \rightarrow (V_L)_{1x=x_o}$

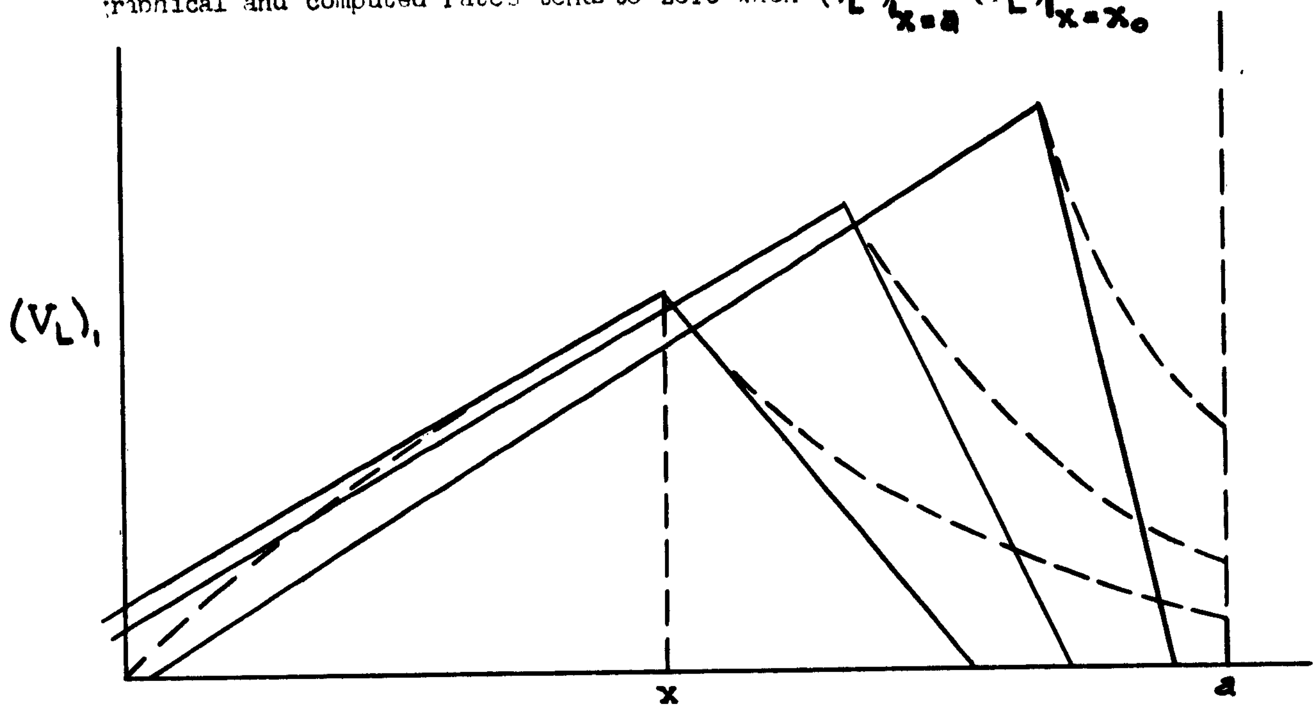


Fig. 4*

Case (b), represented in Fig. 5, shows that if $(V_L)_1$ decreases, the graphical rate will be lower than the computed rate. Their difference tends to zero when $(V_L)_{1x=a} \rightarrow (V_L)_{1x=x_o}$

*Dotted lines on Fig 4, 5, and 6 are temperature distributions $(V_L)_1$. Solid lines are drawn tangent to distributions at the sources.

