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**On The Motion of A
Slightly Deformed Projectile**

A. S. GALBRAITH

DEPARTMENT OF THE ARMY PROJECT No. 503-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108

BALLISTIC RESEARCH LABORATORIES



ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

The equations of motion of a body with slight asymmetries, such as a bomb with a bent tail, are derived, using linear aerodynamics.

The "resonant" effect of coincidence between yaw and spin rates is discussed. Conditions that the spin and yaw be "locked in" (spin rate equals yaw rate) are found, applicable in the simpler cases.

I. INTRODUCTION

Because of manufacturing tolerances, damage in handling, etc., some projectiles differ appreciably in form or weight distribution from the perfect symmetry assumed in ordinary ballistic theory. This report sets up the differential equations of motion of a projectile with certain small asymmetries, and discusses qualitatively some features of the motion.*

To fix the ideas, the analysis will be made for a rigid bomb with a symmetrical body whose principal axis of inertia coincides with the axis of symmetry. The tail will be assumed weightless, with its center of pressure not on the axis of the body and its lines of zero lift and moment not parallel to this axis. If the body is unbalanced, it can be considered aerodynamically as a balanced body with an appropriate fictitious tail, and the analysis will have the same form.

II. SYSTEM OF FORCES AND MOMENTS

Let $\underline{x}_1, \underline{x}_2, \underline{x}_3$ be a right handed system of unit vectors on axes rigidly attached to the projectile, with origin at the center of gravity, \underline{x}_1 pointing in the forward direction along the axis of symmetry (hereafter called the axis), and the other axes so chosen that the center of pressure** of the tail is at $-d\underline{x}_1 + h\underline{x}_2$, where d is the diameter of the body. Let \underline{z} be a unit vector along the trajectory. Let $\underline{\Omega}$ be the vector angular velocity of the projectile, measured with respect to fixed axes. When resolved on the moving axes, $\underline{\Omega} = \Omega_1 \underline{x}_1 + \Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3$. Let \underline{u} be the vector velocity of the center of gravity, $\underline{u} = u \underline{z}$. See Figure 1.

The line of zero normal force for the tail*** will be the unit vector $\underline{\lambda} = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3$, where λ_1 is nearly unity.

*Somewhat less general treatments of the problem have been given before; see Ref. 3.

**Center of pressure for normal force. Since the analysis is intended chiefly to suggest the form of the system of forces and moments, not numerical relations among the members, the fact that the center of pressure for Magnus force, say, is not at this point can be taken care of by the size of the corresponding aerodynamic force coefficient. It is assumed that all centers of pressure for the tail lie on a line through the center of gravity.

***By the normal force on the tail will be meant the difference between the normal force on the complete bomb with straight tail and on the body alone, as measured, for example, in a wind tunnel.

The following forces and moments will be assumed to act:

1. Gravity.

2. (a) A force on the body whose axial and cross components are respectively given by the axial and cross components of the velocity of a representative point on the axis; i.e., the components of $u\mathbf{z} + L\boldsymbol{\Omega} \times \mathbf{x}_1$, where L is a constant length. This force includes axial drag, normal force, and cross-spin (pitching) force.

(b) A corresponding force on the tail, with the resolution along and perpendicular to the line of zero normal force. The velocity components are found by writing the velocity of the center of pressure of the tail as:

$$u(\mathbf{z} \cdot \mathbf{z}') \mathbf{z}' + u [\mathbf{z} - (\mathbf{z} \cdot \mathbf{z}') \mathbf{z}'] + \boldsymbol{\Omega} \times (-td\mathbf{x}_1 + hd\mathbf{x}_2).$$

3. (a) A force on the body given by $\boldsymbol{\Omega} \times (u\mathbf{z} + L\boldsymbol{\Omega} \times \mathbf{x}_1)$, the Magnus force and Magnus cross force. This generalizes the Magnus force on a baseball, $\boldsymbol{\Omega} \times \mathbf{u}$.

(b) A similar force on the tail, given by $\boldsymbol{\Omega} \times \left\{ u(\mathbf{z} \cdot \mathbf{z}') \mathbf{z}' + u [\mathbf{z} - (\mathbf{z} \cdot \mathbf{z}') \mathbf{z}'] + \boldsymbol{\Omega} \times (-td\mathbf{x}_1 + hd\mathbf{x}_2) \right\}$.

4. The moments of these forces, given by $\mathbf{x}_1 \times (u\mathbf{z} + L\boldsymbol{\Omega} \times \mathbf{x}_1)$, etc.

5. Other moments: Spin damping, spin accelerating if the fins are canted, and moments resulting from non-coincidence of the lines of zero normal force and moment of the tail.

To compute the detailed form of these forces and moments, write:

$u\mathbf{z} = u_1 \mathbf{x}_1 + u_2 \mathbf{x}_2 + u_3 \mathbf{x}_3$. Then $u_1/u = \cos \delta$, where δ is the angle of yaw, and $(u_2^2 + u_3^2)/u^2 = \sin^2 \delta$. Then 2(a) gives forces proportional to $u_1 \mathbf{x}_1$ and $(u_2 \mathbf{x}_2 + u_3 \mathbf{x}_3) + L (\boldsymbol{\Omega} \times \mathbf{x}_2 - \boldsymbol{\Omega} \times \mathbf{x}_3)$.

The factors of proportionality must have the dimensions MT^{-1} , and aerodynamic theory and experience indicate that they should be $\rho d^2 uK$, where

ρ is the density of the air and K is a dimensionless constant. Then these forces will be written

$$(1) \quad -\rho d^2 u u_1 K_{DA1} \underline{x}_1 - \rho d^2 u K_{N1} (u_2 \underline{x}_2 + u_3 \underline{x}_3) - \rho d^3 u K_{S1} (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3),$$

the axial drag, normal force, and cross-spin force, with the subscript one indicating that the body is referred to. The minus signs make K_{DA1} and K_{N1} positive.

In working out 2(b) it will be assumed that u_2/u , u_3/u , l_2 , and l_3 are so small that the product of two of them is negligible. Also, l_1 will be taken as unity. The coefficients $-\rho d^2 u K_{DA2}$, $-\rho d^2 u K_{N2}$, and $\rho d^2 u K_{S2}$ will be applied to $u(\underline{x} \cdot \underline{z}') \underline{z}'$, $u [\underline{x} - (\underline{x} \cdot \underline{z}') \underline{z}']$, and $\underline{\Omega} \times (-td\underline{x}_1 + h\underline{x}_2)$ respectively. The result is

$$(2) \quad -\rho d^2 u u_1 K_{DA2} (\underline{x}_1 + l_2 \underline{x}_2 + l_3 \underline{x}_3) - \rho d^2 u K_{N2} (u_2 \underline{x}_2 + u_3 \underline{x}_3) \\ + \rho d^2 u u_1 K_{N2} (l_2 \underline{x}_2 + l_3 \underline{x}_3) - \rho d^3 u K_{S2} (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3) \\ + \rho d^3 u K_{S2} (h/t) (\Omega_3 \underline{x}_1 - \Omega_1 \underline{x}_3).$$

The Magnus forces in 3(a) are proportional to:

$$\Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3), u_1 (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3), \Omega_1 (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3),$$

and $(\Omega_2^2 + \Omega_3^2) \underline{x}_1$. The first gives the ordinary Magnus force,

$\rho d^3 K_{F1} \Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3)$. The second has the same form as the cross-spin

force, except for the factor u_1 instead of u . Since the assumption

$\cos \delta = 1$ will be made later, this force will be included in the cross-spin force of 2(a). The third term gives the Magnus force due to cross-

spin, $\rho d^4 K_{XF1} \Omega_1 (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3)$. The fourth term is of second

order in small quantities; it would give a force $\rho d^4 K (\Omega_2^2 + \Omega_3^2)$

or $\rho d^4 u^2 K (\Omega_2^2/u^2 + \Omega_3^2/u^2)$, where the parenthesis is of the order

of δ^2 , so it will be dropped. Then the Magnus forces on the body will be written

$$(3) \quad -\rho d^3 K_{F1} \Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3) + \rho d^4 K_{XF1} \Omega_1 (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3).$$

In computing the forces from 3(b), the velocities along and normal to the vector $l_1 \underline{x}_1 + l_2 \underline{x}_2 + l_3 \underline{x}_3$ will be kept separate. Thus $(\underline{z} \cdot \underline{z}') \underline{z}'$ will be written $u_1 \underline{x}_1/u$, neglecting small terms, and $\underline{z} - (\underline{z} \cdot \underline{z}') \underline{z}'$ will be written $(u_2/u - u_1 l_2/u) \underline{x}_2 + (u_3/u - u_1 l_3/u) \underline{x}_3$, the term $-(u_2 l_2/u + u_3 l_3/u) \underline{x}_1$ being dropped because it is of second order in small quantities. Then 3(b) gives terms

$$u_1 (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3), -\Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3), \Omega_1 u_1 (l_3 \underline{x}_2 - l_2 \underline{x}_3),$$

$$-td \Omega_1 (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3), \text{ and } -td (h/t) \Omega_1^2 \underline{x}_2; \text{ terms in } \Omega_2^2,$$

Ω_3^2 , and $\Omega_2 (h/t)$ have been dropped because (h/t) is regarded as small.

