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LUMPED ELEMENT IMPEDANCE TRANSFORMERS



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LUMPED ELEMENT IMPEDANCE TRANSFORMERS

by

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Progress report of work done under
Office of Naval Research Contracts

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PREFACE

The work on Lumped Element Impedance Transformers is one of a series resulting from a study carried on in the Electrical Engineering Department of the Carnegie Institute of Technology on exponential line section pulse transformers. In particular this report covers a generalization of the problem considered by T. J. O'Donnell (an earlier report in this series) in "Lumped Circuit Analogs of Tapered Transmission Lines".

This work described in this report has been supported jointly by Carnegie Institute of Technology and Office of Naval Research Contract N7onr 30306

Introduction

The transient properties of tapered transmission lines for use as pulse transformers have been extensively investigated and described in earlier reports of this series. One report has dealt with the possibilities of lumped circuit analogs of tapered transmission lines. Owing to restrictions placed on certain of the network properties incorporated in these analogs there has been reason to believe that more advantageous broad-band transformations could be achieved by a modified analytical attack on the problem. This report is concerned with such a modified attack and shows, in fact, that improved characteristics can be obtained i.e. greater bandwidth for the same transformation ratio, or greater transformation ratio for the same bandwidth, with a specified number of circuit elements.

This report constitutes a dissertation presented by David H. Geipel in practical fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering at Carnegie Institute of Technology.

TABLE OF CONTENTS

Summary	1
Introduction.....	2
Section I ...Determinantal Condition for Impedance Transformation	9
Section IIThe Three Mesh Network.....	13
Section IIIThe Four Mesh Network.....	27
Section IVThe Five Mesh Network.....	43
Section VThe Input Image Impedance.....	56
Section VIThe Voltage Transfer Function.....	60
Section VIIExperimental Results.....	67
Section VIII.....Discussion and Suggestions for Further Study...	73
Appendix I	77
Appendix II	79
Appendix III.....	81
Bibliography.....	83

LIST OF FIGURES

Figure 1	Dietze's Tapered Filter.....	8
Figure 2	Exponentially Tapered Network of O'Donnell.....	8
Figure 3	Tschebyscheff Tapered Network of O'Donnell.....	8
Figure 4	General Three Mesh Network.....	14
Figure 5	Admissible Values of a and b for Three Mesh Case.....	14
Figure 6	Three Mesh Network With $\alpha = 0$	23
Figure 7	Three Mesh Network With $b = 0$	23
Figure 8	Three Mesh Network With $a = 0$	23
Figure 9	Bandwidth Ratio Values for Three Mesh Case.....	24
Figure 10	General Four Mesh Network.....	28
Figure 11	Four Mesh Network With $a = 0$	28
Figure 12	Four Mesh Network With $b = 0$	38
Figure 13	Four Mesh Network With $\alpha = 0$	38
Figure 14	Admissible Values of g_1 for Four Mesh Case.....	40
Figure 15	Bandwidth Ratio as a Function of g_1 and K	41
Figure 16	Plots of Bandwidth Ratio for Four Mesh Case.....	42
Figure 17	General Five Mesh Network.....	44
Figure 18	Admissible Values of a and g for Five Mesh Case.....	44
Figure 19	Five Mesh Network With $a = 0$	44
Figure 20	Five Mesh Network With $b = 0$	54
Figure 21	Five Mesh Network With $\alpha = 0$	54
Figure 22	Plots of Bandwidth Ratio for Five Mesh Case.....	55
Figure 23	Plot of Input Image Impedance in the Pass Band.....	59
Figure 24	Voltage Transfer Function for Mismatched Narrow Band Network,	63

Figure 25 Voltage Transfer Function for Mismatched Medium Band Network.	64
Figure 26 .Voltage Transfer Function for Mismatched Wide Band Network..	65
Figure 27 ..Combined Plots of Transfer Functions for Constant Mismatch.	66
Figure 28Experimental Results With Narrow Band Network.....	70
Figure 29Experimental Results With Medium Band Network.....	71
Figure 30Experimental Results With Wide Band Network.....	72

SUMMARY

This thesis is concerned with the synthesis of lumped element networks consisting of three, four or five meshes which act so as to transform impedances, independent of frequency. The networks are designed over all--without specified impedance conditions at interior junctions--by applying a necessary and sufficient condition for impedance transformation to the impedance determinant of the complete network. Within this framework, steps are taken to insure single band-pass operation; and, wherever possible, simplicity and economy dictate the selection of elements.

The bandwidth ratio, the ratio of upper cutoff frequency to lower cutoff frequency, as a function of impedance transformation ratio is used as a criterion of performance, and expressions for this parameter are given for each of the networks considered. In the three and the five mesh cases, networks corresponding to maximum bandwidth ratio are obtained. The means of finding the value of source resistance which gives optimum matching over the pass-band and the voltage transfer function are indicated by examples involving three, three mesh networks. Experimental verification of the theory and of the efficacy of assuming lossless elements is also included.

In addition to allowing the realization of impedance transforming networks with larger bandwidth ratios and much greater flexibility than hitherto available, this investigation seems to indicate that proper exploitation of the freedom made available by relaxing the condition of impedance matching at interior junctions will lead to superior tapered filters.

INTRODUCTION

The purpose of this thesis is to present a simple means of synthesizing impedance transformers, using only lumped inductors and capacitors, that have wider bandwidths and greater flexibility than those hitherto available. This greater flexibility is manifested in the ability of those networks to permit, for any given impedance transformation ratio, a range of bandwidth ratio values. This is in contrast to the previous network designs which possess a strict one-to-one bandwidth ratio versus impedance transformation ratio characteristic. The approach used in the work that follows depends on treating a network of the required complexity as a complete entity rather than designing basic sub-units which are then cascaded to form the complete network. It was felt that by doing this the constraints implicit in requiring that interior junctions be impedance matched would be relaxed; and, therefore, one would be more free to choose component values that lead to improved network designs.

A great amount of work has been done on all kinds of impedance transforming structures. Modern requirements for broad band impedance matching devices have led in the recent past to a great amount of investigation of tapered transmission lines. Most work has been done on the transmission lines utilizing an exponential taper because of the relative ease of mathematical analysis and physical realizability. In this type of line the capacitance and inductance per unit length vary, reciprocally with each other, as an exponential function of distance along the line.

Burrows⁽¹⁾ and Wheeler⁽²⁾ were among the first to describe the steady

References will be numbered consecutively and appear in the bibliography at the end of the thesis.

state behavior of such lines. More recently Schatz and Williams^{(3),(4)} have analytically and experimentally determined the transient response of these structures and found that they are best suited to transforming extremely short duration pulses with fast rise times and high peak powers. Unfortunately, these structures are physically large and unwieldy especially where larger ratios of transformation are involved. In an effort to avoid this difficulty some work has been done on a transmission line utilizing a helically wound inner conductor in which either the pitch or the turns density is made to vary in an exponential fashion⁽⁵⁾. This is in contrast to the original method of using an exponentially flared tubing as the inner conductor.

In addition to the helical line as a means of circumventing the undesirable length of the exponential line some effort has been directed towards synthesizing lumped element analogs of the tapered transmission line. These are generally referred to as tapered artificial lines or tapered filters.

Tapered filters designed to operate between different impedance levels are not a recent innovation. Patents issued to Norton⁽⁶⁾ and Dietze⁽⁷⁾ indicate methods of designing tapered filters. Norton seems to be the first to have used the idea of determining a dissymmetrical circuit which is equivalent to a symmetrical section plus an ideal transformer. Gladwin⁽⁸⁾, using the equivalences of Norton, shows many applications of these circuits in addition to showing that such networks must be of the band-pass type. Dietze's method consisted simply of cascading symmetrical sections having identical transfer constants but exponentially increasing characteristic impedances. Thus, the transition from a low impedance to a high impedance level is made more gradual with a consequent

improvement in performance. (See Figure (1))

In his recent investigations O'Donnell⁽⁹⁾, following the method of Wheeler and Murnaghan⁽¹⁰⁾, has calculated the ratio of output to input image impedances for Dietze's structure and concluded that it approaches the desired constancy only in the high frequency attenuation band while having the most rapid variation in all but the lowest pass band. This in addition to the absence of a general design procedure induced O'Donnell to investigate other means of synthesizing tapered filters which would have ratios of output to input image impedances that were constant, independent of frequency.

O'Donnell's method consisted of matching on an image impedance basis dissymmetrical T or π sections which have different input and output image impedances as shown in Figures (2) and (3).

In the first case treated by O'Donnell the relationship $\frac{Z_{1k}}{Z_{1k-1}} = r^2$ was enforced. This case is analogous to the exponential taper and required that a symmetrical section of unity transformation ratio with characteristic impedance equal to Z_{11} precede the transforming network in order to eliminate a non-realizable impedance. The second case considered by O'Donnell was predicated on the condition that no non-realizable element would occur in the transforming network proper. The resulting network possessed a taper law which he called the Tschetyscheff taper. More detailed references will be made later to the two networks mentioned above and to their respective characteristics.

In attempting to synthesize a network possessing certain given characteristics one can either elect to use the classical theory of four terminal networks or the more modern techniques which utilize a function theoretic approach. The latter are perhaps best typified by the potential

analogue methods of Darlington⁽¹¹⁾ or the two terminal impedance synthesis of Brune⁽¹²⁾. There are available also the network synthesis methods of Cauer and Bode⁽¹³⁾ which are based on the symmetrical lattice and allow any degree of approximation to given realizable filter characteristics.

Darlington has developed a general synthesis procedure for reactive four terminal ladder networks terminated by arbitrary resistances which will yield approximations to desired insertion loss functions. Fano⁽¹⁴⁾ has indicated a method for designing reactive networks that will provide optimum matching of a resistive source to an arbitrary load impedance.

Unfortunately, the procedures of Cauer and Bode generally apply to symmetrical structures and have little application to the problem at hand. Darlington's potential analogue method as well as his ladder network theory seems to be best suited for treating cases where a structure having a specific frequency characteristic is desired. Moreover, these approaches involve great mathematical complexity and the numerical complications are such as to limit their practical usefulness to circumstances that warrant the expenditure of the necessary time and effort. Another reason for treating the problem with classical network theory is the fact that none of the function theoretic methods allow one to conveniently avoid the use of mutual inductances or ideal transformers.

As previously indicated, the point of departure between this work and that of Norton, Dietze and O'Donnell is founded on the idea that the design of a lumped element impedance transforming structure might better be formulated by treating an N mesh network as a complete unit. Previous investigators designed their networks out of sub-units which were then cascaded in the ways indicated. In all cases conditions are placed on the image impedances which exist at points interior to the complete struc-

ture. It was felt that by considering only the overall network, without any specification of the impedance conditions existing at interior points of the network, these unnecessary constraints could be avoided thereby yielding an improved design.

The method to be followed consists simply of applying the determinantal condition that is necessary and sufficient for impedance transformation to networks having three, four and five meshes respectively. Then, within this framework, a set of relationships between element values which assures single band pass operation is obtained. In all instances where an arbitrary specification of some of the components is possible the choice is made on the basis of simplicity and economy of elements. The parameter of main interest is the bandwidth ratio defined as the ratio of upper and lower cutoff frequencies, and the expression for this parameter is derived for each of the networks considered.

Section I includes the derivation of the basic determinantal equations on which the thesis is based, and the conditions to insure impedance transformation are also indicated.

In Section II, III and IV these conditions are applied to three, four and five mesh networks respectively. In each case, conditions for single band pass operation are developed, and the bandwidth ratio versus transformation ratio characteristic is obtained. A comparison of these results with those of O'Donnell's is also included.

In Section V the shape of the input image impedance curve for frequencies in the pass band is calculated, and dimensionless plots are given for representative value of β .

The value of matching source resistance that gives the optimum voltage transfer characteristic is determined in Section VI along with repre-

sentative plots of the voltage transfer characteristic for three networks of the three mesh type. These networks though designed for the same impedance transformation ratio differ in form and display different bandwidth ratios.

Section VII contains a sample calculation of component values for the three three mesh networks in Section VI along with an experimental verification of the theory.

Section VIII includes a discussion of certain aspects not considered in the thesis proper and some suggestions for further study.

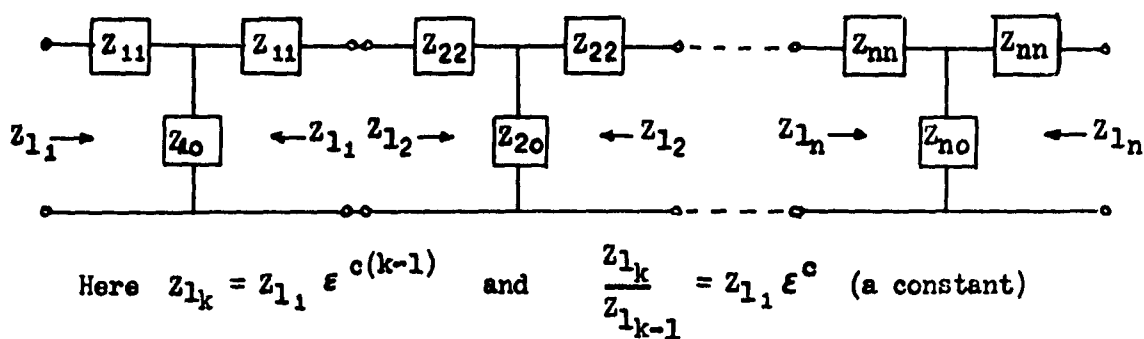


Figure 1 Dietze's Tapered Filter

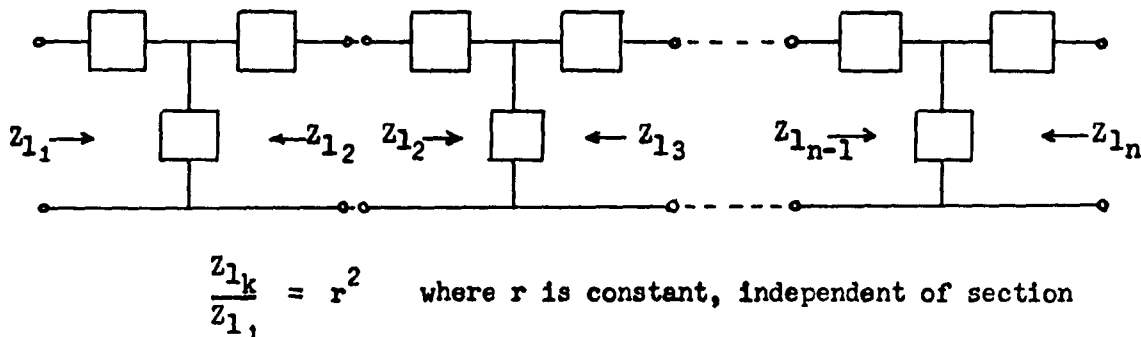
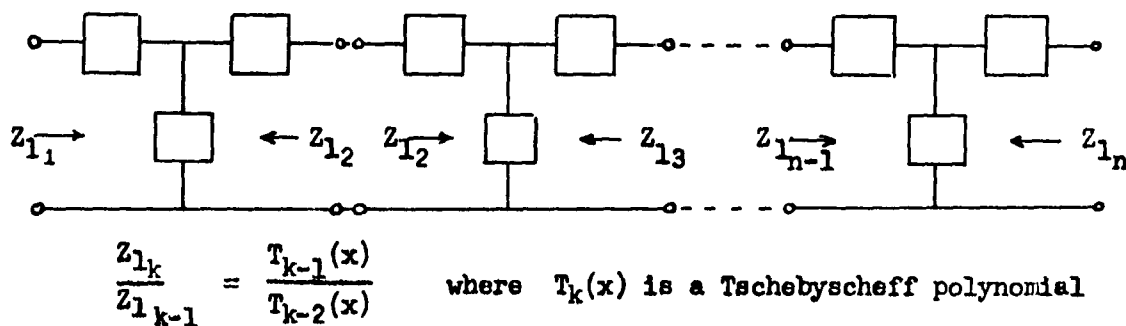


Figure 2 Exponentially tapered network of O'Donnell



of order K and x is equal to $\frac{s^2+1}{s^2-1}$. Overall impedance transformation ratio of n sections is $T_n(x)$.

Figure 3 Tschebyscheff tapered network of O'Donnell

Section I

DERIVATION OF DETERMINENTAL CONDITIONS FOR IMPEDANCE TRANSFORMATION

The following derivations may be found in reference (15). They are included here for completeness and easy reference.

Consider a linear, passive, bilateral, four-terminal, n-mesh network like that shown below.



Using Kirchhoff's voltage law and writing the mesh equations we have

$$\begin{aligned} E_{in} &= Z_1 I_1 - j_{12} I_2 - j_{13} I_3 \dots - j_{1n} I_n \\ 0 &= -j_{21} I_1 + Z_2 I_2 - j_{23} I_3 \dots - j_{2n} I_n \\ 0 &= -j_{31} I_1 - j_{32} I_2 \dots - j_{3n} I_n \\ &\vdots \\ &= \dots \\ &= \dots \\ E_o &= -j_{n1} I_1 - j_{n2} I_2 \dots + Z_n I_n \end{aligned}$$

Where, Z_j is the sum of all of the impedances in the jth mesh, j_{jk} is the sum of all of the impedances which are common to both the jth and kth mesh, and I_j is the current in the jth mesh.

Solving this system of equations for I_1 and I_n yields

$$I_1 = \frac{E_{in} \Delta_{11} + (-1)^n E_o \Delta_{1n}}{\Delta} \quad \text{and} \quad I_n = \frac{(-1)^{n-1} E_{in} \Delta_{n1} - E_o \Delta_{nn}}{\Delta}$$

where Δ is the determinant of the impedance elements and Δ_{jk} is the minor formed from Δ by deleting the jth column and kth row.

Denoting the input and output image impedances by Z_{1_1} and Z_{1_2} respectively, we can say that, by definition, when the network is terminated at the output by Z_{1_2} the impedance seen at the input is Z_{1_1} and placing Z_{1_1} across the input terminals causes the impedance Z_{1_2} to appear at the output terminals. That is, when $E_o = I_n Z_{1_2}$ then $E_{1n} = I_1 Z_{1_1}$, and when $E_{1n} = -I_1 Z_{1_1}$ then $E_o = -I_n Z_{1_2}$.

Considering the first of these statements, when

$$E_o = \frac{Z_{1_2} [(-1)^{n-1} E_{1n} \Delta_{n1} - E_o \Delta_{nn}]}{\Delta} \quad \text{then} \quad E_{1n} = Z_{1_1} \frac{[E_{1n} \Delta_{11} + (-1)^n E_o \Delta_{1n}]}{\Delta}$$

$$\text{Thus, if } E_o(1 + Z_{1_2} \frac{\Delta_{nn}}{\Delta}) = (-1)^{n-1} Z_{1_2} E_{1n} \frac{\Delta_{n1}}{\Delta}, \quad E_{1n}(1 - Z_{1_1} \frac{\Delta_{11}}{\Delta}) = (-1)^n E_o Z_{1_1} \frac{\Delta_{1n}}{\Delta}$$

$$\text{Therefore, } E_o (\Delta + Z_{1_2} \Delta_{nn}) = -E_o \frac{Z_{1_1} Z_{1_2} \Delta_{n1} \Delta_{1n}}{(\Delta - Z_{1_1} \Delta_{11})} \quad \text{or}$$

$$\Delta^2 + \Delta(Z_{1_2} \Delta_{nn} - Z_{1_1} \Delta_{11}) - Z_{1_1} Z_{1_2} (\Delta_{11} \Delta_{nn} - \Delta_{1n} \Delta_{n1}) = 0$$

Now, because of reciprocity, $\Delta_{1n} = \Delta_{n1}$. Also, $(\Delta_{11} \Delta_{nn} - \Delta_{1n}^2) = \Delta \Delta^*$ where Δ^* is found from Δ by deleting the first and last rows and columns. (See reference 16) Hence, $\Delta + Z_{1_2} \Delta_{nn} - Z_{1_1} \Delta_{11} - Z_{1_1} Z_{1_2} \Delta^* = 0$ (1-1)

Applying an exactly parallel method to the requirement that if $E_{1n} = -I_1 Z_{1_1}$ then $E_o = I_n Z_{1_2}$ yields the following equation.

$$\Delta - Z_{1_2} \Delta_{nn} + Z_{1_1} \Delta_{11} - Z_{1_2} Z_{1_1} \Delta^* = 0 \quad (1-2)$$

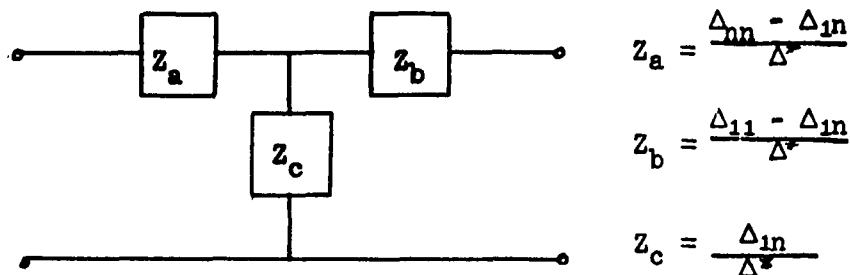
Solving (1-1) and (1-2) simultaneously gives the equations for Z_{1_1} and Z_{1_2} .

$$Z_{1_2} = \sqrt{\frac{\Delta_{11} \Delta}{\Delta_{nn} \Delta^*}} \quad \text{and} \quad Z_{1_1} = \sqrt{\frac{\Delta_{nn} \Delta}{\Delta_{11} \Delta^*}}$$

These equations now permit a statement of the determinantal conditions that are necessary and sufficient to insure a network that will transform impedances independent of frequency. The impedance transformation is $\frac{Z_{1_2}}{Z_{1_1}}$ and if transformation independent of frequency is desired, it is only necessary to make certain that $\frac{\Delta_{11}}{\Delta_{nn}} = K$ (a constant). K is the impedance

transformation ratio.

It can also easily be shown from the foregoing that any such four terminal network can be represented as a T section like that shown below.



This representation will be useful in Section V to find the voltage transfer function when the network is mismatched.

An alternative approach could be used because for an n-node network a system of equations involving admittances, similar to those for the mesh analysis, can be written. Similar results are obtained for the input and output image admittances by exactly parallel considerations. That is, Y_{11}

and Y_{12} are given by $Y_{11} = \sqrt{\frac{\Delta_{nn} \Delta}{\Delta_{11} \Delta^*}}$ and $Y_{12} = \sqrt{\frac{\Delta_{11} \Delta}{\Delta_{nn} \Delta^*}}$, and

the requirement of admittance transformation independent of frequency amounts again to the stipulation that $\frac{\Delta_{11}}{\Delta_{nn}} = K$. The various minors of Δ are the same as previously defined except that Δ is now made up of the various branch and node admittances.

In all of what follows the networks will be considered on the mesh basis. All that can be done on the mesh basis using impedances can be duplicated by considering node equations involving admittances. Indeed, there is quite a bit more flexibility available when the nodal analysis is used because it is frequently easier topologically to include certain mutual coupling admittances, in higher order networks, than it is to achieve an equivalent coupling impedance in a mesh structure. In order to partially

restrict the content of this thesis, however, the additional degrees of freedom available through the use of a nodal analysis will not be exploited. Thus, the remainder of this thesis will be directed towards a study of networks whose elements are regarded as impedances and whose structure is such that the condition $\frac{\Delta_{11}}{\Delta_{nn}} = K$ is fulfilled.

Section II

THE THREE MESH NETWORK

In this chapter the determinantal condition required for impedance transformation is applied to a three mesh network. Constraints are introduced so that single-band-pass operation is assured, and the networks thus evolved are analysed to determine their bandwidth ratio characteristics. Consider the impedance determinant Δ for the three mesh network shown in Figure (4).

$$\Delta = \begin{vmatrix} Z_1 & -j_{12} & -j_{13} \\ -j_{12} & Z_2 & -j_{23} \\ -j_{13} & -j_{23} & Z_3 \end{vmatrix} \quad \text{Where,}$$

$$Z_1 = j_{11} + j_{12} + j_{13}$$

$$Z_2 = j_{22} + j_{12} + j_{23}$$

$$Z_3 = j_{33} + j_{13} + j_{23}$$

The condition that $\frac{Z_{12}}{Z_{11}} = \frac{\Delta_{11}}{\Delta_{nn}} = K$ corresponds to the requirement that $(Z_2 Z_3 - j_{23}^2) = K(Z_1 Z_2 - j_{12}^2)$.

This equation will be satisfied and a simpler geometric configuration will result if j_{13} is made equal to $\sqrt{K} j_{12}$. Then $Z_3 = K Z_1$, which requires that

$$j_{13} + j_{13} + j_{23} = K j_{11} + K j_{12} + K j_{13} \text{ or since } j_{13} = \sqrt{K} j_{12}$$

$$j_{23} = K j_{11} + (K - \sqrt{K}) j_{12} + (K-1) j_{13}$$

j_{23} will certainly be realizable if j_{11} , j_{12} and j_{13} are realizable and if $K \geq 1$. (Throughout the thesis K is regarded as being greater than or equal to unity. If $K < 1$ is desired, it is only necessary to turn a network designed for $K \geq 1$ end for end.)