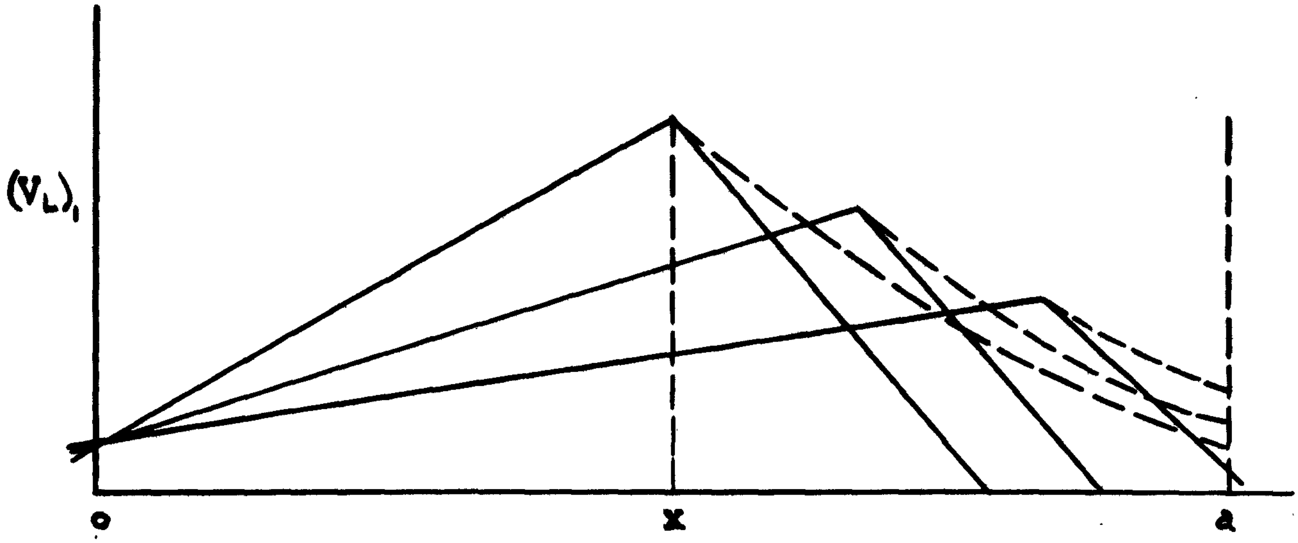


Fig. 5

In case (c), Figure 6, graphical and computed rates will coincide. As the moving planes $\underline{x}$ and $\underline{2a} - \underline{x}$ approach $\underline{x} = \underline{a}$, the rate of solidification decreases and tends to a constant. A constant temperature $(V_L)_1$, at any position of the plane source moving from $\underline{x} = 0$ (or $(V_L)_2$ at the plane source moving from $\underline{x} = \underline{2a}$) is the correct assumption.

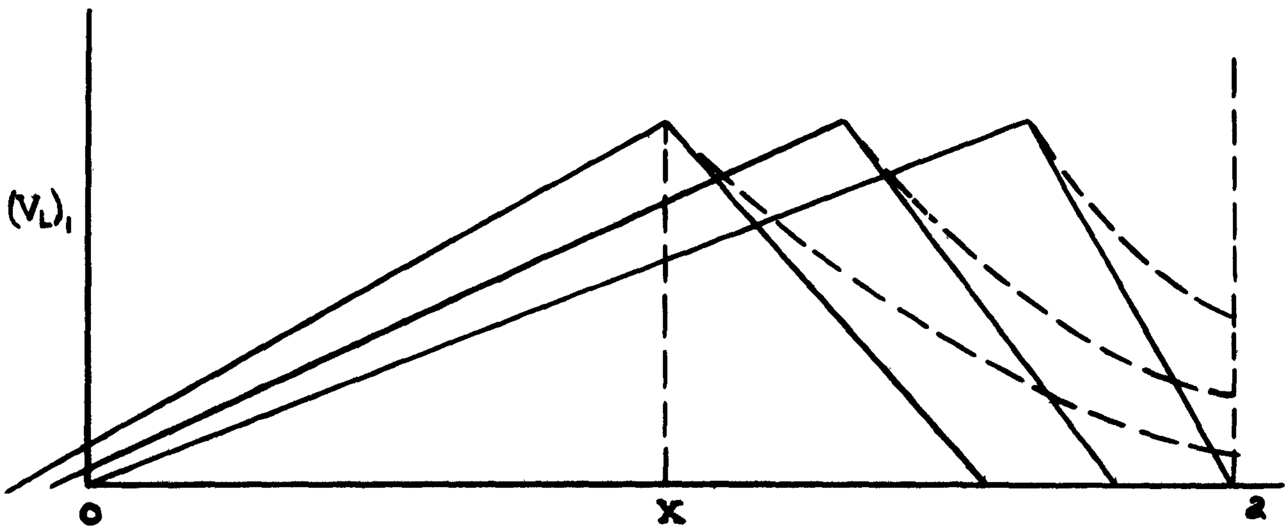


Fig. 6

We may state that in a freezing medium, initially at temperature ϕ, and with its bounding surfaces $\underline{x} = 0$, and $\underline{x} = \underline{2a}$, kept at zero, the positions of the surfaces of separation of solid and liquid phases are given by the two functions of τ

$$\frac{x}{a} = 2k_0\sqrt{\tau} \qquad 0 \leq \tau \leq \tau_0$$
$$\frac{x}{a} = -b + 2k_2\sqrt{\tau-\tau_2} \qquad \tau_0 \leq \tau \leq \tau_a \qquad (33)$$

The constants and the ranges of τ of (33) will vary for each numerical case and are to be determined from the data of the problem according to the conditions given in this section.

In the forms of (33), the thermal diffusivity $\underline{k}$, included in the dimensionless variable $\tau = \frac{k}{a^2}t$, is not undetermined. We assume $\underline{k}$ constant and equal to the average diffusivity of the medium in the temperature range V to ϕ, i.e. in the molten material. This constant is to be determined experimentally, and methods of measurement may be based on solutions given in the following sections.