The first term will be absorbed in the K_{S2} term as before, the second gives the ordinary Magnus force on the tail, the third gives the Magnus force due to the tail's line of zero lift not being parallel to the axis, the fourth is the Magnus cross-force, and the fifth is a Magnus force resulting from the fact that the tail has a velocity $hd \Omega_1 \underline{x}_3$ across the air stream when the missile is spun. Then the Magnus forces on the tail will be written

$$-pd^3 \Omega_1 K_{F2} (u_3 \underline{x}_2 - u_2 \underline{x}_3) - pd^4 \Omega_1 K_{XF2} (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3)$$

(4)

$$+ pd^3 u_1 \Omega_1 K_{F2} (l_3 \underline{x}_2 - l_2 \underline{x}_3) - pd^4 (h/t) t K_{XF2} \Omega_1^2 \underline{x}_2.$$

In adding (1), (2), (3), (4), terms of the same form will be combined and a coefficient K with no numerical subscript will be assigned. It is not intended to imply, for example, that K_F below is $K_{F1} + K_{F2}$, but merely that a force proportional to $\Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3)$ may be expected to appear. ($K_{F1} + K_{F2}$ may be a useful first estimate.) Then the aerodynamic force on the projectile will be assumed to be:

$$-pd^2 uu_1 K_{DA} \underline{x}_1 - pd^2 uK_N (u_2 \underline{x}_2 + u_3 \underline{x}_3) + pd^3 uK_S (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3)$$

$$-pd^3 K_F \Omega_1 (u_3 \underline{x}_2 - u_2 \underline{x}_3) + pd^4 K_{XF} \Omega_1 (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3)$$

(5)

$$\begin{aligned}
& + \rho d^2 uu_1 K_{L2} (\ell_2 \underline{x}_2 + \ell_3 \underline{x}_3) + \rho d^3 u_1 \Omega_1 K_{F2} (\ell_3 \underline{x}_2 - \ell_2 \underline{x}_3) \\
& - \rho d^4 \beta t K_{XF2} \Omega_1^2 \underline{x}_2 + \rho d^3 u \beta K_{S2} (\Omega_3 \underline{x}_1 - \Omega_1 \underline{x}_3).
\end{aligned}$$

Here $\beta = h/t$ and $K_{N2} - K_{DA2} = \bar{K}_{L2}$, the lift coefficient of the tail.

The moments of the forces about the center of gravity can be found thus: The forces (1) and (3), acting on the body, are assumed to act through points on the axis. Then their moments can be found by taking the vector products with \underline{x}_1 , raising the powers of d by one, and changing the aerodynamic coefficients to allow for the positions of the centers of pressure. If the new K 's are given the additional subscript M , the result is:

$$\rho d^3 u K_{MN1} (u_3 \underline{x}_2 - u_2 \underline{x}_3) - \rho d^4 u K_{MS1} (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3)$$

(6)

$$- \rho d^4 K_{MF1} \Omega_1 (u_2 \underline{x}_2 + u_3 \underline{x}_3) + \rho d^5 K_{MF1} \Omega_1 (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3).$$

The forces (2) and (4) are assumed to act through points on the vector $-td \underline{x}_1 + hd \underline{x}_2$. Therefore, this vector must be used instead of \underline{x}_1 to find the moments. The result is:

$$\begin{aligned}
& - \rho d^3 ut K_{N2} (u_3 \underline{x}_2 - u_2 \underline{x}_3) + \rho d^4 ut K_{S2} (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3) \\
& - \rho d^4 \Omega_1 t K_{F2} (u_2 \underline{x}_2 + u_3 \underline{x}_3) + \rho d^5 \Omega_1 t K_{XF2} (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3) \\
& + \rho d^3 uu_1 t K_{L2} (\ell_3 \underline{x}_2 - \ell_2 \underline{x}_3) - \rho d^4 \Omega_1 t K_{F2} (\ell_2 \underline{x}_2 + \ell_3 \underline{x}_3) \\
& + \beta \rho d^3 \left\{ (-ut K_{N2} u_3 + uu_1 t K_{L2} \ell_3 - dut K_{S2} \Omega_2 + d \Omega_1 t K_{F2} u_2 \right. \\
& - d^2 \Omega_1 t K_{XF2} \Omega_3 - du_1 \Omega_1 t K_{F2} \ell_2 - \beta du \Omega_1 t K_{S2} \underline{x}_1 - du \Omega_1 t K_{S2} \underline{x}_2 \\
& \left. + (uu_1 t K_{DA2} + d^2 t^2 \Omega_1^2 K_{XF2}) \underline{x}_3 \right\}.
\end{aligned}$$

Some small terms, e.g., in $\beta^2 K_{S2}$ and $\beta K_{F2} l_2$, have been kept; not because they are effective but because they are easily seen physically.

When these moments are added to those in (6), corresponding terms will be lumped together; e.g., $\rho d^3 u(K_{M1} - tK_{N2}) (u_3 \underline{x}_2 - u_2 \underline{x}_3)$ will be written $\rho d^3 u K_M (u_3 \underline{x}_2 - u_2 \underline{x}_3)$. The notation will be according to the following scheme:

Force Coeff.	Associated Moment Coeff.
K_N	K_M
K_S	K_H
K_F	K_T
K_{XF}	K_{XT}

The coefficients of moments due to asymmetry, (those containing l_2, l_3 , or h/t) will be left as they are; not because tK_{S2} , for example, is necessarily the correct coefficient, but to show the derivation and help in making rough estimates. The system of moments, given by 4. above, is then:

$$\rho d^3 \left\{ u K_M (u_3 \underline{x}_2 - u_2 \underline{x}_3) - d u K_H (\Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3) \right. \\ \left. - d \Omega_1 K_T (u_2 \underline{x}_2 + u_3 \underline{x}_3) - d^2 \Omega_1 K_{XT} (\Omega_3 \underline{x}_2 - \Omega_2 \underline{x}_3) \right. \\ \left. + u u_1 t K_{L2} (l_3 \underline{x}_2 - l_2 \underline{x}_3) - d u_1 \Omega_1 t K_{F2} (l_2 \underline{x}_2 + l_3 \underline{x}_3) \right\}$$

(7)

$$+ \beta \rho d^3 \left\{ (u u_1 t K_{L2} l_3 - u t K_{N2} u_3 - d u t K_{S2} \Omega_2 + d \Omega_1 t K_{F2} u_2 \right. \\ \left. - d^2 \Omega_1 t K_{XF2} \Omega_3 - d u_1 \Omega_1 t K_{F2} l_2) \underline{x}_1 \right. \\ \left. - d u \Omega_1 t K_{S2} \underline{x}_2 + (u u_1 t K_{DA2} + d^2 t^2 \Omega_1^2 K_{XF2}) \underline{x}_3 \right\} .$$

In addition, there will be assumed: (a) Spin damping moments, proportional to $\Omega_1 \underline{x}_1$ for the body and $\{ \Omega \cdot (-t d \underline{x}_1 + h d \underline{x}_2) \}$ $(-t d \underline{x}_1 + h d \underline{x}_2)$ for the tail, the aerodynamic coefficients being called

K_{A1} and K_{A2} . (b) A spin generating moment on the tail, proportional to:

$\left\{ \epsilon - (r/u) \left[\Omega \cdot (-tdx_1 + hdx_2) \right] \right\} (-tdx_1 + hdx_2)$, where ϵ is the angle of cant of the fins and r is the distance from the axis of the tail to the center of pressure of one fin; the coefficient will be K_E . (c) If the axis of zero moment of the tail does not coincide with the axis of zero lift, an additional moment $\rho d^3 u^2 K_c (n_1 x_1 + n_2 x_2 + n_3 x_3)$ will be assumed, where $n_2^2 + n_3^2 = 1$ approximately and n_1 is small. When second order terms are dropped, these moments are:

$$(8) \quad \rho d^3 u \left\{ (uK_E \epsilon - rK_E \Omega_1 - dK_A \Omega_1 + uK_c n_1) x_1 + (-u\beta K_E \epsilon + r\beta K_E \Omega_1 + d\beta K_A \Omega_1 + uK_c n_2) x_2 + uK_c n_3 x_3 \right\}.$$

The second and third terms in each parenthesis will combine, as will the first and fourth, so only the first and second will be written. In any actual problem these combinations must not be forgotten. Then (7) and (8) give the moments considered hereafter.