Under the conditions mentioned above, the determinant Δ becomes

$$\Delta = \begin{vmatrix} Z_1 & -j_{12} & -j_{13} \\ -j_{12} & Z_2 & -\sqrt{K} j_{12} \\ -j_{13} & -\sqrt{K} j_{12} & K Z_1 \end{vmatrix} \quad \text{and} \quad Z_{11} = \sqrt{\frac{\Delta_{nn}}{\Delta_{11} \Delta^*}} = \frac{1}{\sqrt{K}} \sqrt{\frac{\Delta}{\Delta^*}}$$

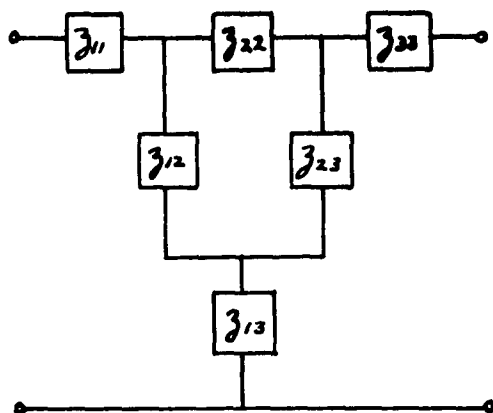


Figure 4

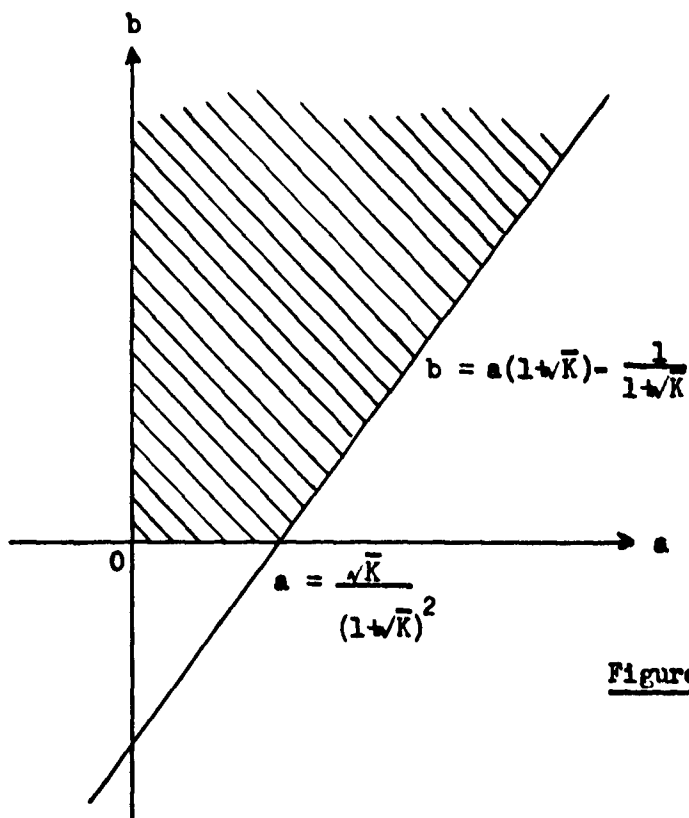


Figure 5

This thesis is primarily concerned with structures displaying single band pass operation. Since a pass band occurs when Z_{11} is real, it is desirable to minimize the number of poles or zeros of Z_{11} . This implies that, since Δ and Δ^* are both ultimately rational functions of frequency, it would be desirable to enforce the condition that Δ^* be contained as a factor of Δ . Furthermore, this consideration suggests that aside from economical considerations it would be wise to make the impedance elements in the network correspond to simple configurations. For example, it would be imprudent to make j_{12} or j_{13} represent a parallel resonant circuit.

$$\begin{aligned} \text{Now, } \Delta &= K Z_1^2 Z_2 - 2\sqrt{K} j_{12} j_{13} - Z_2 j_{12}^2 - 2K Z_1 j_{12}^2 \\ \Delta &= K Z_2 \left(Z_1^2 - \frac{j_{12}^2}{K} \right) - 2K j_{12}^2 \left(Z_1 + \frac{j_{12}}{\sqrt{K}} \right), \text{ and } \Delta^* = Z_2. \end{aligned}$$

$$\begin{aligned} \text{Thus, } Z_{11} &= \frac{1}{\sqrt{K}} \sqrt{\frac{K Z_2 \left(Z_1^2 - \frac{j_{12}^2}{K} \right) - 2K j_{12}^2 \left(Z_1 + \frac{j_{12}}{\sqrt{K}} \right)}{Z_2}} \\ Z_{11} &= j_{12} \sqrt{\left(\frac{Z_1}{j_{12}} \right)^2 - \left(\frac{j_{12}}{\sqrt{K} j_{12}} \right)^2 - 2 \left[\frac{Z_1 + \frac{j_{12}}{\sqrt{K}}}{Z_2} \right]} \quad (2-1) \end{aligned}$$

In order that Δ^* be a factor of Δ in this case requires only that

$$Z_1 + \frac{j_{12}}{\sqrt{K}} = \phi Z_2 \quad (\text{where } \phi \text{ is a real constant}) \text{ or that}$$

$$j_{11} + j_{12} + j_{13} + \frac{j_{12}}{\sqrt{K}} = \phi (j_{11} + j_{12} + j_{13}).$$

Since $j_{13} = \sqrt{K} j_{12}$, this in turn becomes $j_{11} + j_{12} + (1 + \frac{1}{\sqrt{K}}) j_{12} = \phi j_{11} + \phi (1 + \sqrt{K}) j_{12}$

Keeping the series impedances in the first two meshes of the same kind, ie

$j_{11} = \alpha j_{12}$, and expressing j_{13} as $j_{13} = a j_{11} + b j_{12}$ we have

$$j_{12} \left[\alpha + a \left(1 + \frac{1}{\sqrt{K}} \right) \right] + \left(1 + b + \frac{b}{\sqrt{K}} \right) j_{12} = \phi j_{12} + \phi (1 + \sqrt{K}) j_{12}$$

Assuming that $\left(\frac{j_{11}}{j_{12}} \right)$ is a function of frequency, which is of course the only case of interest, this condition will be satisfied if and only if

$$\left[\alpha + a \left(1 + \frac{1}{\sqrt{K}} \right) \right] = \phi \quad \text{and} \quad \left(1 + b + \frac{b}{\sqrt{K}} \right) = \phi (1 + \sqrt{K})$$

Equation (2-1) becomes, upon substituting the values of $Z_{1,1}$ and $j_{1,2}$,

$$Z_{1,1} = j_{1,2} \sqrt{(\alpha \frac{j_{1,2}}{j_{1,1}} + 1 + a \frac{j_{1,2}}{j_{1,1}} + b)^2 - \frac{1}{K}(a \frac{j_{1,2}}{j_{1,1}} + b)^2 - 2\phi} \quad \text{or}$$

$$Z_{1,1} = j_{1,2} \sqrt{[(a+\alpha)^2 - \frac{a^2}{K}](\frac{j_{1,2}}{j_{1,1}})^2 + 2[(1+b)(a+\alpha) - \frac{ab}{K}](\frac{j_{1,2}}{j_{1,1}}) + [(1+b)^2 - \frac{b^2}{K} - 2\phi]} \quad (2-2)$$

In order to determine the frequencies that the network passes it is only necessary to determine the regions where $Z_{1,1}$ is real. In the networks to be considered, these regions will lie between adjacent zeros of the real function of $(\frac{j_{1,2}}{j_{1,1}})$ given under the radical sign in equation (2-2). Zeros occur when

$$\frac{j_{1,2}}{j_{1,1}} = \frac{-2[(1+b)(a+\alpha) - \frac{ab}{K}] \pm \sqrt{4[(1+b)(a+\alpha) - \frac{ab}{K}]^2 - 4[(a+\alpha)^2 - \frac{a^2}{K}][(1+b)^2 - \frac{b^2}{K} - 2\phi]}}{2[(a+\alpha)^2 - \frac{a^2}{K}]}$$

$$\frac{j_{1,2}}{j_{1,1}} = \frac{-[(a+\alpha)(1+b) - \frac{ab}{K}] \pm \sqrt{2\phi(a+\alpha)^2 - \frac{2ab(a+\alpha)(1+b)}{K} + \frac{1}{K}b^2(a+\alpha)^2 + \frac{1}{K}a^2(1+b)^2 - 2\phi\frac{a^2}{K}}}{[(a+\alpha)^2 - \frac{a^2}{K}]}$$

or finally when

$$\frac{j_{1,2}}{j_{1,1}} = \frac{-[(1+b)(a+\alpha) - \frac{ab}{K}] \pm \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - (a+\alpha)b]^2}}{[(a+\alpha)^2 - \frac{a^2}{K}]}$$

In the case where $j_{1,2}$ represents a capacitor and $j_{1,1}$ represents an inductor, i.e., where $j_{1,2} = \frac{1}{j\omega C_0}$ and $j_{1,1} = j\omega L_0$, the zeros of $Z_{1,1}$ occur when

$$\frac{j_{1,2}}{j_{1,1}} = \frac{j\omega L_0}{\frac{1}{j\omega C_0}} = -\frac{\omega}{\omega_0} = -\pi^2 = \frac{-[(1+b)(a+\alpha) - \frac{ab}{K}] \pm \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - (a+\alpha)b]^2}}{[(a+\alpha)^2 - \frac{a^2}{K}]}$$

where $\omega_0 = \frac{1}{\sqrt{L_0 C_0}}$. If upper cutoff frequency = f_2 and lower cutoff frequency = f_1 , the bandwidth ratio β is defined as $\frac{f_2}{f_1}$ and is given by

$$\beta^2 = \left(\frac{f_2}{f_1}\right)^2 = \left(\frac{\pi_2}{\pi_1}\right)^2 = \frac{[(1+b)(a+\alpha) - \frac{ab}{K}] + \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - (a+\alpha)b]^2}}{[(1+b)(a+\alpha) - \frac{ab}{K}] - \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - (a+\alpha)b]^2}}$$

It will be more convenient in the work that follows to express β as

$$\beta^2 = \frac{1 + \sqrt{M}}{1 - \sqrt{M}} \quad (2-3)$$

where M is given by
$$M = \frac{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - (a+\alpha)b]^2}{[(1+b)(a+\alpha) - \frac{ab}{K}]^2} \quad (2-4)$$

Thus, a network has been arrived at that has an impedance transformation ratio K and a bandwidth ratio β provided the impedance elements are related as follows,

$$j_{11} = \sqrt{K} j_{12} \quad j_4 = \alpha j_{12}$$

$$j_{13} = a j_{11} + b j_{12} \quad j_{33} = K j_{11} + (K - \sqrt{K}) j_{12} + (K-1) j_{13}$$

and the constants α, a, b and ϕ satisfy the two equations

$$\alpha + a(1 + \frac{1}{\sqrt{K}}) = \phi \quad (2-5)$$

$$(1 + b + \frac{b}{\sqrt{K}}) = \phi(1 + \sqrt{K}) \quad (2-6)$$

It can be seen that, by using the two equations above, both ϕ and α can be eliminated from the expression for M thereby making M a function of the two variable constants a and b. The values of a and b can be arbitrarily selected subject to the limitation that $\alpha \geq 0$, $a \geq 0$ and $b \geq 0$. The function M can now be analysed to determine whether there exists an admissible choice of a and b that will yield an optimum bandwidth ratio for each value of K.

Substituting into (2-4) the value of $(\alpha + a) = \phi - \frac{a}{\sqrt{K}}$, M becomes

$$M = \frac{2\phi[(\phi - \frac{a}{\sqrt{K}})^2 - \frac{a^2}{K}] + \frac{1}{K}[a(1+b) - b(a+\alpha)]^2}{[(1+b)(\phi - \frac{a}{\sqrt{K}}) - \frac{ab}{K}]^2} \quad (2-7)$$

which may in turn be written as
$$M = 1 - \frac{[(1+b)^2 - 2\phi - \frac{b^2}{K}][(\phi - \frac{a}{\sqrt{K}})^2 - \frac{a^2}{K}]}{[(1+b)(\phi - \frac{a}{\sqrt{K}}) - \frac{ab}{K}]^2} \quad (2-8)$$

But $\phi = \frac{b}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}$ and thus

$$(1+b)^2 - 2\phi - \frac{b^2}{K} = \frac{(K-1)}{K}b^2 + 2(1 - \frac{1}{K})b + 1 - \frac{2}{1+\sqrt{K}} = \frac{(K-1)}{K}[b^2 + 2b\frac{\sqrt{K}}{1+\sqrt{K}} + \frac{K}{(1+\sqrt{K})^2}]$$

$$\text{or } (1+b)^2 - 2\phi - \frac{b^2}{K} = \frac{K-1}{K} \left(b + \frac{\sqrt{K}}{1+\sqrt{K}}\right)^2.$$

$$\text{Also, } \left[(1+b)\left(\phi - \frac{a}{\sqrt{K}}\right) - \frac{ab}{K}\right] = (1+b)\phi - \frac{a}{\sqrt{K}}(1+b + \frac{b}{\sqrt{K}}) = (1+b)\phi - \frac{a}{\sqrt{K}}(1+\sqrt{K})$$

Hence, substituting these results into (2-8) and noting that $\phi = \frac{1}{\sqrt{K}}(b + \frac{\sqrt{K}}{1+\sqrt{K}})$

$$M = 1 - \frac{K-1}{K} \frac{(b + \frac{\sqrt{K}}{1+\sqrt{K}})^2 \left[\left(\phi - \frac{a}{\sqrt{K}}\right)^2 - \frac{a^2}{K}\right]}{\frac{1}{K}(b + \frac{\sqrt{K}}{1+\sqrt{K}})^2 \left[1+b - a\left(\frac{1+\sqrt{K}}{\sqrt{K}}\right)\right]^2} = 1 - \frac{(K-1)\left[\left(\phi - \frac{a}{\sqrt{K}}\right)^2 - \frac{a^2}{K}\right]}{\left[1+b - a\left(\frac{1+\sqrt{K}}{\sqrt{K}}\right)\right]^2}$$

$$\text{Now } \left(\phi - \frac{a}{\sqrt{K}}\right)^2 - \frac{a^2}{K} = \phi^2 - 2\phi \frac{a}{\sqrt{K}} = \phi\left(\phi - \frac{2a}{\sqrt{K}}\right) = \frac{(b + \frac{\sqrt{K}}{1+\sqrt{K}})(b - 2a + \frac{\sqrt{K}}{1+\sqrt{K}})}{K}$$

$$\text{so that } M = 1 - \frac{(K-1)(b + \frac{\sqrt{K}}{1+\sqrt{K}})(b - 2a + \frac{\sqrt{K}}{1+\sqrt{K}})}{K \left[1+b - a\left(\frac{1+\sqrt{K}}{\sqrt{K}}\right)\right]^2} = 1 - \frac{(K-1)b_1(b_1 - 2a)}{K \left[b_1 + \frac{1}{1+\sqrt{K}} - a\left(\frac{1+\sqrt{K}}{\sqrt{K}}\right)\right]^2} \quad (2-9)$$

$$\text{where } b_1 = b + \frac{\sqrt{K}}{1+\sqrt{K}}.$$

Now that M has been obtained as a function of a and b (or b_1) only, the conditions previously mentioned ($\alpha \geq 0$, $a \geq 0$, and $b \geq 0$) can be studied in order to see how a and b may vary. These constraints are necessary if all of the impedance elements are to be realizable. Certainly the admissible values of a and b must lie only in the first quadrant of the a, b plane. Now since $\alpha = \phi - a \frac{1+\sqrt{K}}{\sqrt{K}}$ and $\phi = \frac{b}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}$ it is necessary, for $\alpha \geq 0$, that

$$\frac{b}{\sqrt{K}} \geq a \frac{1+\sqrt{K}}{\sqrt{K}} - \frac{1}{1+\sqrt{K}} \quad \text{or that } b \geq a(1+\sqrt{K}) - \frac{\sqrt{K}}{1+\sqrt{K}}$$

Thus, the region of admissible values of a and b corresponds to the shaded domain shown in Figure (5).

It is of interest to establish whether or not there are any allowed values of a and b which, for any given K, make M and therefore β a maximum. This can be done by finding out whether or not there are any interior points of the domain at which $\frac{\partial M}{\partial a} = 0$ and $\frac{\partial M}{\partial b} = 0$. Such points may repre-

sent values of a and b at which β is a maximum. If there are no such points then the maximum M that can be achieved corresponds to some point lying on the boundary. Using equation (2-9) for M and setting $\frac{\partial M}{\partial b} = \frac{\partial M}{\partial b_1} = 0$ yields the following equation

$$2(b_1 - a) \left[b_1 + \frac{1}{1+\sqrt{K}} - a \left(\frac{1+\sqrt{K}}{\sqrt{K}} \right) \right] = 2b_1(b_1 - 2a)$$

$$b_1^2 - b_1 a + \frac{1}{1+\sqrt{K}} (b_1 - a) - \frac{a(b_1 - a)(1+\sqrt{K})}{\sqrt{K}} = b_1^2 - 2ab_1$$

or
$$b_1 \left(\frac{1}{1+\sqrt{K}} - \frac{a}{\sqrt{K}} \right) = a \left(\frac{1}{1+\sqrt{K}} - \frac{a(1+\sqrt{K})}{\sqrt{K}} \right). \text{ Hence, } b_1 = a \frac{[1 - \frac{a(1+\sqrt{K})^2}{\sqrt{K}}]}{[1 - \frac{a(1+\sqrt{K})}{\sqrt{K}}]}$$

Setting $\frac{\partial M}{\partial a} = 0$ yields the equation

$$-2(b_1 + \frac{1}{1+\sqrt{K}} - a \frac{1+\sqrt{K}}{\sqrt{K}}) = 2(b_1 - 2a) \left(- \frac{1+\sqrt{K}}{\sqrt{K}} \right)$$

Solving this for a yields

$$a = \frac{b_1}{1+\sqrt{K}} - \frac{\sqrt{K}}{(1+\sqrt{K})^2} = \frac{1}{1+\sqrt{K}} (b_1 - \frac{\sqrt{K}}{1+\sqrt{K}})$$

If the two conditions, $\frac{\partial M}{\partial a} = 0$ and $\frac{\partial M}{\partial b_1} = 0$, are to hold simultaneously,

it is necessary that $b_1 [1 - \frac{a(1+\sqrt{K})}{\sqrt{K}}] = a [1 - \frac{a(1+\sqrt{K})^2}{\sqrt{K}}]$ or that, substituting for a ,

$$b_1 (1 - \frac{b_1}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}) = \frac{1}{1+\sqrt{K}} (b_1 - \frac{\sqrt{K}}{1+\sqrt{K}}) [1 - \frac{1+\sqrt{K}}{\sqrt{K}} b_1 + 1].$$

Solving this for b_1 leads to the expression $b_1 = \frac{2b_1}{1+\sqrt{K}} - 2 \frac{\sqrt{K}}{(1+\sqrt{K})^2}$

with the result that $b_1 = \frac{-2\sqrt{K}}{(\sqrt{K}-1)(\sqrt{K}+1)} = \frac{-2\sqrt{K}}{(K-1)}$.

Thus, $b + \frac{\sqrt{K}}{1+\sqrt{K}} = b_1 = \frac{-2\sqrt{K}}{(K-1)}$ and $b = - \frac{\sqrt{K}}{1+\sqrt{K}} (1 + \frac{2}{\sqrt{K}-1}) = \frac{-\sqrt{K}}{\sqrt{K}-1}$.

It is apparent that for $K > 1$ there is no $b \geq 0$ such that $\frac{\partial M}{\partial b}$ and $\frac{\partial M}{\partial a}$ are simultaneously zero. Consequently, there are no interior points (a, b) where β is a maximum. Attention will therefore be directed to a study

of the cases where a and b lie on a segment of the boundary line. These boundary cases correspond physically to networks having one or two less elements than the general three mesh network and are therefore simpler and more economical structures.

The first of the boundary cases to be considered corresponds to making $\alpha = 0$. From equation (2-6), $b = \sqrt{K} \phi - \frac{\sqrt{K}}{1+\sqrt{K}}$ and from equation (2-5) $\phi = \frac{1+\sqrt{K}}{\sqrt{K}} a$. Hence, $b = (1+\sqrt{K})a - \frac{\sqrt{K}}{1+\sqrt{K}}$. Also, as shown in Figure (5)

$$a \geq \frac{\sqrt{K}}{(1+\sqrt{K})^2} . \text{ Since } b_1 = b + \frac{\sqrt{K}}{1+\sqrt{K}}, b_1 = (1+\sqrt{K})a ; \text{ and substituting}$$

this into equation (2-9) for M yields

$$M = 1 - \frac{K-1}{K} \frac{(1+\sqrt{K})a - \frac{\sqrt{K}}{1+\sqrt{K}}}{\left[\frac{a(1+\sqrt{K})(\sqrt{K}-1)}{\sqrt{K}} + \frac{1}{1+\sqrt{K}} \right]^2} = 1 - \frac{(K-1)^2}{K} \frac{a^2}{\left[\frac{1}{1+\sqrt{K}} + \frac{K-1}{\sqrt{K}} a \right]^2}$$

It is clear that M will be a maximum when a takes on the smallest permissible value; that is, when $a = \frac{\sqrt{K}}{(1+\sqrt{K})^2}$ or when $b = 0$. Under these conditions M is given by

$$M = 1 - \frac{K(K-1)^2}{K(1+\sqrt{K})^4 \left[\frac{(K-1)}{(1+\sqrt{K})^2} + \frac{1}{1+\sqrt{K}} \right]^2} = 1 - \frac{(K-1)^2}{K(\sqrt{K}+1)^2} = 1 - \frac{(\sqrt{K}-1)^2}{K} = \frac{2\sqrt{K}-1}{K} \quad (2-10)$$

From the equation given above for M it can be seen that, by allowing a to take on larger and larger values, M can be made to take on as small a value as desired. This means that, by varying a between $a = \frac{\sqrt{K}}{(1+\sqrt{K})^2}$ and $a = \infty$, β may be varied between unity and the value corresponding to M as given by equation (2-10).

The next boundary case of interest is when $b = 0$. Equations (2-5) and (2-6) show that in this case $\alpha = \frac{1}{1+\sqrt{K}} - a \frac{1+\sqrt{K}}{\sqrt{K}}$ and $\phi = \frac{1}{1+\sqrt{K}}$.

Figure (5) shows that a must satisfy $0 \leq a \leq \frac{\sqrt{K}}{(1+\sqrt{K})^2}$. The reduced expres-

sion for M is most conveniently found by using equation(2-4) because when $b = 0$ it reduces to $M = 2\phi + \frac{a^2}{K(a+\alpha)^2} (1-2\phi)$. Since $\phi = \frac{1}{1+\sqrt{K}}$, M becomes $M = \frac{2}{1+\sqrt{K}} + \frac{a^2}{K(a+\alpha)^2} \frac{\sqrt{K}-1}{\sqrt{K}+1}$. It can easily be seen that maximum M occurs when $\alpha = 0$ or when $a = \frac{\sqrt{K}}{(1+\sqrt{K})^2}$, in which case M assumes the same value as in the previous instance where $\alpha = 0$ and $b = 0$ (as it must). The equivalence of both expressions can be realized by noting that $\frac{2\sqrt{K}-1}{K}$ is identically equal to $\frac{2}{1+\sqrt{K}} + \frac{\sqrt{K}-1}{K(\sqrt{K}+1)}$. In addition, by varying a between the limits given above the value of M may be made to vary from $M = \frac{2}{1+\sqrt{K}}$ to $M = \frac{2}{1+\sqrt{K}} + \frac{\sqrt{K}-1}{K(1+\sqrt{K})}$.

The final boundary case is when $a = 0$. Then $b \geq 0$ and $\phi = \alpha = \frac{b}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}$. Equation (2-9), with $a = 0$, gives the result $M = 1 - \frac{(K-1) b_1^2}{K(b_1 + \frac{1}{1+\sqrt{K}})^2}$ or

$$M = 1 - \frac{K-1}{K} \frac{1}{[1 + \frac{1}{(1+\sqrt{K})b_1}]^2}. \text{ Maximum M obviously occurs when } b_1 \text{ takes on}$$

its smallest value. The smallest value b_1 can have is $b_1 = \frac{\sqrt{K}}{1+\sqrt{K}}$ which corresponds to $b = 0$. Then

$$M = 1 - \frac{K-1}{K(1+\frac{1}{\sqrt{K}})^2} = 1 - \frac{(K-1)}{(\sqrt{K}+1)^2} \equiv \frac{2}{1+\sqrt{K}}. \text{ This}$$

checks with the previous result obtained for $b = 0$ and $a = 0$. For $b > 0$

M will assume all values between $M = \frac{1}{K}$ and $M = \frac{2}{1+\sqrt{K}}$ because the limit of M as b approaches infinity is $M = 1 - \frac{K-1}{K} = \frac{1}{K}$.

The conditions and results of the various cases considered are summarized below.

$$\text{Case \#1} \quad \alpha = 0, \quad b = a(1+\sqrt{K}) - \frac{\sqrt{K}}{1+\sqrt{K}}, \quad \text{and} \quad a \geq \frac{\sqrt{K}}{(1+\sqrt{K})^2}$$

$$M = 1 - \frac{(K-1)^2 a^2}{K [(K-1) \frac{a}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}]^2}$$

$$\text{Case \#2} \quad b = 0, \quad \alpha = \frac{1}{1+\sqrt{K}} - a\left(\frac{\sqrt{K}+1}{\sqrt{K}}\right), \quad \text{and } 0 \leq a \leq \frac{\sqrt{K}}{(1+\sqrt{K})^2}$$

$$M = \frac{2}{1+\sqrt{K}} + \frac{a^2}{K(a+\alpha)^2} \frac{\sqrt{K}-1}{\sqrt{K}+1}$$

$$\text{Case \#3} \quad a = 0, \quad b \geq 0, \quad \text{and } \alpha = \beta = \frac{b}{\sqrt{K}} + \frac{1}{1+\sqrt{K}}$$

$$M = \frac{\left[\frac{b^2}{K} + \frac{2b}{\sqrt{K}} + \frac{2}{1+\sqrt{K}}\right]}{(1+b)^2}$$

The circuits corresponding to the separate cases summarized above are shown in Figures (6), (7), and (8). The bandwidth ratio characteristics of the various circuits may be summarized by saying that the circuit for case #1 will, for any K, permit a bandwidth ratio between unity and the maximum value that can be achieved using one of the three mesh networks considered. This maximum value is found by using the M given in equation (2-10). If the desired β is such that $\frac{2}{1+\sqrt{K}} \leq M \leq \frac{2\sqrt{K}-1}{K}$, the circuit for case #2 may be used; whereas if $\frac{1}{K} < M \leq \frac{2}{1+\sqrt{K}}$ is necessary, the circuit for case #3 may be used.

In general, circuits corresponding to case #2 or case #3 should be used wherever possible because these circuits have higher order zeros at $\omega = \infty$ and the upper frequency cutoff characteristics will therefore be sharper than for the circuits of case #1.