The complete exact solution of the problem, for the special case $\phi = V$ (i.e. initial temperature is at melting point of the medium) is given by Lightfoot. In this case solidification proceeds from each boundary $\underline{x} = 0$ and $\underline{x} = \underline{2a}$, independently, and the position of the moving plane $\underline{x}$ (or $\underline{2a - x}$), until solidification is complete, is given by

$$\frac{x}{a} = 2k\sqrt{\tau} \qquad (34)$$

where k is obtained from (2) as for a semi-infinite medium.

In any numerical example $\phi = V$, we can write

$$\tau_a = \left(\frac{1}{2k}\right)^2 \qquad (35)$$

and we verify that for τ_a as in (35) we will obtain

$$V_\phi = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi}{2}\, e^{-(2n+1)^2\pi^2 \tau_a/4}$$

$$= V - \frac{2L\sqrt{\pi}}{c}\, k e^{k^2}\theta(k)\left\{1-\theta(k)\right\} \tag{36}$$

Some numerical solutions for media having the same melting point are shown in Figure 7. The initial temperatures ϕ and ratios $\frac{L}{c}$ are adjusted for equal value of constant k_0 .

4. The Sphere and the Infinite Cylinder

In the solution of the problem of solidification in the finite region 2a , the planes of separation of solid and liquid phases have been regarded as moving plane sources of heat. Their speed was obtained from the condition that the temperature at the sources is equal to the melting point of the medium. The temperature due to the original supply of heat is evaluated from a known solution while the temperature due to latent heat of fusion is determined from analysis. Solification in a sphere, or in a cylinder, gives rise to spherical, or cylindrical surface sources of heat, and the forms of V_L , for these cases, are similar to (24) or (25). As for moving plane surfaces, correct evaluation of the series of integrals can not be obtained by available methods. The temperature (V_L), at the surfaces of separation of solid and liquid phases will be determined from analysis.

The known solutions V_ϕ in the slab of thickness 2a, in the infinite cylinder $a > r \geq 0$, and in the sphere $a > r \geq 0$, initially at ϕ and with boundary surfaces kept at zero, expressed in terms of the dimensionless variable $\tau = \frac{K}{a^2}t$, are as follows:

$$(V_\phi)_1 = \frac{4\phi}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi x}{2a} e^{-(2n+1)^2 \pi^2 \tau_1/4} \tag{37}$$

$$(V_\phi)_2 = 2\phi \sum_{n=0}^{\infty} \frac{J_0(r\alpha_n)}{a_n J_1(a\alpha_n)} e^{-a_n^2 \tau_2} \tag{38}$$

$$(V_\phi)_3 = \frac{2a\phi}{\pi r} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \sin \frac{(n+1)\pi r}{a} e^{-(n+1)^2 \pi^2 \tau_3} \tag{39}$$

where the subscripts 1, 2, and 3, refer to the slab, the cylinder and the sphere, in that order.

The temperature distributions obtained with (37), (38), and (39), for the same medium, with dimension a and initial temperature ϕ , the same for the three regions, will never result as in Figure 8, regardless of τ_1 , τ_2 , and τ_3 . Identical distributions in any given range $x = a - r$, $2a - x = a - r$ and $0 < x \leq a$, may be obtained for particular values of τ_1, τ_2 , and τ_3 , as in Figure 9.

