ON THE DESIGN AND COMPARISON OF CERTAIN DICHOTOMOUS EXPERIMENTS

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TECHNICAL REPORT NO. 24

MAY 10, 1954

PREPARED UNDER CONTRACT N6onr 25140
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH

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1. Introduction and Summary.

It may frequently happen that a researcher, wishing to decide which one of a set of alternatives to accept, finds that there are several experiments available to him which he might perform to guide him in reaching his decision. Thus, he is faced with making a preliminary decision as to which experiment or experiments he is to perform. If he admits the possibility of performing more than one experiment, then the questions of how many, which ones, and in what order, arise. It is such questions that come under the heading of comparison and design of experiments.

In its most general formulation, a sample space, $\mathcal{Z}$, is an ordered quadruple, $(Z, \mathcal{B}, \Omega, P)$, where Z is an arbitrary set, $\mathcal{B}$ is a Borel field of subsets of Z, Ω is an arbitrary set, and P is a function defined on $\mathcal{B} \times \Omega$ with the property that for each $\omega \varepsilon \Omega$, P_ω, the restriction of P to $\mathcal{B} \times (\omega)$, is a probability measure on $\mathcal{B}$. In this setting an experiment is a sub-Borel field of $\mathcal{B}$. If $\mathcal{C}$ is a Borel field of subsets of a set W and T is a $\mathcal{C}-\mathcal{B}$ measurable function from Z to W, then T is a random variable and

$$\mathcal{B}_T = \left\{ B \varepsilon \mathcal{B} : \text{ for some } E \varepsilon \mathcal{C},\ B = \left\{ z : T(z) \varepsilon E \right\} \right\}$$

is a sub-Borel field of $\mathcal{B}$. $\mathcal{B}_T$ is called the experiment associated with the random variable T. Keeping in mind that many random variables may be associated with the same experiment, and therefore to view an experiment as a sub-Borel field is the more basic approach, no confusion will result in this paper from identifying random variable with experiment.

Since the random variables dealt with in this paper are all real valued, to say that an experiment is available to the researcher is to say that there is a real random variable which he can observe and whose distribution is known for each $\omega \varepsilon \Omega$.

While much of the general theory of the design problem has been developed, e.g., by Wald [1] and Magwire [2], actual solutions of particular problems, especially of the sequential type, have not been obtained. This paper stems from work towards solving the design problem for particular cases. Attention is restricted to dichotomous experiments; i.e., Ω is assumed to contain but two elements which will be called hypotheses and denoted by H_1 and H_2. It is supposed that one is required to decide which hypothesis is true and that a loss of one unit is suffered if the false hypothesis is chosen while no loss occurs if the true one is chosen. Further, ζ will denote the a priori probability that H_1 is true and the criterion to be used in comparing experiments will be the Bayes risks associated with the various experiments.

In Section 2 it is supposed that there are two experiments, i.e., random variables, X and Y available and that but one experiment is allowed. Some conditions for uniform inequalities between the Bayes risk associated with X and that associated with Y are obtained. Certain relations between the Kullbach-Leibler information numbers for X and for Y and their Bayes risks are shown. In particular, it is found that a necessary condition that one random variable have a Bayes risk uniformly less or equal that of the other is that its Kullbach-Leibler information numbers are greater or equal those for the other. The case in which the distributions are normal is discussed in some detail and a few remarks are addressed to the matter of viewing the Kullbach-Leibler information numbers, in certain special cases, as functions of that transformation, t, such that the distribution of t(X) under H_1 is the distribution of X under H_2.

Section 3 is devoted to the problem of designs in the case of binomial distributions. It is supposed that the two experiments available, X and Y, are independent and of equal cost, and that it is given that a total of n experiments is to be performed. Two problems are discussed: What is the best division of the n experiments between X's and Y's if one is to decide this matter before experimentation? What is the best sequential design, i.e., the best rule prescribing, as a function of the results of the preceding experiments, which random variable to observe in the next experiment.

In Section 4, instead of considering the performance of a fixed number of experiments, the experimentation is supposed terminated by a particular sequential stopping rule and one is interested in discovering sequential designs which minimize the expected number of experiments that will be performed.

In the final section, 5, a somewhat different purpose of experimentation is introduced. Again, X and Y are two real random variables with known distributions under the two hypotheses. A total of n experiments is allowed and a sequential design, telling which random variable to observe at each step, which will maximize the sum of the n observations is sought.

The design δ_0 which requires, at each step that play which maximizes the expected value of the next observation is considered in particular. For the case in which X and Y have binomial distributions such that X under H_1 and Y under H_2 have the same distributions and X under H_2 and Y under H_1 have the same distributions, the problem is known as the 'Two-armed Bandit'. Brief outlines of two methods of attack on the question of the optimal design for the Two-armed Bandit are given. It is a conjecture of Blackwell's that δ_0 is the optimal design. By both methods this conjecture was found to hold true for small values of n. Each, however, appears to be too cumbersome in the general case to provide a full proof.

2. Some Relations Between Bayes Risks and the Kullbach-Leibler Information Numbers.

2.1 General Results. Of the two hypotheses, H_1 and H_2, let H_1 be true with *a priori* probability ζ and H_2 be true with *a priori* probability $1-\zeta$. Suppose that it is required to decide which of the hypotheses is true, suffering a loss of one if the false hypothesis is chosen and no loss otherwise. Further, suppose that X and Y are real random variables having distribution functions F_i and G_i, respectively, under hypothesis H_i and with the corresponding densities f_i and g_i with respect to a common measure, Ψ, such that $f_i > 0$ if and only if $g_i > 0$. An observation either of X or of Y is allowed to assist in making the decision as to the true hypothesis.

Of course, if but one observation were allowed and one were interested only in comparing X and Y for one particular value of ζ, the preliminary decision as to whether to observe X or Y, i.e., the design problem, reduces to computing the Bayes risk against ζ when using X, $R_X(\zeta)$, and that when Y is used, $R_Y(\zeta)$, and using the random variable corresponding to the smaller risk. Since one is not interested in such a strongly restricted comparison, this criterion will not yield a simple solution, unless $R_X(\zeta) \leq R_Y(\zeta)$ for all ζ, or $R_X(\zeta) \geq R_Y(\zeta)$ for all ζ, $(0 \leq \zeta \leq 1)$, in which case the choice between X and Y is clear. Furthermore, any criterion for choosing between X and Y should agree with this whenever one risk curve lies uniformly on or below the other.

Considering the statistical games based on X and on Y as S-games ([3]), with S_X and S_Y the respective sets of risk vectors, the condition that $R_X(\zeta) \leq R_Y(\zeta)$ for all ζ is equivalent to $S_X \supset S_Y$, i.e., any risk vector attainable using Y can also be attained by using X. Interest in conditions under which $S_X \supset S_Y$ is further increased in view of results of Blackwell's [4] that if such is the case, then regardless of the number of actions open to the researcher or the loss function used, the set of risk vectors attainable with X contains that attainable with Y.

Let $R_X \leq R_Y$ denote that $R_X(\zeta) \leq R_Y(\zeta)$ for all ζ. Throughout the paper it will be found very convenient to consider $\frac{\zeta}{1-\zeta}$ and this will regularly be denoted by η.

<u>Lemma</u> 2.1. Two conditions, each necessary and sufficient, that $R_X(\zeta) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} R_Y(\zeta)$ are:

(i) $$\int_0^\infty \min(u-\eta,0)\,dE(u) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \int_0^\infty \min(u-\eta,0)\,dF(u) \;;$$

(ii) $$\int_0^\infty \min\left(1-\frac{\eta}{u},0\right)dG(u) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \int_0^\infty \min\left(1-\frac{\eta}{u},0\right)dH(u) \;;$$

where E and G are the c.d.f.'s of $f_2(x)/f_1(x)$ under H_1 and H_2, respectively, and F and H are the c.d.f.'s of $g_2(x)/g_1(x)$ under H_1 and H_2, respectively.

Proof. From the well known theory of Bayes solutions (see [3], Chapter 6), the Bayes risk against ζ using X is given by

$$(1)\quad R_X(\zeta) = \zeta \int_{\frac{f_2(x)}{f_1(x)} > \frac{\zeta}{1-\zeta}} f_1(x)d\Psi(x) + (1-\zeta)\int_{\frac{f_2(x)}{f_1(x)} \le \frac{\zeta}{1-\zeta}} f_2(x)d\Psi(x) \quad .$$

With $\eta = \frac{\zeta}{1-\zeta}$, this can be written as

$$(2)\quad \frac{R_X(\zeta)}{1-\zeta} - \eta = \int_{\frac{f_2(x)}{f_1(x)} \le \eta} f_2(x)d\Psi(x) - \eta \int_{\frac{f_2(x)}{f_1(x)} \le \eta} f_2(x)d\Psi(x) \quad .$$

With $E(u) = \int_{\frac{f_2(x)}{f_1(x)} \le u} f_1(x)d\Psi(x)$,

$$(3)\quad \frac{R_X(\zeta)}{1-\zeta} - \eta = \int_0^\eta u\,dE(u) - \eta\int_0^\eta dE(u)$$

$$= \int_0^\infty \min(u-\eta, 0)dE(u) \quad .$$

With $F(u) = \int_{\frac{f_2(x)}{f_1(x)} \le u} f_2(x)d\Psi(x)$,

$$(4)\quad \frac{R_X(\zeta)}{1-\zeta} - \eta = \int_0^\eta dF(u) - \eta\int_0^\eta \frac{1}{u}\, dF(u)$$

$$= \int_0^\infty \min\left(1-\frac{\eta}{u}, 0\right)dF(u) \quad .$$

With the analogous expressions for the risk associated with Y, the conclusion is immediate.

<u>Lemma</u> 2.2. (i) $R_X(\zeta) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} R_Y(\zeta)$ if and only if

$$\int_0^{\eta} \alpha_X\left(\frac{u}{1+u}\right)du \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \int_0^{\eta} \alpha_Y\left(\frac{u}{1+u}\right)du \quad ,$$

where $\alpha_X(\zeta)$ is the probability, under H_1, that in following the Bayes procedure against ζ with X, H_2 will be chosen.

(ii) If $G(u)/u \to 0$ as $u \to 0$, then $R_X(\zeta) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} R_Y(\zeta)$ if and only if

$$\int_0^{\eta} \beta_X\left(\frac{u}{1+u}\right)du \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} \int_0^{\eta} \beta_Y\left(\frac{u}{1+u}\right)du \quad ,$$

where $\beta_X(\zeta)$ is the probability under H_2 that in following the Bayes procedure against ζ with Y, H_1 will be chosen.

<u>Proof</u>. From Equation (3) in the proof of Lemma 2.1,

$$\text{(1)} \qquad \frac{R_X(\zeta)}{1-\zeta} - \eta = \int_0^{\eta} (u-\eta)\,dE(u) \quad .$$

Integrating (1) by parts yields

$$\text{(2)} \qquad \frac{R_X(\zeta)}{1-\zeta} - \eta = -\int_0^{\eta} E(u)\,du \quad .$$

However, $E(u) = \displaystyle\int_{\frac{f_2(x)}{f_1(x)} \leq u} f_1(x)\,d\psi(x) = 1 - \alpha_X\left(\frac{u}{1+u}\right)$. Hence,

$$\text{(3)} \qquad \frac{R_X(\zeta)}{1-\zeta} - \eta = \int_0^{\eta} \alpha_X\left(\frac{u}{1+u}\right)du \quad .$$

From the similar expression involving R_Y, conclusion (i) follows.

A parallel argument from equation (4) in the proof of Lemma 2.1 yields conclusion (ii), noting that $G(u) = \beta_X(\frac{u}{1+u})$.

<u>Theorem</u> 2.1. Two conditions, each necessary and sufficient that $R_X = R_Y$ are:

(i) f_2/f_1 and g_2/g_1 have the same distributions under H_1;

(ii) f_2/f_1 and g_2/g_1 have the same distributions under H_2.

<u>Proof</u>. The sufficiency is immediate from Lemma 2.1. To show the necessity, suppose $R_X = R_Y$, then for all $\eta \geq 0$,

$$(1) \qquad \int_0^\infty \min(u-\eta, 0)dE(u) = \int_0^\infty \min(u-\eta, 0)dF(u) \quad .$$

Now, for any $a > 0$, let $\phi_n(u) = u \min(u-a, 0)$ and let $\gamma_n(u) = -n \min(u-(a+\frac{1}{n}), 0)$, $n = 1, 2, 3, \ldots$. Then

$$(2) \qquad \int_0^\infty (\phi_n(u) + \gamma_n(u))dE(u) = \int_0^\infty (\phi_n(u) + \gamma_n(u))dF(u)$$

for all n. Hence,

$$(3) \quad E(a) + \int_{a < u \leq a+\frac{1}{n}} (1-n(u-a))dE(u) = F(a) + \int_{a < u \leq a+\frac{1}{n}} (1-n(u-a)dF(u) \quad .$$

Letting $n \to \infty$, $E(a) = F(a)$, i.e., the likelihood ratios, f_2/f_1 and g_2/g_1, have the same distribution under H_1. It follows immediately that $\alpha_X(\zeta) = \alpha_Y(\zeta)$, since $E(u) = 1-\alpha_X(\frac{u}{1+u})$. Now $R_X(\zeta) = \zeta\alpha_X(\zeta) + (1-\zeta)\beta_X(\zeta)$; hence, $R_X = R_Y$ and $\alpha_X = \alpha_Y$ implies $\beta_X = \beta_Y$, which is conclusion (ii).

With these conditions that $R_X \leq R_Y$, attention is turned to the relation between the condition $R_X \leq R_Y$ and the Kullback-Leibler information numbers.