The graph shown in Figure (9) portrays the regions of realizable β and K along with three boundary curves. The curve labelled "Curve I" represents the maximum values of β that can be obtained using a three mesh network of the type considered. Curve II displays the values of β at which the transition from a case #2 network to a case #3 network takes place. Curve III shows the lower limit of values of β that are obtainable with

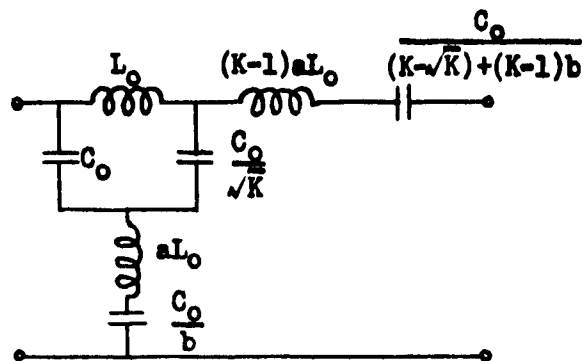


Figure 6

$$\alpha = 0$$

$$b = a(1 + \sqrt{K}) - \frac{\sqrt{K}}{1 + \sqrt{K}}$$

$$a \geq \frac{\sqrt{K}}{(1 + \sqrt{K})^2}$$

$$M = 1 - \frac{(K-1)^2 a^2}{K \left[(K-1) \frac{a}{\sqrt{K}} + \frac{1}{1 + \sqrt{K}} \right]^2}$$

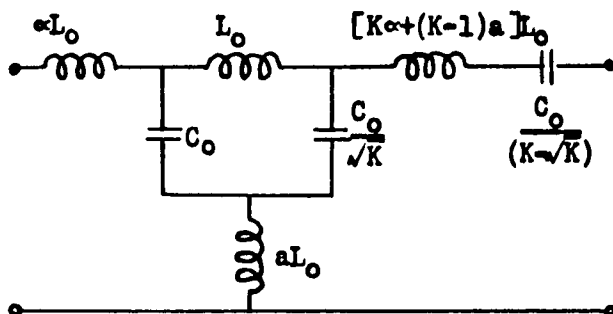


Figure 7

$$b = 0$$

$$\alpha = \frac{1}{1 + \sqrt{K}} - a \left(\frac{1 + \sqrt{K}}{\sqrt{K}} \right)$$

$$0 \leq a \leq \frac{\sqrt{K}}{(1 + \sqrt{K})^2}$$

$$M = \frac{2}{1 + \sqrt{K}} + \frac{a^2}{K(a + \alpha)^2} \frac{(\sqrt{K}-1)}{(\sqrt{K}+1)}$$

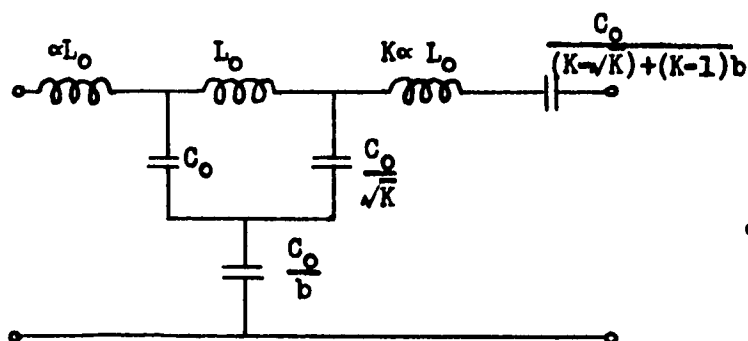


Figure 8

$$a = 0$$

$$b \geq 0$$

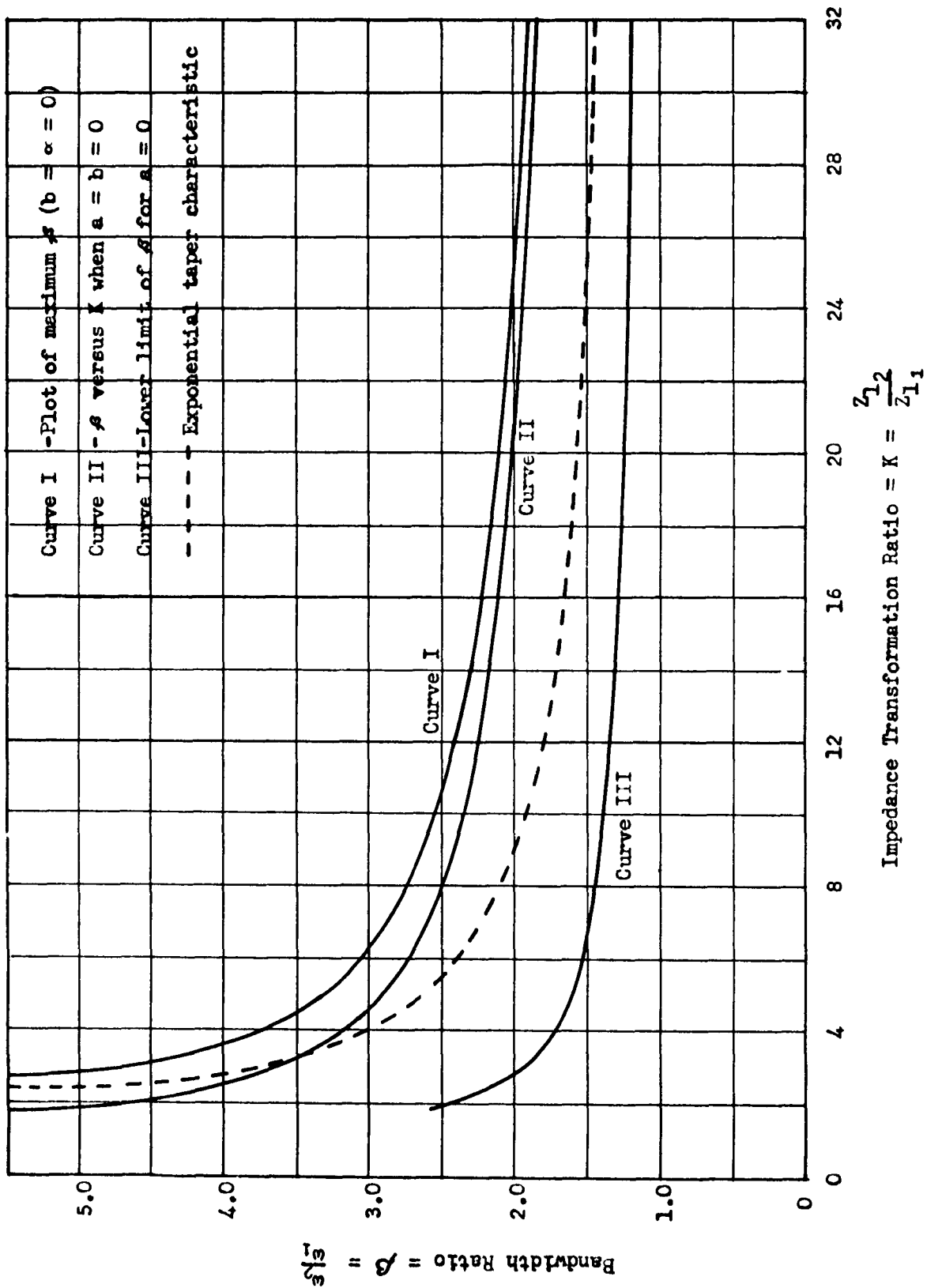
$$\alpha = \phi = \frac{b}{\sqrt{K}} + \frac{1}{1 + \sqrt{K}}$$

$$M = \frac{\left[\frac{b^2}{K} + 2 \frac{b}{\sqrt{K}} + \frac{2}{1 + \sqrt{K}} \right]}{(1+b)^2}$$

Figure 9

-24-

Plot of Values of β Realizable With a Three Mesh Network



networks considered in case #3. Also shown in Figure(9) are the bandwidth ratio characteristics that may be obtained using the designs of O'Donnell. The dashed curve corresponds to the so-called "Exponential Taper" filter while the curve showing the bandwidth ratio characteristic of the "Tschebyscheff Taper" filter is coincident with Curve II.

This identity with Curve II can be shown by taking O'Donnell's expression for β as a function of K for an equivalent Tschebyscheff filter and reducing it to the terms of this thesis. For a Tschebyscheff tapered filter of three meshes (two sections) the impedance transformation ratio is given by the equation $K = [T_2(x)]^2$ where $T_2(x)$ is a second order Tschebyscheff polynomial and is equal to $T_2(x) = (2x^2 - 1)$. Thus, $\sqrt{K} = T_2(x) = (2x^2 - 1)$. x is a parameter used by O'Donnell and is equal to

$$x = \frac{\left(\frac{\omega_2}{\omega_1}\right)^2 + 1}{\left(\frac{\omega_2}{\omega_1}\right)^2 - 1} = \frac{\beta^2 + 1}{\beta^2 - 1} \quad \text{so that } \sqrt{K} = 2 \frac{(\beta^2 + 1)^2}{(\beta^2 - 1)^2} - 1.$$

Now $\sqrt{\frac{1 + \sqrt{K}}{2}} = \frac{\beta^2 + 1}{\beta^2 - 1}$ and solving for β^2 yields

$$\beta^2 = \frac{1 + \sqrt{\frac{1 + \sqrt{K}}{2}}}{\sqrt{\frac{1 + \sqrt{K}}{2}} - 1} = \frac{1 + \sqrt{\frac{2}{1 + \sqrt{K}}}}{1 - \sqrt{\frac{2}{1 + \sqrt{K}}}}.$$

This expression is identically equal to the expression used to plot the transition curve (Curve II) shown in Figure(9). It can also be shown that these identical curves represent the characteristics of circuits that are identical. Thus, the Tschebyscheff filter networks may be regarded as a special case of the more general three mesh networks herein considered.

It should perhaps be noted that the improvement in the maximum

* See page 31 of reference (9a)

value of β , represented by Curve I, is not obtained at the expense of additional impedance elements since the circuit involved has the same number of elements as the Tschebyscheff tapered filter. In addition to the realization of larger bandwidth ratios, the three mesh networks studied provide a much greater degree of flexibility in as much as, for any value of K , a range of values of β may be obtained. This is in contrast to the strict one to one relationship between β and K displayed by the exponential taper and Tschebyscheff taper networks. This feature seems to be rather important since it is quite conceivable that in many instances an impedance transforming network designed to operate between fixed impedance levels with a specified bandwidth ratio would be useful. In this case the theory presented here would permit the design of a structure that combines both functions of impedance transformation and filtering while the exponential or Tschebyscheff tapered networks would, in most cases, require the cascading of a symmetrical filter to obtain the desired bandwidth ratio.

Section III

THE FOUR MESH NETWORK

In this section the same general methods used in Section II will be applied to the synthesis of a four mesh structure having the form indicated in Figure (10). In this case,

$$\Delta = \begin{vmatrix} Z_1 & -j_{12} & 0 & -j_{14} \\ -j_{12} & Z_2 & -j_{23} & 0 \\ 0 & -j_{23} & Z_3 & -j_{34} \\ -j_{14} & 0 & -j_{34} & Z_4 \end{vmatrix} \quad \text{where} \quad \begin{aligned} Z_1 &= j_{11} + j_{12} + j_{14} \\ Z_2 &= j_{12} + j_{23} + j_{24} \\ Z_3 &= j_{23} + j_{34} + j_{35} \\ Z_4 &= j_{34} + j_{45} + j_{46} \end{aligned}$$

As before, if $\frac{Z_{12}}{Z_{11}}$ is to be made equal to a constant K, it is necessary and sufficient that $\frac{\Delta_{11}}{\Delta_{nn}} = K$. This is equivalent to requiring that

$$Z_2 Z_3 Z_4 - Z_4 j_{23}^2 - Z_2 j_{34}^2 = K [Z_1 Z_2 Z_3 - Z_3 j_{12}^2 - Z_1 j_{13}^2] \quad (3-1)$$

The impedances j_{12} , j_{23} , and j_{34} will all be regarded as being of the same type. In other words, j_{12} may be expressed as $(g j_{12})$ and j_{34} may be written as $(g_1 j_{12})$ where g and g_1 are real positive constants.

(3-1) will be satisfied if $Z_4 = K Z_1$ and if $Z_3 = \frac{g_1^2}{K} Z_2$. This implies that

$$j_{44} + j_{34} + j_{14} = K(j_{11} + j_{12} + j_{14}) \quad \text{or} \quad j_{44} = K j_{11} + (K - g_1) j_{12} + (K - 1) j_{14} \quad (3-2)$$

and that

$$j_{33} + j_{23} + j_{34} = \frac{g_1^2}{K} (j_{11} + j_{12} + j_{23}) \quad \text{or} \quad j_{33} + (g + g_1) j_{12} = \frac{g_1^2}{K} j_{11} + \frac{g_1^2}{K} (1 + g) j_{12}$$

$$\text{Thus, } j_{33} = \frac{g_1^2}{K} j_{11} + [\frac{g_1^2}{K} (1 + g) - (g + g_1)] j_{12} \quad (3-3)$$

In order to reduce complexity and to keep the series impedance elements as much alike as possible the coefficient of j_{12} in (3-3) will be made equal to zero. Hence,

$$\frac{g_1^2}{K} (1 + g) = (g + g_1) \quad (3-4)$$

From (3-2), j_{44} will certainly be realizable as long as $K \geq g_1$.

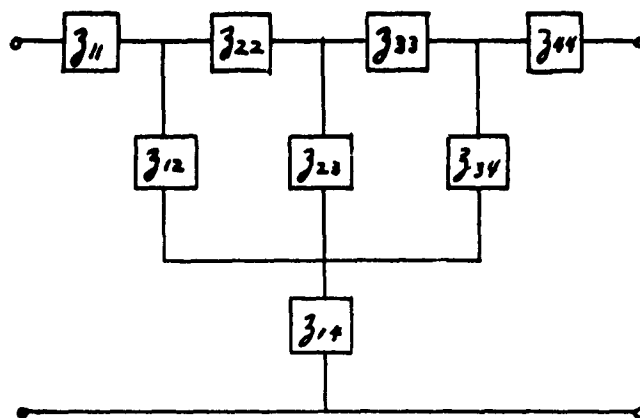
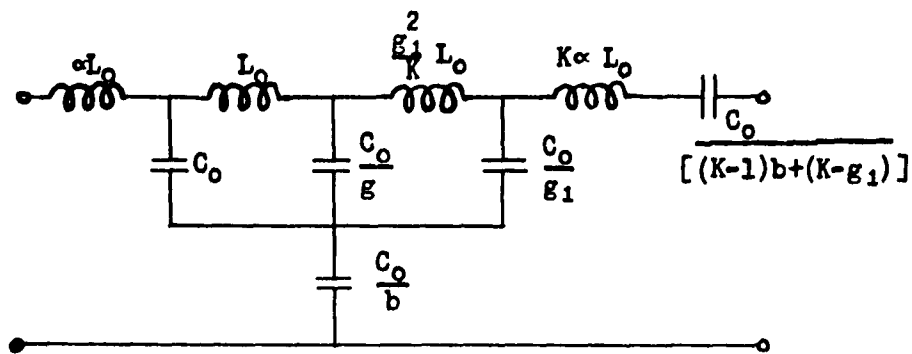


Figure 10



$$g = \frac{g_1(K-g_1)}{(g_1^2-K)}$$

$$b = \frac{(g_1-1)(g_1^2-K) - 2(K-g_1)^2}{2(K-g_1)(K-1)}$$

$$g_1 \leq g_1 < K$$

$$\alpha = \frac{E_1}{K} b + \frac{[K(g_1-K) + g_1(K-g_1)]}{2K(K-1)}$$

Figure 11

Once again, $Z_{11} = \sqrt{\frac{\Delta_{nn} \Delta}{\Delta_{11} \Delta^*}} = \frac{1}{\sqrt{K}} \sqrt{\frac{\Delta}{\Delta^*}}$. Δ^* and Δ are found by ex-

panding the appropriate determinants with the result that $\Delta^* = Z_2 Z_3 - j_{13}^2$ and $\Delta = (K Z_1^2 - j_{14}^2) \Delta^* - Z_1 Z_2 j_{14}^2 - j_{12}^2 (Z_3 Z_4 - j_{14}^2) - 2 j_{12} j_{14} j_{13} j_{14}$. Substituting in the values of Z_3, Z_4, j_{13} , and j_{14} in terms of Z_1, Z_2 , and j_{12} gives the following

$$\Delta = (K Z_1^2 - j_{14}^2) \Delta^* - g_1^2 Z_1 Z_2 j_{12}^2 - K Z_1 \frac{g_1^2}{K} Z_2 j_{12}^2 + g_1^2 j_{12}^2 - 2 g g_1 j_{14} j_{12}^3$$

$$\Delta = (K Z_1^2 - j_{14}^2) \Delta^* - 2 g_1^2 Z_1 Z_2 j_{12}^2 - 2 g g_1 j_{14} j_{12}^3 + g_1^2 j_{12}^4$$

$$\text{Now, } \frac{\Delta}{\Delta^*} = (K Z_1^2 - j_{14}^2) - g_1^2 j_{12}^2 \frac{(2 Z_1 Z_2 + 2 \frac{g}{g_1} j_{12} j_{14} - j_{12}^2)}{(Z_2 Z_3 - j_{13}^2)}$$

$$\text{Finally, } \frac{\Delta}{\Delta^*} = K j_{12}^2 \left[\left(\frac{Z_1}{j_{12}} \right)^2 - \frac{1}{K} \left(\frac{j_{14}}{j_{12}} \right)^2 - 2 \frac{(Z_1 Z_2 + \frac{g}{g_1} j_{12} j_{14} - \frac{j_{12}^2}{2})}{(Z_2^2 - \frac{K g^2}{g_1^2} j_{12}^2)} \right]. \quad (3-5)$$

If Δ^* is to be a factor of Δ , it is only necessary to make

$$Z_1 Z_2 + \frac{g}{g_1} j_{12} j_{14} - \frac{j_{12}^2}{2} = \phi \left(Z_2^2 - \frac{K g^2}{g_1^2} j_{12}^2 \right) \quad \text{or}$$

$$\frac{Z_1}{j_{12}} \frac{Z_2}{j_{12}} + \frac{g}{g_1} \frac{j_{14}}{j_{12}} - \frac{1}{2} = \phi \left[\left(\frac{Z_2}{j_{12}} \right)^2 - \frac{K g^2}{g_1^2} \right]$$

If we again let $j_{12} = \alpha j_{11}$ and $j_{14} = a j_{11} + b j_{12}$, the last equation becomes

$$[(a+\alpha) \frac{j_{11}}{j_{12}} + (1+b)] [\frac{j_{11}}{j_{12}} + 1+g] + \frac{g}{g_1} (a \frac{j_{11}}{j_{12}} + b) - \frac{1}{2} = \phi \left[\left(\frac{j_{11}}{j_{12}} + 1+g \right)^2 - \frac{K g^2}{g_1^2} \right]$$

$$\text{or } (a+\alpha) \left(\frac{j_{11}}{j_{12}} \right)^2 + [(1+b)+(1+g)(a+\alpha) + \frac{g}{g_1} a] \frac{j_{11}}{j_{12}} + [(1+b)(1+g) + \frac{g}{g_1} b - \frac{1}{2}] =$$

$$\phi \left(\frac{j_{11}}{j_{12}} \right)^2 + 2\phi(1+g) \frac{j_{11}}{j_{12}} + \phi \left[(1+g)^2 - \frac{K g^2}{g_1^2} \right]$$

Matching coefficients of $\frac{j_{11}}{j_{12}}$ yields the following equalities

$$(a + \alpha) = \phi \quad (3-6)$$

$$(1+b)+(1+g)(a+\alpha) + \frac{g}{g_1} a = 2\phi(1+g) \quad \text{or} \quad 1+b + \frac{g}{g_1} a = \phi(1+g) \quad (3-7)$$

$(1+b)(1+g) + \frac{g}{g_1}b - \frac{1}{2} = \phi(1+g)^2 - \frac{\phi K g^2}{2g_1}$ or, substituting for $(1+b)$ its value in (3-7),

$$\frac{g}{g_1}[(1+g)a-b] + \frac{1}{2} = \phi K \frac{g^2}{2g_1} \quad (3-8)$$

Under these conditions $\frac{\Delta}{\Delta^*} = K j_{12}^2 \left[\left(\frac{Z_1}{j_{12}} \right)^2 - \frac{1}{K} \left(\frac{j_{12}}{j_{12}} \right)^2 - 2\phi \right]$ and

$$Z_{11} = \frac{1}{\sqrt{K}} \sqrt{\frac{\Delta}{\Delta^*}} = j_{12} \sqrt{(\alpha \frac{j_{12}}{j_{12}} + 1 + a \frac{j_{12}}{j_{12}} + b)^2 - \frac{1}{K} (a \frac{j_{12}}{j_{12}} + b)^2 - 2\phi}.$$

This, it will be noted, is exactly the same form as the expression obtained for the Z_{11} of a three mesh network as given in the equation immediately preceding (2-2). If, as was done in the three mesh networks, j_{12} is made an inductance equal to L_0 and j_{12} is made a capacitance equal to C_0 , the equation for the bandwidth ratio will be identical to that obtained in the three mesh case. That is, from equations (2-3) and (2-4),

$$\mathcal{B}^2 = \frac{1 + \sqrt{M}}{1 - \sqrt{M}} \quad \text{where} \quad M = \frac{2\phi \left[(a+\alpha)^2 - \frac{a^2}{K} \right] + \frac{1}{K} [a(1+b) - b(a+\alpha)]^2}{[(1+b)(a+\alpha) - \frac{ab}{K}]^2} \quad (3-9)$$

The quantities ϕ, α, a , and b are now, of course, subject to different constraints. There are also, as previously indicated, conditions on g and g_1 ; namely, $g_1 \leq K$ and $\frac{g}{K}(1+g) = (g+g_1)$. This latter equation can be solved for g_1 in terms of g and K and yields

$$g_1 = \frac{K}{2(1+g)} \left[1 + \sqrt{1 + \frac{4g(1+g)}{K}} \right].$$

The condition $g_1 \leq K$ can now be examined to determine the requirements, if any, on g . If $g_1 \leq K$, then $\frac{K}{2(1+g)} \left[1 + \sqrt{1 + \frac{4g(1+g)}{K}} \right] \leq K$ or

$$\sqrt{1 + \frac{4g(1+g)}{K}} \leq 2g+1. \text{ Squaring each side, } 1 + \frac{4g(1+g)}{K} \leq 1 + 4g + 4g^2$$

or $\frac{1}{K} \leq 1$ which is satisfied for all $K \geq 1$.

It is again possible in principal to express M in terms of any two of the variables α , a , b , g , g_1 and ϕ since there are four additional equations interrelating these variables. It would then be theoretically possible to determine the values of the selected coordinates at which M is a maximum by the same method used in analysing the three mesh structure. The range of values over which the selected coordinates can vary constitute a region that can be determined by the condition that g , a , b and α all be greater than zero. It could then be determined what if any points of maximum M occur within the admissible region. These points, or if there are none, the boundary of the region would then serve as a guide to the design of a four mesh structure with optimum bandwidth ratio.

Unfortunately, equations (3-6), (3-7), and (3-8) are of such a nature that, when substituted into the expression for M , the two simultaneous equations obtained by making the partial derivatives of M equal to zero are extremely intractable. For example, if M is reduced to a function of b and g_1 only, the two simultaneous equations are cubics in b with coefficients that are complicated non-linear functions of g_1 . Any general algebraic solution of these equations appears quite impractical.

For this reason, attention will be confined to a survey of the boundary cases; i.e., where one or more of the variables a , b and α are zero. This corresponds to networks having one or two less elements than the general case.

Before considering the separate cases it will be convenient for what follows to reduce the general expression for M to a different form. Applying equation (3-6) to the expression for M labelled (3-9) we obtain the following results:

$$M = \frac{2g(g^2 - \frac{a^2}{K}) + [a(1+b) - gb]^2}{[(1+b)g - \frac{ab}{K}]^2} = 1 + \frac{2g(g^2 - \frac{a^2}{K}) + (1+b)^2 \frac{a^2}{K} - g^2(1+b)^2 + \frac{g^2 b^2}{K} - \frac{a^2 b^2}{K^2}}{[(1+b)g - \frac{ab}{K}]^2}$$

$$\text{or } M = 1 - \frac{(g^2 - \frac{a^2}{K})[(1+b)^2 - 2g - \frac{b^2}{K}]}{[(1+b)g - \frac{ab}{K}]^2} \quad (3-10)$$

Using the general constraining equations marked (3-7) and (3-8) it is possible to find a more convenient form for $[(1+b)^2 - 2g - \frac{b^2}{K}]$. The algebraic manipulation involved is considerable and is therefore included in Appendix I. The result is that

$$[(1+b)^2 - 2g - \frac{b^2}{K}] = \frac{K-1}{K} (b + \frac{K-g_1}{K-1})^2. \text{ Since}$$

this is a completely general result it may be substituted into (3-10)

thereby yielding

$$M = 1 - \frac{K-1}{K} \frac{(b + \frac{K-g_1}{K-1})^2 (g^2 - \frac{a^2}{K})}{[(1+b)g - \frac{ab}{K}]^2} \quad (3-11)$$

Because of the irrational nature of the equation giving g_1 as a function of g and K it will be better to express g as a function of g_1 which, by solving (3-4), turns out to be $g = g_1 \frac{K-g_1}{g_1-K}$. For $g \geq 0$ it is necessary that $\sqrt{K} \leq g_1 \leq K$.

Consider now the case when $a = 0$, $\alpha \geq 0$, and $b \geq 0$. Equation (3-11) reduces to

$$M = 1 - \frac{K-1}{K} \frac{(b + \frac{K-g_1}{K-1})^2}{(1+b)^2}. \text{ The equation for } b \text{ as a function}$$

of g_1 is also derived in Appendix I and is $b = \frac{[(g_1-1)(g_1^2-K) - 2(K-g_1)^2]}{2(K-g_1)(K-1)}$.