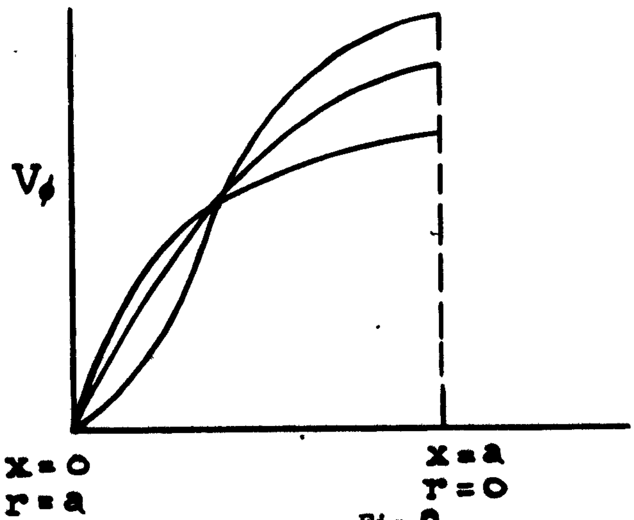


Fig. 8

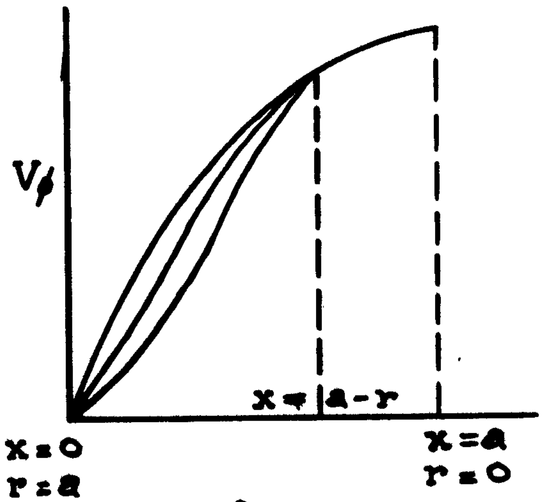


Fig. 9

In the problem of solidification we assume that, in all three regions, the surfaces of separation of solid and liquid phases are at equal distance from the boundary (we may have $\tau_1 \neq \tau_2 \neq \tau_3$) and for these surfaces we write

$$V_\phi + V_L = V \tag{40}$$

At these surfaces we obtain also

$$(V_\phi)_1 = (V_\phi)_2 = (V_\phi)_3$$

$$(V_L)_1 = (V_L)_2 = (V_L)_3 \tag{41}$$

where $(V_L)_1$, in this instance, is the temperature caused by all the heat of fusion flowing in the slab 2a.

We will prove briefly that (41) is a necessary condition in the solution of these problems.

We consider the slab and the infinite cylinder, solid at unit distance from the boundary, or at

$$x = a - r = 1 \tag{42}$$

and at the surfaces of separation, we make

$$(V_\phi)_1 > (V_\phi)_2 \tag{43}$$

The inequality of (43) means that, in the solid part of the slab, the temperature distribution $(V_\phi)_1$ would be higher, at all points, than the temperature distributions $(V_\phi)_2$ in the solid part of the cylinder. During solidification of unit volumes of medium adjacent to the external surfaces, the total flow of original heat, per unit area, would be greater at the cylindrical surface $a > r \geq a - l$ than at the corresponding plane surface $x = a - r$. During solidification, in any region, original heat and latent heat of fusion flow to the boundary concurrently and are subject to the same boundary conditions. With a greater total flow of original heat we would obtain also a greater total flow of latent heat of fusion; the temperature distribution $(V_L)_1$ in the solid part of the slab would be higher, at all points, than the temperature distribution $(V_L)_2$ in the cylinder.

Assumption (43) leads to the conclusion that, either in the slab or the cylinder, the temperature at the surface of separation of solid and liquid phases would result

$$V_\phi + V_L \neq V \tag{44}$$

The same conclusion will be reached reversing the inequality in (44) and for subsequent elements of volume, until solidification is complete. The same result will be reached comparing solidification in a slab and in a sphere.

This conclusion opens the way to the solution of the problems of solidification previously discussed and appears to have been overlooked in past work on the subject.

Since the solution of the problems of solidification for the slab, the infinite cylinder, and the sphere, must satisfy condition (41), the solutions for the cylinder and the sphere can be derived from the solution obtained for the slab.

For solidification in the slab, at a given distance x/a, we evaluate τ from the two forms of (33) and solve (37) for these values of τ and x/a. In (38) and (39) we substitute [illegible] and V_ϕ as obtained from (37). The values τ_a and τ_ϕ given by the (38) and (39) can be obtained graphically and checked numerically.

Figure 10 shows that, initially, the rates of solidification in all three regions coincide. After initial periods, the rates of solidification of a cylinder and a sphere will assume constant values; in a sphere more rapidly than in a cylinder.

5. The Slab, the Infinite Cylinder, and the Sphere, Initial temperature constant and radiation at the surface into a medium at constant temperature

As in the previous cases, the solutions of these problems are derived from the solution obtained for solidification in a slab, with initial temperature constant and bounding surfaces kept at zero.

In a medium, initially at ϕ and with radiation at $x = -a$ and $x = +a$ into a medium at zero, the temperature distribution V_ϕ is given by

$$V_\phi = \phi \sum_{n=1}^{\infty} \frac{2A \cos \frac{\beta_n x}{a}}{[\beta_n + A + A^2] \cos \beta_n} e^{-\beta_n^2 \tau} \tag{45}$$ *

where $A = ah$, h being the heat transfer coefficient of the surface, and $\beta_1, \beta_2 \ldots \beta_n$ the positive roots of