The mean information per observation of X for discriminating between H_1 and H_2 when H_i is true is defined by Kullbach and Leibler, [5], [6], to be

$$I_X(1:2) = \int_{-\infty}^{\infty} f_1(x)\log\frac{f_1(x)}{f_2(x)}\,d\Psi(x) \quad , \quad \text{for } i=1, \tag{2.1.1}$$

and

$$I_X(2:1) = \int_{-\infty}^{\infty} f_2(x)\log\frac{f_2(x)}{f_1(x)}\,d\Psi(x) \quad , \quad \text{for } i=2.$$

The mean divergence between H_1 and H_2 per observation of X they then define to be

$$J_X = I_X(1:2) + I_X(2:1) \quad . \tag{2.1.2}$$

$I_X(1:2)$ and $I_X(2:1)$ will be referred to as the K-L numbers for X. The K-L numbers and the divergence for Y are similarly defined.

It is noted in passing that if the distribution of X is of the exponential type, i.e., $f_i(x) = \beta(\omega_i)e^{\omega_i x}$, then

$$I_X(1:2) = \log\frac{\beta(\omega_1)}{\beta(\omega_2)} + (\omega_1-\omega_2)E_{\omega_1}[X] \quad ; \tag{2.1.3}$$

$$I_X(2:1) = \log\frac{\beta(\omega_1)}{\beta(\omega_2)} + (\omega_2-\omega_1)E_{\omega_2}[X] \quad ; \text{ and}$$

$$J_X = (\omega_1-\omega_2)(E_{\omega_1}[X]-E_{\omega_2}[X]) \quad .$$

Thus, J_X is an interesting measure of the 'distance' between H_1 and H_2 relative to the random variable X, being the product of two often considered measures.

If $I_X(1:2) > I_Y(1:2)$ and $I_X(2:1) > I_Y(2:1)$, one would say that, in the Kullbach-Leibler sense, X is the more informative. The question that arises is that of the relation between being more informative in the Kullbach-Leibler sense and being more informative in the sense of uniformly smaller Bayes risks. It will be seen in the remainder of this section that the two are not equivalent, but that interesting relations do exist.

<u>Theorem</u> 2.2. $R_X = R_Y$ implies equality of the corresponding K-L numbers for X and for Y.

<u>Proof</u>. With E and F as defined in Lemma 2.1,

$$(1)\qquad I_X(1:2) = \int_{-\infty}^{\infty} f_1(x) \log \frac{f_1(x)}{f_2(x)}\, d\Psi(x) = -\int_0^{\infty} \log u\, dE(u) \quad , \text{ and}$$

$$I_Y(1:2) = \int_{-\infty}^{\infty} g_2(x) \log \frac{g_2(x)}{g_1(x)}\, d\Psi(x) = -\int_0^{\infty} \log u\, dF(u) \quad .$$

By Theorem 2.1, $E = F$ and hence $I_X(1:2) = I_Y(1:2)$.

In the same way,

$$(2)\qquad I_X(2:1) = \int_{-\infty}^{\infty} f_2(x) \log \frac{f_2(x)}{f_1(x)}\, d\Psi(x) = \int_0^{\infty} u \log u\, dE(u) \quad , \text{ and}$$

$$I_Y(2:1) = \int_{-\infty}^{\infty} g_2(x) \log \frac{g_2(x)}{g_1(x)}\, d\Psi(x) = \int_0^{\infty} u \log u\, dF(u) \quad .$$

Hence, $I_X(2:1) = I_Y(2:1)$ also.

<u>Theorem</u> 2.3. If $R_X \leq R_Y$, then the K-L numbers for X are greater or equal to the corresponding K-L numbers for Y.

<u>Proof</u>. Again with E and F as defined in Lemma 2.1,

$$(1)\qquad \int_0^{\infty} u\,dE(u) = \lim_{\eta\to\infty} \int_0^{\eta} u\,dE(u) = \lim_{\eta\to\infty} \int_{\frac{f_2(x)}{f_1(x)} \leq \eta} f_2(x)\,d\Psi(x) = 1 \quad .$$

Similarly, $\int_0^{\infty} u\,dF(u) = 1$. Hence, for ϕ any linear function,

$$(2)\qquad \int_0^{\infty} \phi(u)\,dE(u) = \int_0^{\infty} \phi(u)\,dF(u) \quad .$$

By Lemma 2.1,

$$(3)\qquad \int_0^{\infty} \min(u-\eta, 0)\,dE(u) \leq \int_0^{\infty} \min(u-\eta, 0)\,dF(u) \quad .$$

It is easily seen, then, that for any concave function, ϕ,

$$(4) \qquad \int_0^\infty \phi(u)dE(u) \leq \int_0^\infty \phi(u)dF(u) \quad .$$

In particular, for $\phi(u) = \log u$,

$$(5) \qquad I_X(1:2) = -\int_0^\infty \log u dE(u) \geq -\int_0^\infty \log u dF(u) = I_Y(1:2) \quad ;$$

while for $\phi(u) = -u \log u$,

$$(6) \qquad -I_X(2:1) = -\int_0^\infty u \log u dE(u) \leq -\int_0^\infty u \log u dF(u) = -I_Y(2:1) \quad .$$

Equations (5) and (6) yield the conclusion of the theorem.

In the matter of converses to Theorems 2.2 and 2.3, no general theorems were obtained. In each special case investigated, equality of the corresponding K-L numbers was found to be equivalent to equality of the risks, but a uniform inequality of the K-L numbers failed to imply a uniform inequality between the risks.

2.2 The Case of Normal Distributions. Attention is now turned to the particular case in which both X and Y have normal distributions under each hypothesis. Since, for normal distributions, both the risk function and the K-L numbers are invariant under affine transformations, there is no loss of generality in treating the situation given by the following diagram:

	X	Y
H_1	$N(0,1)$	$N(0,1)$
H_2	$N(\mu, \sigma^2)$	$N(m,v)$

where $\mu \geq 0$, $m \geq 0$, and $\sigma^2 \geq v$.

The K-L numbers for X are:

(2.2.1) $$I_X(1:2) = \tfrac{1}{2}[\log \sigma^2 - 1 + \frac{1}{\sigma^2} + \frac{\mu^2}{\sigma^2}] \quad , \text{ and}$$

$$I_X(2:1) = \tfrac{1}{2}[\log \frac{1}{\sigma^2} - 1 + \sigma^2 + \mu^2] \quad .$$

Those for Y are, of course, the same with the obvious substitutions.

<u>Theorem</u> 2.4. The following three statements are equivalent.

(i) $R_X = R_Y$.

(ii) $I_X(1:2) = I_Y(1:2)$ and $I_X(2:1) = I_Y(2:1)$.

(iii) $\sigma^2 = v$ and $\mu = m$.

<u>Proof</u>. By Theorem 2.2, (i) implies (ii). Further, (iii) clearly implies (i). Hence, it is necessary only to show that (ii) implies (iii).

Assuming (ii) to be true, then

(1) $$\log \frac{\sigma^2}{v} + \frac{1}{\sigma^2} - \frac{1}{v} = \frac{m^2}{v} - \frac{\mu^2}{\sigma^2} \quad ,$$

and

(2) $$\log \frac{v}{\sigma^2} + \sigma^2 - v = m^2 - \mu^2 \quad .$$

Suppose (iii) not true, in particular, that $\sigma^2 > v$.

<u>Case</u> I: $\sigma^2 > 1$. Multiplying equation (2) by $-\frac{1}{v}$ and adding to equation (1) it is found that μ^2 is of the same sign as

(3) $$A(\sigma^2, v) = (v+1) \log \frac{\sigma^2}{v} + \frac{v}{\sigma^2} - 1 - \sigma^2 + v \quad .$$

But $A(v,v) = 0$ and $\frac{\partial}{\partial \sigma^2} A(\sigma^2, v) = \frac{1}{\sigma^2}(\sigma^2 - v)(1 - \sigma^2) < 0$. Hence $\mu^2 < 0$, a clear absurdity.

<u>Case</u> II: $\sigma^2 \leq 1$. Multiplying equation (2) by $-\frac{1}{\sigma^2}$ and adding to equation (1) it is seen that m^2 is of the same sign as

(4) $$B(\sigma^2, v) = (\sigma^2 + 1) \log \frac{\sigma^2}{v} + 1 - \frac{\sigma^2}{v} - \sigma^2 + v \quad .$$

But $B(v,v) = 0$ and $\frac{\partial}{\partial\sigma^2} B(\sigma^2,v) = (\log\frac{1}{v} - \frac{1}{v}) - (\log\frac{1}{\sigma^2} - \frac{1}{\sigma^2})$, which is negative, since $\log x - x$ is strictly decreasing for $x \geq 1$. Hence, a similar contradiction is reached: $m^2 \leq 0$.

In either case, it must be concluded that if (ii) is true, then $\sigma^2 = v$ and, as an immediate consequence, $m = \mu$.

It is noted in passing that the same line of argument yields the

Corollary: For $v < \sigma^2$, $I_X(1:2) \geq I_Y(1:2)$ implies that $I_X(2:1) > I_Y(2:1)$, while for $v > \sigma^2$, $I_X(2:1) \geq I_Y(2:1)$ implies that $I_X(1:2) > I_Y(1:2)$.

For a further analysis of the case of normal distributions, assume σ^2 and μ fixed, $\sigma^2 > 1$, and consider the (v,m^2) plane. One can immediately determine the region in which $I_X(1:2) \geq I_Y(1:2)$ and that in which $I_X(2:1) \geq I_Y(2:1)$. From equations

(2.2.1) $$I_X(1:2) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} I_Y(1:2)$$

if and only if

(2.2.2) $$m^2 \left\{ \begin{matrix} < \\ = \end{matrix} \right\} h_1(v) = v(\log\sigma^2 + \frac{\mu^2+1}{\sigma^2} - v\log v - 1 \quad .$$

$I_X(2:1) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} I_Y(2:1)$ if and only if

(2.2.3) $$m^2 \left\{ \begin{matrix} < \\ = \end{matrix} \right\} h_2(v) = \mu^2 + \sigma^2 - \log\sigma^2 + \log v - v \quad .$$

That $h_1(v) \leq h_2(v)$ for $v \leq \sigma^2$ with equality only at $v = \sigma^2$ is a consequence of Theorem 2.4 and corollary. (It can be shown similarly that for $v > \sigma^2$, $h_2(v) < h_1(v)$ for all v for which $h_2(v) \geq 0$).

Together with Theorem 2.3, these results yield the result that for $v \leq \sigma^2$,

(2.2.4) $$\left\{(v,m^2) : R_X \leq R_Y\right\} \subset \left\{(v,m^2) : m^2 \leq h_1(v)\right\} \quad .$$

To investigate more fully the relation of these two sets, in particular, to see if, perchance, they are equal, the risk functions must be computed. From this point on, both v and σ^2 will be assumed to be greater than 1. For the particular case under consideration, the probabilities of the two types of errors when using the Bayes procedure against ζ based on X, α_X and β_X, are easily computed.

$$(2.2.5) \qquad \alpha_X(\zeta) = \Pr\left(X^2 - \left(\frac{X-\mu}{\sigma}\right)^2 > 2\log\frac{\zeta\sigma}{1-\zeta} \,\Big|\, H_1\right)$$

$$= 1-\Pr\left(\left|X+\frac{\mu}{\sigma^2-1}\right| < \frac{\sigma}{\sigma^2-1}\sqrt{\mu^2+(\sigma^2-1)\log\eta^2\sigma^2}\,\Big|\, H_1\right),$$

where $\eta = \frac{\zeta}{1-\zeta}$ and it is to be understood that $\alpha_X(\zeta) = 1$ if $\mu^2 + (\sigma^2-1)\log\eta^2\sigma^2 < 0$.

$$(2.2.6) \qquad \beta_X(\zeta) = \Pr\left(X^2 - \left(\frac{X-\mu}{\sigma}\right)^2 < 2\log\frac{\zeta\sigma}{1-\zeta} \,\Big|\, H_2\right).$$

Since the distribution of $\frac{X-\mu}{\sigma}$ under H_2 is the same as that of X under H_1, (2.2.6) can be expressed as

$$(2.2.7) \qquad \beta_X(\zeta) = \Pr\left(\left|X+\frac{\mu}{\sigma^2-1}\right| < \frac{1}{\sigma^2-1}\sqrt{\mu^2+(\sigma^2-1)\log\eta^2\sigma^2}\,\Big|\, H_1\right),$$

where again $\eta = \frac{\zeta}{1-\zeta}$ and it is to be understood that $\beta_X(\zeta) = 0$ whenever $\mu^2 + (\sigma^2-1)\log\eta^2\sigma^2 < 0$.

Since $R_X(\zeta) = \zeta\alpha_X(\zeta)+(1-\zeta)\beta_X(\zeta)$, it is seen that for $\mu^2+(\sigma^2-1)\log\sigma^2\eta^2 \leq 0$, $R_X(\zeta) = \zeta$. The computation of $\alpha_Y(\zeta)$ and $\beta_Y(\zeta)$ will clearly yield the same expressions as (2.2.5) and (2.2.7) with the obvious substitutions of parameters. Thus, for $m^2+(v-1)\log\eta^2 v \leq 0$, $R_Y(\zeta) = \zeta$.

<u>Lemma</u> 2.3. $\frac{d}{d\zeta} R_X(\zeta) < 1$ for $\log \eta^2 > -(\frac{\mu^2}{\sigma^2-1} + \log \sigma^2)$.