The force of gravity will be represented by the vector

$$(9) \quad \underline{g} = g_1 x_1 + g_2 x_2 + g_3 x_3.$$

III. EQUATIONS OF MOTION

These will be:

$$m (d/dt) (u_1 x_1 + u_2 x_2 + u_3 x_3) = \text{sum of the forces, and}$$

$$(d/dt) (A \Omega_1 x_1 + B \Omega_2 x_2 + B \Omega_3 x_3) = \text{sum of moments about center of gravity,}$$

where m is the mass, A is the axial moment of inertia, and B the moment of inertia about a transverse axis through the center of gravity. Before writing down the equations in component form, the force equations will be divided by m and the moment equations by B . Wherever B divides an aerodynamic coefficient it will be replaced by mk^2d^2 , where k is the transverse

radius of gyration in calibers; and any combination $(\rho d^3/m)K$ will be replaced by J . The equations of motion are then, using (5), (7), (8) and (9),

$$(a) \quad \dot{u}_1 - u_2 \ell_3 + u_3 \Omega_2 = - (J_{DA}/d) uu_1 + u\beta J_{S2} \Omega_3 + \varepsilon_1,$$

$$(b) \quad \dot{u}_2 - u_3 \Omega_1 + u_1 \Omega_3 = -u(J_M/d) u_2 - uJ_S \Omega_3 - J_F \Omega_1 u_3 \\ + dJ_{XF} \Omega_1 \Omega_2 + uu_1 (J_{L2}/d) \ell_2 + u_1 J_{F2} \Omega_1 \ell_3 - \beta tdJ_{XF2} \Omega_1^2 + \varepsilon_2,$$

$$(c) \quad \dot{u}_3 - u_1 \Omega_2 + u_2 \Omega_1 = -u (J_M/d) u_3 + uJ_S \Omega_2 + J_F \Omega_1 u_2 \\ + dJ_{XF} \Omega_1 \Omega_3 + uu_1 (J_{L2}/d) \ell_3 - u_1 J_{F2} \Omega_1 \ell_2 - u\beta J_{S2} \Omega_1 + \varepsilon_3;$$

(10)

$$(d) \quad \dot{\Omega}_1 = (B/Ak^2d^2) \left\{ u^2 J_E \epsilon - udJ_A \Omega_1 + \beta [uu_1 tJ_{L2} \ell_3 - utJ_{N2} u_3 \right. \\ \left. - utdJ_{S2} \Omega_2 + \Omega_1 tdJ_{F2} u_2 - \Omega_1 td^2 J_{XF2} \Omega_3 - u_1 \Omega_1 tdJ_{F2} \ell_2 \right. \\ \left. - \beta u \Omega_1 tdJ_{S2} \right\},$$

$$(e) \quad \dot{\Omega}_2 - \Omega_1 \Omega_3 (1 - A/B) = \left[uJ_M u_3 - udJ_H \Omega_2 - \Omega_1 dJ_T u_2 \right. \\ \left. - \Omega_1 d^2 J_{XT} \Omega_3 + uu_1 tJ_{L2} \ell_3 - u_1 \Omega_1 tdJ_{F2} \ell_2 - u^2 \beta J_E \epsilon \right. \\ \left. + ud \beta J_A \Omega_1 - \beta u \Omega_1 tdJ_{S2} \right] / k^2 d^2,$$

$$(f) \quad \dot{\Omega}_3 + \Omega_1 \Omega_2 (1 - A/B) = \left[-uJ_M u_2 - udJ_H \Omega_3 - \Omega_1 dJ_T u_3 \right. \\ \left. + \Omega_1 d^2 J_{XT} \Omega_2 - uu_1 tJ_{L2} \ell_2 - u_1 \Omega_1 tdJ_{F2} \ell_3 + u^2 K_C n_3 \right]$$

$$+ \beta (u u_1 t J_{DA2} + \Omega_1^2 d^2 t^2 J_{XF2}) / k^2 d^2.$$

To these must be added the equations for g_1, g_2, g_3 , obtained by differentiating

$$g_1 x_1 + g_2 x_2 + g_3 x_3 = \text{constant:}$$

$$(g) \quad \dot{g}_1 - g_2 \Omega_3 + g_3 \Omega_2 = 0,$$

(10)

$$(h) \quad \dot{g}_2 - g_3 \Omega_1 + g_1 \Omega_3 = 0,$$

$$(i) \quad \dot{g}_3 - g_1 \Omega_2 + g_2 \Omega_1 = 0;$$

and the equation

(10)

$$(j) \quad u_1^2 + u_2^2 + u_3^2 = u^2.$$

Equations (10) give the motion of the projectile under the assumptions made.

The algebraic signs have been chosen thus:

K_M, K_H, K_F, K_{XF} are positive if the corresponding forces act ahead of the center of gravity; e.g., K_M is negative for a statically stable bomb. Aerodynamic coefficients for the tail alone are positive, provided the Magnus force is given in direction by (spin vector) \times (velocity vector), and acts behind the under of gravity.

Before proceeding with the analysis, dimensionless variables will be introduced. Let:

$$v = u/d, \quad v_i = u_i/d, \quad i = 1, 2, 3,$$

$$\omega_i = \Omega_i/v, \quad \sigma_i = g_i d/u^2, \quad i = 1, 2, 3, \quad \text{and } \alpha = \int_0^t v dt.$$

Since the system of aerodynamic forces and moments is only plausible for small yaws, let $u_1/u = \cos \theta = 1$. Let primes denote differentiation with respect to u . Then the equations of motion become:

$$(a) \quad v_1'/v - (v_2 \omega_3 - v_3 \omega_2)/v = -J_{DA} + \beta J_{S2} \omega_3 + G_1,$$

$$(b) \quad v_2'/v - (v_3 \omega_1 - v_1 \omega_3)/v = -J_N v_2/v - J_S \omega_3 - J_F \omega_1 v_3/v \\ + J_{XF} \omega_1 \omega_2 + J_{L2} \ell_2 + J_{F2} \omega_1 \ell_3 - \beta t J_{XF2} \omega_1^2 + G_2,$$

(11)

$$(c) \quad v_3'/v - (v_1 \omega_2 - v_2 \omega_1)/v = -J_N v_3/v + J_S \omega_2 + J_F \omega_1 v_2/v \\ + J_{XF} \omega_1 \omega_3 + J_{L2} \ell_3 - J_{F2} \omega_1 \ell_2 = \beta J_{S2} \omega_1 + G_3;$$

$$(d) \quad \omega_1' + \omega_1 v'/v = (B/Ak^2) \left[J_E \epsilon - J_A \omega_1 + \beta t (J_{L2} \ell_3 - J_{N2} v_3/v \right. \\ \left. - J_{S2} \omega_2 + \omega_1 J_{F2} v_2/v - \omega_1 J_{XF2} \omega_3 - \omega_1 J_{F2} \ell_2 - \beta \omega_1 J_{S2}) \right],$$

$$(e) \quad \omega_2' + \omega_2 v'/v - \omega_1 \omega_3 (1 - A/B) = \left[J_M v_3/v - J_H \omega_2 - \omega_1 J_T v_2/v \right. \\ \left. - \omega_1 J_{XT} \omega_3 + t J_{L2} \ell_3 - \omega_1 t J_{F2} \ell_2 - \beta J_E \epsilon + \beta J_A \omega_1 - \beta \omega_1 t J_{S2} \right] / k^2,$$

$$(f) \quad \omega_3' + \omega_3 v'/v + \omega_1 \omega_2 (1 - A/B) = \left[-J_M v_2/v - J_H \omega_3 - \omega_1 J_T v_3/v \right. \\ \left. + \omega_1 J_{XT} \omega_2 - t J_{L2} \ell_2 - \omega_1 t J_{F2} \ell_3 + J_C n_3 + \beta t J_{DA2} + \beta \omega_1^2 t^2 J_{XF2} \right] / k^2;$$

$$(g) \quad G_1' + 2G_1 v'/v - G_2 \omega_3 + G_3 \omega_2 = 0,$$

$$(h) \quad G_2' + 2G_2 v'/v - G_3 \omega_1 + G_1 \omega_3 = 0,$$

$$(i) \quad G_3' + 2G_3 v'/v - G_1 \omega_2 + G_2 \omega_1 = 0;$$

$$(j) \quad v^2 = v_1^2 + v_2^2 + v_3^2,$$

$$(k) \quad vv' = v_1 v_1' + v_2 v_2' + v_3 v_3'$$

Now velocities are in calibers/sec., angular velocities in radians/caliber of travel, and derivatives are taken with respect to calibers of travel; and $(v_2/v)^2 + (v_3/v)^2 = \sin^2 \delta$.