$$\beta \tan \beta = ah \tag{46}$$

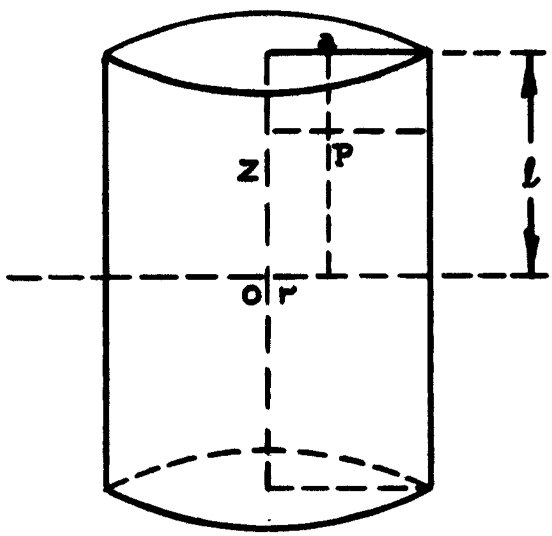
The temperature distribution obtained with (46) for any value of n and τ and the temperature distribution obtained from (37) for the slab, for any value of τ, will never cross each other, as in Figure 8. The difference between temperature gradients at corresponding points, through the entire region, will always be ≥ 0, or always ≤ 0. We conclude that, when in the region $-a < x < +a$, at constant initial temperature and radiating at $x = -a$ and at $x = +a$ into a medium at constant temperature, we find the surfaces of separation of solid and liquid phases at the same distance from the boundary as in the slab of (37), condition (41) will be satisfied. The solution of the problem for this case is obtained by the method outlined in section 4.

Condition (41) applies also to an infinite cylinder and to a sphere, initially at constant temperature and cooled by radiation into a medium at constant temperature. The two problems can be solved by the same method and forms of V_ϕ for these cases can be found in textbooks.

* Carslaw and Jaeger, Conduction of Heat In Solids, p. 100

6. The Finite Cylinder and the Parallelepiped

In the cylinder of Figure 11 initially at constant temperature ϕ and with boundary surfaces kept at zero, the surface of separation of solid and liquid phases will be at a point



$$P \begin{cases} \frac{l-z}{l} = \frac{1}{4} \\ \frac{a-r}{a} = \frac{2}{3} \end{cases} \tag{47}$$

Fig 11

when $(V_\phi)_P$ is equal to the temperature V_ϕ obtained in the slab of (37) at the surface of separation $\frac{x}{l} = \frac{1}{4}$. In general, solidification in a finite cylinder occurs with the higher of the two temperatures V_ϕ obtained for the slab at the two positions given by the coordinates of the point as in (47).

The known solution V_ϕ in a finite cylinder, initially at ϕ and with boundary surfaces kept at zero, expressed in terms of the dimensionless variable $\tau = \frac{k}{a^2} t$, is as follows:

$$V_\phi = \frac{8\phi}{\pi} \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{a_m J_1(a\alpha_n)} e^{-\alpha_n^2 \tau} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos \frac{(2m+1)\pi z}{2l} e^{-(2m+1)^2 \pi^2 \left(\frac{a}{2l}\right)^2 \tau} \tag{48}$$

Solution V_ϕ for a finite cylinder, initially at ϕ and cooled by radiation into a medium at constant temperature is given in textbooks.

Solidification in parallelepiped is solved by the same method.

Figure 12 shows surfaces of separation in a finite cylinder. The numerical data are the same as for (b) of Figure 7.