<u>Proof</u>. It is first shown that

$$(1) \qquad \frac{d}{d\zeta} R_X(\zeta) = \alpha_X(\zeta) - \beta_X(\zeta) \quad .$$

Setting $A = \sqrt{\mu^2 + (\sigma^2-1)\log \eta^2\sigma^2}$,

$$(2) \quad \frac{d}{d\zeta} \alpha_X(\zeta) = -\frac{\sigma}{\zeta(1-\zeta)A} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(\frac{\mu}{\sigma^2-1})^2 - \frac{1}{2}\frac{\sigma^2 A^2}{(\sigma^2-1)^2}} [e^{\frac{\mu\sigma A}{\sigma^2-1}} + e^{-\frac{\mu\sigma A}{\sigma^2-1}}].$$

$$(3) \quad \frac{d}{d\zeta} \beta_X(\zeta) = \frac{1}{\zeta(1-\zeta)A} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\mu\sigma}{\sigma^2-1})^2 - \frac{1}{2}\frac{A^2}{\sigma^2-1}} [e^{\frac{\mu\sigma A}{\sigma^2-1}} + e^{-\frac{\mu\sigma A}{\sigma^2-1}}] \; .$$

From (2) and (3) one obtains, after some simplifications, that

$$(4) \quad \zeta \frac{d}{d\zeta} \alpha_X(\zeta) + (1-\zeta) \frac{d}{d\zeta} \beta_X(\zeta) =$$

$$\frac{e^{\frac{\mu\sigma A}{\sigma^2-1}} + e^{-\frac{\mu\sigma A}{\sigma^2-1}}}{\zeta(1-\zeta)A\sqrt{2\pi}} \left\{ -\sigma\zeta e^{-\frac{\sigma^2 \log \eta\sigma}{\sigma^2-1}} + (1-\zeta) e^{-\frac{\log \eta\sigma}{\sigma^2-1}} \right\} \; .$$

Since $\eta = \frac{\zeta}{1-\zeta}$, the bracketed quantity can be written as

$$(5) \qquad \frac{1}{1+\eta} \left\{ -e^{-\frac{\sigma^2 \log \eta\sigma}{\sigma^2-1} + \log \sigma\eta} + e^{-\frac{\log \eta\sigma}{\sigma^2-1}} \right\} = 0 \; .$$

Thus

$$\frac{d}{d\zeta} R_X(\zeta) = \zeta \frac{d}{d\zeta} \alpha_X(\zeta) + \alpha_X(\zeta) + (1-\zeta) \frac{d}{d\zeta} \beta_X(\zeta) - \beta_X(\zeta)$$

$$= \alpha_X(\zeta) - \beta_X(\zeta) \quad .$$

Now if $\log \eta^2 > -(\frac{\mu^2}{\sigma^2-1} + \log \sigma^2)$, then $\mu^2 + (\sigma^2-1)\log \eta^2\sigma^2 > 0$ and therefore both $\alpha_X(\zeta) < 1$ and $\beta_X(\zeta) > 0$, which establishes the lemma.

The preceding proof that $\frac{d}{d\zeta} R_X(\zeta) = \alpha_X(\zeta) - \beta_X(\zeta)$ for this special case is included as it illustrates a situation in which certain terms are shown to be zero. Similar situations will arise later and the method here used will be referred to. The fact that the derivative of the risk curve is so related to α and β is, however, a very general result for statistical games with two states of nature, two actions, and a 0 or 1 loss function. The fact is essentially demonstrated by Blackwell and Girshick ([3], Section 6.3) and from their discussion it is clear that a rigorous proof can easily be given the proposition that $\frac{d}{d\zeta} R(\zeta) = \alpha(\zeta) - \beta(\zeta)$ whenever the left member exists (as it does almost everywhere).

Lemma 2.3, and its analogue for Y, show that

$$(2.2.8) \qquad R_X(\zeta) \begin{cases} = \zeta & \text{if } \log \eta^2 \leq -(\frac{\mu^2}{\sigma^2-1} + \log \sigma^2) \ , \\ < \zeta & \text{if } \log \eta^2 > -(\frac{\mu^2}{\sigma^2-1} + \log \sigma^2) \ . \end{cases}$$

And

$$R_Y(\zeta) \begin{cases} = \zeta & \text{if } \log \eta^2 \leq -(\frac{m^2}{v-1} + \log v) \ , \\ < \zeta & \text{if } \log \eta^2 > -(\frac{m^2}{v-1} + \log v) \ . \end{cases}$$

From (2.2.8) it is clear that a necessary condition that $R_X \leq R_Y$ is that

$$\frac{\mu^2}{\sigma^2-1} + \log \sigma^2 \geq \frac{m^2}{v-1} + \log v \ , \text{ or}$$

$$(2.2.9) \qquad m^2 \leq h_3(v) = (v-1)(\frac{\mu^2}{\sigma^2-1} + \log \frac{\sigma^2}{v}) \ .$$

As a consequence of Theorem 2.3, it must be true that $h_3(v) \leq h_1(v)$ for $1 \leq v \leq \sigma^2$. But it is easily verified by differentiation of $h_1 - h_3$ that equality holds only for $v = \sigma^2$. Thus, any pair (v, m^2) with $h_3(v) < m^2 < h_1(v)$ provides an example in which X is more informative in the sense of the Kullbach-Leibler information numbers but $R_X \leq R_Y$ fails to hold.

Thus far the results have been necessary conditions that $R_X \leq R_Y$. The most restrictive of these is that $m^2 \leq h_3(v)$. The principal result of the remainder of this section will be that $m^2 \leq h_3(v)$ is also a sufficient condition that $R_X \leq R_Y$.

Lemma 2.4. For fixed $v > 1$, $R_Y(\zeta)$ is a non-increasing function of m for each ζ.

Proof. From the expressions already derived for α_Y and β_Y, it follows that, with $L = \frac{\sqrt{v}}{v-1}\sqrt{m^2 + (v-1)\log \eta^2 v}$,

$$(1) \quad \sqrt{2\pi}\left(\frac{R_Y(\zeta)}{1-\zeta} - \eta\right) = -\eta \int_{-\frac{m}{v-1}-L}^{-\frac{m}{v-1}+L} e^{-\frac{1}{2}t^2}\,dt + \int_{-\frac{m\sqrt{v}}{v-1}-L}^{-\frac{m\sqrt{v}}{v-1}+L} e^{-\frac{1}{2}t^2}\,dt ,$$

$$= -\eta \int_{-L}^{L} e^{-\frac{1}{2}\left(t-\frac{m}{v-1}\right)^2} dt + \frac{1}{\sqrt{v}} \int_{-L}^{L} e^{-\frac{1}{2v}\left(t-\frac{mv}{v-1}\right)^2} dt .$$

Let $a \overset{s}{\sim} b$ denote that a and b are of the same sign. Then,

$$(2) \quad \frac{\partial}{\partial m} R_Y(\zeta) \overset{s}{\sim} \frac{\partial}{\partial m} \int_{-L}^{L} \left[\frac{1}{\sqrt{v}} e^{-\frac{1}{2v}\left(t-\frac{mv}{v-1}\right)^2} - \eta\, e^{-\frac{1}{2}\left(t-\frac{m}{v-1}\right)^2}\right] dt ,$$

$$= \frac{\partial L}{\partial m}\left(\frac{1}{\sqrt{v}} e^{-\frac{1}{2v}\left(L-\frac{mv}{v-1}\right)^2} - \eta\, e^{-\frac{1}{2}\left(L-\frac{m}{v-1}\right)^2} + \frac{1}{\sqrt{v}} e^{-\frac{1}{2v}\left(L+\frac{mv}{v-1}\right)^2} - \eta\, e^{-\frac{1}{2}\left(L+\frac{m}{v-1}\right)^2}\right)$$

$$+ \int_{-L}^{L} \frac{\partial}{\partial m}\left(\frac{1}{\sqrt{v}} e^{-\frac{1}{2v}\left(t-\frac{mv}{v-1}\right)^2} - \eta\, e^{-\frac{1}{2}\left(t-\frac{m}{v-1}\right)^2}\right) dt .$$

By a reduction paralleling that used in the proof of Lemma 2.3, the first term of the last member of (2) is found to be zero, while if the indicated differentiation in the second term is carried out, the resulting integral can be evaluated and one finds that

$$(3) \quad \frac{\partial}{\partial m} R_Y(\zeta) \overset{s}{\sim} \sqrt{v}\,[e^{-\frac{1}{2v}(L+\frac{mv}{v-1})^2} - e^{-\frac{1}{2v}(L-\frac{mv}{v-1})^2}]$$

$$- \eta[e^{-\frac{1}{2}(L+\frac{m}{v-1})^2} - e^{-\frac{1}{2}(L-\frac{m}{v-1})^2}] \quad .$$

Multiplying the first term on the right by $1 = \frac{1}{v} + \frac{v-1}{v}$, (3) can be reduced to

$$(4) \qquad \frac{\partial}{\partial m} R_Y(\zeta) \overset{s}{\sim} \frac{v-1}{\sqrt{v}} [e^{-\frac{1}{2v}(L+\frac{mv}{v-1})^2} - e^{-\frac{1}{2v}(L-\frac{mv}{v-1})^2}] \ ,$$

since the remaining terms reduce to zero in the same way in which the first term of (2) did so. Since $L \geq 0$, it follows from (4) that $\frac{\partial}{\partial m} R_Y(\zeta) \leq 0$ and the proof is done.

It can now be concluded that there are two non-negative single-valued functions, say ϕ_1 and ϕ_2, of v, for $1 \leq v \leq \sigma^2$, such that for $m^2 \leq \phi_1(v)$, $R_X \leq R_Y$ and for $m^2 \geq \phi_2(v)$, $R_X \geq R_Y$. The possibility that $\phi_1 \equiv 0$ or that $\phi_2 \equiv +\infty$ is not at this point excluded.

Let ϕ be a non-negative, differentiable function of v $(v > 1)$ with $\phi(\sigma^2) \doteq \mu^2$. Now set $m = \phi(v)$ and consider $R_Y(\zeta)$ as a function of v. From equation (1) in the proof of Lemma 2.4,

$$(2.2.10) \quad \sqrt{2\pi}\,(\frac{R_Y(\zeta)}{1-\zeta} - \eta) = \int_{-L}^{L} \frac{1}{\sqrt{v}} e^{-\frac{1}{2v}(t-vC)^2} dt - \eta \int_{-L}^{L} e^{-\frac{1}{2}(t-C)^2} dt \quad ,$$

where

$$C = \frac{1}{v-1} \sqrt{\phi(v)} \ ,$$

and

$$L = \frac{\sqrt{v}}{v-1}\sqrt{m^2+(v-1)\log \eta^2 v} = \sqrt{v}\sqrt{C^2+\frac{\log \eta^2 v}{v-1}} \quad .$$

Differentiating with respect to v yields the equation

(2.2.11)
$$\frac{\sqrt{2\pi}}{1-\zeta}\frac{\partial}{\partial v} R_Y(\zeta) = \frac{\partial L}{\partial v}\Big[\frac{1}{\sqrt{v}}e^{-\frac{1}{2v}(L-vC)^2} - \eta e^{-\frac{1}{2}(L-C)^2}$$
$$+\frac{1}{\sqrt{v}}e^{-\frac{1}{2v}(L+vC)^2} - \eta e^{-\frac{1}{2}(L+C)^2}\Big]$$
$$+\int_{-L}^{L}\frac{\partial}{\partial v}\Big(\frac{1}{\sqrt{v}}e^{-\frac{1}{2v}(t-vC)^2} - \eta e^{-\frac{1}{2}(t-C)^2}\Big)dt \quad .$$

The first term of the right member reduces to zero as in the preceding proofs. Then, carrying out the differentiation indicated in the last term and rearranging,

(2.2.12)
$$\frac{\sqrt{2\pi}}{1-\zeta}\frac{\partial}{\partial v} R_Y(\zeta) = \frac{dC}{dv}\Big[\frac{1}{\sqrt{v}}\int_{-L}^{L}(t-vC)e^{-\frac{1}{2v}(t-vC)^2}dt$$
$$-\eta\int_{-L}^{L}(t-C)e^{-\frac{1}{2}(t-C)^2}dt\Big]$$
$$+\int_{-L}^{L}\frac{1}{2v^2\sqrt{v}}(t^2-v^2C^2-v)e^{-\frac{1}{2v}(t-vC)^2}dt \quad .$$

Evaluating the integrals in the first term and proceding in the same manner as in going from (3) to (4) in the proof of Lemma 2.4, one has,

$$\frac{\sqrt{2\pi}}{1-\zeta}\frac{\partial}{\partial v} R_Y(\zeta) = \frac{dC}{dv}\frac{v-1}{\sqrt{v}}\Big[e^{-\frac{1}{2v}(L+vC)^2} - e^{-\frac{1}{2v}(L-vC)^2}\Big]$$
$$+\frac{1}{2v^2\sqrt{v}}\int_{-L}^{L}(t^2-v^2C^2-v)e^{-\frac{1}{2v}(t-vC)^2}dt \quad .$$

Or,

$$(2.2.13)\quad \frac{\partial}{\partial v} R_Y(\zeta) \simeq \frac{dC}{dv}\frac{v-1}{\sqrt{v}}\, e^{-\frac{L^2+v^2C^2}{2v}}\,[e^{-LC} - e^{LC}]$$

$$+ \frac{1}{2v^2\sqrt{v}} \int_{-L}^{L} (t^2 - v^2C^2 - v)\, e^{-\frac{1}{2v}(t-vC)^2}\, dt \quad ,$$

where

$$\frac{dC}{dv} = \frac{(v-1)\phi'(v) - 2\phi(v)}{2(v-1)^2\sqrt{\phi(v)}} \quad .$$

The second term of the right member of (2.2.13), the integral, is negative for all η, i.e., for all $L \geq 0$, since for small L the integrand is negative and as L increases beyond the point at which the maximum value of the integrand is zero, the value of the integral increases monotonically to a limit whose value is easily found by a direct integration to be zero.