IV. THE YAWING MOTION

Let $(v_2 + iv_3)/v = -\zeta$, $\omega_2 + i\omega_3 = \mu$, and $G_2 + iG_3 = \Gamma$. Then from (11) (b) and (c), (e) and (f), and (h) and (i),

$$(a) \quad \zeta' + \zeta (v'/v + i\omega_1) + i\mu(v_1/v) = \zeta (-J_N + i\omega_1 J_F) - \mu(iJ_S + \omega_1 J_{XF}) \\ - (J_{I2} - iJ_{F2} \omega_1)(\ell_2 + i\ell_3) + i\beta J_{S2} \omega_1 + \beta t J_{XF2} \omega_1^2 - \Gamma,$$

(12)

$$(b) \quad \mu' + \mu v'/v + i\mu\omega_1(1-A/B) = \zeta (iJ_M + \omega_1 J_T) - \mu(J_H - i\omega_1 J_{XT}) \\ - (\ell_2 + i\ell_3)(itJ_{I2} + \omega_1 tJ_{F2}) + iJ_C n_3 + \beta(itJ_{DA2} + i\omega_1^2 t^2 J_{XF2}) \\ - J_E \epsilon + J_A \omega_1 - \omega_1 t J_{S2},$$

$$(c) \quad \Gamma' + 2\Gamma v'/v + i\omega_1 \Gamma - iG_1 \mu = 0$$

Then the complex numbers ζ , μ , Γ represent the sine of the angle of yaw, the component of the angular velocity perpendicular to the x_1 axis (cross component), and the component of gravity perpendicular to the x_1 axis, respectively.

It is now convenient to change to axes not fixed in the projectile. The substitutions

$$(13) \quad \zeta = \xi \exp \left(i \int_0^a \omega_1 da \right), \quad p = \mu \exp \left(i \int_0^a \omega_1 da \right),$$

$$G = \Gamma \exp \left(i \int_0^a \omega_1 da \right),$$

rotate the axes back as fast as the projectile spins forward. Then by proper choice of the zero point of a , the axes can be taken as (say) forward along the projectile's axis, and approximately down in a vertical plane containing the axis, and to the left, horizontally, for small yaws and over moderately long pieces of the trajectory.

Equations (12) become

$$(14) \quad (a) \quad \zeta' + \zeta v'/v + ipv_1/v = \zeta \left(-J_N + i\omega_1 J_F \right) - p \left(iJ_S + \omega_1 J_{IF} \right) - G$$

$$- \left[\left(J_{L2} - i\omega_1 J_{F2} \right) \left(\ell_2 + i\ell_3 \right) - i\beta\omega_1 J_{S2} - \beta\omega_1^2 t J_{IF2} \right] \exp \left(i \int_0^a \omega_1 da \right),$$

$$(b) \quad p' + p(v'/v - i\omega_1 A/B) = \left\{ \zeta \left(iJ_M + \omega_1 J_T \right) - p \left(J_H - i\omega_1 J_{IT} \right) \right.$$

$$+ \left[- \left(\ell_2 + i\ell_3 \right) \left(itJ_{L2} + \omega_1 tJ_{F2} \right) + iJ_C n_3 \right.$$

$$\left. \left. + \beta \left(itJ_{DA2} + i\omega_1^2 t^2 J_{IF2} - J_E \epsilon + J_A \omega_1 - \omega_1 tJ_{S2} \right) \right] \exp \left(i \int_0^a \omega_1 da \right) \right\} / k^2,$$

$$(c) \quad G' + 2Gv'/v - iGp = 0.$$

To get the differential equation for the yawing motion, differentiate (14)(a), substitute for p' and G' from (14)(b) and (c), and then substitute for p from (14)(a). Products of J 's will be omitted from coefficients of ζ and ζ' in the final equation, but retained (to the second order) in the right member. The exponential terms in (14)(a) and (b) will be denoted by q_1 and q_2 for convenience. Since $v_1/v = u_1/u = \cos \delta$, it will be set equal

to unity.* The coefficient of p in (14)(a) will be taken as constant, since $(J_{IT}/\omega_1)^2$ can hardly be appreciable in most cases. The expressions q_1 and q_2 will be treated as of second order. It is shown in Ref. 2 that good approximations to G_1 and G are

$$G_1 = - (g/v^2 d) \sin \theta,$$

$$G = (g/v^2 d) \sin \theta \zeta + (g/v^2 d) \cos \theta,$$

where θ is the angle from the horizontal to the trajectory, positive upwards; these approximations will be used in (14)(c) to compute G' and $(g/v^2 d)$ will be called γ and assumed small. All J 's will be treated as constants; their derivatives can hardly be larger than second order, except in very unusual cases. The quotient v'/v will be small for shells and bombs, but may be fairly large for rockets, so it will be left in wherever it appears. The equation for the yawing motion then becomes

$$\begin{aligned} & \zeta'' + \zeta' \left[2v'/v + J_N + J_H/k^2 + \gamma \sin \theta - i\omega_1 A/B - i\omega_1 (J_F + J_{IT}/k^2) \right] \\ & + \zeta \left[- J_H/k^2 + (v'/v)' + (v'/v)^2 + (v'/v)(J_N + J_H/k^2) - \omega_1^2 (A/B) J_F \right. \\ (15) & \\ & \left. - i\omega_1 (A/B)(v'/v + J_N + \gamma \sin \theta) + i\omega_1 J_F/k^2 - i\omega_1 (v'/v) J_F \right] \\ & = \gamma \cos \theta (v'/v - J_H/k^2 - \gamma \sin \theta + i\omega_1 A/B + iJ_{IT}/k^2) \\ & + q_1 (v'/v - i\omega_1 A/B) + q_1' - iq_2. \end{aligned}$$

(The reader may note that in the simple case of vertical fire in a vacuum, this equation is incorrect, because the term $\gamma^2 \sin^2 \theta$ in the coefficient of ζ' has been dropped. If this term is put in, the equation will give the correct result.)

*This is not really necessary here; the important assumption is that $(v_1/v)'$ is negligible. However, in combining certain force terms earlier this assumption was made, and certainly for the motions of greatest interest the yaw is small.

Except for some changes in notation, and the addition of the terms in q_1 and q_2 , this equation is identical with equation (1.23) of Ref. 1, as corrected in Ref. 2.

The yawing motion will have (usually) a transient oscillatory part, compounded of two damped circular motions with rates

$$(16) \quad (A\omega_1/2B)(1 \pm \sigma),$$

where $\sigma^2 = 1 - 1/s$, $s = A^2 \omega_1^2 k^2 / 4B^2 J_M$, and $1/s$ is < 1 ; and a "steady" yaw, a particular integral of equation (15). The formulas (16), and conditions for the damping of the transient motion, are discussed in Ref. 1.

V. RESONANCE

The steady yaw can be broken into two parts. The first is computed by putting $q_1 = q_2 = 0$, and is shown in Ref. 1 to give a small yaw causing drift.* The second can be computed by putting $\gamma = 0$ in the right member of (15). In general, no convenient analytical expression is available for this result. However, it is useful to consider the case of a projectile moving at nearly constant velocity, with ω_1 changing slowly.

If γ is set equal to zero in the right member of (15), the equation can be written

$$(17) \quad \zeta'' + (C + iD) \zeta' + (E + iF) \zeta = Q \exp\left(\int i\omega_1 da\right).$$

If a solution is sought in the form $\zeta = \exp\left(\int r da\right)$, where r is a slowly changing function, substitution gives.

$$\zeta = Q \exp\left(\int i\omega_1 da\right) / \left[r^2 + r + (C + iD)r + (E + iF) \right].$$

Now if the variation in Q , C , D , E and F can be neglected, this is a solution of (17) if $r = i\omega_1$. But the denominator is the left member of the Riccati equation obtained when seeking the expression for the transient oscillatory yaw. Thus if ω_1 is one of the natural rates (16), the denominator is very small; the phenomenon is similar to the resonance of a damped

*If a shell is fired at large angles of elevation, the yaw near the summit of the trajectory may be too large for the simple theory of this report.

oscillating system in one degree of freedom; the magnitude of ζ is restricted only by the amount of damping. This resonance can be illustrated by plotting $|Q / [-\omega_1^2 + i\omega_1(C + iD) + E + iF]|$ as a function of ω_1 , assuming that ω_1' is negligible.