Inclosures:
1. Figure 2
2. Figure 7
3. Figure 10
4. Figure 12

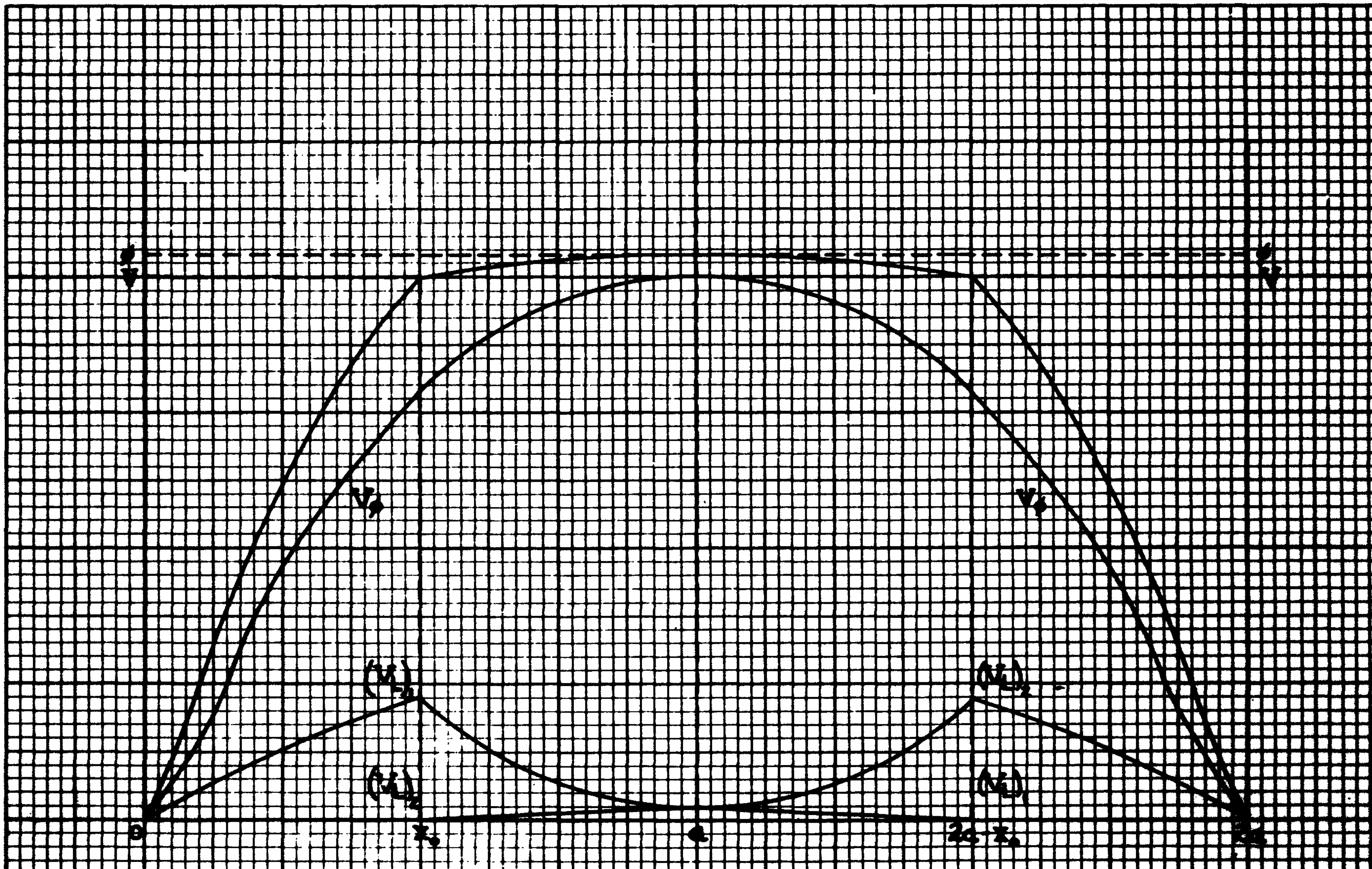