It may be noted at this point that for $\frac{dC}{dv} = 0$, i.e., for $\phi(v) = \frac{\mu^2}{(\sigma^2-1)^2}(v-1)^2$, the derivative $\frac{\partial}{\partial v} R_Y(\zeta)$ is negative for all ζ. Combined with Lemma 2.4 this yields

$$\text{(i)}\quad R_X \leq R_Y \quad \text{for } 1 \leq v \leq \sigma^2 \quad \text{and} \quad m^2 \leq \frac{\mu^2}{(\sigma^2-1)^2}(v-1)^2 \quad ,$$

and

$$\text{(ii)}\quad R_X \geq R_Y \quad \text{for } v \geq \sigma^2 \quad \text{and} \quad m^2 \geq \frac{\mu^2}{(\sigma^2-1)^2}(v-1)^2 \quad .$$

However, let ϕ be the function h_3 defined by (2.2.10). It is asserted that for this choice of ϕ the right member of (2.2.13) is less than or equal to zero for all $L \geq 0$. To show this, note first that with $\phi = h_3$, $\frac{dC}{dv} = -\frac{1+vC^2}{2v(v-1)C}$. Now consider the right member of (2.2.13) as a function, Ψ, of L for $L \geq 0$. $\Psi(0) = \Psi(+\infty) = 0$. Thus, to show that Ψ is negative for all L, it will suffice to show that there is an L' such that $\Psi'(L) \leq 0$ for $1 < L'$ and $\Psi'(L) \geq 0$ for all $L \geq L'$.

$$(2.2.14)\qquad \psi'(L) = \frac{1}{v\sqrt{v}}\, e^{-\frac{1}{2v}(L+vC)^2}\Big[\frac{dC}{dv}(v-1)\Big\{(L-vC)e^{2LC}-(L+vC)\Big\}$$

$$+\frac{L^2-v^2C^2-v}{2v}(e^{2LC}+1)\Big]$$

$$\overset{s}{\sim} -\frac{1+vC^2}{2vC}[L(e^{2LC}-1)-vC(e^{2LC}+1)]+\frac{L^2}{2v}(e^{2LC}+1)$$

$$-\frac{1+vC^2}{2}(e^{2LC}+1)$$

$$\overset{s}{\sim} L(e^{2LC}+1)-\frac{1+vC^2}{C}(e^{2LC}-1)$$

$$= e^{2LC}(L-\frac{1+vC^2}{2})+L+\frac{1+vC^2}{C} \quad .$$

Denote this last expression in (2.2.14) by $\gamma(L)$. Then

$$\gamma'(L) = e^{2LC}[1+2C(1-\frac{1+vC^2}{C})]+1 \quad ,$$

and

$$(2.2.15)\qquad \gamma''(L) = 4C^2e^{2LC}(L-vC) \quad .$$

From (2.2.15) one sees, then, that $\gamma''(L)$ is negative for $L < vC$ and is positive for $L > vC$. Hence, γ is concave on the interval $(0,vC)$ and convex on the interval $(vC,+\infty)$. But $\gamma(0)=0$ and $\gamma(+\infty) = +\infty$. Hence there is an L' such that γ, and therefore ψ', is negative for $L < L'$ and positive for $L > L'$. In this way the proof is complete for the

__Lemma__ 2.5. For $m^2 = h_3(v)$, $v \geq 1$, $R_{\underline{v}}(\zeta)$ is for each ζ a non-increasing function of v.

Combining the results of Lemmas 2.3, 2.4, and 2.5, it is seen that the following theorem, giving restricted necessary and sufficient conditions for uniform inequalities between the risks, holds.

<u>Theorem</u> 2.5. For $1 \leq v \leq \sigma^2$.

$$\left\{(v,m^2) : R_X \leq R_Y\right\} = \left\{(v,m^2) : m^2 \leq h_3(v)\right\} .$$

For $v \geq \sigma^2 \geq 1$,

$$\left\{(v,m^2) : R_X \geq R_Y\right\} = \left\{(v,m^2) : m^2 \geq h_3(v)\right\} .$$

It would at this point be pleasant if a choice of ϕ could be made such that for $m^2 = \phi(v)$, $v > 1$, $R_Y(\zeta)$ would be for each ζ a non-decreasing function of v, i.e., such that $\psi(L) \geq 0$ for all $L \geq 0$. Two necessary conditions for such to be the case are immediate, namely, $\psi'(L) \leq 0$ for all sufficiently large L, and $\frac{dC}{dv} < 0$. But from (2.2.14),

$$\psi'(L) \overset{s}{\sim} \frac{dC}{dv}(v-1)[(L-vC)e^{2LC}-(L+vC)]+\frac{L^2-v^2C^2-v}{2v}(e^{2LC}+L) .$$

Now let $-\frac{dC}{dv}(v-1) = P > 0$. Then,

$$\psi'(L) \overset{s}{\sim} e^{2LC}\,\frac{L^2-v^2C^2-v}{2v} - P(L-vC) + \frac{L^2-v^2C^2-v}{2v} + P(L+vC) ,$$

and for given P and v this becomes and remains positive as L increases. Hence, one cannot find a curve along which $R_Y(\zeta)$ is for each ζ non-decreasing in v, except the degenerate case $v = \sigma^2$, where $R_Y(\zeta)$ is uniformly (in ζ) non-decreasing as m decreases.

Now by Lemma 2.3 there is a function, call it h_4, such that for $1 \leq v \leq \sigma^2$,

$$\left\{(v,m^2) : R_X \geq R_Y\right\} = \left\{(v,m^2) : m^2 \geq h_4(v)\right\} .$$

From the preceding paragraph it follows that in general $h_4 \neq h_2$, since for any point (v_1,m_1^2) let R_{Y_1} be the associated risk curve (in the obvious way) and let $B(v_1,m_1^2) = \left\{(v,m^2) : R_Y \geq R_{Y_1}\right\}$. It is asserted that the lower boundary of $B(v_1,m_1^2)$ does not in general coincide with the line of constant $I(2:1)$ through (v_1,m_1^2). Suppose it did. Let (v_1,m_1^2) lie on the curve of h_2, $1 < v_1 < \sigma^2$. Then for (v,m^2) also on the curve of h_2 and $1 \leq v < v_1$, one would have $R_Y \geq R_{Y_1} \geq R_X$. But this would imply that for each ζ, $R_Y(\zeta)$ is non-increasing in v, for $m^2 = h_2(v)$. Since this has been just shown to be impossible, and $h_4 \geq h_2$ according to Theorem 2.3, it must be concluded that $h_4 > h_2$ for $1 \leq v \leq \sigma^2$.

Many of the interesting results of this section can be summarized in the following way. For $1 \leq v \leq \sigma^2$, the four functions h_1, h_2, h_3, and h_4 determine five sets: for $m^2 \leq h_3$, X is more informative than Y both in the Kullbach-Leibler sense and in the sense that $R_X \leq R_Y$; for $h_3 < m^2 \leq h_1$, X is the more informative only in the Kullbach-Leibler sense; for $h_1 < m^2 < h_2$, neither random variable is the more informative in either sense; for $h_2 \leq m^2 < h_4$, Y is the more informative in the Kullbach-Leibler sense only; and for $h_4 \leq m^2$, Y is the more informative in both senses. From the results and methods of this section it can be verified that if $v > \sigma^2 \geq 1$, then for $m^2 \leq h_4$, X is the more informative in each sense; for $h_4 < m^2 \leq h_2$, X is the more informative in the Kullbach-Leibler sense only; for $h_2 < m^2 < h_1$, neither is more informative in either sense; for $h_1 \leq m^2 < h_3$, Y is the more informative in the Kullbach-Leibler sense only, and for $h_3 \leq m^2$, Y is the more informative in both senses. The function h_4 has not as yet been explicitly given.

2.3 <u>The K-L Numbers in Relation to Certain Classes of Distributions</u>.

Attention is next turned to another and interesting point of view with regard to the K-L information numbers. Suppose that the densities under consideration are elements of a class, $\{f_\omega : \omega \varepsilon \Omega\}$, of densities positive on the same set. Assume that there is an Abelian group, T, of transformations of the domain of the f_ω's and a corresponding group, $\bar{T}$, of transformations of Ω such that if X has density f_ω, then for $t \varepsilon T$, t(X) has density $f_{\bar{t}(\omega)} = \mu(t^{-1}) f_\omega$, that is, $d\Psi(t^{-1}x) = \mu(t^{-1}) d\Psi(x)$. Finally, assume that given ω_1 and ω_2 in Ω, there is a $t \varepsilon T$ such that $\omega_2 = \bar{t}(\omega_1)$.

<u>Theorem</u> 2.6. The K-L numbers are functions only of the transformation that carries f_1 into f_2 and not of f_1 and f_2 individually.

<u>Proof</u>. Choose a $t \varepsilon T$ such that $f_2(x) = f_1(t^{-1}x)\mu(t^{-1})$. Then

$$(1) \qquad I(1:2) = \int_{-\infty}^{\infty} f_1(x) \log \frac{f_1(x)}{f_1(t^{-1}x)\mu(t^{-1})} \, d\Psi(x)$$

$$= -\log \mu(t^{-1}) + \int_{-\infty}^{\infty} f_1(x) \log \frac{f_1(x)}{f_1(t^{-1}x)} \, d\Psi(x) \quad .$$

To show that the value of the integral is a function of t only and does not depend on f_1, choose any $f_o \varepsilon \{f_\omega\}$ and let $f_1(x) = f_o(s^{-1}x)\,\mu(s^{-1})$. Then, with $y = s^{-1}x$, $t^{-1}y = t^{-1}s^{-1}x = s^{-1}t^{-1}x$, and (1) can be rewritten as

$$(2) \qquad I(1:2) = -\log \mu(t^{-1}) + \int_{-\infty}^{\infty} f_o(y) \log \frac{f_o(y)}{f_o(t^{-1}y)} \, d\Psi(y) \quad .$$

A similar proof holds for I(2:1) and the proof is complete.

Whenever it happens that equality of the K-L numbers for X and for Y implies that the <u>same</u> transformation that carries f_1 into f_2 also carries g_1 into g_2, it will follow also that for some $t \in T$, Y and $t(X)$ have the same distributions under each hypothesis; this will be denoted $Y \overset{d}{=} t(X)$. In such cases it is clear that equality of K-L numbers implies equality of risks. That all the conditions on the group T given above are not necessary for equality of the K-L numbers to imply equality of risks appears immediately from the case of normal distributions where the group is not Abelian, the 'Jacobians' are not constants, and the correspondence between transformations and K-L numbers is not 1-1 but still, equality of K-L numbers implies equality of risks.

<u>Lemma 2.6</u>. $Y \overset{d}{=} t(X)$ implies $R_X = R_Y$ and if the likelihood ratios, f_2/f_1 and g_2/g_1, are monotone in the same direction, then $R_X = R_Y$ implies that $Y \overset{d}{=} t(X)$.

<u>Proof</u>. The first statement is clear. Without loss of generality, let X and Y have the common density h under H_1 and densities f and g respectively under H_2. It then suffices to show that $f = g$, for then the same transformation that carries h into f carries h into g.

From Theorem 2.1, if $R_X = R_Y$, then

$$(1) \qquad \int_{\frac{f(x)}{h(x)} \le \eta} h(x)\,d\Psi(x) = \int_{\frac{g(x)}{h(x)} \le \eta} h(x)\,d\Psi(x) \quad \text{for all} \quad \eta \ge 0 .$$

Let

$$\left\{x : \frac{f(x)}{h(x)} \le \eta\right\} = \left\{x : x \le \gamma_1(\eta)\right\},$$

and

$$\left\{x : \frac{g(x)}{h(x)} \le \eta\right\} = \left\{x : x \le \gamma_2(\eta)\right\} .$$

Since all densities are assumed positive together, it follows from (1) that $\gamma_1 = \gamma_2$. Hence

(2) $$\left\{x : f(x) \leq \eta\, h(x)\right\} = \left\{x : g(x) \leq \eta\, h(x)\right\} .$$

If $f(x_o) > g(x_o)$, then an η can be found such that $f(x_o) < \eta h(x_o) < g(x_o)$, contradicting (2). If $f(x_o) < g(x_o)$, then a similar contradiction arises. Therefore, $f = g$.

As an example of such a class of densities and transformations as is being considered in this section, consider the Γ-distributions:

(2.3.1) $$f_{\omega}(x) = \frac{\omega^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\omega x} \quad , \qquad (\omega > 0) \quad ,$$

with $T = \left\{t_c : t_c(x) = cx\right\}$; $\mu(t_c) = c$; and $\bar{t}_c(\omega) = c\omega$.

Suppose the density of X has parameter ω_1 and that of Y has parameter λ_1 under H_1. Let $\omega_2 = \omega_1$, then

(2.3.2) $$I_X(1{:}2) = \alpha[\log \frac{\omega_1}{\omega_2} - \frac{\omega_1 - \omega_2}{\omega_1}] = \alpha[\log \frac{1}{a} - 1 + a] \quad ;$$

and

$$I_X(2{:}1) = \alpha[\log \frac{\omega_2}{\omega_1} - \frac{\omega_2 - \omega_1}{\omega_2}] = \alpha[\log a - 1 + \frac{1}{a}] \quad .$$

Equations (2.3.2) give the K-L numbers explicitly as functions of a, where a corresponds to the transformation carrying f_{ω_1} into f_{ω_2}. If $\lambda_2 = b\lambda_1$, then the expressions for the K-L numbers for Y are given by (2.3.2) with a replaced by b. In the question of equality of information numbers, then,

(2.3.3) $$I_X(1{:}2) = I_Y(1{:}2) \text{ if and only if } \log \frac{b}{a} = b - a \; ;$$

and

$$I_X(2{:}1) = I_Y(2{:}1) \text{ if and only if } \log \frac{a}{b} = \frac{1}{b} - \frac{1}{a} \; .$$

Equality of information numbers in this case implies that $ab(b-a) = b-a$ and hence that either (i) $a = b$, or (ii) $ab = 1$. If (ii) but not (i) holds, then $\log b - b = \log \frac{1}{b} - \frac{1}{b}$. This can easily be shown not to be true for $b \neq 1$. Therefore $a = b$ and an example is provided in which the relation between the K-L numbers and the group of transformations is 1-1. Also note that if the K-L numbers are equal then for some $c > 0$, $\lambda_1 = c\omega_1$ and $\lambda_2 = c\omega_2$, i.e., Y and cX have the same distribution.