From this,

$$(b + \frac{K-g_1}{K-1}) = \frac{1}{K-1} \left[\frac{(g_1-1)(g_1^2-K) - 2(K-g_1)^2}{2(K-g_1)} + (K-g_1) \right] = \frac{(g_1-1)(g_1^2-K)}{2(K-1)(K-g_1)} \text{ and}$$

$$(1+b) = \frac{[(g_1-1)(g_1^2-K) - 2(K-g_1)^2]}{2(K-1)(K-g_1)} + 1 = \frac{(g_1-1)(g_1^2-K) - 2[(K-g_1) - (K-1)](K-g_1)}{2(K-1)(K-g_1)}$$

or $(1+b) = \frac{(g_1-1)[g_1^2 - K + 2(K-g_1)]}{2(K-1)(K-g_1)}$. Substituting these results into the equation for M gives

$$M = 1 - \frac{K-1}{K} \left[\frac{g_1^2 - K}{g_1^2 - K + 2(K-g_1)} \right]^2 = 1 - \frac{K-1}{K} \left[\frac{1}{1 + \frac{2(K-g_1)}{g_1^2 - K}} \right]^2. \quad (3-12)$$

In order to maximize M it is necessary to make the quantity in brackets as small as possible. This will be achieved if g_1 is allowed to assume its lowest value. The range of values over which g_1 may vary can be determined by noting that b , α , and g must all be positive if the networks are to be realizable. It was shown earlier that if $g \geq 0$ then $\sqrt{K} \leq g_1 \leq K$. Using the equation above for b as a function of g_1 and K it can be seen that for $b \geq 0$ $(g_1-1)(g_1^2-K) \geq 2(K-g_1)^2$. Expanding this we obtain the requirement that $g_1^3 - 3g_1^2 + 3Kg_1 + K - 2K^2 \geq 0$. Thus, g_1 must be such that $g_1 \geq g'_1$ where g'_1 satisfies the above expression when the equality sign is used. To determine for what values of g_1 $\alpha \geq 0$ consider that when $a = 0$, $\alpha = \phi$. Appendix I includes an equation for ϕ in terms of b and g_1 ; that is, $\phi = \frac{g_1 b}{K} + \frac{[K(g_1-1) + g_1(K-g_1)]}{2K(K-1)}$. Since it is already required that $1 < \sqrt{K} \leq g_1 \leq K$, all terms are positive and $\alpha = \phi$ can never be negative. Thus, the only restrictions on g_1 are $g'_1 \leq g_1 < K$. (g_1 is never allowed to actually take on the value K and K is always considered greater than unity, except in limiting cases, because division by $(K-g_1)$ and $(K-1)$ occurs frequently). The allowed values of g_1 , when $a = 0$, constitute the region shown in Figure (14) that is bounded by the straight line $g_1 = K$ and the curve $g_1 = g'_1$. The curve is obtained by solving numerically the defining equation of g_1 given above.

Because of the freedom in the choice of g_1 , M , and therefore β ,

may vary continuously between a minimum and a maximum. The maximum value of M can be found by inserting the appropriate value of $g_1 = g_1'$ into equation (3-12). The minimum value of M can be found by allowing g_1 to approach K in equation (3-12) and noting that the result is $M = \frac{1}{K}$. Plots of M versus g_1 for representative values of K are shown in Figure (15).

The next case of interest is when $b = 0$, $\alpha \geq 0$, and $a \geq 0$. Under these circumstances the general expression for M given in equation (3-11) reduces to

$$M = 1 - \frac{(K-g_1)^2}{K(K-1)} \left(\frac{\phi^2 - \frac{a^2}{K}}{\phi^2} \right) \quad \text{Equations (3-7) and (3-8),}$$

under the condition that $b = 0$, can be used to determine α , a , ϕ and $(\phi^2 - \frac{a^2}{K})$ as functions of g_1 alone. This is done in Appendix II with the following results.

$$\phi = \frac{2K g_1 - g_1^2 - K}{2 K (K-1)}$$

$$a = \frac{[-g_1^3 + 3g_1^2 - 3Kg_1 - K + 2K^2]}{2 (K-1) (K-g_1)}$$

$$\alpha = \frac{[g_1^3(1+K) - 6Kg_1^2 + (5K^2+K)g_1 - 2K^3]}{2 (K-1) (K-g_1)}$$

$$\left(\phi^2 - \frac{a^2}{K} \right) = \frac{(g_1^2 - K)[4K(K-g_1) - (g_1^2 - K)^2]}{4K^2 (K-1) (K-g_1)^2}$$

Substituting into the expression given above for M the equations for ϕ^2 and $(\phi^2 - \frac{a^2}{K})$

$$M = 1 - \frac{(K-g_1)^2}{K(K-1)} \cdot \frac{(g_1^2 - K)[4K(K-g_1) - (g_1^2 - K)^2]}{4K^2 (K-1) (K-g_1)^2} \cdot \frac{4K^2 (K-1)^2}{[2Kg_1 - g_1^2 - K]^2}$$

$$M = 1 - \frac{(g_1^2 - K) [4K(K-g_1) - (g_1^2 - K)^2]}{K [K(K-1) - (K-g_1)^2]^2} \quad (3-13)$$

The expression for a shows that for $a \geq 0$ g_1 must be such that $g_1 \leq g_1'$

where g_1' is the same as previously defined. Referring to the equation for α makes it apparent that, if α is to be greater than or equal to zero, it is necessary that g_1 obey the condition $g_1 \geq g_1''$ where g_1'' is defined as $(1+K) g_1''^3 - 6K g_1''^2 + (5K^2+K) g_1'' - 2K^3 = 0$. Thus, the admissible values of g_1 , for the case where $b = 0$, form a region in the (K, g_1) plane which is bounded above by the curve $g_1 = g_1'$ and bounded below by the curve $g_1 = g_1''$. This region is also shown on Figure (14). Equation (3-13) is not an obvious monotonic function of g_1 as was the expression for M in the case where $a = 0$. However, for selected values of K , the variation in β as g_1 goes from $g_1 = g_1'$ to $g_1 = g_1''$ can be easily plotted. This is done on Figure (15), and it can be seen that only for larger values of K does β monotonically increase with increasing g_1 .

Finally, there is to be considered the case when $\alpha = 0$, $a \geq 0$, and $b \geq 0$. If $\alpha = 0$, then $a = 0$ and the general expression for M , equation (3-11), reduces to

$$M = 1 - \frac{K-1}{a^2} \frac{(b + \frac{K-g_1}{K-1})^2 a^2 \frac{K-1}{K}}{[1 + b - \frac{b}{K}]^2} = 1 - \frac{(b + \frac{K-g_1}{K-1})^2}{(b + \frac{K}{K-1})^2}.$$

Using equations (3-7) and (3-8) with $\alpha = 0$ allows one to find b and a as functions of g_1 . This is carried out in Appendix III with the consequence that

$$a = \frac{(g_1^2 - K)}{2(K-1)(K-g_1)^2} [2(K-g_1) + (g_1^2 - K)]$$

and $b = \frac{[g_1(K-1)(g_1^2 - K) - 2(K-g_1)^3]}{2(K-1)(K-g_1)^2}$. Also included in Appendix III

is a calculation of the quantity $\left[\frac{(b + \frac{K-g_1}{K-1})}{(b + \frac{K}{K-1})} \right]$ in terms of g_1 and K

alone. That is,

$$\left[\frac{(b + \frac{K-g_1}{K-1})}{(b + \frac{K}{K-1})} \right] = \frac{1}{1 + \frac{2(K-g_1)^2}{(K-1)(g_1^2 - K)}}$$

$$\text{Hence, } M = 1 - \frac{1}{\left[1 + \frac{2(K-g_1)^2}{(K-1)(g_1-K)} \right]^2} \quad (3-14)$$

From the equation given above for a it can be seen that the condition $a \geq 0$ does not impose any additional restriction on g_1 since g_1 is always required to be such that $\sqrt{K} \leq g_1 \leq K$, and therefore all terms are positive. The condition $b \geq 0$ however requires that $g_1(K-1)(g_1^2-K) \geq 2(K-g_1)^3$. By expanding these terms this condition becomes

$$(1+K)g_1^3 - 6Kg_1^2 + (5K^2 + K)g_1 - 2K^3 \geq 0$$

which is equivalent to stipulating that $g_1 \geq g_1''$. g_1'' is the solution of the above expression when the equality sign prevails. Thus, the limitations on g_1 stated above may be interpreted to mean that when $\alpha = 0$ g_1 must lie in a region of the (K, g_1) plane bounded below by the curve $g_1 = g_1''$ and above by straight line $g_1 = K$.

Expressions for the bandwidth ratio of the four mesh networks considered have been obtained in each of the separate cases by determining M as a function of g_1 and K where M and β are related by the equation $\beta = \sqrt{\frac{1 + \sqrt{M}}{1 - \sqrt{M}}}$. g_1 is an independent variable subject only to the requirement that it be between certain limits, and it is this freedom in the selection of g_1 which is responsible for the flexibility that these circuits possess.

The regions of admissible values of g_1 for each of the three cases are shown on Figure (14) and are delineated by plots of the defining equations for g_1' and g_1'' . The derived equations for M were used to illustrate on Figure (15) the bandwidth ratio as a function of g_1 for representative values of K . Each of the curves on Figure (15) consist of three segments each of which corresponds to one of the three cases $a = 0$, $b = 0$, and $\alpha = 0$.

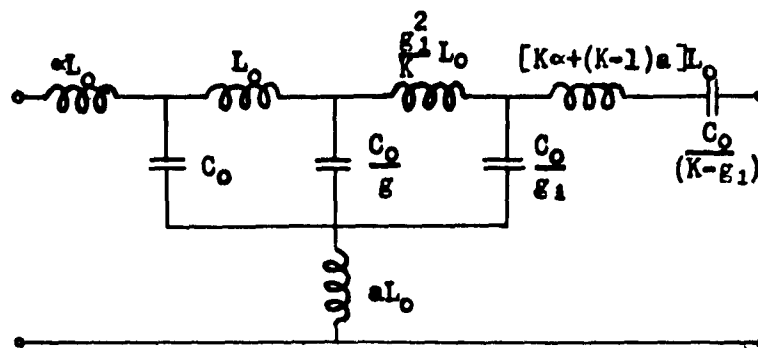
The networks associated with each of the three cases are shown schematically in Figures (11), (12) and (13), along with the equations for the component values.

The considerable amount of algebraic manipulation required to find the various reduced expressions for M as a function of g_1 and K alone was checked in the following manner. Arbitrary admissible values of K and g_1 were used to compute α , a , Q , and b . These were then substituted in the original general expression for M . The resulting value was then compared to the value obtained by using the appropriate reduced expression involving K and g_1 only. In addition, it can be seen from Figure (15) that the reduced expressions checked with each other at the end points of the allowed interval of g_1 . In other words, the curves are all continuous at $g_1 = g_1'$ and $g_1 = g_1''$.

Figure (16) displays the range values that β may assume for any given K using one of the four mesh structures herein considered. O'Donnell's curves of β versus K are also included so that comparison can easily be made. Curve I is the plot of β as a function of K when $a = 0$ and $b = 0$. Curve II is the lowest value of β obtainable with a network having $b = 0$. This latter curve is only equivalent to the case $b = 0$ and $\alpha = 0$ for larger values of K as can be seen from Figure (15). Curve III is the plot of

$$\beta = \sqrt{\frac{\sqrt{K} + 1}{\sqrt{K} - 1}} \quad \text{which is the lowest attainable value of } \beta \text{ when } a = 0.$$

Figure (16) may be interpreted in the following manner. Values of β between Curves I and III may be realized by making $a = 0$. Values of β between Curves I and II may be realized by making $b = 0$, and values of β between Curve II and $\beta = 1$ can be achieved by using the case where $\alpha = 0$. The dashed line corresponds to the β versus K relationship displayed by O'Donnell's exponential taper network having four meshes and shows that



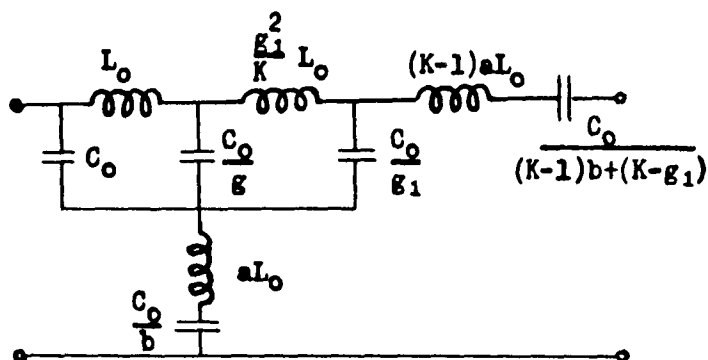
$$g = \frac{g_1(K-g_1)}{(g_1^2 - K)}$$

$$a = \frac{[-g_1^3 + 3g_1^2 - 3Kg_1 - K + 2K^2]}{2(K-1)(K-g_1)}$$

$$g_1'' \leq g_1 \leq g_1'$$

$$\alpha = \frac{[(1+K)g_1^3 - 6Kg_1^2 + (5K^2 + K)g_1 - 2K^3]}{2(K-1)(K-g_1)}$$

Figure 12



$$g = \frac{g_1(K-g_1)}{(g_1^2 - K)}$$

$$a = \frac{(g_1^2 - K)[2(K-g_1) + g_1^2 - K]}{2(K-g_1)^2(K-1)}$$

$$g_1'' \leq g_1 < K$$

$$b = \frac{[g_1(K-1)(g_1^2 - K) - 2(K-g_1)^3]}{2(K-g_1)^2(K-1)}$$

Figure 13

higher values of β may be achieved by using this network when values of K less than $K = 14$ are used. It should be mentioned, however, that the exponential taper network uses one more element than the network which corresponds to Curve I. The β characteristic of the Tschebyscheff taper network is, as it was in the three mesh case, identical to the characteristic for the case when $a = 0$ and $b = 0$ which is shown in Figure (16) as Curve I. One can thus conclude that from the standpoint of maximum bandwidth ratio the four mesh structures derived in this section are an improvement over the networks of O'Donnell only for larger values of K . However, as in the three mesh case the four mesh networks possess a flexibility that O'Donnell's structures do not have. That is, any value of β between unity and the maximum value as given by Curve I is realizable by using either one or another of the three cases considered in this section. The practical utility of this feature was discussed in Section II.

Figure 14

Plot of the Admissable Values of g_1

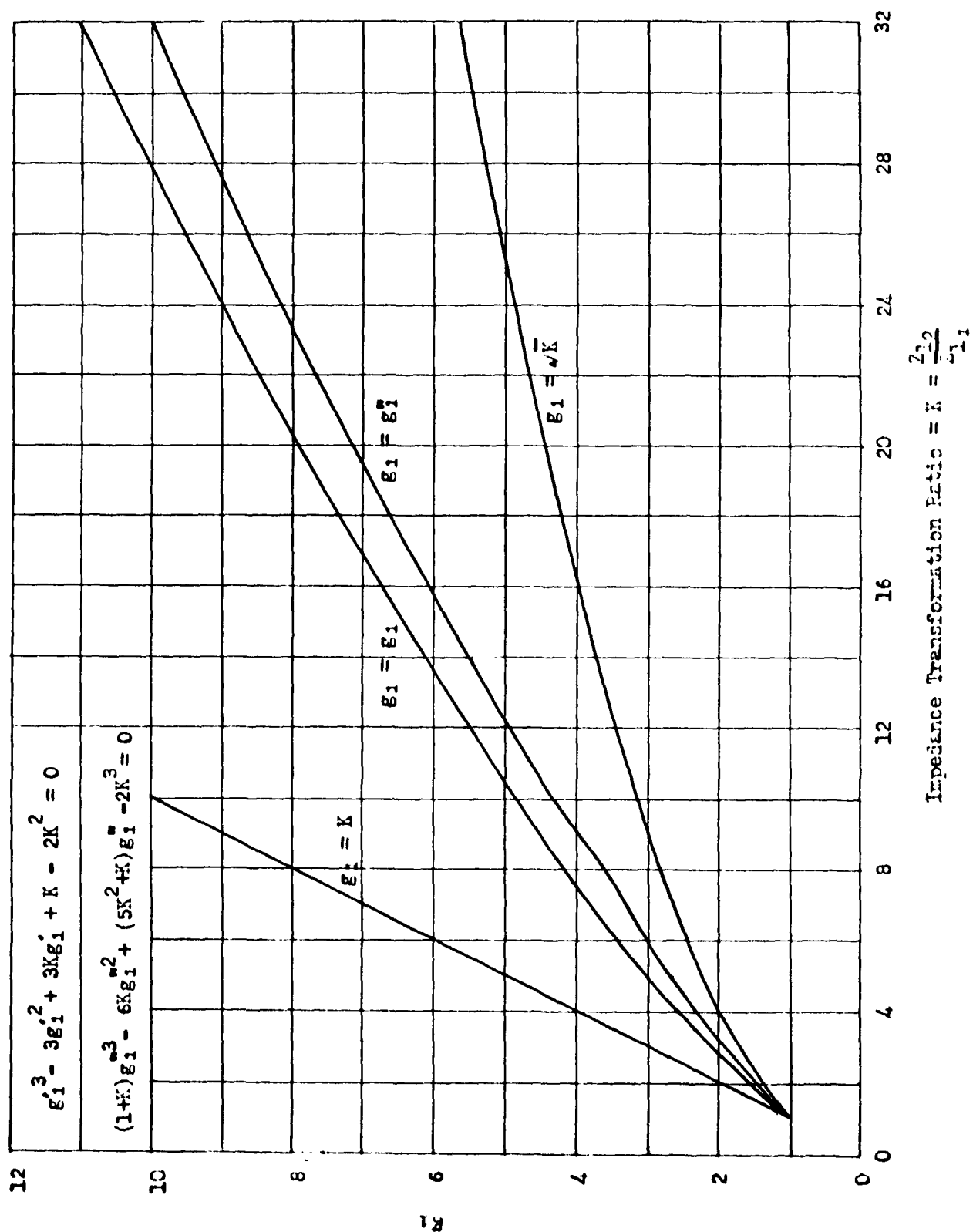


Figure 15

-41-

Plot of Bandwidth Ratio Versus g_1 for Selected Values of K

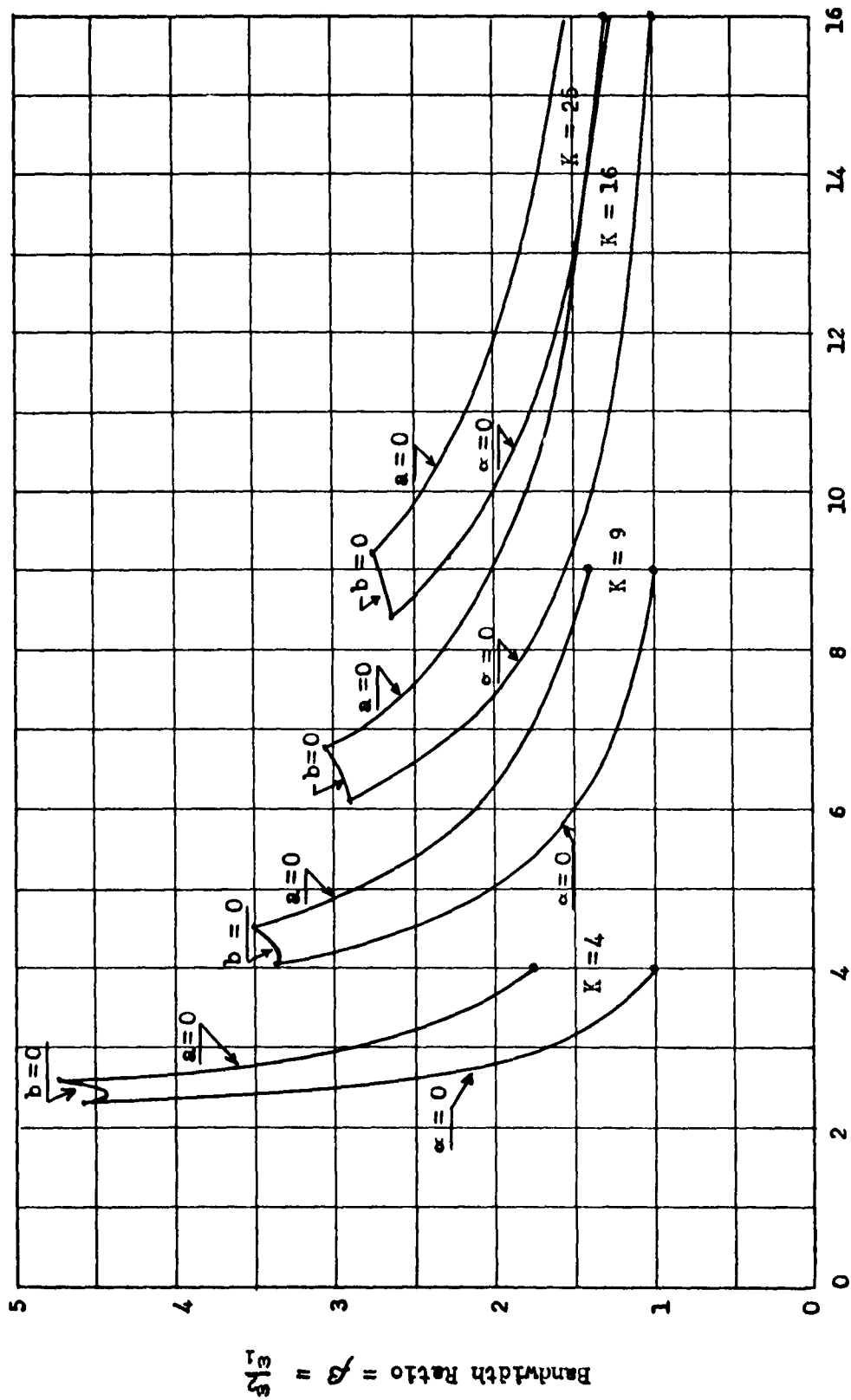
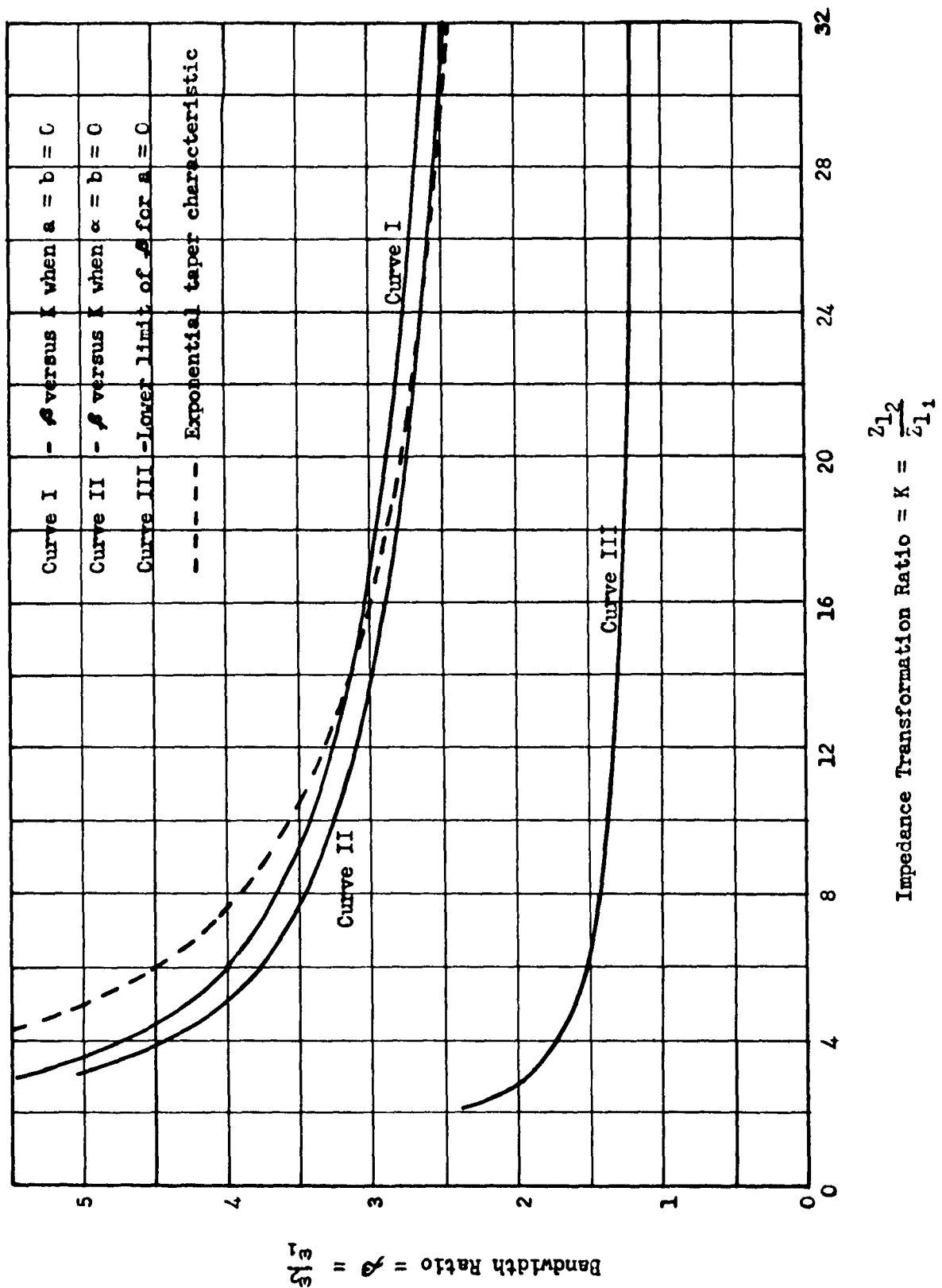


Figure 16

-42-

Plot of Values of β Realizable With a Four Mesh Network



Section IV

THE FIVE MESH NETWORK

In this section methods similar to those used in Sections II and III will be employed to obtain expressions for the bandwidth ratio and component values of the five mesh structure shown in Figure (17).