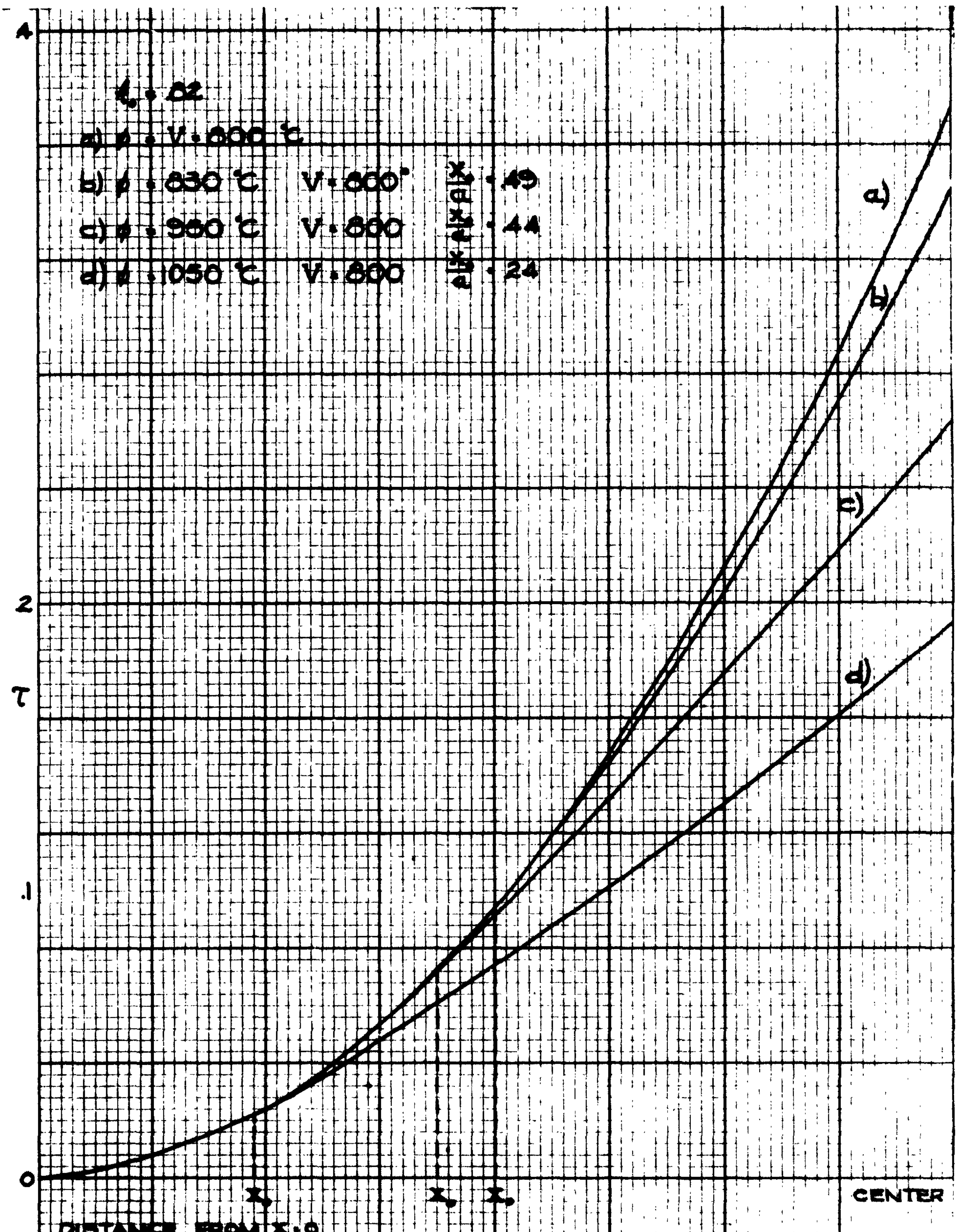
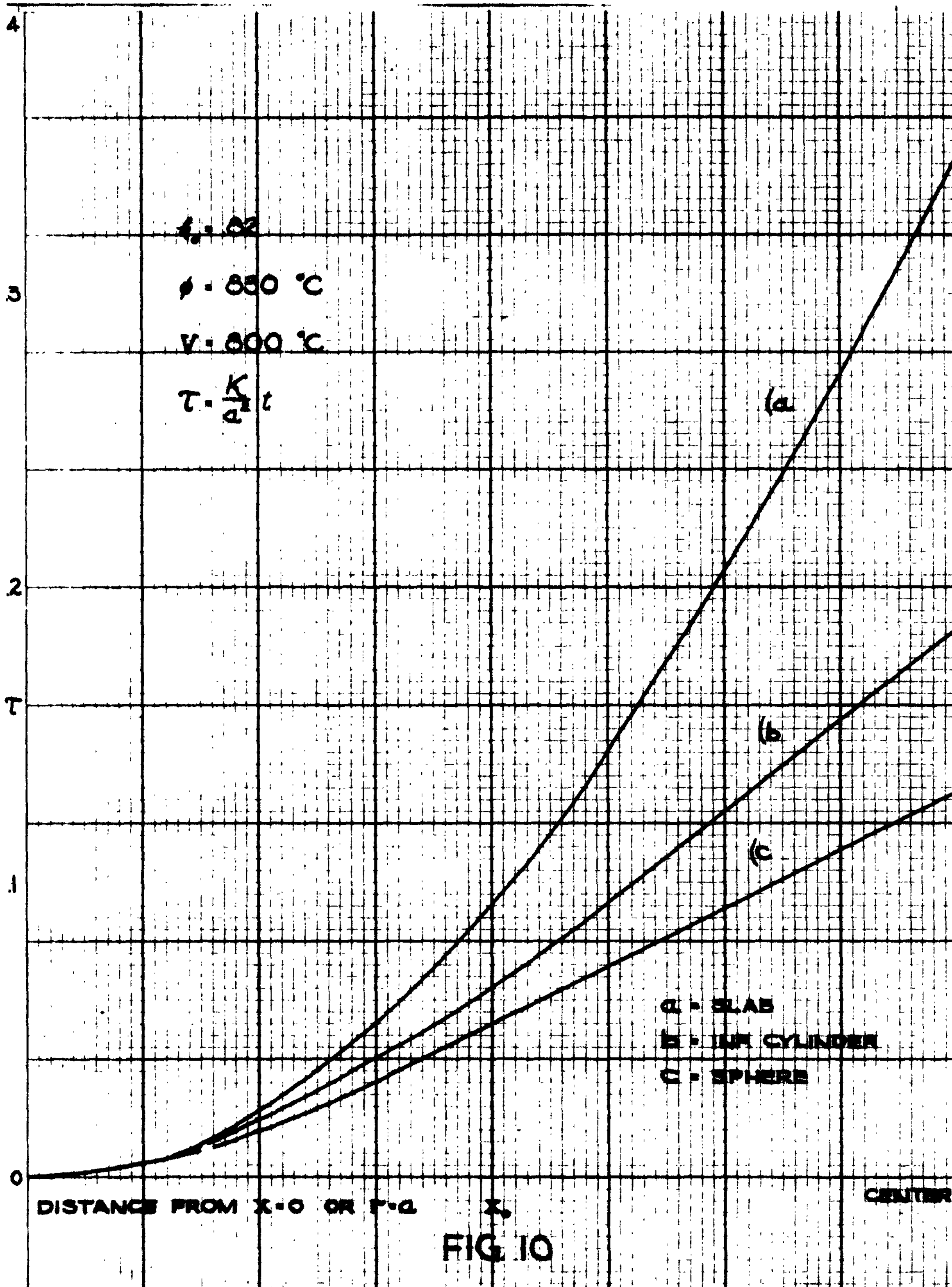