3. Designs for a Binomial Testing Problem.

3.1. The Problem. In this section consideration is given to specific design problems in which the random variables have binomial distributions. Again it is supposed that there are two hypotheses, H_1 and H_2, with ζ the *a priori* probability that H_1 is the true hypothesis, and that one must decide which of the two hypotheses is true with a loss of one if the decision is incorrect and no loss if it is correct. There are available two random variables, X and Y, having binomial distributions with parameters p and q, respectively, under H_1 and parameters q and p, respectively, under H_2.

		X	Y
(ζ)	H_1	p	q
$(1-\zeta)$	H_2	q	p

Suppose that the observations are independent, the total number of observations to be taken, n, is fixed, and that the cost of observations is independent of the true hypothesis, the random variable observed, and of the result of the observation.

The first problem considered is that of non-sequential designs, i.e., before experimentation it must be decided which of the observations are to be of X and which of Y.

The second problem treated is that of sequential designs, i.e., rules which for each $j < n$ tell one, as a function of the information available after the j^{th} experiment, which random variable to observe on the $j+1^{st}$ experiment.

In each case the principle of choice among possible designs is, of course, that of minimizing the Bayes risk.

3.2. <u>Non-sequential Designs</u>. Since the observations are assumed to be independent, the non-sequential design problem reduces to determining for each ζ, the optimum number of observations of X.

Let $R_r(\zeta)$ denote the risk against ζ if X is observed r times and Y n-r times. Assume for definiteness, and without loss of generality, that $p > q$ and note that by the evident symmetry, $R_r(\zeta) = R_{n-r}(1-\zeta)$. Furthermore, there is no loss of generality if it is assumed that $p(1-p) > q(1-q)$, for if not, one would, by interchanging p and 1-p, q and 1-q, and X and Y, find oneself in the assumed case.

As before, it will be convenient to consider $\eta = \frac{\zeta}{1-\zeta}$ rather than ζ itself much of the time.

For general n, the solution is characterized by a division of the interval [0,1] into intervals with the property that for ζ in a given interval a certain number of observations of X, i.e., a certain value of r, is optimal. In some of the intervals the optimum value of r is not unique.

The general equation for the value of $R_r(\zeta)$ is given by

$$(3.2.1) \qquad R_r(\zeta) = \sum_{k=0}^{n} \sum_{i=\max(k-n+r)}^{\min(k,r)} \binom{r}{i}\binom{n-r}{k-i} \min\left\{\begin{matrix} p^{r-i}(1-p)^i q^{n-r-k+i}(1-q)^{k-i}\zeta \\ q^{r-i}(1-q)^i p^{n-r-k+i}(1-p)^{k-i}(1-\zeta) \end{matrix}\right\}$$

The preceding characterization of the solution follows from the fact that R_r is piecewise linear for each value of r.

The turning points of R_r occur for those ζ for which the two quantities whose minimum is taken in (3.2.1) are equal, i.e., for $\eta = \left(\frac{q}{p}\right)^{2r-n}\left(\frac{q(1-p)}{p(1-q)}\right)^{k-2i}$. Since $\frac{q(1-p)}{p(1-q)} < 1$, the first turning point occurs for that k and i which maximize $k-2i$, which is $k = n-r$ and $i = 0$. Thus the first turning point of R_r is $\left(\frac{q}{p}\right)^r\left(\frac{1-p}{1-q}\right)^{n-r}$, which is a decreasing function of r. For all ζ such that $\eta < \left(\frac{q}{p}\right)^r\left(\frac{1-p}{1-q}\right)^{n-r}$, $R_r(\zeta) = \zeta$.

The functions R_r can now be compared for small ζ, or equivalently, for small η.

$$(3.2.2) \qquad R_r(\zeta) = \zeta \qquad \text{for all } r \text{ and for } \eta \le \left(\frac{q}{p}\right)^n,$$

$$\left.\begin{matrix} R_n(\zeta) < \zeta & \\ R_r(\zeta) = \zeta & \text{for } r < n \end{matrix}\right\} \text{ for } \left(\frac{q}{p}\right)^n < \eta < \left(\frac{q}{p}\right)^{n-1}\left(\frac{1-p}{1-q}\right).$$

Thus there is complete indifference for $0 \le \eta \le \left(\frac{q}{p}\right)^n$ and $\left(\frac{q}{p}\right)^n$ is the left end point of an interval in which the unique optimum value of r is n.

To push the analysis a bit further along the η axis, consider the equations giving the second segments of R_r.

$$(3.2.3) \qquad R_r(\zeta) = q^r(1-p)^{n-r} + \zeta[1-p^r(1-q)^{n-r} - q^r(1-p)^{n-r}] \ .$$

The intersection of these lines, for $r < n$, with that for $r = n$ occur at

$$\eta_r = \frac{q^r}{p^r}\,\frac{(1-p)^{n-r}-q^{n-r}}{(1-q)^{n-r}-p^{n-r}} \quad . \tag{3.2.4}$$

Now setting $t = n-r-1$, $\eta_r > \eta_{r+1}$ if and only if

$$\frac{(1-p)^{t+1}-q^{t+1}}{(1-q)^{t+1}-p^{t+1}} > \frac{q}{p}\,\frac{(1-p)^t-q^t}{(1-q)^t-p^t} \quad . \tag{3.2.5}$$

Since $1-q > p$, both denominators are positive and one can obtain the equivalent relation

$$[(1-p)^t+(1-p)^{t-1}q+\ldots+q^t][(1-q)^{t-1}p+(1-q)^{t-2}p+\ldots+p^t]$$

$$> [(1-q)^t+(1-q)^{t-1}p+\ldots+p^t][(1-p)^{t-1}q+(1-p)^{t-2}q^2+\ldots+q^t] \quad .$$

Since adding $(1-q)^t \sum_{k=0}^{t}(1-p)^{t-k}q^k$ to the left side and $(1-p)^t \sum_{k=0}^{t}(1-q)^{t-k}p^k$ to the right side will yield an identity, $\eta_r > \eta_{r+1}$ is equivalent to

$$(1-p)^t \sum_{k=0}^{t}(1-q)^{t-k}p^k - (1-q)^t \sum_{k=0}^{t}(1-p)^{t-k}q^k > 0 \quad .$$

This in turn can be written as

$$\sum_{k=0}^{t}(1-p)^{t-k}(1-q)^{t-k}[(1-p)^k p^k - (1-q)^k q^k] > 0 \quad . \tag{3.2.6}$$

But (3.2.6) clearly holds, as $p(1-p) > q(1-q)$, and hence, η_r is strictly decreasing in r.

The intersection of the second segment of R_n with that of R_{n-1} is at $\eta_{n-1} = (\frac{q}{p})^{n-1} > (\frac{q}{p})^n$. And since the second turning point of R_n occurs at $\eta = (\frac{q}{p})^{n-1}(\frac{1-q}{1-p}) > \eta_r$ and that of R_{n-1} is at $\eta = (\frac{q}{p})^{n-2} > \eta_{n-1}$, the solution for a somewhat larger range of η can be given.

(3.2.7) For $0 < \eta < (\frac{q}{p})^n$ there is indifference between all r's;

for $(\frac{q}{p})^n < \eta < (\frac{q}{p})^{n-1}$ n is the optimal value for r;

and for $(\frac{q}{p})^{n-1} < \eta < u$ n-1 is the optimal value for r.

The value of u in (3.2.7) is dependent upon conditions of the form

$$p^s(1-p)^t \left\{ \begin{matrix} < \\ > \end{matrix} \right\} q^s(1-q)^t.$$

Not only does the value of u vary with cases, but the optimal value of r in the interval having u as its left end point will also vary with cases. It would appear that in the attempt to gain a complete solution, one shortly becomes bogged down in a morass of special cases.

Certain solutions for small values of n were computed and are given below as they appear in relation to the η-axis for $0 < \eta < 1$, which corresponds to $0 < \zeta < 1/2$. By the symmetry about $\zeta = 1/2$ noted before, the solution for all ζ can easily be determined from these. (I denotes indifference).

n = 1

optimal r		I		1	
η	0		$\frac{q}{p}$		1

n = 2: $1 < (1-p)^2 + (1-q)^2$:

optimal r		I		2		1		2	
η	0		$(\frac{q}{p})^2$		$\frac{q}{p}$		$\frac{q}{p}\frac{1+p-q}{1-p+q}$		1

$1 > (1-p)^2 + (1-q)^2$:

optimal r		I		2		1		0	
η	0		$(\frac{q}{p})^2$		$\frac{q}{p}$		$\frac{1-p}{1-q}$		1

$n = 3$: $\quad (1-p)^2+(1-q)^2 < 1-\frac{pq}{2}$:

optimal r		I		3		2		1		0		3		2		
η	0		$(\frac{q}{p})^3$		$(\frac{q}{p})^2$		$\frac{q(1-p)}{p(1-q)}$		$(\frac{1-p}{1-q})^2$		A		$\frac{q(1-q)}{p(1-p)}$		1	

or

optimal r		I		3		2		1		3		2	
η	0		$(\frac{q}{p})^3$		$(\frac{q}{p})^2$		$\frac{q(1-p)}{p(1-q)}$		A		$\frac{q(1-q)}{p(1-p)}$		1

according as $A = \frac{q^2(3-2q)-q(1-p)^2}{p^2(3-2p)-p(1-q)^2}$ is greater or less than $(\frac{1-p}{1-q})^2$.

$1-\frac{pq}{2} < (1-p)^2+(1-q)^2 < 1+pq$:

optimal r		I		3		2		3		2
η	0		$(\frac{q}{p})^3$		$(\frac{q}{p})^2$		A		$\frac{q(1-p)}{p(1-q)}$		1

$1+pq < (1-p)^2+(1-q)^2 < 1+\frac{pq}{1-p-q}$:

optimal r		I		3		2		3		2		0	
η	0		$(\frac{q}{p})^3$		$(\frac{q}{p})^2$		A		$\frac{q(1-p)}{p(1-q)}$		C		1

where $C = \frac{1-(1-q)^3-(1-p)^2-2pq(1-p)}{1-(1-p)^3-(1-q)^2-2pq(1-q)}$.

These are the solutions for what appears to be about half of the cases for $n=3$.

Thus, for small values of η the solution has been found for all cases, while for the remaining η's there is no apparent pattern and the solutions (to say nothing of their computation) even for small n lead one to the conclusion that it is just about hopeless to seek a complete general solution. It should be noted that the symmetric choice of the parameters above is clearly a help rather than a hinderance; nearly any choice of parameters will yield a similar morass of cases. The exceptions are those choices of the parameters for which, for $n = 1$, $R_o \leq R_1$, or $R_o \geq R_1$. In such cases, the optimum value of r is zero or n, respectively, for all n and ζ.

3.3 Sequential Designs. Suppose that there is a total of n experiments to be performed, or observations to be taken. Let ζ denote the *a priori* probability that H_1 is true and ζ_j the *a posteriori* probability after having observed the results of the first j experiments. Now to obtain the optimal sequential design one must decide after the j^{th} observation, as a function of the information obtained in the previous experiments, which is contained in ζ_j, and the number of observations remaining, n-j, whether to observe X or Y on the $j+1^{st}$ experiment.

Let $f_n(\zeta)$ be the Bayes risk if the optimal sequential design for n experiments were used. If, now, n+1 experiments were contemplated and X were observed first, then the optimal design followed for the remaining n experiments, the risk would be

$$(3.3.1) \qquad g(X,n,\zeta) = f_n\left(\frac{p\zeta}{p\zeta+(1-\zeta)q}\right)(p\zeta+(1-\zeta)q)$$
$$+f_n\left(\frac{(1-p)\zeta}{(1-p)\zeta+(1-q)(1-\zeta)}\right)((1-p)\zeta+(1-q)(1-\zeta)) \quad .$$

If Y were observed first and then the optimal design followed for the remaining n steps, the risk would be

$$(3.3.2) \qquad g(Y,n,\zeta) = f_n\left(\frac{q\zeta}{q\zeta+(1-\zeta)p}\right)(q\zeta+(1-\zeta)p)$$

$$+ f_n\left(\frac{(1-q)\zeta}{(1-q)\zeta+(1-p)(1-\zeta)}\right)((1-q)\zeta+(1-p)(1-\zeta)) \quad .$$

Hence, the following functional equation is obtained.

$$(3.3.3) \qquad f_{n+1}(\zeta) = \min(g(X,n,\zeta),g(Y,n,\zeta)) \quad .$$

The design problem is to determine those ζ for which $g(X,n,\zeta) < g(Y,n,\zeta)$, $g(X,n,\zeta) = g(Y,n,\zeta)$, and $g(X,n,\zeta) > g(Y,n,\zeta)$, respectively. If there were n+1 observations remaining to be taken, then for ζ in the first set, X should be observed next, for ζ in the third set, Y should be observed next, while for ζ in the second set there is indifference between X and Y, since one would do equally well starting with either.

For $n = 1$, the sequential and non-sequential designs coincide and f_1 is easily found. In theory, one can then, by use of the equation (3.3.3), compute f_n for any n. This method is so complicated as to be practically prohibitive. A method is given below for obtaining the sequential designs without having to compute each of the risk functions. Even this method bogs down in cases as n increases; however, for given values of p and q, it would be possible to use.