Graphs to illustrate resonance in the motion of a fin-stabilized missile appear in Figures 3 and 4. To avoid a multiplicity of cases, several simplifying restrictions have been made in preparing these graphs. The procedure was this: v'/v and ω_1' were assumed negligible, and q_1' was written $i\omega_1 q_1$. Then

$$Q = - \left[t - i\omega_1(1-A/B) \right] (\lambda_2 + i\lambda_3) J_{L2} + J_C n_3 + \beta \left[t J_{DA2} + i(J_E \epsilon - \omega_1 J_A) \right] + \omega_1 \left\{ \left[t - i\omega_1(1-A/B) \right] \left[iJ_{F2}(\lambda_2 + i\lambda_3) + \beta(iJ_{S2} + \omega_1 t J_{XF2}) \right] \right\} = Q_1 + \omega_1 Q_2, \text{ say.}$$

Now a fin-stabilized projectile has $J_M < 0$; hence, by the definitions after (16), σ is > 1 . Then ω_1 can be equal to only one natural rate, that of the same sign.* If this occurs, (16) shows $\omega_1^2 = (-J_M/k^2)/(1-A/B)$, which for ordinary missiles is of the order of 10^{-4} . The values of ω_1 of most interest will then be near .01, so the term $\beta\omega_1 J_{XF2}$ in Q_2 was dropped. Also, if ω_1' is very small, $J_E \epsilon - \omega_1 J_A$, the principal part of ω_1' , (cf. (11)(d)) will be very small, and so can be omitted from Q_1 . Since t is likely to be 2 or more, $t - i\omega_1(1-A/B)$ was replaced by t . Then the approximations

$$Q_1 = - t J_{L2} (\lambda_2 + i\lambda_3) + J_C n_3 + \beta t J_{DA},$$

$$Q_2 = it(J_{F2} + \beta J_{S2}),$$

are reasonable. These approximations were not needed in making the graphs, but the argument above shows that useful results can be obtained near the resonant frequency by making the graphs as described below.

*If the projectile is spin-stabilized with $J_M > 0$, ω_1 cannot be equal to either rate.

The effect of Q_1 is shown conveniently by plotting the ratio of

$$|Q_1 / [-\omega_1^2 + i\omega_1(C + iD) + (E + iF)]|$$

to its value when $\omega_1 = 0$. This can be written as

$$|(E + iF)\omega_1 = \sqrt{[-\omega_1^2 + i\omega_1(C + iD) + E + iF]}|$$

Now if the rate $(A\omega_1/2B)(1+\sigma)$ is set equal to N , it follows that

$$-J_H/k^2 - N^2 = N\omega_1 A/B.$$

If $J_N(1-A/B) + J_H/k^2 + J_T/k^2$ is put equal to λ , the ratio can be written

$$[1 - (\omega_1/N)(A/B)] / \left\{ [1 - (\omega_1/N)(A/B) - (\omega_1/N)^2(1-A/B)]^2 + (\omega_1/N)^2(\lambda/N)^2 \right\}^{1/2},$$

which is plotted in Figure 3 as a function of ω_1/N . (λ/N is replaced by λ/N at $\omega_1 = N$.) This graph indicates how the effect of some eccentricities is amplified by spin.

The effect of Q_2 can be shown by plotting

$$(\omega_1/N) / (\text{same denominator as before}),$$

as is done in Figure 4. The size of the yaw produced can be found by multiplying the ordinate by $|Q_2|$; i.e., by $|t(J_{F2} + \mu J_{S2})|$.

VI. LOCKING IN

A tendency for the spin and yaw to lock in; i.e., for the projectile to spin at the resonant rate, so that the angle between the plane of yaw and the plane through the axis and the vector hdx_2 is constant, can be shown in certain rather idealized cases.

Consider a projectile on a straight trajectory, v'/v constant, with a circular yawing motion of constant rate. (For small spins, the rates (16) are not very sensitive to changes in the spin rate.) Figure 2 shows the Eulerian angles δ , β , ψ , and the axes X_1 , X_2 , X_3 fixed in direction in space, with origin at the c.g. and X_1 along the trajectory. If δ is small, $\Omega_1 = \dot{\beta} \cos \delta + \dot{\psi} = \dot{\beta} + \dot{\psi}$ approximately, so $\omega_1 = \beta' + \psi'$. From Figure 2 it can be seen that the complex yaw ζ can be written

$$\zeta = \delta e^{i(\beta - \pi/2)} = -i \delta e^{i\beta},$$

and further that

$$v_2/v = \delta \sin \psi, \quad v_3/v = \delta \cos \psi.$$

Now it can be seen that the component of angular velocity perpendicular to the axis is

$$i \dot{\zeta} = i e^{i\theta} (\dot{\delta} + i\delta\dot{\theta}).$$

Since $\dot{\delta} = 0$ here, the component wanted is

$$- \delta\dot{\theta}(\cos\theta + i\sin\theta),$$

from which

$$\omega_2 = \delta \dot{\theta} \cos \psi, \quad \omega_3 = - \delta \dot{\theta} \sin \psi.$$

Substitution for v_2 , v_3 and the ω 's in (11)(d), and neglect of $\dot{\theta}''$, give

$$(18) \quad \psi'' = \psi' (a_1 + a_2 \sin \psi) + a_3 \sin \psi + a_4 \cos \psi + a_5,$$

$$\text{where } a_1 = -v'/v - (J_A + hJ_{F2}l_2 + \beta hJ_{S2})/k_a^2,$$

$$a_2 = (h\delta/k_a^2)(J_{F2} + \theta' J_{IF2}),$$

$$a_3 = \theta' a_2,$$

$$a_4 = - (h\delta/k_a^2)(J_{M2} + \theta' J_{S2}),$$

$$a_5 = - \theta'(v'/v) + [J_E \epsilon - \theta' J_A + h(J_{I2}l_3 - J_{F2}l_2 - \beta J_{S2})] / k_a^2,$$

where $k_a^2 = B/Ak^2$; k_a is the axial radius of gyration in calibers.

Equation (18) has the solution

$$(19) \quad a_3 \sin \psi + a_4 \cos \psi + a_5 = 0,$$

$$\text{or } \sin(\psi + \eta) = -a_5 / [a_3^2 + a_4^2]^{1/2},$$

$$\text{where } \sin \eta = a_4 / [a_3^2 + a_4^2]^{1/2}, \quad \cos \eta = a_3 / [a_3^2 + a_4^2]^{1/2},$$

provided that the inequality

$$(20) \quad a_5^2 \leq a_3^2 + a_4^2$$

holds. This inequality is then necessary and sufficient that a solution $\psi = \text{constant}$ exist; i.e., that a "locked-in" state be possible. Then for $h\delta$ sufficiently small, locking in will not occur.

It is convenient to define the angle χ by the equations $\sin \chi = a_5 / (a_3^2 + a_4^2)^{1/2}$, $\cos \chi = (a_3^2 + a_4^2 - a_5^2)^{1/2} / (a_3^2 + a_4^2)^{1/2}$. Then

(19) is satisfied by

$$\psi = -\eta - \chi \text{ and } \psi = -\eta + \chi - \pi.$$

If δ' is small, as is often the case, $(a_3^2 + a_4^2)^{1/2} = h\delta J_{N2}/k_a^2$ approximately. Then $\sin \eta = -1$, $\eta = -\pi/2$ approximately. Thus

$$\psi = \pi/2 - \chi$$

or

$$\psi = \chi - \pi/2,$$

or from Figure 2, the plane of the bent tail differs from the plane of yaw by the angle χ .

Stability of the Locked-In State. Let ψ_0 be a constant solution of (18), and let $\psi = \psi_0 + \tau$ be another solution, with τ small. Then if the term in τ^2 is neglected, (19) gives

$$\tau'' - \tau'(a_1 + a_2 \sin \psi_0) - \tau(a_3 \cos \psi_0 - a_4 \sin \psi_0) = 0.$$

The discriminant of the characteristic equation is $(a_1 + a_2 \sin \psi_0)^2 + 4(a_3 \cos \psi_0 - a_4 \sin \psi_0)$. Then for stability; i.e., for τ to damp out, it is necessary and sufficient that $-(a_1 + a_2 \sin \psi_0)$ be greater than the numerical value of the real part of the square root of the discriminant. For this to be true, $4(a_3 \cos \psi_0 - a_4 \sin \psi_0)$ must be negative. Substitution shows that if $\psi_0 = -\eta - \chi$, $(a_3 \cos \psi_0 - a_4 \sin \psi_0) = (a_3^2 + a_4^2 - a_5^2)^{1/2}$ while if $\psi_0 = -\eta + \chi - \pi$ the negative of this appears. Hence stability is possible only in the latter case, and the necessary and sufficient conditions are:

$$(21) \quad \psi_0 = -\eta + \chi - \pi, \quad a_1 + a_2 \sin \psi_0 < 0.$$

If $\eta = -\pi/2$ approximately, $\psi_0 = \chi - \pi/2$; or if χ is small, the nose is yawed in about the same direction as the tail is bent.

An oversimplified physical picture is given in Figure 5, where $h_2 = h_3 = v' = 0$, and the normal force on the tail is assumed to dominate the others. Then $\eta = -\pi/2$ approximately. The projectile is viewed from the rear along the trajectory; A, T and P are the projections of the nose, the trajectory and the center of pressure of the tail on the plane of the paper. Directed arcs represent moments.