For the network shown the impedance determinant Δ is equal to

$$\Delta = \begin{vmatrix} Z_1 & -j_{12} & 0 & 0 & -j_{15} \\ -j_{12} & Z_2 & -j_{23} & 0 & 0 \\ 0 & -j_{23} & Z_3 & -j_{34} & 0 \\ 0 & 0 & -j_{34} & Z_4 & -j_{45} \\ -j_{15} & 0 & 0 & -j_{45} & Z_5 \end{vmatrix}$$

The minors Δ_{11} and Δ_{nn} are equal to

$$\Delta_{11} = Z_5 \Delta^* - j_{45}^2 (Z_2 Z_3 - j_{23}^2)$$

$$\Delta_{nn} = Z_1 \Delta^* - j_{12}^2 (Z_3 Z_4 - j_{34}^2)$$

Therefore, the condition that $\frac{\Delta_{11}}{\Delta_{nn}} = K$ requires that

$$Z_5 \Delta^* - j_{45}^2 (Z_2 Z_3 - j_{23}^2) = K [Z_1 \Delta^* - j_{12}^2 (Z_3 Z_4 - j_{34}^2)]$$

Following the method used in the previous cases, the impedance elements j_{23} , j_{34} , and j_{45} will be made the same kind of an impedance as j_{12} . Hence we can write $j_{23} = g j_{12}$, $j_{34} = g_1 j_{12}$, and $j_{45} = g_2 j_{12}$ where g , g_1 , and g_2 are real positive constants. Thus,

$$Z_5 = K Z_1 + \frac{j_{12}^2 Z_3 (Z_2 g_2^2 - K Z_4) + j_{12}^4 (K g_1^2 - g^2 g_2^2)}{\Delta^*}$$

In order to keep Z_5 as simple as possible the conditions $Z_4 = \frac{g_2^2}{K} Z_2$ and $K g_1^2 = g^2 g_2^2$ will be enforced. With this stipulation $Z_5 = K Z_1$. The condition that $Z_4 = \frac{g_2^2}{K} Z_2$ implies that

$$(j_{44} + j_{34} + j_{45}) = \frac{g_2^2}{K} (j_{11} + j_{23} + j_{12})$$

Substituting for j_{23} , j_{34} , and j_{45} and noting that $\frac{g_2^2}{K} = \frac{g_1^2}{g}$ this equation becomes

$$j_{44} + (g_1 + g_2) j_{12} = \frac{g_1^2}{g} [j_{11} + (1+g) j_{12}] \quad \text{or}$$

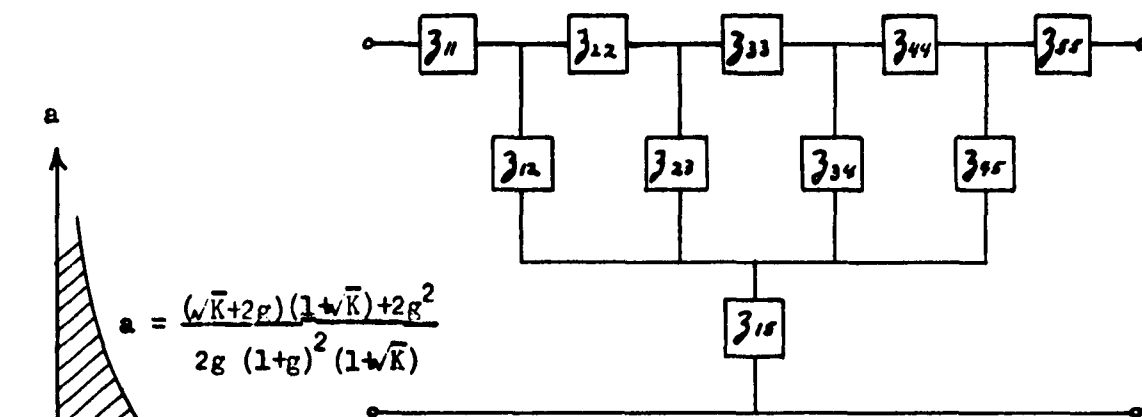


Figure 17

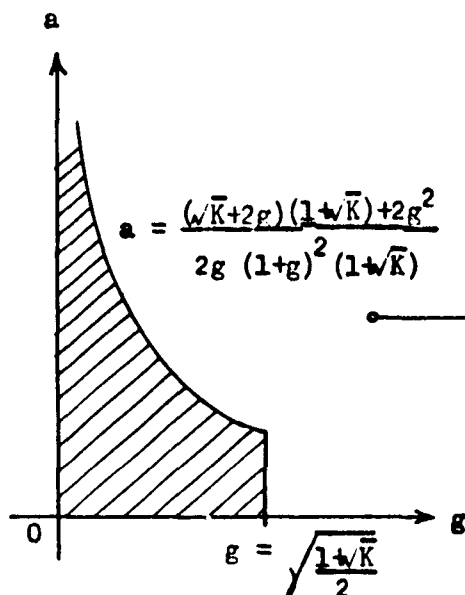
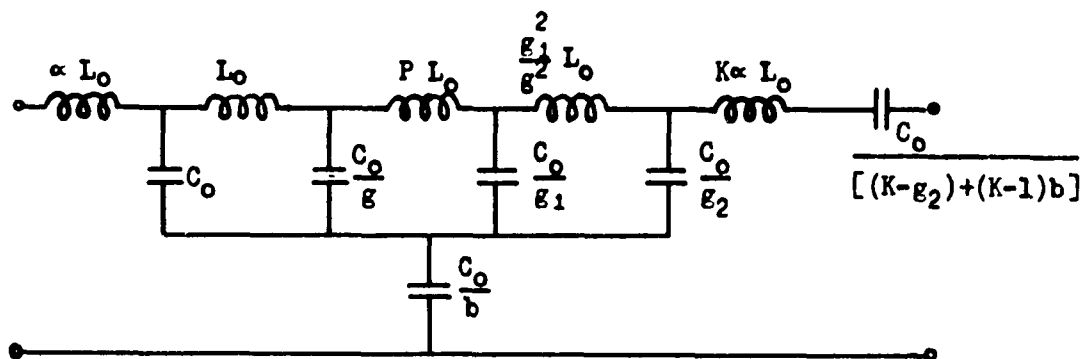


Figure 18



$$0 < g \leq \sqrt{\frac{1+\sqrt{K}}{2}}$$

$$g_1 = g \frac{(g+\sqrt{K})}{(1+g)}$$

$$g_2 = \sqrt{K} \frac{(g+\sqrt{K})}{(1+g)}$$

$$b = \frac{\sqrt{K}}{\sqrt{K}+1} \frac{(1+\sqrt{K}-2g^2)}{2g(1+g)}$$

$$\alpha = \frac{1+b}{1+g}$$

$$P = 2g^2 \alpha$$

$$M = 1 - \frac{(1+\sqrt{K})^2(K-1)}{[(\sqrt{K}+2g)(1+\sqrt{K})+2g^2]^2}$$

Figure 19

$$j_{yy} = \frac{g_1^2}{g} j_{xx} + \left[\frac{g_1^2}{g} (1+g) - (g_1 + g_2) \right] j_{xz} \quad (4-1)$$

In order to make all of the interior series impedance elements similar we will let $\frac{g_1^2}{g} (1+g) = (g_1 + g_2)$. Since $g_2 = \sqrt{K} \frac{g_1}{g}$, this becomes

$$\frac{g_1^2}{g} (1+g) = 1 + \frac{\sqrt{K}}{g} \quad \text{or} \quad g_1 = g \frac{(g + \sqrt{K})}{(1 + g)} \quad (4-2)$$

$$\text{and, of course,} \quad g_2 = \sqrt{K} \frac{g_1}{K} = \sqrt{K} \frac{(g + \sqrt{K})}{(1 + g)} \quad (4-3)$$

The requirement that $Z_5 = K Z_1$ means that $(j_{xx} + j_{yy} + j_{zz})$ is equal to $K(j_{xx} + j_{yy} + j_{zz})$ or that $j_{zz} = K j_{xx} + (K - g_2) j_{xz} + (K - 1) j_{yz}$ (4-4). Since j_{xx} , j_{xz} , and j_{yz} are assumed realizable and $K \geq 1$, j_{zz} will certainly be realizable if $g_2 \leq K$. This will be true if $(g + \sqrt{K}) \leq \sqrt{K} (1 + g)$, and therefore it is always true because K is always greater than or equal to unity. Hence, j_{zz} is realizable for any value of g .

The expression for Δ can be obtained by substituting into the determinant the values of j_{xx} , j_{yy} , and j_{zz} in terms of j_{xz} , the values of Z_5 and Z_4 in terms of Z_1 and Z_2 and expanding by minors. Doing this one obtains the following

$$\Delta = (K Z_1^2 - j_{zz}^2) \Delta^* - \left[\frac{2 K g_1^2}{g} Z_1 Z_2 Z_3 j_{xz}^2 - 2 K g_1^2 Z_1 j_{xz}^4 + 2 \sqrt{K} g_1^2 j_{xz}^4 j_{yz} - g_2^2 Z_3 j_{xz}^4 \right]$$

where $\Delta^* = Z_2 (Z_3 Z_4 - g_1^2 j_{xz}^2) - g^2 \frac{g_2^2}{K} Z_2 j_{xz}^2$. Using the relations $\frac{g_2^2 g^2}{K} = g_1^2$

and $Z_4 = \frac{g_2^2}{K} Z_2 = \frac{g_1^2}{g^2} Z_2$ this becomes $\Delta^* = g_1^2 Z_2 \left(\frac{Z_2 Z_3}{g^2} - 2 j_{xz}^2 \right)$

$$\text{Thus, } \frac{\Delta}{\Delta^*} = (K Z_1^2 - j_{zz}^2) - 2 K g_1^2 j_{xz}^2 \frac{\left[\frac{Z_1 Z_2 Z_3}{g^2} - Z_1 j_{xz}^2 + \frac{j_{yz}^2}{\sqrt{K}} - Z_3 \frac{j_{xz}^2}{2 g^2} \right]}{g_1^2 Z_2 \left(\frac{Z_3 Z_2}{g^2} - 2 j_{xz}^2 \right)} \quad \text{or}$$

$$\frac{\Delta}{\Delta^*} = K j_{12}^2 \left[\left(\frac{Z_1}{j_{12}} \right)^2 - \frac{1}{K} \left(\frac{j_{12}}{j_{12}} \right)^2 - 2 \frac{\left(\frac{Z_1 Z_2 Z_3}{g^2} - Z_1 j_{12}^2 + j_{12}^2 \frac{j_{12}}{\sqrt{K}} - Z_3 \frac{j_{12}^2}{2g^2} \right)}{Z_2 \left(\frac{Z_3 Z_2}{g^2} - 2 j_{12}^2 \right)} \right] \quad (4-5)$$

In order to make Δ^* a factor of Δ it is only necessary that

$$\frac{1}{g^2} \frac{Z_1}{j_{12}} \frac{Z_2}{j_{12}} \frac{Z_3}{j_{12}} - \frac{Z_1}{j_{12}} + \frac{1}{\sqrt{K}} \frac{j_{12}}{j_{12}} - \frac{1}{2g^2} \frac{Z_3}{j_{12}} = \phi \frac{Z_2}{j_{12}} \left(\frac{Z_2}{j_{12}} \frac{Z_3}{j_{12}} \frac{1}{g^2} - 2 \right) \quad (4-6)$$

As was done in the previous sections, the series elements j_{11} and j_{13} will be regarded as being similar to j_{12} . That is, $j_{11} = \alpha j_{12}$ and $j_{13} = P j_{12}$. Also, j_{15} will be made such that $j_{15} = a j_{12} + b j_{12}$. The quantities α , P , a , and b are all real, positive constants. Now,

$$Z_1 = j_{11} + j_{12} + j_{15} = (a + \alpha) j_{12} + (1 + b) j_{12}$$

$$Z_2 = j_{12} + j_{12} + j_{12} = j_{12} + (1 + g) j_{12}$$

$$Z_3 = j_{13} + j_{13} + j_{13} = P j_{12} + (g + g_1) j_{12}$$

Substituting these equations into (4-6) and denoting $\frac{j_{12}}{j_{12}}$ by x leads to

$$\frac{1}{g^2} [(a + \alpha)x + (1 + b)] [x + (1 + g)] [Px + (g + g_1)] - [(a + \alpha)x + (1 + b)] + \frac{ax + b}{\sqrt{K}} - \frac{[Px + (g + g_1)]}{2g^2} =$$

$$\phi [x + (1 + g)] \left[\frac{(x + 1 + g)(Px + g + g_1)}{g^2} - 2 \right]$$

If both sides of this equation are expanded and arranged in descending powers of x and if coefficients of like powers of x are made equal, the four equations shown below are obtained.

$$\frac{P}{g^2}(a + \alpha) = \frac{\phi P}{g^2} \quad \text{or} \quad (a + \alpha) = \phi \quad (4-7)$$

$$1 + b = \phi (1 + g) \quad (4-8)$$

$$\phi + \frac{a}{\sqrt{K}} = \frac{P}{2g^2} \quad (4-9)$$

$$1 + b + \frac{b}{\sqrt{K}} = \frac{g + g_1}{2g^2} \quad (4-10)$$

When these equations hold,

$$Z_{11} = \sqrt{\frac{\Delta_{nn} \Delta}{\Delta_{11} \Delta^*}} = \frac{1}{\sqrt{K}} \sqrt{\frac{\Delta}{\Delta^*}} = j_{12} \sqrt{\left(\frac{Z_1}{j_{12}}\right)^2 - \frac{1}{K} \left(\frac{j_{12}}{j_{12}}\right)^2 - 2\phi}$$

which becomes upon putting in the values of $\frac{Z_1}{j_{12}}$ and $\frac{j_{12}}{j_{12}}$

$$Z_{11} = \sqrt{[(a+\alpha) \frac{j_{12}}{j_{12}} + (1+b)]^2 - \frac{1}{K} [a \frac{j_{12}}{j_{12}} + b]^2 - 2\phi}$$

This is again precisely the same form for Z_{11} , as was obtained in the three mesh case and the four mesh case. In order to find the bandwidth ratio in this instance, assuming j_{11} is an inductance equal to L_0 and j_{12} is a capacitance equal to C_0 , use may be made of the mathematics immediately following equation (2-2) in Section II. The result is that the bandwidth ratio is given by

$$\beta = \sqrt{\frac{1 + \sqrt{M}}{1 - \sqrt{M}}} \quad \text{where} \quad M = \frac{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}(a-\alpha b)^2}{[(1+b)(a+\alpha) - \frac{ab}{K}]^2}$$

and the quantities a , α , b , and ϕ are interrelated by equations (4-7), (4-8), (4-9), and (4-10).

Because of the fact that $(a+\alpha)$ is again equal to ϕ use may be made of the algebra in Section III which alters the form of M from that given above to

$$M = 1 - \frac{[(1+b)^2 - 2\phi - \frac{b^2}{K}](\phi^2 - \frac{a^2}{K})}{[(1+b)\phi - \frac{ab}{K}]^2} \quad (4-11)$$

Since equation (4-2) shows that g_1 is a function of g and K , equations (4-8) and (4-9) show that both ϕ and b are functions of g and K only. Thus, M may be regarded as a function of only two "free" variables, namely a and g . (K is always considered as being fixed). The values that a and g can take on may be found by requiring that all of the elements that make up the network be realizable. Thus, it is necessary that a , b , α , P , and g all be greater than or equal to zero. From (4-9), $P \geq 0$ if $\alpha \geq 0$, and if $a \geq 0$. From equation (4-10), $b \geq 0$ only when $(g+g_1) \geq 2g^2$.

Since $g_1 = g \frac{(g + \sqrt{K})}{(1 + g)}$, we require that $1 + \frac{(g + \sqrt{K})}{(1 + g)} \geq 2g$ or that $(1 + \sqrt{K}) \geq 2g^2$.

Thus, $b \geq 0$ when $g \leq \sqrt{\frac{1 + \sqrt{K}}{2}}$. The condition that $\alpha \geq 0$ is the same as the condition $a \leq \phi$. From equation (4-8), $\phi = \frac{1+b}{1+g}$ and from equation (5-10)

$$b = \frac{\sqrt{K}[g + g_1 - 2g^2]}{(\sqrt{K} + 1) 2 g^2} = \frac{\sqrt{K} [1 + \sqrt{K} - 2g^2]}{(\sqrt{K} + 1) 2g(1+g)}. \text{ Hence, } \phi = \frac{(\sqrt{K} + 1) 2g(1+g) + \sqrt{K}(1 + \sqrt{K} - 2g^2)}{2 g (1+g)^2 (1 + \sqrt{K})}$$

$$\text{or } \phi = \frac{\sqrt{K}(1 + \sqrt{K}) + 2g(\sqrt{K} + 1 + g)}{2g (1+g)^2 (1 + \sqrt{K})} = \frac{(\sqrt{K} + 1)(\sqrt{K} + 2g) + 2g^2}{2g (1+g)^2 (1 + \sqrt{K})}$$

One can interpret these derived conditions by saying that, for all of the elements to be realizable, the admissible values of a and g must lie in the closed region of the (a, g) plane bounded by the lines $a = 0$, $g = 0$, $g = \sqrt{\frac{1 + \sqrt{K}}{2}}$, and the curve $a = \phi = \frac{(\sqrt{K} + 2g)(1 + \sqrt{K}) + 2g^2}{2g (1+g)^2 (1 + \sqrt{K})}$. This

region is sketched in Figure (18).

The next step is to determine whether there are any interior points of this region at which M is a maximum. This, of course, corresponds to finding whether there is a maximum bandwidth ratio. In the three mesh case the search for a maxima in M was carried out by noting whether there were any points in the given region at which the first two partial derivatives of M were simultaneously equal to zero. In this case the absence of any interior points of maximum M is most conveniently demonstrated by considering $\frac{\partial M}{\partial a}$ and $\frac{\partial^2 M}{\partial a^2}$ instead of solving simultaneously $\frac{\partial M}{\partial a} = 0$ and $\frac{\partial M}{\partial g} = 0$. The expressions for $\frac{\partial M}{\partial a}$ and $\frac{\partial^2 M}{\partial a^2}$ are quite simply obtained since b and ϕ are functions of g and K only.

The approach to be used depends on the fact that if there is a maxima in M at some point in the region then it is necessary that at that point $\frac{\partial M}{\partial a} = 0$ and $\frac{\partial^2 M}{\partial a^2} < 0$. If both these conditions are not satisfied at any point interior to the admissible region of the (a, g) plane, the max-

imum value of M will be achieved at some point on the boundary of this region.

Using equation (4-11) and remembering that b and ϕ are not functions of a we find that

$$\frac{\partial M}{\partial a} = -[(1+b)^2 - 2\phi - \frac{b^2}{K}] \frac{-\frac{2a}{K}[\phi(1+b) - \frac{ab}{K}]^2 + \frac{2b}{K}(\phi^2 - \frac{a^2}{K})[\phi(1+b) - \frac{ab}{K}]}{[\phi(1+b) - \frac{ab}{K}]^4}$$

$$\text{or } \frac{\partial M}{\partial a} = -\frac{2\phi}{K}[(1+b)^2 - 2\phi - \frac{b^2}{K}] \frac{[b\phi - a(1+b)]}{[\phi(1+b) - \frac{ab}{K}]^3}. \text{ If } \frac{\partial M}{\partial a} = 0 \text{ then } a = \phi \frac{b}{1+b}.$$

Thus, for every value of g in the allowed region there is a point (a, g) where $\frac{\partial M}{\partial a} = 0$ since the restriction on a was that $a \leq \phi$ and clearly $\frac{b}{1+b} < 1$ for all b. It remains to show $\frac{\partial^2 M}{\partial a^2} > 0$ at all points where $\frac{\partial M}{\partial a} = 0$.

$$\text{Now, } \frac{\partial^2 M}{\partial a^2} = \frac{-2\phi}{K}[(1+b)^2 - 2\phi - \frac{b^2}{K}] \frac{-(1+b)[\phi(1+b) - \frac{ab}{K}]^3 + [\phi(1+b) - \frac{ab}{K}]^2 [\phi b - a(1+b)] \frac{3b}{K}}{[\phi(1+b) - \frac{ab}{K}]^6}$$

$$\text{or } \frac{\partial^2 M}{\partial a^2} = \frac{-2\phi}{K}[(1+b)^2 - 2\phi - \frac{b^2}{K}] \frac{-\phi(1+b)^2 + \frac{ab(1+b)}{K} + \frac{3\phi b^2}{K} - \frac{3ab(1+b)}{K}}{[\phi(1+b) - \frac{ab}{K}]^4}$$

Substituting for a its value of $\phi \frac{b}{1+b}$ yields

$$\frac{\partial^2 M}{\partial a^2} = \frac{2\phi^2}{K} \frac{[(1+b)^2 - 2\phi - \frac{b^2}{K}][(1+b)^2 - \frac{b^2}{K}]}{[\phi(1+b) - \frac{ab}{K}]^4}. \text{ The only term in this expres-}$$

sion which could possibly make $\frac{\partial^2 M}{\partial a^2}$ negative is the term $[(1+b)^2 - 2\phi - \frac{b^2}{K}]$.

From equation (4-8),

$$2(b-\phi)+1 = 2(b-\frac{1+b}{1+g})+1 = \frac{2(gb-1)}{1+g}+1 = \frac{2gb+g-1}{1+g}.$$

$$\text{Since } b = \frac{\sqrt{K} [1+\sqrt{K}-2g^2]}{2g(1+g)(1+\sqrt{K})},$$

$$2(b-\phi)+1 = \frac{\sqrt{K}(1+\sqrt{K}-2g^2)+(\sqrt{K}+1)(g^2-1)}{(1+g)^2(1+\sqrt{K})} = \frac{(K-1)-g^2(\sqrt{K}-1)}{(1+g)^2(1+\sqrt{K})}$$

$$\text{or} \quad 2(b-\phi) + 1 = \frac{\sqrt{K}-1}{\sqrt{K}+1} \frac{(1+\sqrt{K}-g^2)}{(1+g)^2} \cdot \text{Now,}$$

$$\begin{aligned} [(1+b)^2 - 2\phi - \frac{b^2}{K}] &= \frac{K-1}{K} b^2 + 2(b-\phi) + 1 = \frac{(K-1)(1+\sqrt{K}-2g^2)^2}{(\sqrt{K}+1)^2 4g^2 (1+g)^2} + \frac{\sqrt{K}-1}{\sqrt{K}+1} \frac{1+\sqrt{K}-g^2}{(1+g)^2} \\ &= \frac{\sqrt{K}-1}{\sqrt{K}+1} \frac{1}{(1+g)^2} \left[\frac{(1+\sqrt{K})^2}{4g^2} - (1+\sqrt{K}) + g^2 + (1+\sqrt{K}-g^2) \right] = \frac{K-1}{4g^2 (1+g)^2} \cdot \end{aligned}$$

From this it can be seen that $[(1+b)^2 - 2\phi - \frac{b^2}{K}] > 0$ for all $K > 1$. This means that $\frac{\partial^2 M}{\partial a^2}$ is always positive at the points where $\frac{\partial M}{\partial a} = 0$ and there can therefore be no interior points of the region at which M is a maximum. Since the highest values of M occur on the boundary, attention will be directed to a derivation of expressions for M that apply to the various segments of the boundary.

The reduced expression for $[(1+b)^2 - 2\phi - \frac{b^2}{K}]$ just derived allows the general equation for M , equation (4-11), to be written as

$$M = 1 - \frac{(K-1) (\phi^2 - \frac{a^2}{K})}{(1+g)^2 4g^2 [\phi(1+b) - \frac{ab}{K}]^2} \quad (4-12)$$

The part of the boundary corresponding to $g = 0$ is of no practical interest because if $g = 0$ then b and ϕ are infinite. The next part of the boundary to be considered is the segment corresponding to $a = 0$.

$$\text{When } a = 0, 0 < g \leq \sqrt{\frac{1+\sqrt{K}}{2}} \text{ and } M = 1 - \frac{K-1}{4g^2 (1+g)^2 (1+b)^2} \quad (4-13)$$

$$\text{From equation (4-10), } b = \frac{\sqrt{K}}{1+\sqrt{K}} \frac{1+\sqrt{K}-2g^2}{2g(1+g)}. \text{ Hence, } (1+b)^2 = \frac{[(\sqrt{K}+2g)(1+\sqrt{K})+2g^2]^2}{4g^2 (1+g)^2 (1+\sqrt{K})^2}$$

$$\text{and} \quad M = 1 - \frac{(1+\sqrt{K})^2 (K-1)}{[(\sqrt{K}+2g)(1+\sqrt{K})+2g^2]^2} \quad (4-14)$$

It can be seen from this that M can be made to vary between some minimum

and some maximum value by letting g vary between zero and $g = \sqrt{\frac{1+\sqrt{K}}{2}}$.

Of course, g can only approach zero as a limit in which case M approaches, as a limit, the value $M = \frac{1}{K}$. When g takes on its maximum value of

$$g = \sqrt{\frac{1+\sqrt{K}}{2}}, M \text{ becomes } M = 1 - \frac{(K-1)}{[1+\sqrt{K} + 2\sqrt{\frac{1+\sqrt{K}}{2}}]^2} \quad (4-15)$$

Physically this corresponds to a strict ladder network because then

$\beta_{1,2} = a\beta_{1,1} + b\beta_{1,2}$ is zero since a and b are both zero. Thus, it is possible to achieve any value of β between the two extremes represented by the two limiting values of M given above by making $a = 0$ and selecting the appropriate value of g . The range of β values obtainable when $a = 0$ is shown in Figure (22) as the region between Curve II and Curve III.

The second case of interest is when $b = 0$; i.e., when $g = \sqrt{\frac{1+\sqrt{K}}{2}}$. Equation (4-12) yields the fact that when $b = 0$ M is given by

$$M = 1 - \frac{(K-1)(\phi^2 - \frac{a^2}{K})}{4g^2(1+g)^2\phi^2} = 1 - \frac{K-1}{2(1+\sqrt{K})} \frac{(\phi^2 - \frac{a^2}{K})}{[1 + \sqrt{\frac{1+\sqrt{K}}{2}}]^2 \phi^2}$$

Since a is the only variable and ϕ is not a function of a , it can easily be seen that minimum M occurs when a is as small as possible and maximum M occurs when a is as large as possible. The sketch in Figure (18) shows that for this case a may vary between zero and some maximum value which is also equal to ϕ . Thus, the minimum value of M when $b = 0$ is

$$M = 1 - \frac{(K-1)}{2(1+\sqrt{K})[1 + \sqrt{\frac{1+\sqrt{K}}{2}}]^2} \quad (4-16)$$

This equation can easily be shown to be equivalent to equation (4-15).

This is as it must be because both equations correspond to the same physical configuration. Thus it is possible, by making $b = 0$ and varying the

value of a , to achieve higher values of M and consequently higher values of β than were possible by making $a = 0$. The maximum value of M when $b = 0$ is found by letting $a = g$ in which case M is given by

$$M = 1 - \frac{(K-1)^2}{2K(1+\sqrt{K}) \left[1 + \sqrt{\frac{1+\sqrt{K}}{2}}\right]^2} \quad (4-17)$$

The region of β corresponding to the case when $b = 0$ is illustrated in Figure (22) as that region lying between Curve I and Curve II.