Now it is clear that f_1 is piecewise linear and it is concave. It is easily seen then that both $g(X,1,\zeta)$ and $g(Y,1,\zeta)$, and therefore f_2, are also piecewise linear and concave. Furthermore, the turning points of $g(X,n,\zeta)$ are precisely those ζ such that either $\frac{p\zeta}{p\zeta+(1-\zeta)q}$ or $\frac{(1-p)\zeta}{(1-p)\zeta+(1-q)(1-\zeta)}$ is a turning point of f_1. Likewise, the turning points of $g(Y,n,\zeta)$ are those ζ such that either $\frac{q\zeta}{q\zeta+(1-\zeta)p}$ or $\frac{(1-q)\zeta}{(1-q)\zeta+(1-p)(1-\zeta)}$ is a turning point of f_1. In terms of the variable η, η is a turning point of $g(X,n,\zeta)$ if and only if $\frac{p}{q}\eta$ or $\frac{1-p}{1-q}\eta$ is a turning point of f_1 and is a turning point of $g(Y,n,\zeta)$ if and only if $\frac{q}{p}\eta$ or $\frac{1-q}{1-p}\eta$ is a turning point of f_1.

For $n = 1$, the solution can be expressed by diagram.

optimal choice	: Y : X :
η	$0 \quad \frac{q}{p} \quad 1$

Since the same kind of symmetry about $\zeta = 1/2$ is present as was noted in the preceding section, to give the solution up to $\eta = 1$ is sufficient. The turning points of f_1 are (in terms of η), q/p, 1, and p/q.

Arranging the turning points of $g(X,n,\zeta)$ and $g(Y,n,\zeta)$ in order, one has, for $q(1-q)^2 < p(1-p)^2$,

for $g(X,n,\zeta)$:	$(\frac{q}{p})^2$		$(\frac{q}{p})$	$\frac{q(1-q)}{p(1-p)}$		1
for $g(Y,n,\zeta)$:		$\frac{q(1-p)}{p(1-q)}$			$\frac{1-p}{1-q}$	1

;

while if $q(1-q)^2 > p(1-p)^2$, one has the turning points

for $g(X,n,\zeta)$:	$(\frac{q}{p})^2$		$\frac{q}{p}$		$\frac{q(1-q)}{p(1-p)}$	1
for $g(Y,n,\zeta)$:		$\frac{q(1-p)}{p(1-q)}$		$\frac{1-p}{1-q}$		1

In each case, these turning points divide the interval (0,1) into sub-intervals. If $\eta \leq (\frac{q}{p})^2$ and X is observed, then $\eta_1 = (\frac{\zeta_1}{1-\zeta_1})$ will be less than q/p or less than $\frac{q^2(1-p)}{p^2(1-q)}$ according as the observed value of X is 1 or 0. Since in either case $\eta_1 \leq q/p$, it would be optimal to observe Y at the next stage. Similarly, if Y were observed first, then $\eta_1 < 1$ regardless of which value Y assumed. Hence, it would then be optimal to observe X at the next stage. Now, since for independent observations the order is immaterial, the two risks must coincide for $\eta \leq (q/p)^2$. In a like manner it is found that in each of the two cases which are distinguished by the ordering of the turning points, the interval whose left end point is q/p is also an interval of indifference. Knowing that $g(X,n,\zeta)$ and $g(Y,n,\zeta)$ are each piecewise linear and concave, and that they coincide at $\eta = 1$ as well as on these two intervals of indifference, it is sufficient to determine the solution for $n = 2$. There are two cases for the solution

(3.3.4) For $p(1-p)^2 > q(1-q)^2$:

optimal choice	: I :	X :	I :	X :
η	0	$(\frac{q}{p})^2$	$\frac{q}{p}$	$\frac{q(1-q)}{p(1-p)}$

For $p(1-p)^2 < q(1-q)^2$:

optimal choice	: I :	X :	I :	Y :	
η	0	$(\frac{q}{p})^2$	$\frac{q}{p}$	$\frac{1-p}{1-q}$	1

Now the method for obtaining the solution for $n+1$ from that for n follows that given above with $n = 1$. From the turning points of f_n, determine the turning points of $g(X,n,\zeta)$ and $g(Y,n,\zeta)$ and arrange them in order (considering the necessary cases). Determine those η for which both $\frac{p}{q}\eta$ and $\frac{1-p}{1-q}\eta$ lie in Y- or I-intervals of the solution for n; determine those η for which both $\frac{q}{p}\eta$ and $\frac{1-q}{1-p}\eta$ lie either in an X- or an I-interval of the solution for n. The intersection of these two sets will be the indifference intervals in the solution for $n+1$. From this information, the order of the turning points of the two functions $g(X,n,\zeta)$ and $g(Y,n,\zeta)$, and the concavity of these two functions, most of the solution for $n+1$ can be inferred. For $n = 1$, the entire solution for $n = 2$ is determined with no further work, but for most of the cases for larger values of n, the two functions $g(X,n,\zeta)$ and $g(Y,n,\zeta)$ will have to be computed and compared at a few isolated points.

It is the computation and comparison of these functions at isolated points, as well as the multiplicity of cases for larger n that makes even this method imperfect for obtaining general solutions. However, for given values of p and q this procedure could be used to determine the optimal design for moderate n without undue difficulty.

This section is concluded by giving the solution for $n = 3$ after first remarking that the usefulness of the method is not restricted to problems in which the parameters are symmetric.

(3.3.5) For $p(1-p)^2 > q(1-q)^2$ and $p(1-p)^3 < q(1-q)^3$:

optimal choice : I : X : I : X : I : Y :

$\eta \quad 0 \quad (\frac{q}{p})^3 \quad (\frac{q}{p})^2 \quad \frac{q^2(1-q)}{p^2(1-p)} \quad \frac{q}{p} \quad \frac{q(1-q)}{p(1-p)} \quad \frac{1-p}{1-q} \quad 1$

For $p(1-p)^2 > q(1-q)^2$ and $p(1-p)^3 > q(1-q)^3$:

optimal choice : I : X : I : X : I : X :

$\eta \quad 0 \quad (\frac{q}{p})^3 \quad (\frac{q}{p})^2 \quad \frac{q^2(1-q)}{p^2(1-p)} \quad \frac{q}{p} \quad \frac{q(1-q)}{p(1-p)} \quad \frac{q(1-q)^2}{p(1-p)^2} \quad 1$

For $p(1-p)^2 < q(1-q)^2$ and $q^2(1-q)^3 < p^2(1-p)^3$:

optimal choice : I : X : I : X : I :

$\eta \quad 0 \quad (\frac{q}{p})^3 \quad (\frac{q}{p})^2 \quad \frac{q(1-p)}{p(1-q)} \quad \frac{q^2(1-q)}{p^2(1-p)} \quad \frac{q}{p} \quad \frac{q(1-q)}{p(1-p)} \quad 1$

For $p(1-p)^2 < q(1-q)^2$ and $q^2(1-q)^3 > p^2(1-p)^3$:

optimal choice : I : X : I : Y : X : I :

$\eta \quad 0 \quad (\frac{q}{p})^3 \quad (\frac{q}{p})^2 \quad \frac{q(1-p)}{p(1-q)} \quad (\frac{1-p}{1-q})^2 \quad A \quad \frac{q}{p} \quad \frac{q(1-q)}{p(1-p)} \quad 1$

where $A = \dfrac{(1-p)^2(1-p-q)-q^2(1-q)}{(1-q)^2(1-p-q)-p^2(1-p)}$.

4. Some Non-truncated Design Problems.

In the preceding sections attention has been focused entirely on design problems in which the sample size was fixed. A problem in which experimentation is terminated by a sequential stopping rule will now be considered.

4.1. A Mixed Random Walk. Suppose, as in Section 3, that the two random variables, X and Y, have binomial distributions with parameters under the two hypotheses, H_1 and H_2, as given by:

$$\begin{array}{cccc} & & X & Y \\ (\zeta) & H_1 & p & q \\ (1-\zeta) & H_2 & q & p \end{array} \qquad (p > q,\ p(1-p) > q(1-q)).$$

Again, ζ is the a priori probability that H_1 is the true hypothesis and it is given that one must decide which of the hypotheses is the true one with losses as described in the previous sections.

Let an observation of X and an observation of Y have the same cost. A design is now sought which will minimize the expected cost of achieving a Bayes risk from the terminal decision of at most a fixed amount, r. This is equivalent to finding that design which will minimize the expected number of observations required to move the a posteriori probability for H_1 to a position either in the interval $[0,r]$ or in the interval $[1-r,1]$.

Let $\zeta_0 = \zeta$ and ζ_j denote the a posteriori probability for H_1 after having made the first j observations. It will be convenient to consider the problem in terms of the variable $\gamma = \log \eta = \log \frac{\zeta}{1-\zeta}$. Then let

$$a = \log \frac{p}{q}$$

$$b = \log \frac{1-p}{1-q} \quad , \text{ and}$$

$$A = -\log \frac{r}{1-r} \quad .$$

Let n denote the smallest value of j for which either $\zeta_j \leq r$ or $\zeta_j \geq 1-r$. It is seen there are two random walks, both on the γ axis with boundaries at A and -A, one of which is determined by the results of observations of X and the other determined by the results of observations of Y. After having made j observations one finds that the walk has arrived at the point γ_j. Now the choice must be made as to whether it is better that γ_{j+1} should be determined by an observation of X or of Y, i.e., whether the next step should be taken in the X-walk or in the Y walk. A rule is desired prescribing for every situation which walk should be taken in order to minimize the expected value of n.

If at the j+1st step, X is observed, then

$$\text{(4.1.1)} \qquad \gamma_{j+1} = \begin{cases} \gamma_j + a & \text{with probability p under } H_1 \text{ and q under } H_2, \\ \gamma_j - b & \text{with probability 1-p under } H_1 \text{ and 1-q under } H_2. \end{cases}$$

Letting E_X denote expectation when X is observed, it follows that

$$\text{(4.1.2)} \qquad E_X[\gamma_{j+1} - \gamma_j | H_1] = p \log \frac{p}{q} + (1-p)\log \frac{1-p}{1-q} = I_X(1:2),$$

$$E_X[\gamma_{j+1} - \gamma_j | H_2] = q \log \frac{p}{q} + (1-q)\log \frac{1-p}{1-q} = -I_X(2:1),$$

$$\text{and} \quad E_X[\gamma_{j+1} - \gamma_j] = \zeta_j I_X(1:2) - (1-\zeta_j) I_X(2:1) = \zeta_j J_X - I_X(2:1).$$

Since the divergence is always positive, $E_X[\gamma_{j+1} - \gamma_j]$ is an increasing function of γ_j and is zero for

$$(4.1.3) \qquad \zeta_j = 1-\zeta^* = \frac{I_X(2:1)}{J_X} \quad \text{, i.e., for}$$

$$-\gamma^* = \log \frac{I_X(2:1)}{I_X(1:2)} \quad .$$

Similarly, if Y is observed, then

$$(4.1.4) \qquad \gamma_{k+1} = \begin{cases} \gamma_j - a & \text{with probability } q \text{ under } H_1 \text{ and } p \text{ under } H_2, \\ \gamma_j + b & \text{with probability } 1-q \text{ under } H_1 \text{ and } 1-p \text{ under } H_2. \end{cases}$$

Also

$$(4.1.5) \qquad E_Y[\gamma_{j+1} - \gamma_j | H_1] = q \log \frac{q}{p} + (1-q)\log \frac{1-q}{1-p} = I_X(2:1),$$

$$E_Y[\gamma_{j+1} - \gamma_j | H_2] = p \log \frac{q}{p} + (1-p)\log \frac{1-q}{1-p} = -I_X(1:2),$$

and $E_Y[\gamma_{j+1} - \gamma_j] = \zeta_j J_X - I_X(1:2)$.

Hence, $E_Y[\gamma_{j+1} - \gamma_j]$ is also an increasing function of γ_j and is zero for

$$(4.1.6) \qquad \zeta_j = \frac{I_X(1:2)}{J_X} = \zeta \quad \text{, i.e., for } \gamma_j = \gamma^*.$$

To verify that $\gamma^* > 0$, i.e., that $I_X(2:1) < I_X(1:2)$, let $\phi(p) = I_X(1:2) - I_X(2:1)$. Then

$$\phi(p) = (p+q)\log \frac{p}{q} + (2-p-q)\log \frac{1-p}{1-q} \text{ , and}$$

$$\phi''(p) = \frac{(1-2p)(p-q)}{p^2(1-p)^2} \quad .$$

With $p > q$ and $p(1-p) > q(1-q)$, then $1-q > p > q$ and $q < 1/2$. Hence, ϕ is negative and concave for $p < q$, zero at $p = q$, convex and positive for $q < p < 1/2$, and concave for $p \geq 1/2$. But $\phi(q) = \log \frac{1-q}{q} > 0$. Therefore, $\phi(p) > 0$ for $1-q > p > q$.

Therefore the γ axis can be divided into four parts and on them one will have,

$$(4.1.7)\quad -E_Y[\gamma_{j+1}-\gamma_j] > -E_X[\gamma_{j+1}-\gamma_j] > 0 \quad \text{for} \quad -A<\gamma_j<-\gamma^*,$$

$$-E_Y[\gamma_{j+1}-\gamma_j] > E_X[\gamma_{j+1}-\gamma_j] > 0 \quad \text{for} \quad -\gamma^*<\gamma_j<0,$$

$$E_X[\gamma_{j+1}-\gamma_j] > -E_Y[\gamma_{j+1}-\gamma_j] > 0 \quad \text{for} \quad 0<\gamma_j<\gamma^*, \text{ and}$$

$$E_X[\gamma_{j+1}-\gamma_j] > E_Y[\gamma_{j+1}-\gamma_j] > 0 \quad \text{for} \quad \gamma^*<\gamma_j<A.$$

Thus, for $\gamma_j > 0$ the X-wald yields an expected step greater in magnitude that the Y-walk and the expected step is in the 'right' direction, i.e., towards the nearest boundary, A. For $\gamma_j < 0$, the Y-wald enjoys the same advantage, the nearest boundary being -A.