Figure 5 shows that, in the stable case, an increase in χ causes an opposing increase in the moment due to the normal force, while in the unstable case this moment increases χ still more.

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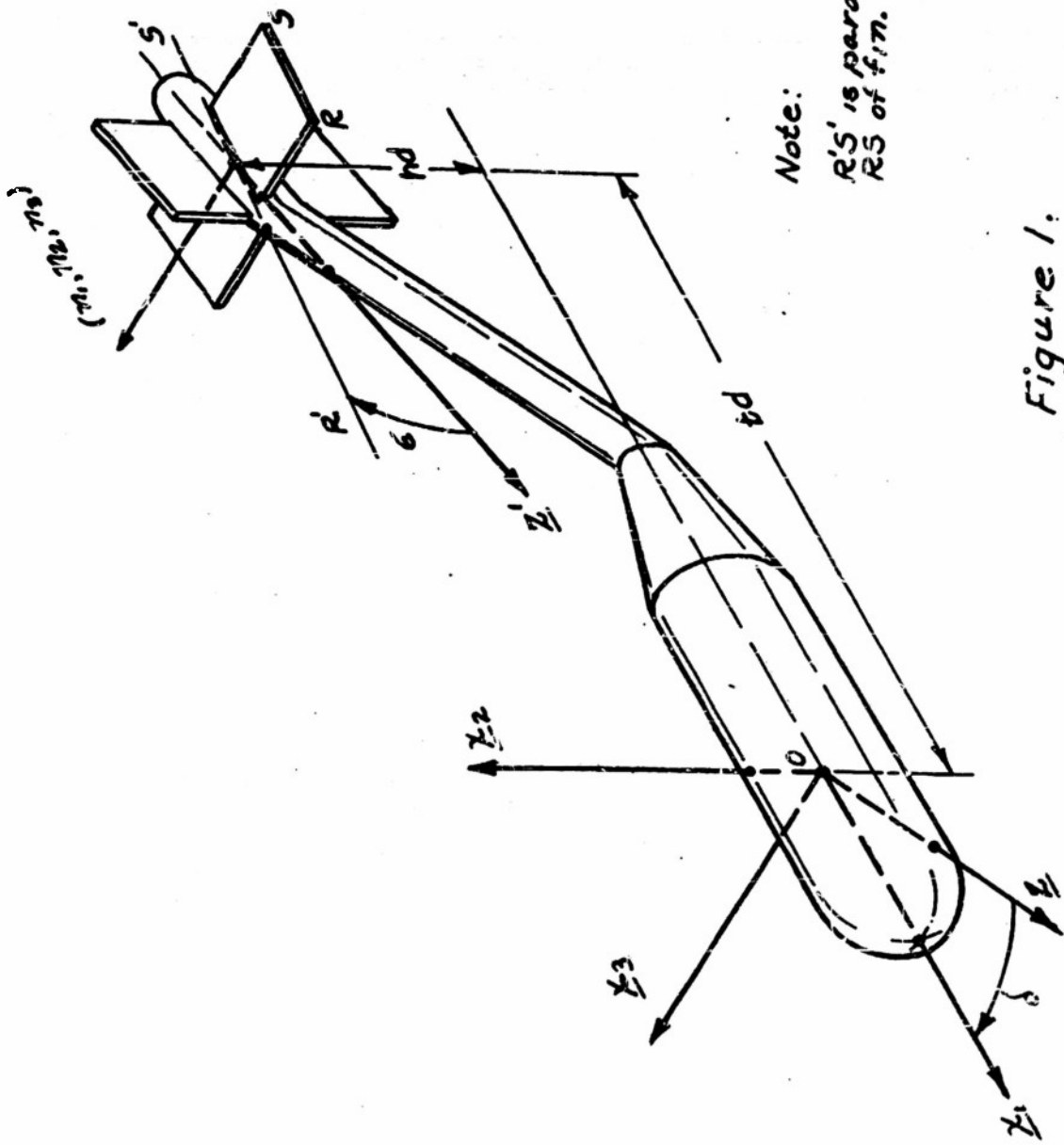


Figure 1.

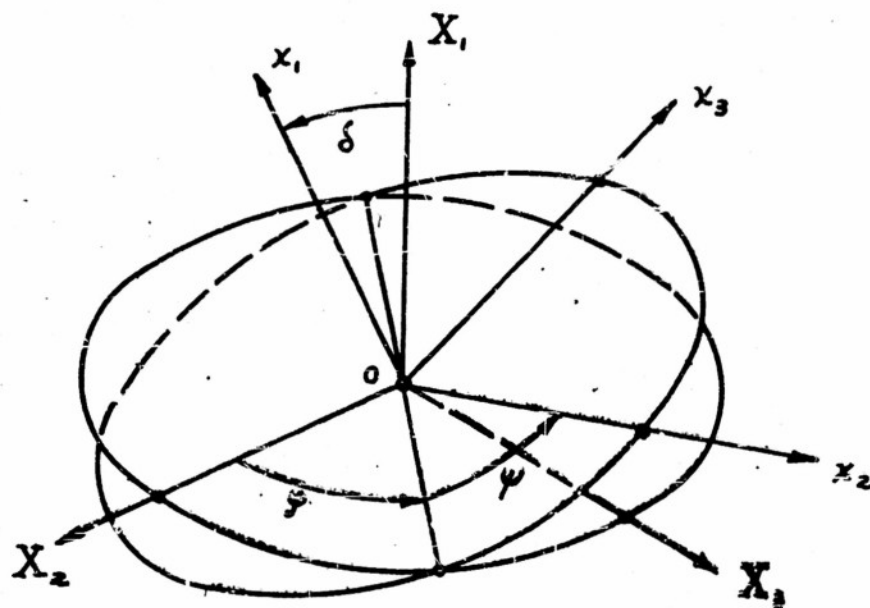


Figure 2.

FIG. 3(a) — GRAPH OF MAGNIFICATION FACTOR $F_1(\omega_1/N)$

$$F_1(\omega_1/N) = \left\{ \frac{1 - (\omega_1/N)(A/B)}{1 - (\omega_1/N)^2 + (\omega_1/N)^2 (\lambda/N)^2} \right\}^{1/2}$$

λ/N is taken as constant with $N \equiv \omega_1$

- $A/B = 1; \lambda/N = 0.05$ ———
- $A/B = 1; \lambda/N = 1$ ———
- $A/B = 0.05; \lambda/N = 0.05$ - - -
- $A/B = 0.05; \lambda/N = 1$ - - -