The final case of interest for the five mesh networks is when $\alpha = 0$. Then $a = g$, and the general expression for M labelled (4-12) reduces to

$$M = 1 - \frac{(K-1)^2}{4g^2K(1+g)^2 \left[1+b-\frac{b}{K}\right]^2} \quad (4-18)$$

Using the equation for b ,

$$(1+b-\frac{b}{K}) = \frac{(K-1)b}{K} + 1 = \frac{K-1}{K} \frac{\sqrt{K}(1+\sqrt{K}-2g^2)}{2g(1+g)(1+\sqrt{K})} + 1 = \frac{(\sqrt{K}-1)(1+\sqrt{K}-2g^2)+2g\sqrt{K}(1+g)}{2g(1+g)\sqrt{K}}$$

or $(1+b-\frac{b}{K}) = \frac{K-1+2g(g+\sqrt{K})}{2g(1+g)\sqrt{K}}$. Substituting this into (4-18) gives

$$M = 1 - \frac{(K-1)^2}{[(K-1)+2g(g+\sqrt{K})]^2} \quad (4-19)$$

The maximum value of M occurs when g takes on its maximum value of

$$g = \sqrt{\frac{1+\sqrt{K}}{2}} \text{ in which case}$$

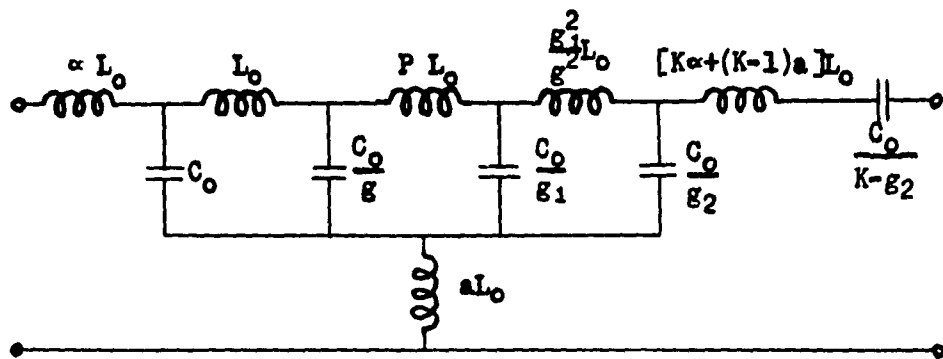
$$M = 1 - \frac{(K-1)^2}{K \left[1+\sqrt{K}+2\sqrt{\frac{1+\sqrt{K}}{2}}\right]^2} \quad (4-20)$$

This can be shown to be identical to equation (4-17). This must be true since both expressions hold for the case when $b = 0$ and $\alpha = 0$. As g is allowed to approach zero the value of M approaches zero so that by taking the appropriate value of g between zero and $g = \sqrt{\frac{1+\sqrt{K}}{2}}$ any value of

β between unity and the value corresponding to the M given by (4-20) may be obtained. These values make up the region shown in Figure (22) that lies beneath Curve I.

The physical configuration of the networks corresponding to the three separate cases considered and expressions for the component values are given in Figures (19), (20), and (21). In each case it is possible, after selecting the desired value of β and K , to calculate all of the components in terms of either L_0 or C_0 . L_0 and C_0 are determined upon specifying the nominal impedance level $Z_0 = \sqrt{\frac{L_0}{C_0}}$ and the nominal design frequency $f_0 = \frac{\omega_0}{2\pi}$.

As in the four mesh case, the maximum value of β that can be achieved by using a five mesh network of the type considered is less than the value of β obtainable with an exponential taper network of five meshes until relatively large values of K are used. However, the values of β realizable with the above mentioned network are greater, for every value of K , than those possessed by the Tschebyscheff taper networks which utilize the same number of elements. Also, the possibility of varying β continuously between unity and some maximum value, at each value of K , is again in evidence and therefore represents an improvement over the strict one-to-one β versus K relationships of the Tschebyscheff and exponential tapers.



$$g = \sqrt{\frac{1 + \sqrt{K}}{2}}$$

$$g_1 = g \frac{(g + \sqrt{K})}{(1 + g)}$$

$$g_2 = \sqrt{K} \frac{(g + \sqrt{K})}{(1 + g)}$$

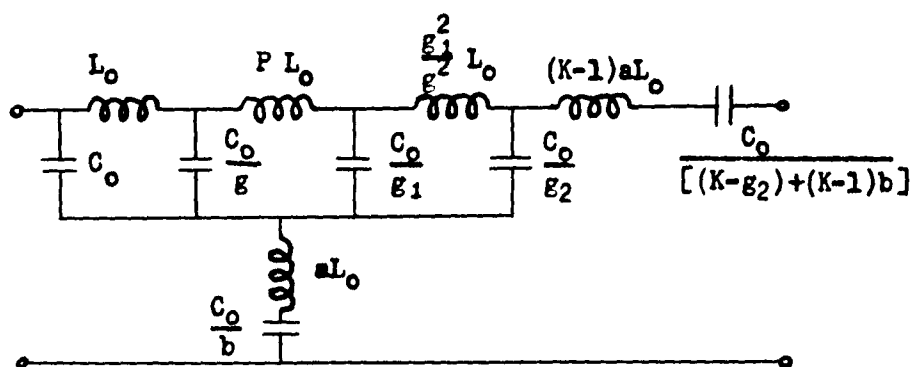
$$0 \leq a \leq \frac{1 + \sqrt{K} + 2g}{2g(1+g)^2}$$

$$P = 2g^2 \left(\frac{1}{1+g} + \frac{g}{\sqrt{K}} \right)$$

$$\alpha = \frac{1}{1+g} - a$$

$$M = 1 - \frac{(K-1)^2 \left[1 - \frac{a^2}{K} (1+g)^2 \right]}{4g^2(1+g)^2}$$

Figure 20



$$0 < g \leq \sqrt{\frac{1 + \sqrt{K}}{2}}$$

$$g_1 = g \frac{(g + \sqrt{K})}{(1 + g)}$$

$$g_2 = \sqrt{K} \frac{(g + \sqrt{K})}{(1 + g)}$$

$$a = \frac{[(\sqrt{K} + 2P)(1 + \sqrt{K}) + 2g^2]}{2g(1+g)^2(1 + \sqrt{K})}$$

$$b = \frac{\sqrt{K}(1 + \sqrt{K} - 2g^2)}{2g(1+g)(1 + \sqrt{K})}$$

$$P = 2g^2 \left(1 + \frac{1}{\sqrt{K}} \right) a$$

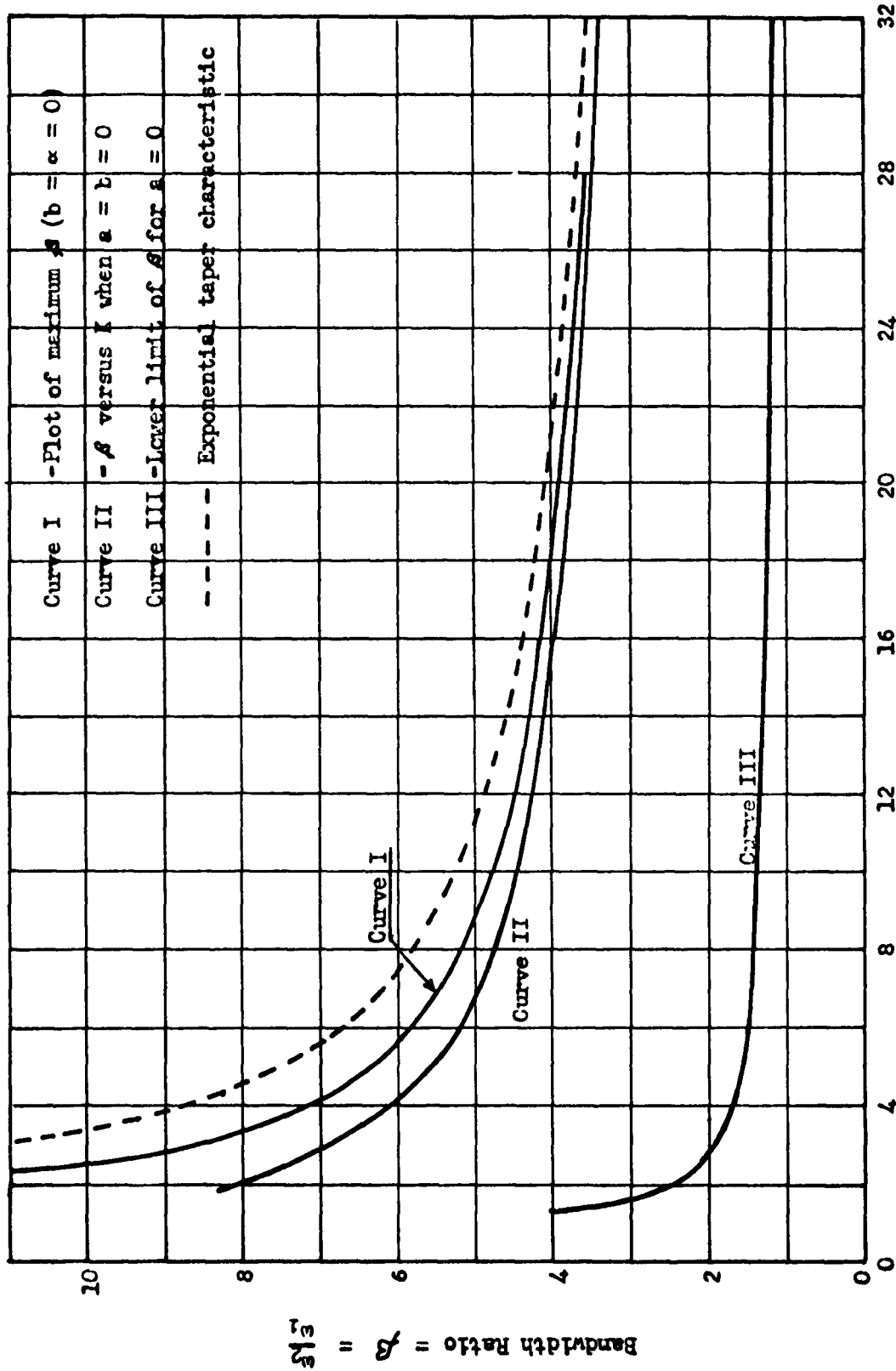
$$M = 1 - \frac{(K-1)^2}{[(K-1) + 2g\sqrt{K} + 2g^2]^2}$$

Figure 21

Figure 22

-55-

Plot of Values of β Realizable With a Five Mesh Network



Section V

THE INPUT IMAGE IMPEDANCE

Until this point in the thesis attention has been directed solely to determining the bandwidth ratio characteristics and the component values of the various network configurations. Expressions for the input image impedance were derived for the three, four, and five mesh networks and were used to find the frequencies at which the image impedance became real. These expressions it turned out were all of the same form. That is, regardless of the number of meshes considered,

$$Z_{1,1} = j_{1,1} \sqrt{[(a+\alpha) \frac{j_{1,1}}{j_{1,2}} + (1+b)]^2 - \frac{1}{K}(a \frac{j_{1,1}}{j_{1,2}} + b)^2 - 2\phi}$$

It is assumed that these networks are to be driven by a voltage source having a purely resistive impedance and terminated in a purely resistive load. The source and load impedances are assumed to be constant, independent of frequency. Since the image impedance as given above is a function of frequency, the question arises as to what the source and load resistances should be to provide an optimum impedance match over the pass band. In order to arrive at a solution to this problem it will be useful to determine the shape of the image impedance in the pass band. Of course, it is only necessary to consider $Z_{1,1}$ and R_s because $Z_{1,2} = K Z_{1,1}$ and $R_L = K R_s$. (R_L and R_s are the load and source resistances respectively.)

Now, $j_{1,1}$ and $j_{1,2}$ were previously designated as $j_{1,1} = j\omega L_0$ and $j_{1,2} = \frac{1}{j\omega C_0}$. Using the previous notation that $\omega^2 L_0 C_0 = \left(\frac{\omega}{\omega_0}\right)^2 = \pi^2$

$$Z_{1,1} = \frac{1}{j\omega C_0} \sqrt{\pi^4 [(a+\alpha)^2 - \frac{a^2}{K}] - 2[(a+\alpha)(1+b) - \frac{ab}{K}] \pi^2 + [(1+b)^2 - \frac{b^2}{K}] - 2\phi}$$

$$\text{or } Z_{11} = \frac{\sqrt{\frac{L_0}{C_0} [(a+\alpha)^2 - \frac{a^2}{K}]^{\frac{1}{2}}}}{j\omega L_0 C_0} \sqrt{\pi^4 - 2\pi^2 \frac{[(a+\alpha)(1+b) - \frac{ab}{K}]}{[(a+\alpha)^2 - \frac{a^2}{K}]} + \frac{[(1+b)^2 - \frac{b^2}{K} - 2\phi]}{[(a+\alpha)^2 - \frac{a^2}{K}]}}$$

Denoting $\sqrt{\frac{L_0}{C_0}}$ by Z_0 this equation can be written as

$$\frac{Z_{11}}{Z_0} = \frac{\sqrt{[(a+\alpha)^2 - \frac{a^2}{K}]}}{\pi} \sqrt{(\pi^2 - \pi_1^2)(\pi_2^2 - \pi^2)} \text{ where } \pi_1 \text{ and } \pi_2$$

are related to the upper and lower cutoff frequencies by $\pi_1 = \frac{2\pi f_1}{\omega_0}$ and $\pi_2 = \frac{2\pi f_2}{\omega_0}$. In terms of α , a , ϕ , and b

$$\pi_1^2 = \frac{[(1+b)(a+\alpha) - \frac{ab}{K}] - \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}(a-\alpha b)^2}}{[(a+\alpha)^2 - \frac{a^2}{K}]} \quad (5-1)$$

$$\pi_2^2 = \frac{[(1+b)(a+\alpha) - \frac{ab}{K}] + \sqrt{2\phi[(a+\alpha)^2 - \frac{a^2}{K}] + \frac{1}{K}(a-\alpha b)^2}}{[(a+\alpha)^2 - \frac{a^2}{K}]} \quad (5-2)$$

In any given case a , α , π_1 and π_2 are fixed quantities and therefore Z_{11} may be plotted as a function of the dimensionless frequency π . As an aid in constructing this plot and for other purposes it is convenient to determine the maximum value of Z_{11} . This will be done by setting $\frac{dZ_{11}}{d\pi}$ equal to zero. For

$$\frac{dZ_{11}}{d\pi} = \frac{d}{d\pi} \left[\frac{(\pi^2 - \pi_1^2)(\pi_2^2 - \pi^2)}{\pi^2} \right] = \pi^2 [(\pi_2^2 - \pi^2)2\pi - 2\pi(\pi^2 - \pi_1^2)] - 2\pi(\pi^2 - \pi_1^2)(\pi_2^2 - \pi^2) = 0,$$

$\pi^4 = \pi_1^2 \pi_2^2$ or $\pi = \sqrt{\pi_1 \pi_2}$. When π takes on this value Z_{11} is equal to

$Z_{11, \max}$ which is

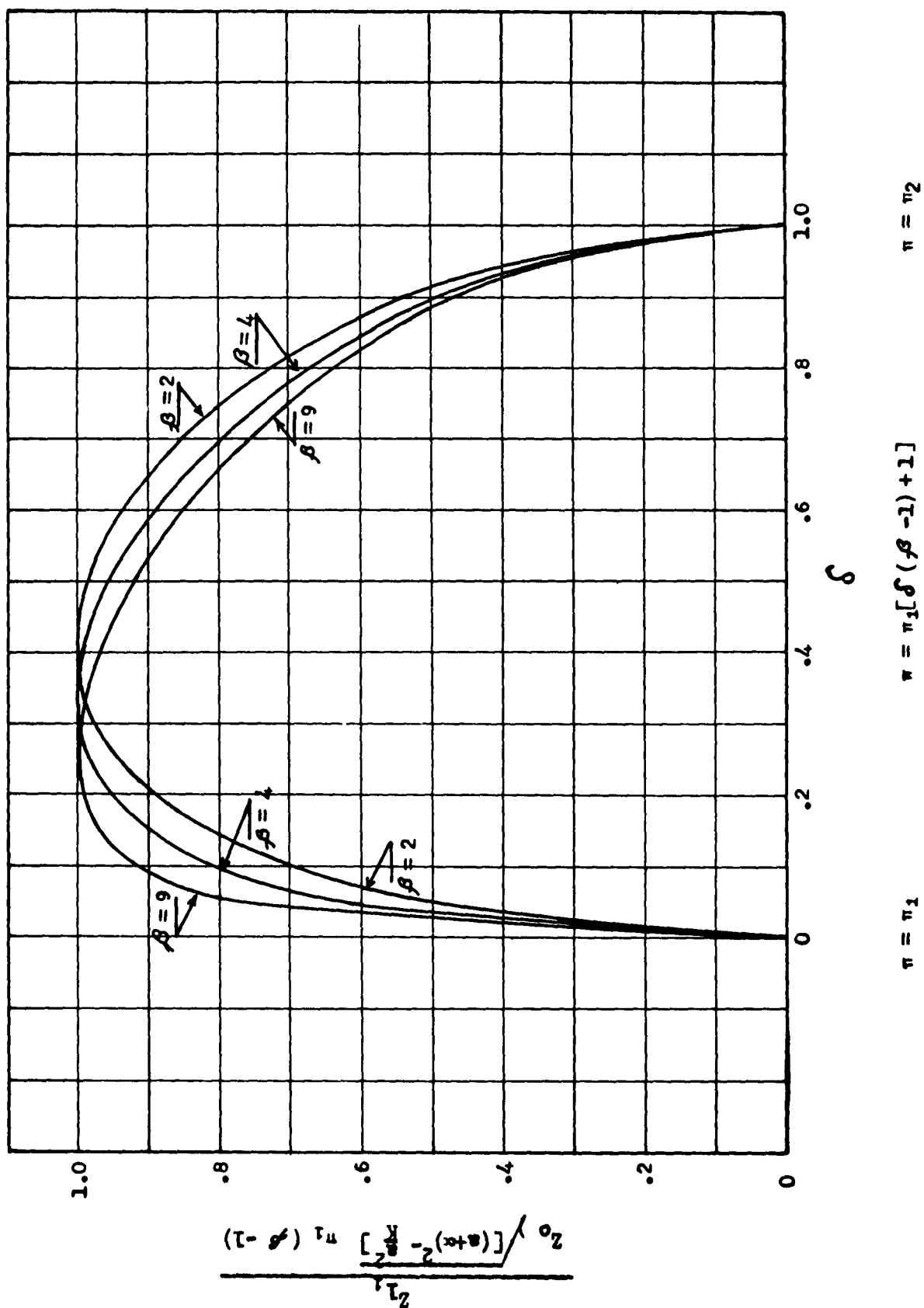
$$Z_{11, \max} = Z_0 \sqrt{[(a+\alpha)^2 - \frac{a^2}{K}]} \sqrt{\frac{(\pi_1 \pi_2 - \pi_1^2)(\pi_2^2 - \pi_1 \pi_2)}{\pi_1 \pi_2}}$$

$$\text{or } Z_{11, \max} = Z_0 \sqrt{[(a+\alpha)^2 - \frac{a^2}{K}]} (\pi_2 - \pi_1) \quad (5-3)$$

Figure 23

Dimensionless Plot of Input Image Impedance for Frequencies in the Pass Band

-59-

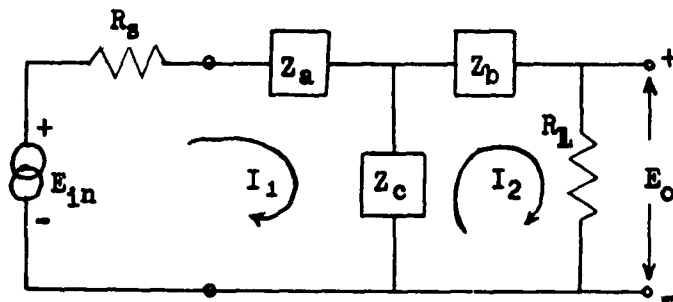


Section VI

THE VOLTAGE TRANSFER FUNCTION

In this section the voltage transfer function will be derived for three circuits of the three mesh type which are subject to various degrees of mismatch. Upon graphing these results the degree of mismatch yielding the best response will become apparent; this in turn will assist in the selection of the proper source resistance for optimum matching.

Using the equivalent T section given in Section I, the impedance transforming network terminated in a purely resistive load R_L and driven by a source having a voltage E_{in} and an internal resistance R_s may be represented as shown below.



$$Z_a = \frac{\Delta_{nn} - \Delta_{1n}}{\Delta^*}$$

$$Z_b = \frac{\Delta_{11} - \Delta_{1n}}{\Delta^*}$$

$$Z_c = \frac{\Delta_{1n}}{\Delta^*}$$

Writing the circuit equations

$$E_{in} = (R_s + Z_a + Z_c)I_1 - Z_c I_2$$

$$0 = -Z_c I_1 + (R_L + Z_b + Z_c)I_2$$

Substituting for Z_a , Z_b , and Z_c their determinantal equivalents and solving for I_2 ,

$$E_o = I_2 R_L = \frac{R_L Z_c E_{in}}{(R_s + \frac{\Delta_{nn}}{\Delta^*})(R_L + \frac{\Delta_{11}}{\Delta^*}) - \frac{\Delta_{1n}^2}{\Delta^{*2}}} = \frac{R_L E_{in} \frac{\Delta_{1n}}{\Delta^*}}{R_s R_L + R_L \frac{\Delta_{nn}}{\Delta^*} + R_s \frac{\Delta_{11}}{\Delta^*} + \frac{\Delta_{11}\Delta_{nn} - \Delta_{1n}^2}{\Delta^{*2}}}$$

Since $\Delta_{11}\Delta_{nn} - \Delta_{1n}^2 = \Delta \Delta^*$ (see Section I), we can write

$$\frac{E_o}{E_{in}} = \frac{R_L \Delta_{1n}}{R_s R_L \Delta^* + R_L \Delta_{nn} + R_s \Delta_{11} + \Delta}$$

Now, $\frac{\Delta_{11}}{\Delta_{nn}} = K$ and $Z_{11} = \frac{1}{\sqrt{K}} \sqrt{\frac{\Delta^*}{\Delta^*}}$. Hence, $\frac{E_o}{E_{in}} = \frac{R_L \Delta_{in}}{(R_s R_L + K Z_{11}^2) \Delta^* + 2K R_s \Delta_{nn}}$.

Because this is an impedance transformer $R_L = K R_s$ and

$\frac{E_o}{E_{in}} = \frac{R_s \Delta_{in}}{(R_s^2 + Z_{11}^2) \Delta^* + 2R_s \Delta_{nn}}$. It has been shown that Z_{11} can be written as

$Z_{11} = Z_o \sqrt{Q} \sqrt{\frac{(\pi^2 - \pi_1^2)(\rho^2 \pi_1^2 - \pi^2)}{\pi^2}}$ and that the maximum value of Z_{11} is

$Z_{11, \max} = Z_o \sqrt{Q} \pi_1 (\rho - 1)$ where $Q = [(a + \alpha)^2 - \frac{\rho^2}{K}]$. Let $R_s = \psi Z_{11, \max}$. (ψ is a constant less than or equal to unity and may be regarded as the mismatch factor.)

Substituting the above quantities into the equation for $\frac{E_o}{E_{in}}$ gives the result $\frac{E_o}{E_{in}} = \frac{\Delta_{in}}{2\Delta_{nn} + Z_o \sqrt{Q} [\psi \pi_1 (\rho - 1) + \frac{(\pi^2 - \pi_1^2)(\rho^2 \pi_1^2 - \pi^2)}{\psi \pi^2 \pi_1 (\rho - 1)}] \Delta^*}$.

For the three mesh network in Section II,

$$\begin{aligned} \Delta_{in} &= \sqrt{K} j_{12}^2 - Z_2 j_{12} = j_{12}^2 [\sqrt{K} - \left(\frac{j_{12}}{j_{12}}\right) \left(\frac{Z_2}{j_{12}}\right)] \\ \Delta_{nn} &= Z_1 Z_2 - j_{12}^2 = j_{12}^2 \left[\left(\frac{Z_1}{j_{12}}\right) \left(\frac{Z_2}{j_{12}}\right) - 1 \right] \\ \text{and } \Delta^* &= Z_2 = j_{12} \left(\frac{Z_2}{j_{12}}\right) \end{aligned}$$

Thus, $\frac{E_o}{E_{in}} = \frac{[\sqrt{K} - \frac{j_{12}}{j_{12}} \frac{Z_2}{j_{12}}]}{2[\frac{Z_1}{j_{12}} \frac{Z_2}{j_{12}} - 1] + \frac{Z_o \sqrt{Q}}{j_{12}} [\psi \pi_1 (\rho - 1) + \frac{(\pi^2 - \pi_1^2)(\rho^2 \pi_1^2 - \pi^2)}{\psi \pi^2 \pi_1 (\rho - 1)}] \frac{Z_2}{j_{12}}}$

Also from Section II, $\frac{Z_2}{j_{12}} = \frac{j_{12}}{j_{12}} + 1 + \sqrt{K} = (-\pi^2 + 1 + \sqrt{K})$

$\frac{j_{12}}{j_{12}} = a \frac{j_{12}}{j_{12}} + b = (-a\pi^2 + b)$

$\frac{Z_1}{j_{12}} = \alpha \frac{j_{12}}{j_{12}} + 1 + a \frac{j_{12}}{j_{12}} + b = [-(a + \alpha)\pi^2 + (1 + b)]$

and $\frac{Z_0}{j\omega} = \frac{\sqrt{L_0}}{\sqrt{C_0}} j\omega C_0 = j\omega \sqrt{L_0 C_0} = j \frac{\omega}{\omega_0} = j\pi$. Using the above equalities

$$\frac{\frac{E_0}{E_{in}}}{\frac{\sqrt{K}}{2}} = \frac{1 - \frac{1}{\sqrt{K}} (a\pi^2 - b)[\pi^2 - (1 + \sqrt{K})]}{[(a + \alpha)(\pi^2 - \frac{1+b}{a+\alpha})[\pi^2 - (1 + \sqrt{K}) - 1] - j\pi \frac{\sqrt{Q}}{2}[\pi^2 - (1 + \sqrt{K})][\psi \pi_1(\psi - 1) + \frac{(\pi^2 - \pi_1^2)(\psi^2 \pi_1^2 - \pi^2)}{\psi \pi_1 \pi^2(\psi - 1)}]}$$

From this expression the quantity $\left| \frac{E_0}{E_{in}} \right| / \frac{\sqrt{K}}{2}$ may be plotted as a function of π and ψ merely by selecting an appropriate combination of a , α , and b as given in Section II. This calculation was carried out for three representative circuits and the results are displayed in Figures (24), (25), and (26). In each case ψ was made to take on the values $\psi = .6$, $\psi = .8$, and $\psi = 1.0$; it can be seen that in each case the value $\psi = .8$ yielded the best response. For $\psi = 1.0$ the response is extremely flat over the center portion of the pass band, but the response is poor in that the effective pass band is narrower than in the case that $\psi = .8$. On the other hand, while the curve for $\psi = .6$ rises at a lower frequency and falls at a higher frequency, the considerable dip in the response at mid-band may be quite undesirable. Thus, one must compromise in the selection of ψ to obtain a response which possesses the widest effective bandwidth while retaining the required degree of flatness.