These considerations lead to the conjecture that, at least for a small relative to A, the optimal design is to take the X-walk on the $j+1^{st}$ step when $\gamma_j > 0$ and the Y-walk otherwise. (It should be remarked that if $p(1-p) < q(1-q)$, the same results will hold with X and Y interchanged).

Now let $X\infty$ denote that design which requires that X be used at each step and $Y\infty$ that design which requires that Y be used at each step. Denote by $E[n|X\infty, \gamma, H_i]$ the expected number of steps in the X-walk with γ as its starting point when H_i is the true hypothesis. Using Wald's well known approximations,

$$(4.1.8)\qquad E[n|X\infty,\gamma,H_1] = \frac{A-\gamma-2A\left\{\dfrac{e^{2Au}-e^{(A+\gamma)u}}{e^{2Au}-1}\right\}}{p\log\frac{p}{q}+(1-p)\log\frac{1-p}{1-q}} ,$$

$$E[n|X\infty,\gamma,H_2] = \frac{A-\gamma-2A\left\{\dfrac{e^{2At}-e^{(A+\gamma)t}}{e^{2At}-1}\right\}}{q\log\frac{p}{q}+(1-q)\log\frac{1-p}{1-q}} ,$$

$$E[n|Y\infty,\gamma,H_1] = \frac{A-\gamma-2A\left\{\dfrac{e^{-2At}-e^{-(A+\gamma)t}}{e^{-2At}-1}\right\}}{-(q\log\frac{p}{2}+(1-q)\log\frac{1-p}{1-q})} , \quad \text{and}$$

$$E[n|Y\infty,\gamma,H_2] = \frac{A-\gamma-2A\left\{\dfrac{e^{-2Au}-e^{-(A+\gamma)u}}{e^{-2Au}-1}\right\}}{-(p\log\frac{p}{q}+(1-p)\log\frac{1-p}{1-q})} ;$$

$$\text{where } u\neq 0 \text{ satisfies } \quad pe^{u\log\frac{p}{q}}+(1-p)e^{u\log\frac{1-p}{1-q}} = 1,$$

$$\text{and } t\neq 0 \text{ satisfies } \quad qe^{t\log\frac{p}{q}}+(1-q)e^{t\log\frac{1-p}{1-q}} = 1.$$

It is easily seen that $u = -1$ and $t = 1$. Then recognizing the denominators in the above expressions as K-L numbers, it follows that

$$(4.1.9)\qquad E[n|Y\infty,\gamma]-E[n|X\infty,\gamma]$$

$$= \zeta\left\{E[n|Y\infty,\gamma,H_1]-E[n|X\infty,\gamma,H_1]\right\}-(1-\zeta)\left\{E[n|Y\infty,\gamma,H_2]-E[n|X\infty,\gamma,H_2]\right\}$$

$$= \frac{I_X(1:2)-I_X(2:1)}{I_X(1:2)I_X(2:1)}\left\{\zeta\left(A-\gamma-2A\,\frac{e^{-2A}-e^{-A-\gamma}}{e^{-2A}-1}\right)+(1-\zeta)\left(A-\gamma-2A\,\frac{e^{2A}-e^{A+\gamma}}{e^{2A}-1}\right)\right\} .$$

Noting that the first factor is positive, then by adding 1-1 to the fraction in the last term and rearranging, it is found that

(4.1.10) $\quad E[n|Y\infty,\gamma]-E[n|X\infty,\gamma]$

$$\overset{s}{\sim}\zeta(A-\gamma)-(1-\zeta)(A+\gamma)-2A\zeta\,\frac{e^{-2A}-e^{-A-\gamma}}{e^{-2A}-1}+2A(1-\zeta)\,\frac{e^{A+\gamma}-1}{e^{2A}-1}$$

$$=-\gamma+A(2\zeta-1)+2A\,\frac{e^{A+\gamma}-1}{e^{2A}-1}-2A\zeta\left\{\frac{e^{-2A}-e^{-A-\gamma}}{e^{-2A}-1}+\frac{e^{A+\gamma}-1}{e^{2A}-1}\right\}\ .$$

It can be easily verified from (4.1.10) that the difference is zero for $\gamma=0$ and $\gamma=A$. It will be shown to be non-negative for $0\le\gamma\le A$.

Noting that $\zeta=\dfrac{e^{\gamma}}{1+e^{\gamma}}$ and denoting the last member of (4.1.10) by $\psi(\gamma)$, then

(4.1.11)
$$\psi''(\gamma)=2Ae^{\gamma}\left\{\frac{1-e^{2\gamma}}{(1+e^{\gamma})^4}+\frac{e^{A}}{e^{2A}-1}-\frac{1}{1+e^{\gamma}}\left(\frac{e^{A+\gamma}}{e^{2A}-1}-\frac{e^{-A-\gamma}}{e^{-2A}-1}\right)\right.$$
$$\left.-\frac{2}{(1+e^{\gamma})^2}\left(\frac{e^{-A-\gamma}}{e^{-2A}-1}+\frac{e^{A+\gamma}}{e^{2A}-1}\right)-\frac{1-e^{2\gamma}}{(1+e^{\gamma})^4}\left(\frac{e^{-2A}-e^{-A-\gamma}}{e^{-2A}-1}+\frac{e^{A+\gamma}-1}{e^{2A}-1}\right)\right\}\ .$$

Simplifying (4.1.11) and removing positive factors yields

(4.1.12)
$$\psi''(\gamma)\overset{s}{\sim}(2+e^{A}+e^{-A})+e^{\gamma}-(2+e^{A}+e^{-A})e^{2A}+e^{3\gamma}\ .$$

At $\gamma=0$, the right side of (4.1.12) is positive. At $\gamma=A$, it equals $e^{-A}+2+e^{A}-2e^{2A}$, which is negative at least for $A\ge\log\frac{3}{2}$. By differentiation the right side is found to be decreasing in an interval $[0,\gamma'')$ and increasing for $\gamma>\gamma''$. Hence, $\psi''(\gamma)$ is first positive and then negative as γ increases from 0 to A ($A\ge\log\frac{3}{2}$); i.e., ψ is first convex and then concave. It remains only to show that $\psi'(0)\ge0$ to assure that $\psi(\gamma)\ge0$ for $0\le\gamma\le A$.

$$\psi'(0)\overset{s}{\sim}4+e^{2A}(A-2)+2Ae^{A}-2Ae^{-A}-e^{-2A}(A+2)\ .$$

Denoting the right member by $S(A)$, successive differentiation shows that $S(0) = S'(0) = S''(0) = S'''(0)$ while the fourth derivative at zero is positive for all $A \geq 0$. Hence it can be concluded that $\psi'(0) > 0$ for $A > 0$ and by the evident symmetry in the problem it follows that

$$(4.1.13) \qquad E[n|Y\infty, \gamma] - E[n|X\infty, \gamma] \begin{cases} > 0 & \text{for } \gamma > 0 \\ < 0 & \text{for } \gamma < 0 . \end{cases}$$

Thus the design which requires the use of X at the j+1st step if $\gamma_j > 0$ and Y if $\gamma_j < 0$, coincides with the design requiring the use of the random variable corresponding to the smaller of $E[n|X\infty, \gamma_j]$ and $E[n|Y\infty, \gamma_j]$. It also coincides with the following design given in terms of the K-L information numbers. Let $J_X(\zeta) = \zeta I_X(1:2) + (1-\zeta) I_X(2:1)$. Then $J_Y(\zeta) = \zeta I_X(2:1) + (1-\zeta) I_X(1:2)$ and $J_X(\zeta) > J_Y(\zeta)$ for $\zeta > 1/2$. Hence, the design just described could also be expressed by the rule: at the j+1st step use the random variable corresponding to the larger of the numbers $J_X(\zeta_j)$ and $J_Y(\zeta_j)$.

Denote this thrice-described design by M. While M has not yet been shown to be the optimum design, it can be shown to be better than either $X\infty$ or $Y\infty$. This comes as a special case of the next result, which concludes this section.

By a <u>stationary</u> design will be meant a design in which the choice at the j+1st step is a function only of the <u>a posteriori</u> probability after the jth step ζ_j.

<u>Lemma</u> 4.1. Let X and Y have densities f_1 and g_1, respectively, under hypothesis H_1 such that both $\log f_2/f_1$ and $\log g_2/g_1$ assume positive and negative values with positive probability. Let D_1 and D_2 be two stationary

designs and D that design which requires, at the $j+1^{st}$ step, the random variable corresponding to the smaller of $E[n|D_1, \gamma_j]$ and $E[n|D_2, \gamma_j]$. Then $E[n|D, \gamma] \leq \min\left\{E[n|D_1,\gamma], E[n\ D_2,\gamma]\right\}$.

Proof. For any set K in the interval $[-A,A]$, let $\tilde{K}$ denote its complement in $[-A,A]$. Let

$$(1) \qquad \Gamma_1 = \left\{\gamma : \text{ for } \gamma_j = \gamma,\ D_1 \text{ requires X at the } j+1^{st} \text{ step}\right\}.$$

Let

$$T_X(\zeta) = \frac{\zeta f_1(X)}{\zeta f_1(X)+(1-\zeta)f_2(X)} \quad , \text{ and}$$

$$T_X(\gamma) = \log\frac{T_X(\zeta)}{1-T_X(\zeta)} \quad , \text{ where } \quad \gamma = \log\frac{\gamma}{1-\gamma} \quad .$$

Then

$$(2) \qquad E[n|D_1,\gamma] = \begin{cases} 1+E_\gamma[n|D_1,T_X(\gamma)] & \text{if} \quad \gamma \varepsilon \Gamma_1, \\ 1+E_\gamma[n|D_1,T_Y(\gamma)] & \text{if} \quad \gamma \varepsilon \tilde{\Gamma}_1. \end{cases}$$

Let $H(\gamma) = \min\left\{E_\gamma[n|D_1,\gamma], E_\gamma[n|D_2,\gamma]\right\}$ then

$$(3) \qquad H(\gamma) = \begin{cases} E_\gamma[n|D_1,\gamma] & \text{for} \quad \gamma \varepsilon (H), \\ E_\gamma[n|D_2,\gamma] & \text{for} \quad \gamma \varepsilon \widetilde{(H)}, \end{cases}$$

$$\geq \begin{cases} 1+E_\gamma[n|D_1,T_X(\gamma)] & \text{for} \quad \gamma \varepsilon \Gamma_1 \cap (H) \cup \Gamma_2 \cap \widetilde{(H)}, \\ 1+E_\gamma[n|D_1,T_Y(\gamma)] & \text{for} \quad \gamma \varepsilon \tilde{\Gamma}_1 \cap (H) \cup \tilde{\Gamma}_2 \cap \widetilde{(H)}. \end{cases}$$

Now let $\Gamma = \Gamma_1 \cap (H) \cup \Gamma_2 \cap \widetilde{(H)}$. Then

$$(4) \qquad E[n/D,\gamma] = \begin{cases} 1+E_\gamma[n\ D,T_X(\gamma)] & \gamma \varepsilon \Gamma \quad , \\ 1+E_\gamma[n\ D,T_Y(\gamma)] & \gamma \varepsilon \Gamma \quad . \end{cases}$$

Then if one sets $G(\gamma) = H(\gamma) - E[n|D,\gamma]$,

$$(5) \qquad G(\gamma) \geq \begin{matrix} E_\gamma[G(T_X(\gamma))] & \gamma \varepsilon \Gamma , \\ E_\gamma[G(T_Y(\gamma))] & \gamma \varepsilon \tilde{\Gamma}. \end{matrix}$$

Now it is asserted that $G \geq 0$, for suppose that it assumes its minimum at γ_0. Then

$$(6) \qquad G(\gamma_0) = \begin{cases} E_{\gamma_0}[G(T_X(\gamma_0))] & \gamma_0 \varepsilon \Gamma , \\ E_{\gamma_0}[G(T_Y(\gamma_0))] & \gamma_0 \varepsilon \tilde{\Gamma} . \end{cases}$$

But for any random variable, Z, if $E[Z] = \min Z$, then $Z = \min Z$ with probability 1. Therefore, $G(\gamma_0) = G(T_X(\gamma_0))$ for $\gamma \varepsilon \Gamma$. Since

$$(7) \qquad \begin{aligned} T_X(\gamma_0) &= \gamma + \log \frac{f_2(X)}{f_1(X)} , \\ G(\gamma_0) &= G(\gamma + \log \frac{f_2(X)}{f_1(X)}). \end{aligned}$$

Similarly, if $\gamma_0 \varepsilon \tilde{\Gamma}$,

$$G(\gamma_0) = G(\gamma_0 + \log \frac{g_2(Y)}{g_1(Y)}) .$$

Since both $\log \frac{f_2(X)}{f_1(X)}$ and $\log \frac{g_2(Y)}{g_1(Y)}$ are negative with positive probability, it is seen that by a finite number of applications of the above reasoning a point $\gamma' \leq -A$ can be reached such that $G(\gamma_0) = G(\gamma')$. But $G(\gamma) = 0$ for $|\gamma| \geq A$. Hence, $G(\gamma) \geq 0$. In view of the definition of G, the proof is complete.

It is clear that the same analysis would apply to any finite number of stationary designs.

5. The 'Two-armed Bandit'.

5.1. General Results. The statistical problem which goes under this general title is that of finding a design which will maximize the sum of n independent observations in the following situation: let X and Y be real valued random variables having c.d.f.'s F_1 and G_1, respectively, under hypothesis H_i $(i = 1,2)$ and ζ be the *a priori* probability that H_1 is the true hypothesis. The problem is to devise a sequential design which will maximize the expected value of the sum of n observations, each of which is to be an observation either of X or Y.