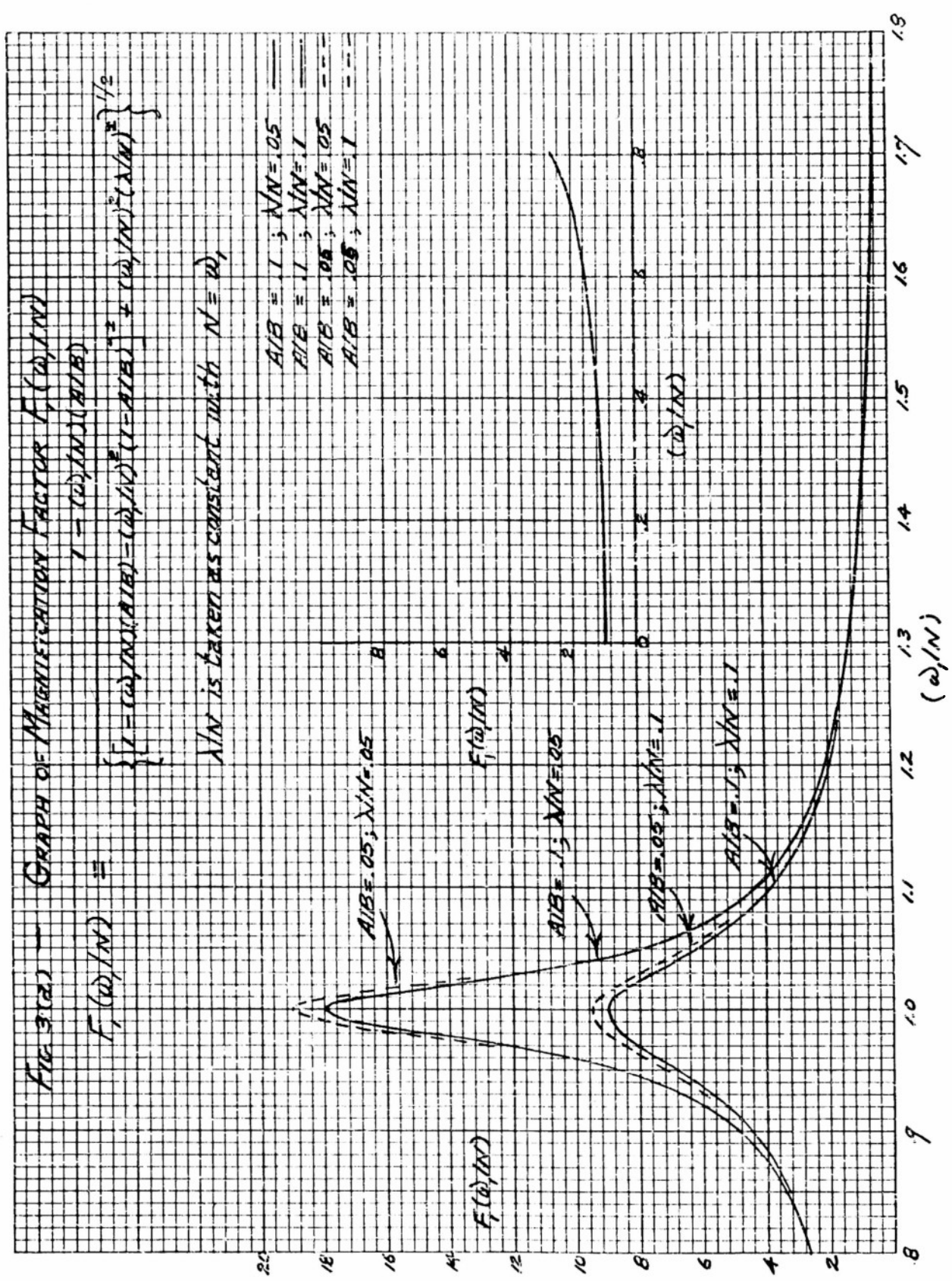


FIG. 3(b)

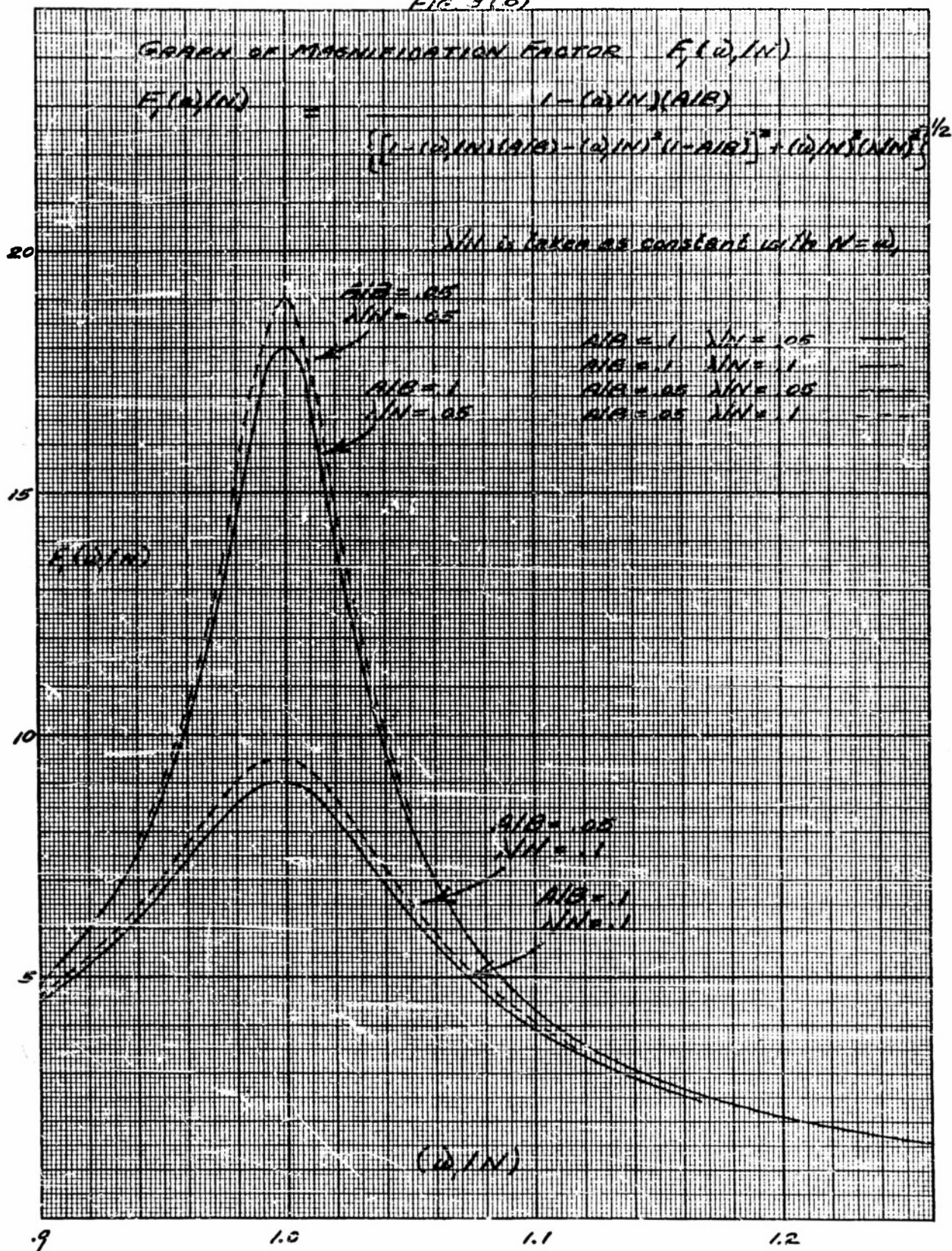
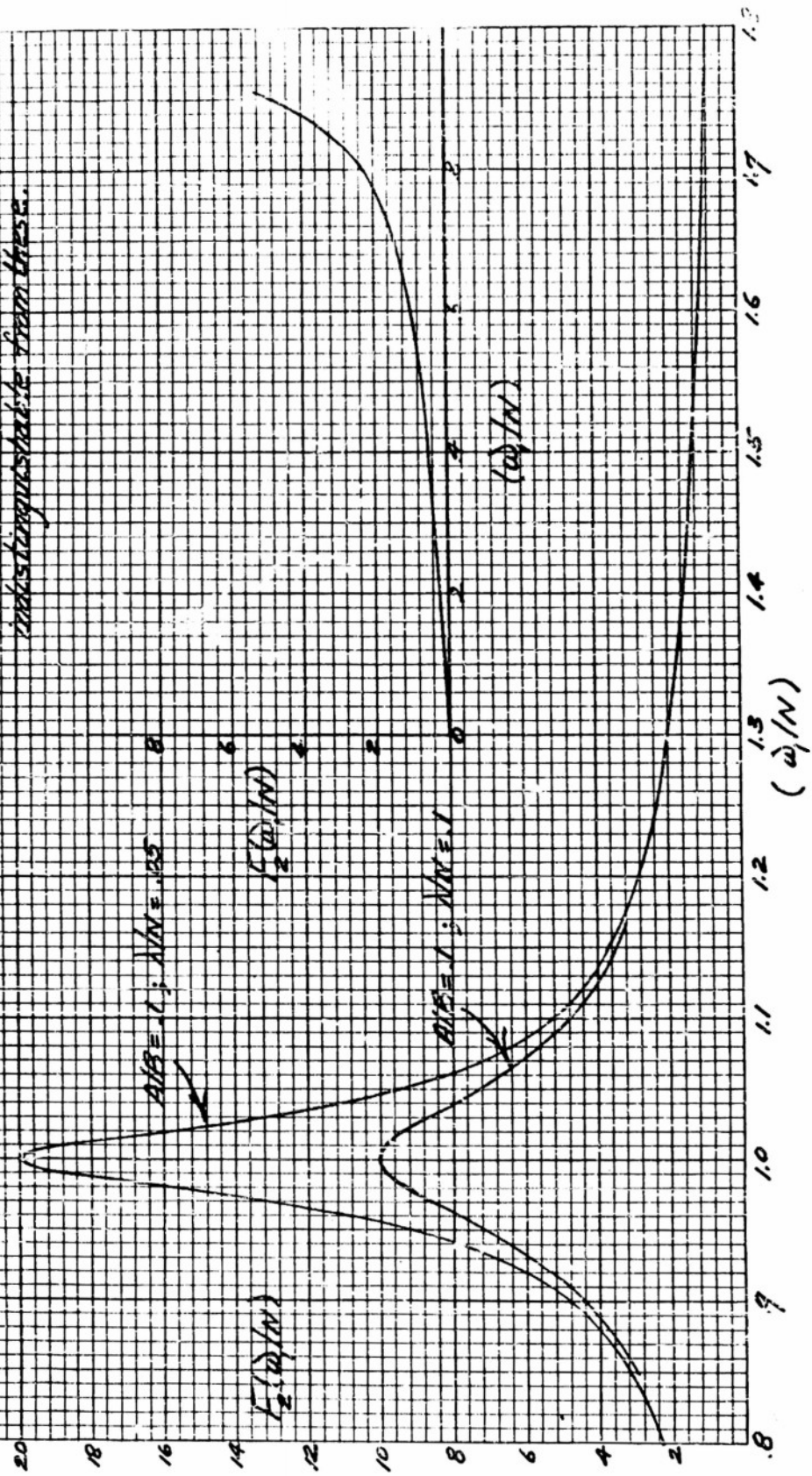