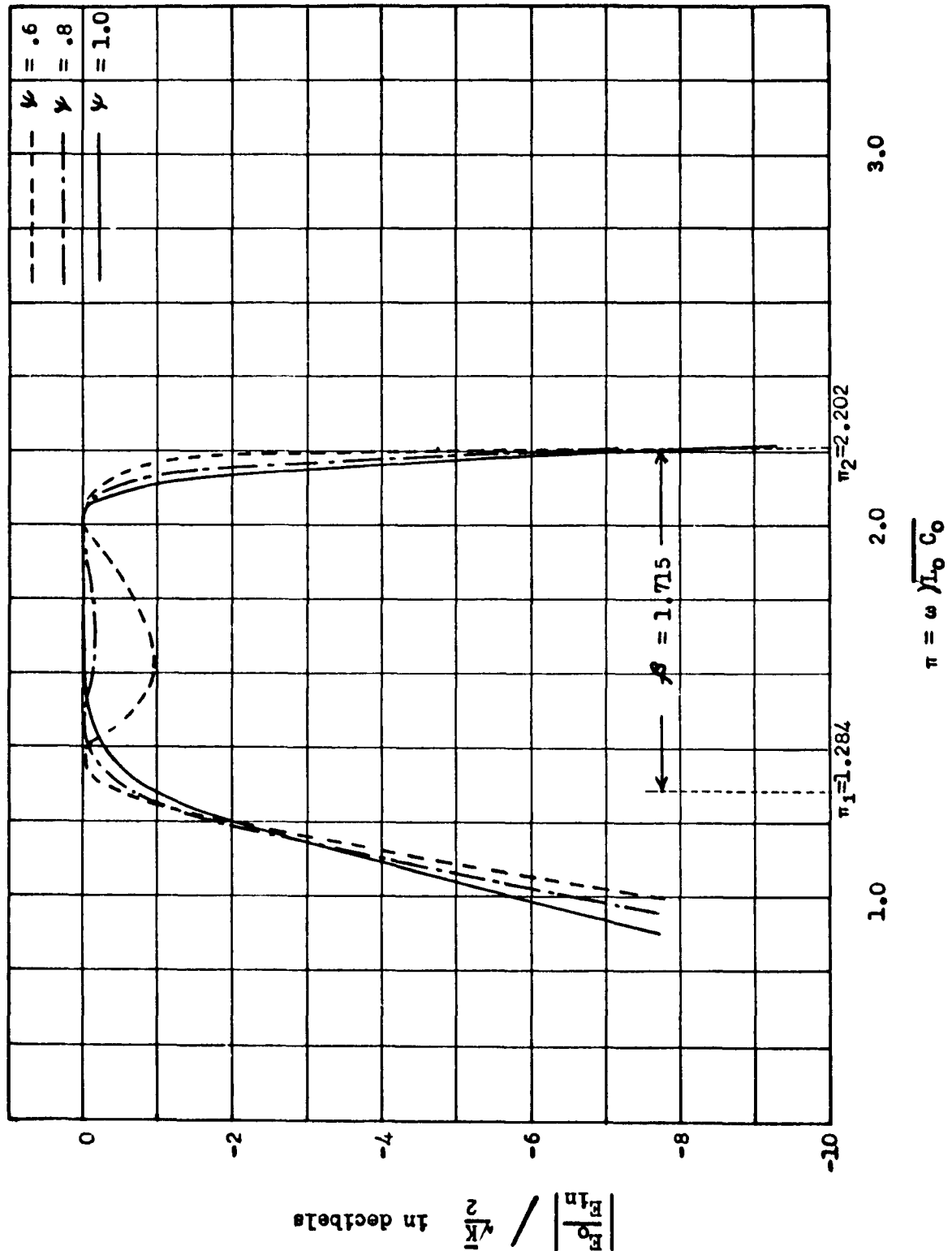
The curves for $\psi = .8$ are combined in Figure (27) to show the difference in bandwidth possessed by these circuits. The values of a , α , and b for the circuits corresponding to the various voltage transfer curves are given at the head of each figure. These same circuits were constructed and used to experimentally check the theory, and the curves in Figure (27) are reproduced for comparison with the experimental results in Section VII.

Figure 24

-63-

Plot of Voltage Transfer Function of Mismatched Three Mesh Network

With $a = 0$, $\alpha = 1.0$, $b = 2.25$ and $K = 9$



Plot of Voltage Transfer Function of Mismatched Three Mesh Network

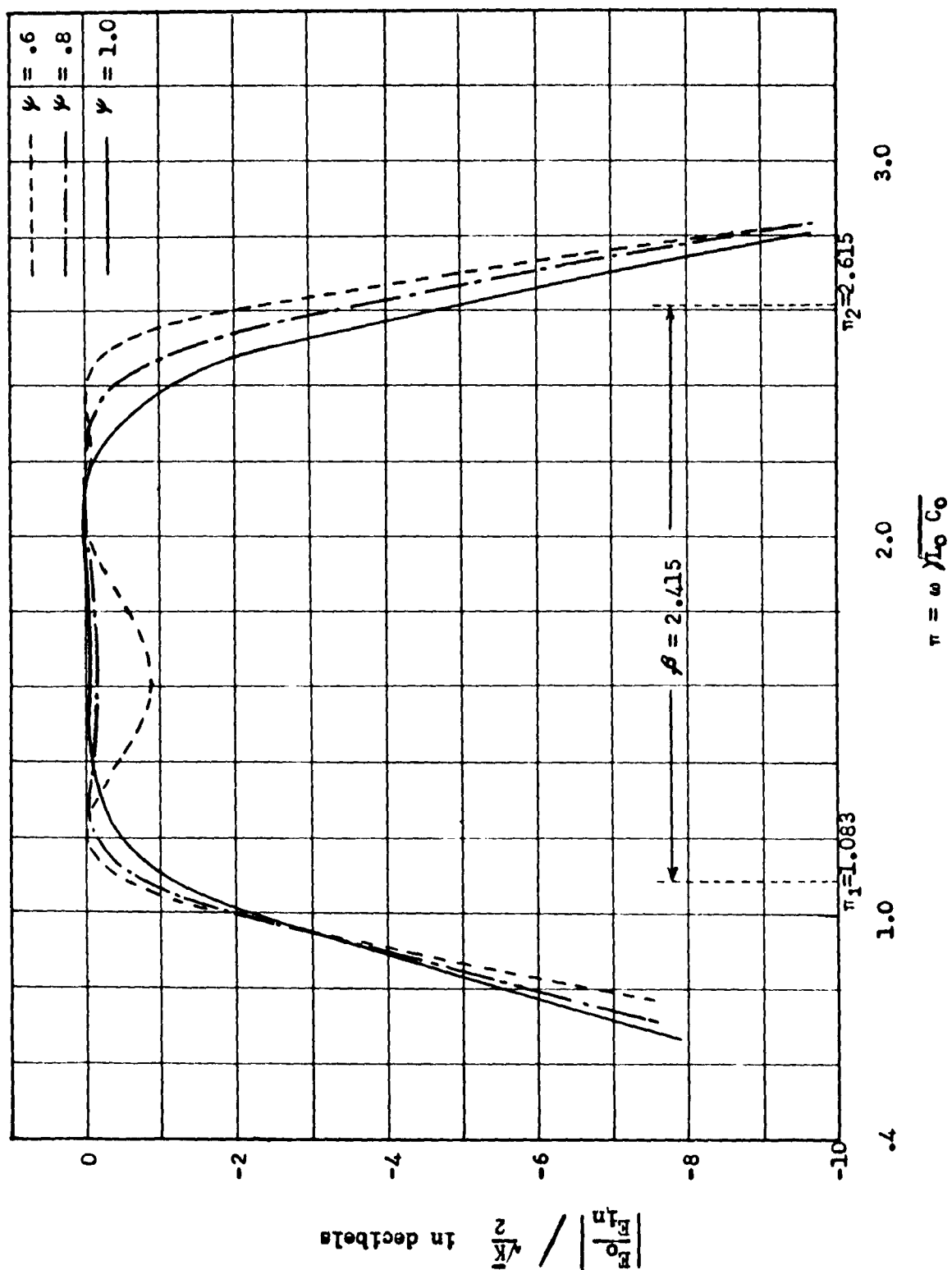
With $a = b = 0$, $\alpha = .25$ and $K = 9$ 

Figure 26

-65-

Plot of Voltage Transfer Function of Mismatched Three Mesh Network

With $\alpha = b = 0$, $a = \frac{3}{16}$, and $K = 9$

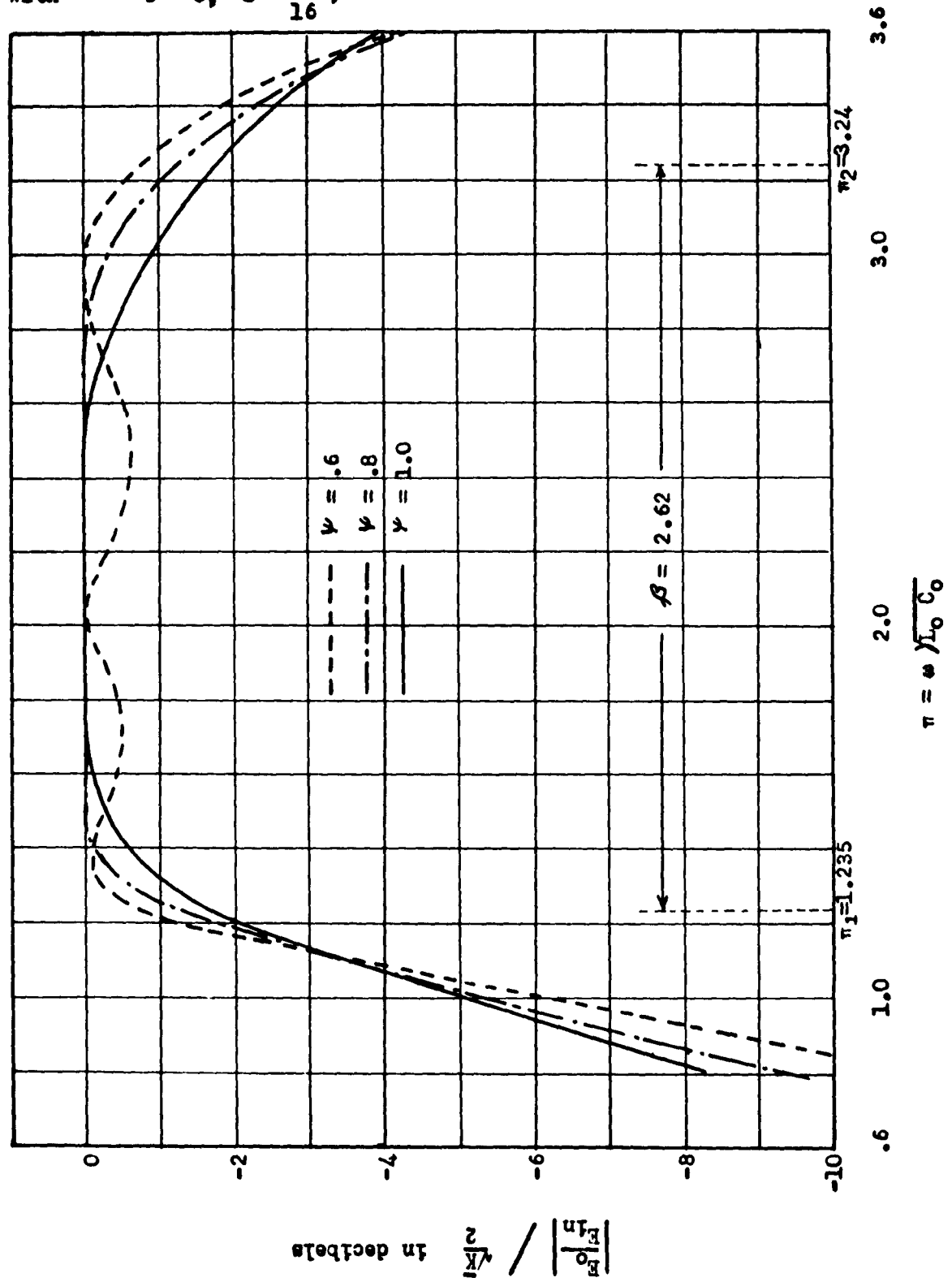
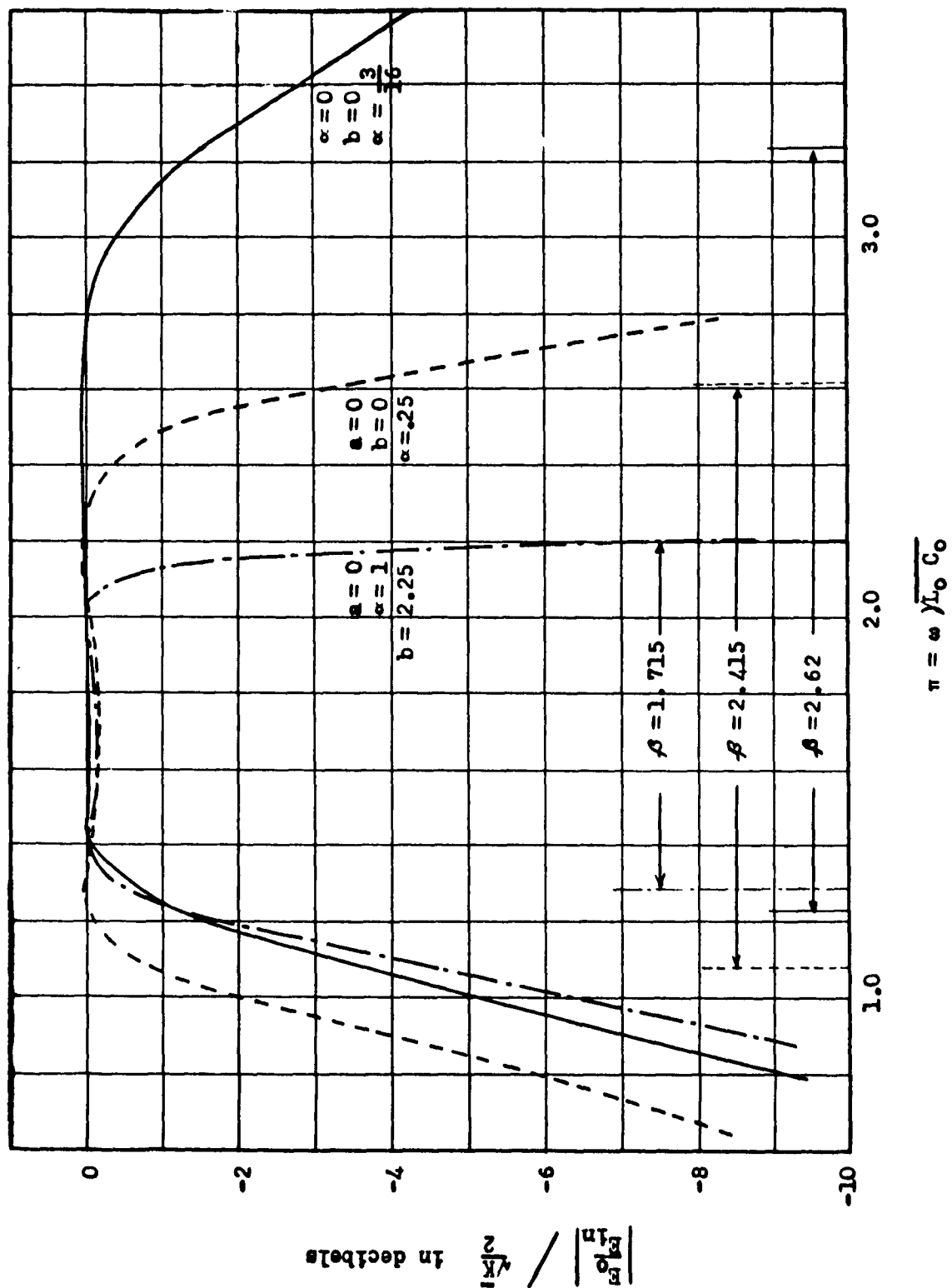


Figure 27

Combined Plot of Voltage Transfer Functions of the Three Mesh Networks

When Each is Mismatched by the Same Degree. $K = 9$ and $\mu = .8$



Section VII

EXPERIMENTAL RESULTS

In order to check experimentally the derived theory, three networks of the three mesh type were designed and constructed. It was felt that, in addition to checking the theory, the experimental results would give a good indication of how much deviation from the expected results occurred due to the fact that the actual impedance elements are neither lossless nor constant with respect to frequency as assumed.

The first step in the design of the networks, which are the same as those for which the voltage transfer function was computed in Section VI, was to select the value of the impedance transformation ratio K . A compromise between a high value of K , which leads to less practical values for β , and a low value of K , which would too closely resemble a symmetrical structure, was effected by choosing $K = 9$. The values for a , α , and b were selected as follows. In the first circuit a was made zero and α was made equal to unity. Hence, from Section II,

$$b = \sqrt{K} \alpha - \frac{\sqrt{K}}{1 + \sqrt{K}} = 2.25.$$

In the second circuit both a and b were made zero, and this choice required that $\alpha = \frac{1}{1 + \sqrt{K}} = .25$. This circuit is equivalent to a Tschebyscheff taper network having three meshes. The condition $\alpha = 0$ and $b = 0$ was enforced in the third circuit so that an example of maximum bandwidth would result. This condition made a equal to $a = \frac{\sqrt{K}}{(1 + \sqrt{K})^2} = \frac{3}{16}$.

The remaining quantities to be specified are

$$Z_0 = \frac{\sqrt{L_0}}{\sqrt{C_0}} \text{ and } \omega_0 = \frac{1}{\sqrt{L_0 C_0}}.$$

The choice of ω_0 was dictated by such considerations as practicability

of element values and ease of measurement with the result that ω_0 was made equal to $\omega_0 = 2\pi f_0$ where f_0 is 7.5 Mc. . The choice of Z_0 in each case was such that $R_g = \psi Z_{1\max} = 50$ when $\psi = .8$.

As an example, the calculations leading to final component values will be carried out for the first circuit where $a = 0$, $\alpha = 1$, and $b = 2.25$.

$$\frac{1}{\sqrt{L_0 C_0}} = \omega_0 = 2\pi f_0 = 2\pi 7.5 \times 10^6 = 47.2 \times 10^6$$

Now, $Z_{1\max} = Z_0 \sqrt{(a+\alpha)^2 - \frac{a^2}{K^2}} (\pi_2 - \pi_1) = \frac{50}{.8} = 62.5$. π_1 and π_2 are found from (5-1) and (5-2) respectively to be $\pi_1 = 1.285$ and $\pi_2 = 2.2$.

Hence,

$$Z_0 = \frac{62.5}{\sqrt{(a+\alpha)^2 - \frac{a^2}{K^2}} (\pi_2 - \pi_1)} = \frac{62.5}{.915} = 68$$

Thus, $L_0 = Z_0^2 C_0$ and $L_0 = \frac{1}{\omega_0^2 C_0}$. Solving for C_0 and L_0 ,

$$C_0 = \frac{1}{\omega_0 Z_0} = \frac{1}{(68) 47.2 \times 10^6} = 310 \times 10^{-12} \text{ farads}$$

$$L_0 = (68)^2 310 \times 10^{-12} = 1.434 \times 10^{-6} \text{ henries}$$

All of the remaining element values may be determined from these values for C_0 and L_0 . The circuit corresponding to the values given above is shown in Figure (28). The element values for the other two circuits were computed in a similar fashion and are shown in Figures (29) and (30).

The actual method of measurement consisted of measuring the open-circuit voltage of the source then connecting the circuit and measuring the voltage developed across the load resistor R_L for different frequencies. The results of the test on each circuit are shown in Figures (28), (29) and (30). It will be noted that on the graphs $\pi = 2$ is labelled 14.75 Mc.

instead of 15.0 Mc. as it was designed to be. This is due to a slight correction in the frequency of the Q-meter used to measure the inductances and capacitances.

The graphs show a very close agreement between the theoretical and experimental curves. For the most part, the difference is on the order of .3db or 3 % in the pass band. This close agreement becomes more important when it is realized that the inductances were relatively low Q coils. The low self-resonant frequency associated with air-core chokes of the large sizes required made them unfeasible so that it was necessary to employ ferrite core coils that had Q_s as low as fifteen, but were more constant with frequency. Thus, it appears that the assumption of elements being lossless is not too unrealistic.

Figure 28

-70-

Experimental Results With Three Mesh Network Having $a = 0$, $\alpha = 1$,
and $b = 2.25$

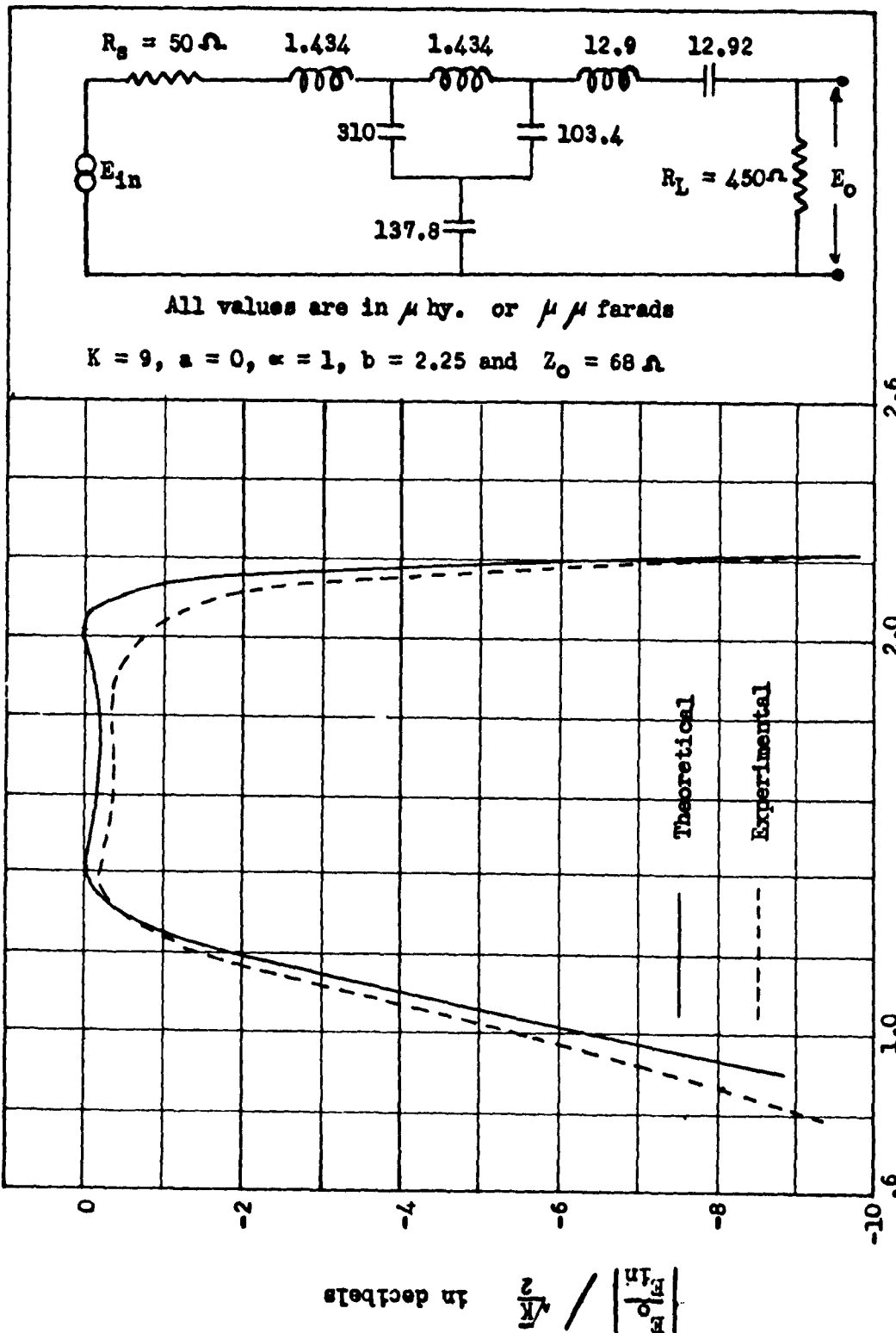
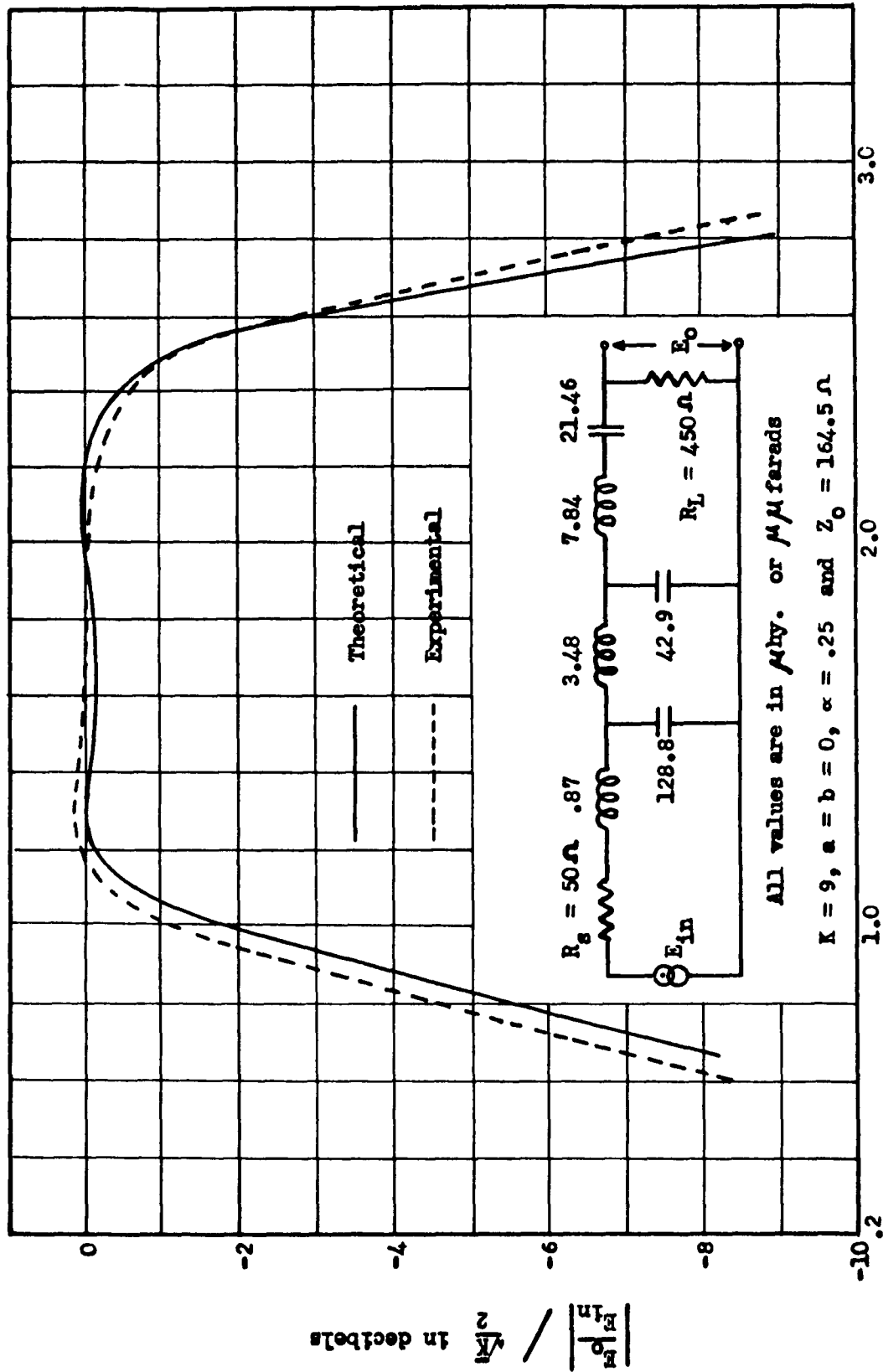