Let f_i and g_i be the densities corresponding to F_i and G_i with respect to the measure Ψ. Let $W_n(\zeta, \delta^*)$ denote the expected value of the sum of the n observations if ζ is the *a priori* probability for H_1 and the optimal design, δ^*, is used. If one observed X first and then continued for n-1 steps following the optimal rule, then the expected sum would be

$$(5.1.1) \qquad A_n = \zeta \int_{-\infty}^{\infty} tf_1(t)d\Psi + (1-\zeta)\int_{-\infty}^{\infty} tf_2(t)d\Psi$$
$$+ \int_{-\infty}^{\infty} W_{n-1}\left(\frac{\zeta f_1(t)}{\zeta f_1(t)+(1-\zeta)f_2(t)}, \delta^*\right)(\zeta f_1(t)+(1-\zeta)f_2(t))d\Psi.$$

Similarly, if Y were observed first and the optimal rule followed for the remaining n-1 steps, the expected sum would be

$$(5.1.2) \qquad B_n = \zeta \int_{-\infty}^{\infty} tg_1(t)d\Psi + (1-\zeta)\int_{-\infty}^{\infty} tg_2(t)d\Psi$$
$$+ \int_{-\infty}^{\infty} W_{n-1}\left(\frac{\zeta g_1(t)}{\zeta g_1(t)+(1-\zeta)g_2(t)}, \delta^*\right)(\zeta g_1(t)+(1-\zeta)g_2(t))d\Psi .$$

Hence, $W_n(\zeta, \delta^*) = \max(A_n, B_n)$.

A natural design to be considered is that which requires that one maximize step by step, i.e., after the j^{th} observation the *a posteriori* probability, ζ_j, is computed and at the next step observe the random variable corresponding to the maximum of $\int_{-\infty}^{\infty} t(\zeta_j f_1(t)+(1-\zeta_j)f_2(t))d\Psi$ and $\int_{-\infty}^{\infty} t(\zeta_j g_1(t)+(1-\zeta_j)g_2(t))d\Psi$. Denote this stepwise maximization design by δ_o.

<u>Theorem</u> 5.1. If the likelihood ratios f_2/f_1 and g_2/g_1 have the same distributions under H_1 and also under H_2, then δ_0 is the optimal design.

<u>Proof</u>. Since,

$$(1) \quad \zeta_1 = \begin{cases} \dfrac{1}{1+\dfrac{1-\zeta}{\zeta}\dfrac{f_2(x)}{f_1(x)}} & \text{if X is observed first,} \\[2ex] \dfrac{1}{1+\dfrac{1-\zeta}{\zeta}\dfrac{g_2(x)}{g_1(x)}} & \text{if Y is observed first,} \end{cases}$$

and the likelihood ratios have the same distributions, the distribution of ζ_1 is independent of which random variable is observed first. Hence, the expected value of the optimal yield from the last n-1 steps is independent of the choice for the first step. One can, therefore, maximize the expected sum of n observations by choosing at the first step the random variable having the larger expected value and continuing with the optimal design for the remaining steps.

Since all the random variables are assumed to be independent, the same argument shows that, given ζ_j, it is optimal to follow δ_0 for the $j+1^{st}$ step.

An example in which the likelihood ratios are distributed alike is:

	X	Y
H_1	$N(0,1)$	$N(\mu,1)$
H_2	$N(\mu,1)$	$N(0,1)$.

If the above example is modified to destroy the symmetry, e.g.,

	X	Y	
H_1	$N(0,1)$	$N(\lambda,1)$	
H_2	$N(\mu,1)$	$N(0,1)$	$(\mu \neq \lambda,\ \mu > 0,\ \lambda > 0)$,

then δ_c is not optimal. If it were, then for $n = 2$ and ζ such that $(1-\zeta)\mu = \zeta\lambda$ there would be indifference as to the first step. If X is observed at the first step, then the expected sum of the two observations is

$$(5.1.3) \qquad (1-\zeta)\mu + (1-\zeta)\mu \Pr(\zeta_1\lambda < (1-\zeta_1)\mu \mid H_2) + \zeta\lambda \Pr(\zeta_1\lambda > (1-\zeta_1)\mu \mid H_1),$$

or

$$(1-\zeta)\mu + (1-\zeta)\mu \int_{\frac{1}{\mu}\log\frac{\zeta\lambda}{(1-\zeta)\mu} - \frac{\mu}{2}}^{\infty} \frac{1}{2\pi} e^{-\frac{t^2}{2}}\,dt + \zeta\lambda \int_{-\infty}^{\frac{1}{\mu}\log\frac{\zeta\lambda}{(1-\zeta)\mu} + \frac{\mu}{2}} \frac{1}{2\pi} e^{-\frac{t^2}{2}}\,dt .$$

If, on the other hand, Y is observed first, the expected sum would be

$$(5.1.4) \qquad \zeta\lambda + \zeta\lambda \int_{-\frac{1}{\lambda}\log\frac{\zeta\lambda}{(1-\zeta)\mu} - \frac{\lambda}{2}}^{\infty} \frac{1}{2\pi} e^{-\frac{t^2}{2}}\,dt + (1-\zeta)\mu \int_{-\infty}^{-\frac{1}{\lambda}\log\frac{\zeta\lambda}{(1-\zeta)\mu} + \frac{\lambda}{2}} \frac{1}{2\pi} e^{-\frac{t^2}{2}}\,dt .$$

But since $(1-\zeta)\mu = \zeta\lambda$, if δ_o were optimal there would be indifference as to the first step and hence

$$(5.1.5) \qquad \int_{-\frac{\mu}{2}}^{\infty} e^{-\frac{t^2}{2}}\,dt + \int_{-\infty}^{\mu/2} e^{-\frac{t^2}{2}}\,dt = \int_{-\frac{\lambda}{2}}^{\infty} e^{-\frac{t^2}{2}}\,dt + \int_{-\infty}^{\lambda/2} e^{-\frac{t^2}{2}}\,dt ,$$

which implies that $\lambda = \mu$.

5.2. <u>The 'Two-armed Bandit' in the Binomial Case</u>. A special case of the Two-armed Bandit of widespread interest is that in which the random variables have binomial distributions with parameters given by:

	X	Y
H_1	p	q
H_2	q	p

A second example in which the likelihood ratios are distributed alike is furnished here in the case $p+q=1$. Hence, for that case, δ_o is the optimal design. Indeed, it is a conjecture of Blackwell's that in any case δ_o is the optimal design.

Before considering the question of δ_o being optimal it will be shown that it has the desirable property of being consistent.

Theorem 5.2. Following the design δ_o, the expected value of the average of the first n observations converges to $\max(p,q)$ as $n \to \infty$.

Proof. Assume that $p > q$. Then,

(1) if $\zeta > 1/2$,

$$W_n(\zeta, \delta_o) = P_\zeta(X=1) + W_{n-1}\left(\frac{\zeta p}{P_\zeta(X=1)}, \delta_o\right)P_\zeta(X=1) + W_{n-1}\left(\frac{(1-p)\zeta}{P_\zeta(X=0)}, \delta_o\right)P_\zeta(X=0);$$

(2) if $\zeta < 1/2$,

$$W_n(\zeta, \delta_o) = P_\zeta(Y=1) + W_{n-1}\left(\frac{q\zeta}{P_\zeta(Y=1)}, \delta_o\right)P_\zeta(Y=1) + W_{n-1}\left(\frac{(1-q)\zeta}{P_\zeta(Y=0)}, \delta_o\right)P_\zeta(Y=0);$$

where $P_\zeta(Z=c) = \zeta P(Z=c|H_1) + (1-\zeta)P(Z=c|H_2)$.

Let $a_n(\zeta, \delta_o) = \frac{1}{n} W_n(\zeta, \delta_o)$. Then $a_n(\zeta, \delta_o)$ is monotone increasing in n and is bounded from above by p for all n. Let $a(\zeta, \delta_o) = \lim_{n\to\infty} a_n(\zeta, \delta_o)$. Since $a_n(\zeta, \delta_o)$ is convex and continuous in ζ for $0 \le \zeta \le 1$, $a(\zeta, \delta_o)$ is also. Further, since $na_n(\zeta, \delta_o)$ satisfies (1) and (2),

$$(3)\quad a(\zeta, \delta_o) = \begin{cases} a\left(\frac{p\zeta}{P_\zeta(X=1)}, \delta_o\right)P_\zeta(X=1) + a\left(\frac{(1-p)\zeta}{P_\zeta(X=0)}, \delta_o\right)P_\zeta(X=0), & \zeta > 1/2 \\ a\left(\frac{q\zeta}{P_\zeta(Y=1)}, \delta_o\right)P_\zeta(Y=1) + a\left(\frac{(1-q)\zeta}{P_\zeta(Y=0)}, \delta_o\right)P_\zeta(Y=0), & \zeta < 1/2. \end{cases}$$

Suppose that the minimum of $a(\zeta, \mathcal{S}_0)$ is assumed at $\zeta_0 > 1/2$. Then it also assumes its minimum at $p\zeta_0/P_{\zeta_0}(X=1) > \zeta_0$. By iteration, it assumes its minimum at $\dfrac{p^n \zeta_0}{p^n\zeta_0 + q^n(1-\zeta_0)}$ which tends to 1 as $n \to \infty$. Hence, ζ_0 could be taken to be 1. If on the other hand, $\zeta_0 < 1/2$, the analogous procedure shows that ζ_0 could be taken to be 0. Thus, the minimum of $a(\zeta, \mathcal{S}_0)$ is assumed either at 0 or 1. But $a(0, \mathcal{S}_0) = a(1, \mathcal{S}_0) = p$, which establishes the theorem.

If one lets Z_n be the average of the first n observations then if $\mathcal{S}_0$ is used, $E(Z_n) \to p$. Furthermore, $E(Z_n) \leq E(Z_{n+1})$ and it is seen that the sequence $\{Z_n\}$ forms a lower semimartingale. From the results of martingale theory [7] it can be concluded also that $Z_n \to p$ with probability 1.

5.3 The Question of Optimal Design. Company was joined with that sizeable group who have jousted with the problem of finding the optimal design for the Two-armed Bandit problem as described in the preceding section. Efforts were directed to proving Blackwell's conjecture that $\mathcal{S}_0$ is optimal and, while not meeting with complete success, the belief remains that the conjecture is correct. A brief outline of two lines of attack used concludes Section 5.

Let $X\mathcal{S}_0$ denote that design calling for X first followed by the use of $\mathcal{S}_0$ for the remaining steps. Consider $W_m(\zeta, X\mathcal{S}_0) - W_m(\zeta, Y\mathcal{S}_0)$; it is equal to $(2\zeta-1)(p-q)$ for $m=1$. The induction hypothesis was made that the difference was positive for $\zeta > 1/2$ for all $m < n$. In computing this difference for small values of n it appeared that the case $p^n(1-p) < q^n(1-q)$ gave the smallest difference. For ζ near 1/2, but greater than 1/2, and

$p^n(1-p) < q^n(1-q)$, $\mathcal{L}_0$ coincides with the rule, $\mathcal{R}$, which requires that when an observation is a success (1 is observed) the same random variable is to be observed at the next step, but if a failure occurs (0 is observed) then at the next step observe that random variable which has failed the fewest times or, if the number of failures are the same, observe next the one which has succeeded the greatest number of times; in case of ties, observe X next. It is easily established by induction that

$$W_n(\zeta, X\mathcal{R}) - W_n(\zeta, Y\mathcal{R}) = (2\zeta - 1)(p^n - q^n) \quad . \tag{5.3.1}$$

Efforts were directed towards showing that

$$W_n(\zeta, X\mathcal{L}_0) - W_n(\zeta, Y\mathcal{L}_0) > W_n(\zeta, X\mathcal{R}) - W_n(\zeta, Y\mathcal{R}) \quad ,$$

at least for ζ near 1/2 and greater than 1/2, by considering the adjustments which would have to be made in play according to $X\mathcal{R}$ and $Y\mathcal{R}$ to make them coincide with play according to $X\mathcal{L}_0$ and $Y\mathcal{L}_0$, respectively. For adjustments in $Y\mathcal{R}$ required at points where $Y\mathcal{R}$ called for Y but $Y\mathcal{L}_0$ called for X and for the symmetric points in adjusting the play starting with X, it was possible to establish that the adjustments were of the proper sign. For the other types of adjustments the attempt to prove that the signs were such as would accomplish the proof was unsuccessful. However, it appeared in the work that the difficult adjustments did not arise for $n \leq 5$ and that for n up to 8 they could be satisfactorily accounted for; hence, the conjecture holds for $n < 9$.

A second attack was made along the following line. For $\zeta > 1/2$ but near 1/2, let k_1 be that number such that if at least k_1 successes precede the first failure then the random variable which failed is observed at the next step; let k_2 be that number such that if there have been at least $k_1 + k_2$

successes before the second failure then the random variable which failed is observed at the next step; etc. Then the contribution to $W_n(\zeta, X\mathcal{S}_o)$ and to $W_n(\zeta, Y\mathcal{S}_o)$ from those sequences containing no failures, one failure, two failures, and three failures was computed and combined successively to obtain the expression for their contribution to $W_n(\zeta, X\mathcal{S}_c)-W_n(\zeta, Y\mathcal{S}_c)$. From the early work it appeared that an inductive pattern would persist in these expressions, as sequences containing more and more failures were added, which would allow one to write out $W_n(\zeta, X\mathcal{S}_o)-W_n(\zeta, Y\mathcal{S}_o)$ in terms of the k_i and verify that it was positive. Upon reaching sequences containing three and four failures the attempt to force the contributions into the previously noted patterns has been unsuccessful.

6. <u>Acknowledgements</u>.

I wish to express by indebtedness to Professor S. Karlin with whom this research was carried out and who gave help and guidance throughout the work, and to Professor M. A. Girshick for his suggestions as to areas of investigation. To the Office of Naval Research I am grateful for financial aid.

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