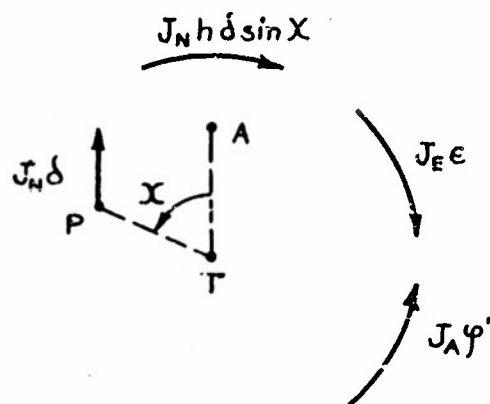
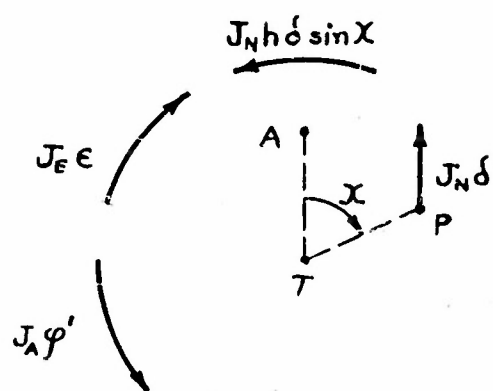
FIG. 4 - GRAPH OF MAGNIFICATION FACTOR $F_2(\omega/N)$

$$F_2(\omega/N) = \left\{ \left[1 - \frac{(\omega/N)^2}{(1-A/B)^2} \right]^2 + \left(\frac{\omega/N}{1-A/B} \right)^2 \right\}^{1/2}$$

$N\omega$ is taken as constant with $N=10$.

Curves for $A/B = .05$ are almost indistinguishable from these.

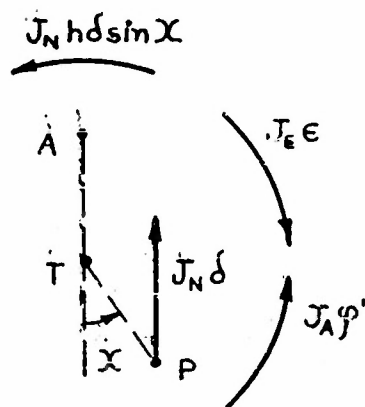
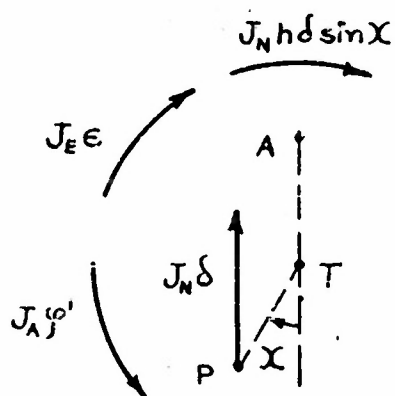




Stable

$$\alpha_5 > 0$$

$$\alpha_5 < 0$$



Unstable

Figure 5

REFERENCES

1. On the Motion of a Projectile With Small or Slowly Changing Yaw, J. L. Kelley and E. J. McShane, BRL Report No. 446, Dec. 1941.
2. Effect of Gravity on Yawing Motion, C. H. Murphy, BRL TN No. 713, June 1952.
3. On the Motion of Projectiles With Slight Configurational Asymmetries, J. D. Nicolaides, BRL Report No. 858, June 1953.

LIST OF SYMBOLS

Underlined letters represent vectors; e.g., \underline{u} is a vector of length u .

- A: axial moment of inertia.
- a_i : coefficients in "locking-in" equation; defined after equation (18).
- B: transverse moment of inertia.
- C, D, E, F: combinations of aerodynamic symbols, etc. See equations (15) and (17).
- d: maximum diameter of projectile.
- G: $(g/v^2d) (\sin \theta + \cos \theta)$; cross component of gravitational acceleration, essentially.
- G_i : $G_i = g_i d/u^2$.
- \underline{g} : vector acceleration of gravity; $\underline{g} = g_1 \underline{x}_1 + g_2 \underline{x}_2 + g_3 \underline{x}_3$.
- h: distance from axis to center of pressure of tail, in calibers.
- J: $J = \rho d^3 K/m$.
- K: dimensionless aerodynamic coefficient.
- K_A --spin damping moment,
- K_C --moment due to non-coincidence of lines of zero lift and moment of tail,
- K_{DA} --axial drag,
- K_E --spin accelerating moment due to canted fins,
- K_F --Magnus force,
- K_H --damping (pitching) moment,
- K_L --lift,
- K_M --static moment,
- K_N --normal force,
- K_S --cross-spin (pitching) force,
- K_T --Magnus moment,
- K_{XF} --Magnus force due to cross-spin,

LIST OF SYMBOLS (Con't)

K_{XT} -- Magnus moment due to cross-spin.

Subscripts 1 and 2 refer to body and tail respectively. An extra subscript M is a temporary notation to indicate the moment of a force.

k : transverse radius of gyration, in calibers; $k^2 = B/md^2$.

k_a : axial radius of gyration, in calibers.

L : distance of representative point from c.g.

l_1 : direction cosines of line of zero normal force of tail.

m : mass of projectile.

N : (nutational) yawing rate; $N = (A\omega_1/B) (1+\sigma)$.

n_1 : moment caused by non-coincidence of lines of zero lift and moment of tail acts along $n_1x_1 + n_2x_2 + n_3x_3$.

p : cross component of angular velocity; $p = \mu \exp(i \int_0^a \omega_1 da)$.

Q : right member of yawing equation (15), when gravity is neglected, is

$$Q \exp(i \int_0^a \omega_1 da); \text{ see equation (17).}$$

q_1, q_2 : terms in equations of motion (14a) and (14b) respectively that are due to asymmetry.

r : a solution of the yawing equation (17) is sought in the form $\exp(\int_0^a rd a)$.

s : ballistic (gyroscopic) stability factor; $s = A^2 \omega_1^2 k^2 / 4B^2 J_M$.

t : distance along axis from c.g. to center of pressure of tail, in calibers.

\underline{u} : vector velocity of projectile; $\underline{u} = u_1x_1 + u_2x_2 + u_3x_3$.

v : $v = u/d$, $v_1 = u_1/d$.

X_1 : axes parallel to fixed directions in space, with origin at c.g.

\underline{x}_1 : unit vectors along axes x_1 , which are fixed in projectile.

\underline{z} : unit vector along trajectory; $\underline{u} = u\underline{z}$.

\underline{z}' : unit vector along line of zero lift of tail.

a : distance travelled in calibers; $a = \int_0^t v dt$.

LIST OF SYMBOLS (Con't)

β : angle of misalignment of tail; $\beta = h/t$.

Γ : $G_2 + iG_3$.

γ : $g/v^2 d$.

δ : angle of yaw, from trajectory to axis.

ϵ : angle of cant of fins.

ζ : complex yaw; $\zeta = \xi \exp(i \int_0^a \omega_1 da)$.

η : angle defined by $\sin \eta = a_4 / (a_3^2 + a_4^2)^{1/2}$, $\cos \eta = a_3 / (a_3^2 + a_4^2)^{1/2}$.

θ : angle from horizontal to trajectory, positive up.

λ : approximately measures total damping; $\lambda = J_N(1-A/B) + (J_H + J_t)/k^2$.

μ : cross angular velocity; $\mu = \omega_2 + i\omega_3$.

χ : angle defined by $\sin \chi = a_5 / (a_3^2 + a_4^2)^{1/2}$, $\cos \chi = (a_3^2 + a_4^2 - a_5^2)^{1/2} / (a_3^2 + a_4^2)^{1/2}$.

ξ : complex yaw; $\xi = -(v_2 + iv_3)/v$.

ρ : density of the air.

σ : $\sigma^2 = 1 - 1/s$.

τ : small perturbation of ψ .

θ : Eulerian angle of orientation.

ψ : Eulerian angle of spin.

$\underline{\Omega}$: angular velocity vector; $\underline{\Omega} = \Omega_1 \underline{x}_1 + \Omega_2 \underline{x}_2 + \Omega_3 \underline{x}_3$.

ω_1 : Ω_1/v .

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