Figure 29

-71-

Experimental Results With Three Mesh Network Having $a = b = 0$ and

$\alpha = .25$

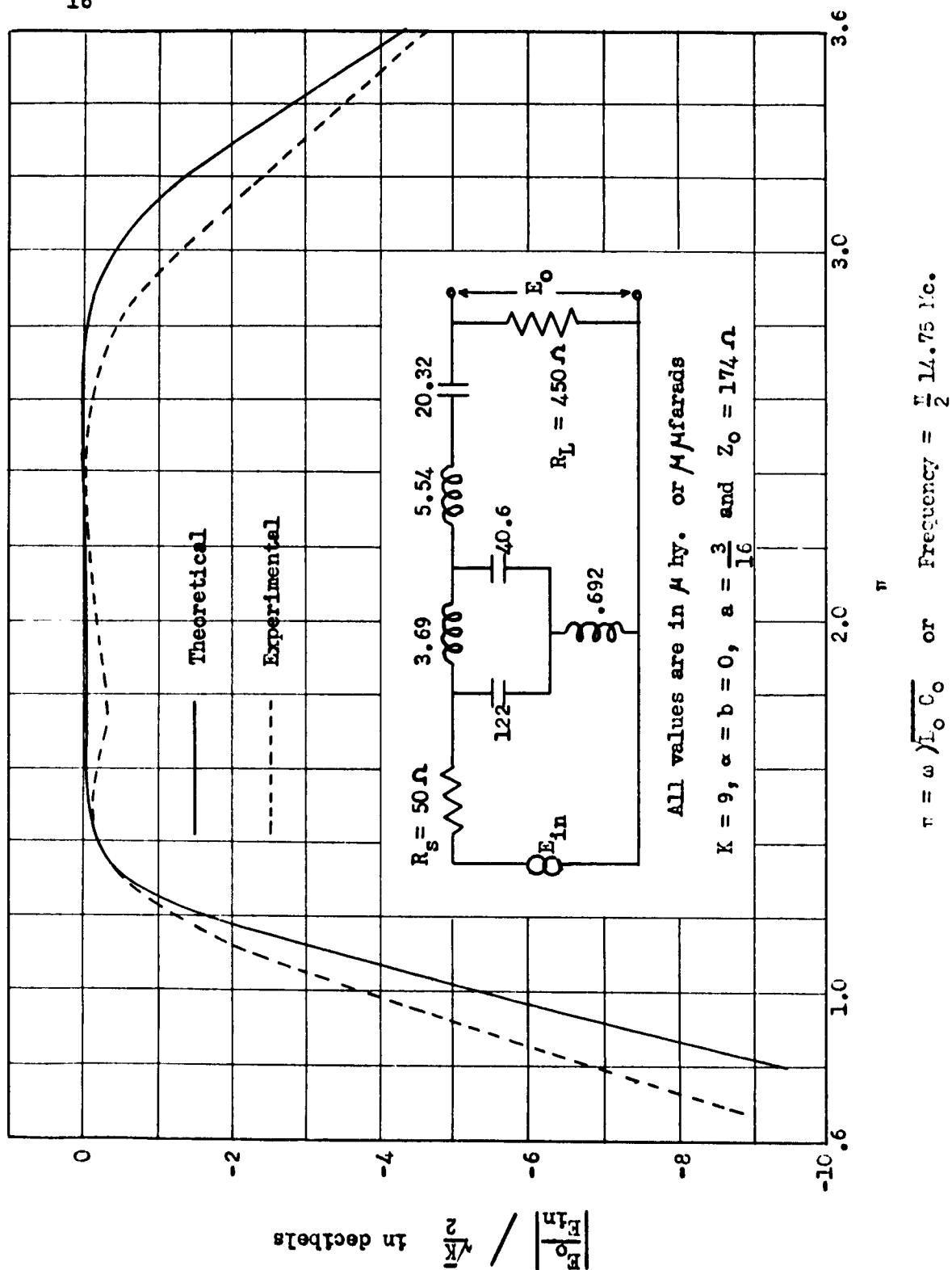


$$\pi = \omega \sqrt{L_o C_o} \quad \text{or} \quad \text{Frequency} = \frac{\pi}{2} 14.75 \text{ Mc.}$$

Figure 30

Experimental Results With Three Mesh Network Having $\alpha = b = 0$, and

$$a = \frac{3}{16}$$



Section VIII

DISCUSSION AND SUGGESTIONS FOR FURTHER STUDY

The specific bandwidth ratio versus impedance transformation ratio characteristics ascribable to the various circuits are discussed at the points in the thesis where they are derived. It may be said in general though that the approach used throughout the thesis, namely the design of the overall network, has enabled the design of circuits that are quite a bit more flexible than hitherto available, and, in some instances, permit the realization of larger bandwidth ratios using the same number of elements.

There are also certain aspects of the networks considered which have not yet been mentioned but which are points of some interest. The first of these considerations is in the form of an inherent limitation to the application of lumped element, impedance transforming networks. It will be remembered that from applying the condition $\frac{\Delta_{11}}{\Delta_{nn}} = K$ we obtained the requirement that the total impedance in the last mesh be K times as large as the total impedance in the first mesh; ie., $Z_n = K Z_1$. This can be interpreted to mean that the capacitors used at the receiving end will be on the order of K times as small as those at the sending end and the inductors at the receiving end will be K times as large as those at the sending end. It seems reasonable to believe that this same situation is inherent in any impedance transforming network because of its very nature. Frequently this factor governs how large of an impedance transformation ratio can be made while still using components of reasonable size. For example, at an early point in the development calculations were carried out for a circuit which would act as a voltage transformer,

giving a voltage gain of twenty, for frequencies in the first six channels of the television band. Under the assumption of a source resistance of 50Ω the coils and condensers at the receiving end had quite practical sizes. The coil and condenser making up the last series impedance, however, were on the order of one milli-henry and ten milli-micro-micro farads respectively - which are completely impractical. Thus, when large values of K are contemplated one must be concerned with practicability as well as realizability of elements. This limitation is independent of the number of meshes employed.

Another aspect of these networks that may be of interest is what might be called an inversion property. In Section II and all succeeding work the elements $j_{1,1}$ and $j_{1,2}$ were taken to be an inductance L_0 and a capacitance C_0 respectively. If $j_{1,1}$ and $j_{1,2}$ are chosen instead to be a capacitance C_0 and an inductance L_0 respectively, the variable $\pi = \frac{\omega}{\omega_0}$ is replaced by $\frac{1}{\pi}$ everywhere it occurs. This has the effect of inverting all of the frequency characteristics with respect to $\pi = \frac{\omega}{\omega_0} = 1$. The new lower and upper cut-off frequencies, f_1'' and f_2'' , are related to the original lower and upper cutoff frequencies, f_1 and f_2 , as follows:

$$f_1'' = \frac{L_0 C_0}{4\pi f_2} \quad \text{and} \quad f_2'' = \frac{L_0 C_0}{4\pi f_1}$$

Physically, this change in $j_{1,1}$ and $j_{1,2}$ amounts to changing all of the original capacitors to inductors and all of the original inductors to capacitors. Of course, the parameters a , α , b etc. are unchanged as is the bandwidth ratio. The practical utility of this feature is that inversion of the original voltage transfer function about $\pi = 1$ may lead to a more desirable response, and, depending on the original values of π_1 and π_2 , converting the network in the described fashion may lead to more convenient values of L_0 and C_0 . It is to be realized that this conversion is not equivalent to finding the dual

of the original network which is also possible.

There are two areas of work not touched upon in this thesis that would seem to be interesting subjects for further study. The first of these concerns the synthesis of impedance transforming networks using nodal analysis. It was briefly mentioned in Section I that the determinantal condition for image admittance transformation is equivalent to that for image impedance transformation. The advantage of using the nodal analysis lies in the fact that for networks of four nodes or more there is no difficulty in constructing a planer network such that all of the elements on or above the main diagonal in the admittance determinant are non-zero and independent of each other. Thus, more degrees of freedom are available than in the case of a similar order mesh system. If these additional degrees of freedom are properly exploited it would seem quite possible that superior performance could be achieved. Of course, comparison of two networks should be on the basis of equal number of elements in each network and not on the basis that both networks have the same number of nodes or meshes.

The second suggestion for future study involves multiple pass band operation of impedance transformers. In all of the preceding work, steps were taken to insure that single band pass operation was achieved. It was for this purpose that Δ^* was made a factor of Δ . If Δ^* is not a factor of Δ or if the polynomial in π describing $Z_{1,1}$ is made of higher order by introducing more complicated elements, $Z_{1,1}$ will have more than one band of frequencies where it is real and there will therefore be more than a single pass band. One such network was designed and it had three pass bands of nearly equal widths separated by stop bands. Though the actual voltage transfer function was not obtained for this circuit

and therefore the sharpness of the response is not known, it would seem that this type of circuit might be of some use in situations requiring impedance transformation over separate bands of frequencies.

Appendix I

DERIVATION OF REDUCED EXPRESSION FOR $[(1+b)^2 - 2\phi - \frac{b^2}{K}]$

Use will be made of equations (3-7) and (3-8) on page 29 which apply to the general case and are given below.

$$(A) \quad 1+b+\frac{g}{g_1} a = \phi(1+g), \quad (B) \quad \frac{g}{g_1} [(1+g)a-b] + \frac{1}{2} = \phi \frac{Kg^2}{g_1^2}$$

Also, we have the relation $\frac{g_1^2}{K}(1+g) = (g+g_1)$ or $g = g_1 \frac{K-g_1}{g_1^2-K}$

Substituting into (B) the value of $\frac{g^2}{K}$ and using (A)

$$\frac{g}{g_1} [(1+g)a-b] + \frac{1}{2} = \frac{\phi g^2(1+g)}{(g+g_1)} = \frac{g^2}{(g+g_1)} [1+b+\frac{g}{g_1} a] \text{ or}$$

$$\frac{g}{g_1} a [1+g - \frac{g^2}{(g+g_1)}] = [\frac{g^2}{(g+g_1)} + \frac{g}{g_1}] b + \frac{g^2}{(g+g_1)} - \frac{1}{2}$$

Solving for a yields $a = b + \frac{g_1 [2g^2 - (g+g_1)]}{2g[g+g_1+gg_1]}$

Using the equation for g, $\frac{g_1}{g} = \frac{g_1^2-K}{K-g_1}$ and $g+g_1+gg_1 = \frac{g_1^2(K-1)}{g_1^2-K}$. Also,

$$2g^2 - (g+g_1) = \frac{2g_1^2(K-g_1)^2}{(g_1^2-K)^2} - \frac{g_1^2(g_1-1)}{(g_1^2-K)} = \frac{g_1^2}{(g_1^2-K)^2} [2(K-g_1)^2 - (g_1-1)(g_1^2-K)]$$

With these relationships the expression for a reduces to

$$a = b + \frac{[2(K-g_1)^2 - (g_1-1)(g_1^2-K)]}{2(K-g_1)(K-1)} \quad (C)$$

From (A) $\phi = \frac{1+b}{1+g} + \frac{g}{g_1(1+g)} [b + \frac{[2(K-g_1)^2 - (g_1-1)(g_1^2-K)]}{2(K-g_1)(K-1)}]$

$$\phi = b \frac{(g+g_1)}{g_1(1+g)} + \frac{1}{1+g} [1 + \frac{g[2(K-g_1)^2 - (g_1-1)(g_1^2-K)]}{g_1 2(K-g_1)(K-1)}]$$

Substituting $\frac{g+g_1}{g_1(1+g)} = \frac{g_1}{K}$, $1+g = 1 + \frac{g_1(K-g_1)}{(g_1^2-K)} = \frac{K(g_1-1)}{(g_1^2-K)}$, $\frac{g}{g_1} = \frac{K-g_1}{(g_1^2-K)}$,

and $2(K-g_1)^2 - (g_1-1)(g_1^2-K) = -[g_1^3 - 3g_1^2 + 3Kg_1 + K - 2K^2]$ the expression for ϕ' becomes

$$\phi' = \frac{g_1}{K} b + \frac{g_1^2-K}{K(g_1-1)} \left[1 - \frac{(K-g_1)[g_1^3 - 3g_1^2 + 3Kg_1 + K - 2K^2]}{2(g_1^2-K)(K-g_1)(K-1)} \right]$$

or

$$\phi' = \frac{g_1}{K} b + \frac{[2(g_1^2-K)(K-1) - (g_1^3 - 3g_1^2 + 3Kg_1 + K - 2K^2)]}{2K(K-1)(g_1-1)}.$$

The term in the brackets when expanded is

$$2Kg_1^2 - 2K^2 - 2g_1^2 + 2K - g_1^3 + 3g_1^2 - 3Kg_1 - K + 2K^2$$

or

$$2Kg_1^2 + g_1^2 + K - g_1^3 - 3Kg_1 = (g_1-1)[K(2g_1-1) - g_1^2].$$

Hence,

$$\phi' = \frac{g_1}{K} b + \frac{[K(2g_1-1) - g_1^2]}{2K(K-1)} \quad (D)$$

Consider now the quantity $[(1+b)^2 - 2\phi' - \frac{b^2}{K}]$

$$\begin{aligned} [(1+b)^2 - 2\phi' - \frac{b^2}{K}] &= 1 + 2b + b^2 \left(\frac{K-1}{K} \right) - 2 \left[\frac{g_1}{K} b + \frac{[K(2g_1-1) - g_1^2]}{2K(K-1)} \right] \\ &= b^2 \frac{K-1}{K} + 2b \frac{K-g_1}{K} + \left[1 - \frac{[K(2g_1-1) - g_1^2]}{K(K-1)} \right] \\ &= b^2 \frac{K-1}{K} + 2b \frac{K-g_1}{K} + \frac{(K-g_1)^2}{K(K-1)} \end{aligned}$$

Thus, $[(1+b)^2 - 2\phi' - \frac{b^2}{K}] = \frac{K-1}{K} [b + \frac{K-g_1}{K-1}]^2$ which is the desired result.

In addition to this result, use is made of equations (C) and (D). (C) is used to give the expression for b when a is zero, and (D) is used, in a slightly different form, to show that $\phi' \geq 0$ for all admissible g_1 and K when $a = 0$.

Appendix II

DERIVATION OF ϕ , a , α , AND $(\phi^2 - \frac{a^2}{K})$ AS FUNCTIONS OF K AND g_1 WHEN $b = 0$

The general equations (3-7) and (3-8) on page 29 reduce when $b = 0$ to

$$(E) \quad 1 + \frac{K}{g_1} a = \phi (1+g) \quad \text{and} \quad (F) \quad \frac{K}{g_1} (1+g)a + \frac{1}{2} = \phi \frac{Kg_1^2}{g_1^2}$$

Also, of course, $g = g_1 \frac{K-g_1}{g_1^2-K}$.

Substituting $a = \frac{g_1}{g} [\phi(1+g)-1]$ into equation (F) yields

$$\frac{K}{g_1} (1+g) \frac{g_1}{g} [\phi(1+g)-1] + \frac{1}{2} = \phi \frac{Kg_1^2}{g_1^2}$$

Thus, $\phi(1+g)^2 - (1+g) + \frac{1}{2} = \phi \frac{g^2(1+g)}{(g+g_1)}$ or $\phi(1+g) \left[\frac{g+g_1+gg_1}{g+g_1} \right] = (1+g) - \frac{1}{2}$

Now $(1+g) = \frac{K(g_1-1)}{(g_1^2-K)}$, $g+g_1+gg_1 = \frac{g_1^2(K-1)}{(g_1^2-K)}$, and $g+g_1 = \frac{g_1^2(g_1-1)}{(g_1^2-K)}$.

Hence, $\phi \frac{K(g_1-1)}{(g_1^2-K)} \frac{g_1^2(K-1)}{(g_1^2-K)} \frac{(g_1^2-K)}{g_1^2(g_1-1)} = \frac{[2Kg_1-2K-(g_1^2-K)]}{2(g_1^2-K)}$ or

$$\phi = \frac{[2Kg_1-g_1^2-K]}{2K(K-1)} \quad (G)$$

To find a , $a = \frac{g_1}{g} [\phi(1+g)-1] = \frac{g_1^2-K}{K-g_1} \left[\frac{(2Kg_1-g_1^2-K)}{2K(K-1)} \frac{K(g_1-1)}{(g_1^2-K)} - \frac{1}{2} \right]$

$$a = \frac{[(2Kg_1-g_1^2-K)(g_1-1)-2(K-1)(g_1^2-K)]}{2(K-1)(K-g_1)}$$

Expanding and reducing the term in brackets

$$a = \frac{[-g_1^3 + 3g_1^2 - 3Kg_1 - K + 2K^2]}{2(K-1)(K-g_1)} \quad (H)$$

Using (G) and (H)

$$\alpha = \phi - a = \frac{[(K-g_1)(2Kg_1-g_1^2-K) - K(-g_1^3+3g_1^2-3Kg_1-K+2K^2)]}{2K(K-1)(K-g_1)}$$

$$\text{or} \quad \alpha = \frac{[g_1^3(1+K) - 6Kg_1^2 + (5K^2+K)g_1 - 2K^3]}{2K(K-1)(K-g_1)} \quad (I)$$

Using expressions for ϕ and a

$$\begin{aligned} \phi^2 - \frac{a^2}{K} &= \frac{(2Kg_1-g_1^2-K)^2}{4K^2(K-1)^2} - \frac{[(2Kg_1-g_1^2-K)(g_1-1) - 2(K-1)(g_1^2-K)]^2}{4K(K-1)^2(K-g_1)^2} \\ &= \frac{(2Kg_1-g_1^2-K)^2[(K-g_1)^2-K(g_1-1)^2] + 4K(2Kg_1-g_1^2-K)(g_1-1)(g_1^2-K)(K-1) - 4K(K-1)^2(g_1^2-K)^2}{4K^2(K-1)^2(K-g_1)^2} \\ &= \frac{(2Kg_1-g_1^2-K)^2(g_1^2-K)(1-K) + 4K(2Kg_1-g_1^2-K)(g_1-1)(g_1^2-K)(K-1) - 4K(K-1)^2(g_1^2-K)^2}{4K^2(K-1)^2(K-g_1)^2} \end{aligned}$$

Factoring out of the above expression the term $(g_1^2-K)(K-1)$ leaves

$$\phi^2 - \frac{a^2}{K} = \frac{(g_1^2-K) [- (2Kg_1-g_1^2-K)^2 + 4K(2Kg_1-g_1^2-K)(g_1-1) - 4K(K-1)(g_1^2-K)]}{4K^2(K-g_1)^2(K-1)}$$

By direct expansion and collection of terms the quantity in the square brackets becomes equal to $[-g_1^4 - K^2 - 8K^2g_1 + 6Kg_1^2 + 4K^3]$

$$\text{Thus,} \quad \phi^2 - \frac{a^2}{K} = \frac{(g_1^2-K)[4K(K-g_1)^2 - (g_1^2-K)^2]}{4K^2(K-1)(K-g_1)^2} \quad (J)$$

Appendix III

DERIVATION OF a AND b AS FUNCTIONS OF K AND g_1 WHEN $\alpha = 0$ When $\alpha = 0$, $a = g$ and the general expressions (3-7) and (3-8) reduce to

$$1+b+\frac{g}{g_1} a = a(1+g) \quad \text{and} \quad \frac{g}{g_1} [(1+g)a-b] + \frac{1}{2} = \frac{Kg^2}{g_1} a$$

These can be written respectively as

$$(K) \quad \frac{a[g_1(1+g)-g]}{g_1} = 1+b \quad \text{and} \quad \frac{g}{g_1} [g_1(1+g)-Kg] = \frac{g}{g_1} b - \frac{1}{2} \quad (L)$$

$$\text{Hence,} \quad \frac{[g_1(1+g)-Kg]}{[g_1(1+g)-g]} (1+b) = b - \frac{g_1}{2g} \quad \text{and}$$

$$b [g_1(1+g)-Kg] - [g_1(1+g)-g] = -\frac{g_1}{2g} [g_1(1+g)-g] - [g_1(1+g)-Kg]$$

$$\text{or} \quad -b(K-1)g = -\frac{g_1(1+g)(g_1+2g)}{2g} + \frac{g_1+2gK}{2}$$

$$\text{From this,} \quad b = \frac{[g_1^2 + 2gg_1 + g^2g_1 + 2g^2g_1 - gg_1 - 2Kg^2]}{2g^2(K-1)}$$

Substituting for g the value $g = g_1 \frac{K-g_1}{g_1^2-K}$ makes the equation for b reduce to

$$b = \frac{[g_1(K-1)(g_1^2-K) - 2(K-g_1)^3]}{2(K-1)(K-g_1)^2} \quad (M)$$

To determine a , the expression for b as given by (K) is substituted into equation (L). Then,

$$\frac{g}{g_1} [(1+g)-\frac{Kg}{g_1}] a = \frac{g}{g_1} [(1+g-\frac{g}{g_1})a - 1] - \frac{1}{2}$$

$$\text{or} \quad \frac{g^2}{g_1} a(1-K) = -\frac{g}{g_1} - \frac{1}{2} \quad \text{Thus,} \quad a = \frac{g_1^2(\frac{g}{g_1} + \frac{1}{2})}{g^2(K-1)}; \quad \text{and since } \frac{g}{g_1} = \frac{K-g_1}{g_1^2-K},$$

$$a = \frac{(g_1^2 - K)^2 \left(\frac{K - g_1}{g_1^2 - K} + \frac{1}{2} \right)}{(K - g_1)^2 (K - 1)}.$$

Finally,
$$a = \frac{(g_1^2 - K) [2(K - g_1) + (g_1^2 - K)]}{2 (K - g_1)^2 (K - 1)} \quad (N)$$

Also of use in Section III is the quantity $\frac{b + \frac{K - g_1}{K - 1}}{b + \frac{K}{K - 1}}$ when $\alpha = 0$.

From (M),
$$b + \frac{K - g_1}{K - 1} = \frac{1}{K - 1} \left[\frac{g_1(K - 1)(g_1^2 - K) - 2(K - g_1)^3}{2 (K - g_1)^2} + (K - g_1) \right]$$

$$b + \frac{K - g_1}{K - 1} = \frac{1}{K - 1} \left[\frac{g_1(K - 1)(g_1^2 - K)}{2 (K - g_1)^2} \right] = \frac{g_1(g_1^2 - K)}{2 (K - g_1)^2}$$

and
$$b + \frac{K}{K - 1} = \frac{1}{K - 1} \left[\frac{g_1(K - 1)(g_1^2 - K) - 2(K - g_1)^3}{2 (K - g_1)^2} + K \right]$$

$$= \frac{[g_1(K - 1)(g_1^2 - K) + 2g_1(K - g_1)^2]}{2 (K - 1) (K - g_1)^2}$$

Thus,
$$\frac{b + \frac{K - g_1}{K - 1}}{b + \frac{K}{K - 1}} = \frac{g_1(g_1^2 - K) (K - 1)}{g_1(g_1^2 - K) (K - 1) + 2g_1(K - g_1)^2} = \frac{1}{1 + \frac{2 (K - g_1)^2}{(K - 1)(g_1^2 - K)}} \quad (O)$$

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