FIG. 2



.4
.2
.1
0
τ
a) φ = V = 800 °C
b) φ = 830 °C V = 800° x/a = .49
c) φ = 950 °C V = 800 x/a = .44
d) φ = 1050 °C V = 800 x/a = .24
a)
b)
c)
d)
DISTANCE FROM X = 0
CENTER

FIG. 7



$\xi_0 = .02$
$\phi = 850$ °C
$V = 800$ °C
$\tau = \frac{K}{a^2} t$
a
b
c
a = SLAB
b = INF CYLINDER
c = SPHERE
4
3
2
τ
1
0
DISTANCE FROM X = 0 OR r = a
X_0
CENTER

FIG 10

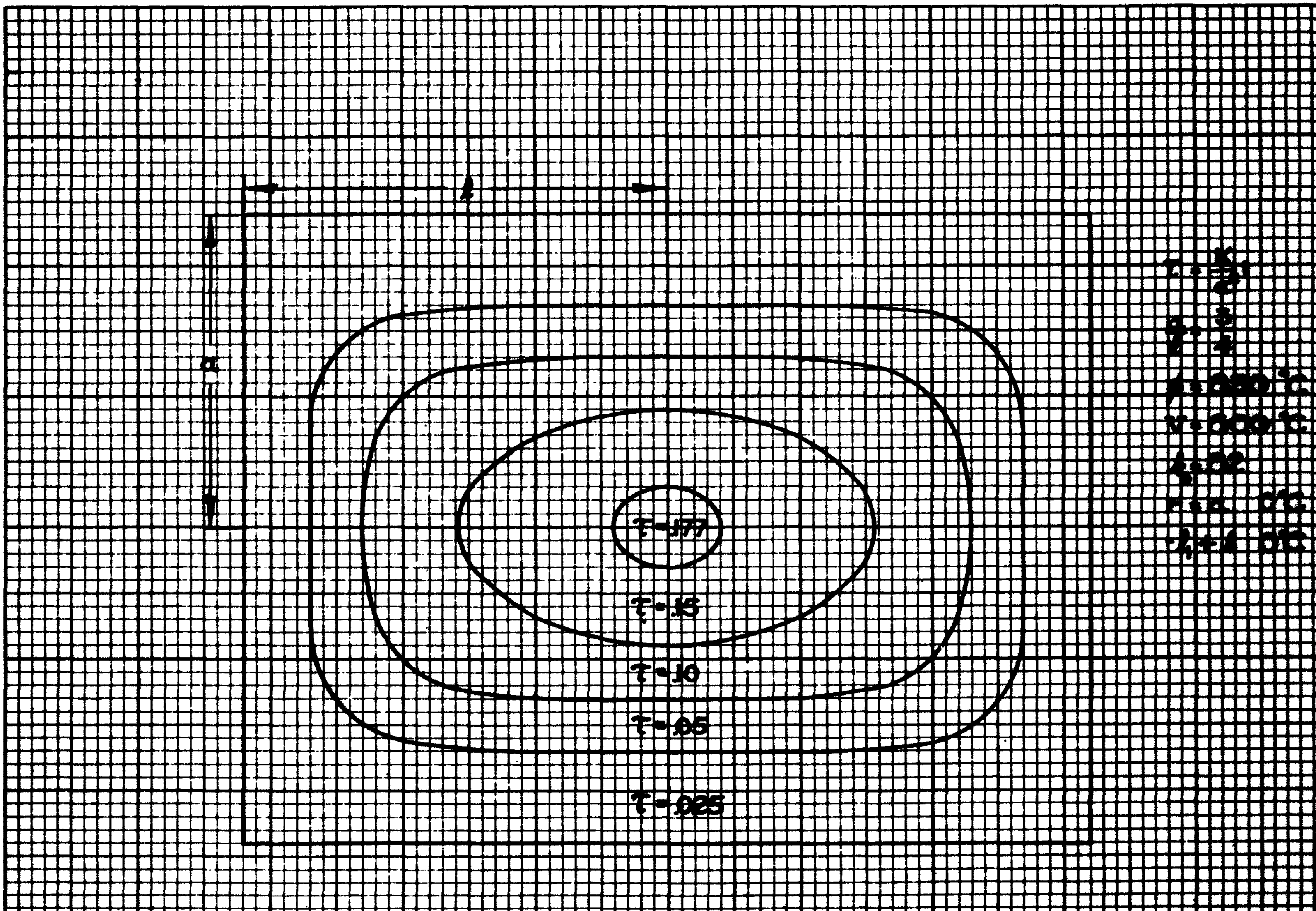


FIG. 12

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