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Bounds On Errors In K_D Caused By Random Errors of Measurement In A Five Station Range

F. D. BENNETT

J. M. BARTOS

DEPARTMENT OF THE ARMY PROJECT No. 503-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108J

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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FDBennett/JMBartos/jw
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ABSTRACT

The maximum effect of random errors in a five station firing range is investigated. Because the data reduction method for drag coefficient, K_D, requires fitting the time-distance data with a cubic polynomial by the method of least squares, the problem becomes that of investigating the maximum effect of random errors in the data on the coefficients of the cubic. This is done in a general way that will apply to a polynomial of any degree and a range of any odd number of stations.

Curvature changes associated with particular distributions of errors are found to cause the largest errors in the least squares coefficients and hence in the drag values.

I. INTRODUCTION

In a recent paper Bergdolt [1] [1] gives a description of the construction and instrumentation of a 12" pressurized range for small caliber projectiles. Some preliminary drag measurements (K_D vs M) are presented and it is noted that the scatter of data, particularly in the neighborhood of the drag curve maximum, is unexpectedly large.

Analysis has been made by Karpov [2] and Charters [3] of the effect of random errors in time and distance on drag data obtained in firing ranges and in combination range-wind tunnels respectively. To simplify the problem both authors assume that a random distance error e_s can be represented as an equivalent random time error e_t by the equation $e_s = \bar{v}\, e_t$ where $\bar{v}$ is the average velocity of the projectile. The treatment of Karpov is based on standard statistical methods assuming a least squares fitting for the time vs. distance data and a normal distribution law for the random errors at each station. Charters follows Karpov in this respect and compares the two types of facility using the Karpov results. Neither author attempts to determine the maximum errors in drag caused by specific distributions of error among the stations of the range.

In this report we find upper bounds for the error in K_D caused by random errors in a five station range as follows: 1) by examining an idealized problem for which all possible distributions can be easily enumerated and 2) by establishing bounds for any possible distribution of errors by an argument based on the results for the idealized problem.

II. STATEMENT OF THE IDEALIZED PROBLEM

We consider here the case where

1) the number n of stations is small e. g. n = 5,

2) errors in time of equal size ϵ occur at each station,

3) + or - signs are equally likely.

For convenience in calculation we consider the stations equally spaced and the origin z = 0 placed at the central station. n is always odd.

III. THEORY

The underlying reason for considering the problem stated above is the intuitive feeling that a certain few distributions of random errors have the greatest effect on the least squares fit of time t versus

[1] Numbers in brackets refer to the bibliography at the end of this report.

where Z is the 4 x 4 matrix above and α, T the corresponding vectors. Since the Z matrix depends only on the coordinates, which remain fixed, the effects of a distribution of errors d_k will be felt only in the α and T vectors. For true time i. e. distribution c) we write

(5) $Z\,\bar{\alpha} = \bar{T}$,

and subtracting (5) from (4) yields

(6) $Z\,(\alpha - \bar{\alpha}) = T - \bar{T}$.

We evaluate $T - \bar{T}$ from (1) and (3) obtaining

$$(7) \quad Z\,(\alpha - \bar{\alpha}) = \epsilon \begin{pmatrix} \sum m_{kj}\, z_j^0 \\ \sum m_{kj}\, z_j^1 \\ \sum m_{kj}\, z_j^2 \\ \sum m_{kj}\, z_j^3 \end{pmatrix}$$

From this we see at once that the numerical error Δa in any coefficient is proportional to ϵ and, once the Z matrix is determined, dependent only on the distribution of errors d_k.

A further result may be obtained more heuristically as follows.

For a projectile of mass m and diameter d fired in atmosphere of density ρ

$$(8) \quad K_D = 2 \frac{m}{\rho\, d^2} \quad \frac{a_2}{a_1}$$

as given on p. 8 of [2]. Comparing rounds fired at the same velocity $v_0 = \dfrac{1}{a_1}$ but different densities we see that

$$(9) \quad \frac{a_2}{\rho} \approx \text{constant}$$

must hold since K_D values are independent of density - ignoring for the moment small Reynold's number or "scale" effects.

Thus with Z and d_k given,

$$(10) \quad \frac{\Delta a_2}{a_2} \approx C\,(d_k) \quad \frac{\epsilon}{\rho}$$

Thus percentage error p_2 in a_2 is 1) proportional to $\frac{\epsilon}{\rho}$ and 2) dependent on a function of d_k. The first conclusion agrees with Karpov's expression for p_2 in [2] middle of p. 9. The latter conclusion is elaborated below.

IV. COMPUTATIONS

To test the dependence of errors in a_1 & a_2 on the error distribution d_k, two cubic curves were chosen as representative of 1 and 5 atmosphere firings respectively. The coefficients selected correspond to those of least squares cubic fitted to data of previous experimental drag firings, modified only by rounding up to a convenient number of figures. Time values were computed for each polynomial at z values of $\pm$ 10', $\pm$ 5', and 0'. The five (z,t) points are then called the true values and correspond to the error distribution 00000. As stated in § II, the choice of symmetrical, equally spaced z's was made to simplify the solution of the least squares normal equations, since with this choice $\sum z^{2n+1} = 0$ for any integral $n > 0$. The least squares cubic for the 5 atmosphere case was determined for each of the 16 error distributions by a Gaussian elimination scheme. A change in sign of ϵ is equivalent to reversing the sign of the error distribution which yields a percentage error in $\frac{2a_2}{a_1}$ of the same magnitude but of opposite sign. As a check on the computations this was verified for 4 such distributions.

The components of the error vectors, $(\alpha - \bar{\alpha})$, were computed by solving (7). Since $(\alpha - \bar{\alpha})$ is dependent only on the z's and ϵ, it applies to both the 1 and 5 atmosphere cases in the determination of the percentage error in $2a_2/a_1$.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

The main results of the computations are summarized in Table I which gives the 16 error distributions, their corresponding $(\alpha - \bar{\alpha})$'s and percentage error in K_D, for the 1 and 5 atmosphere cases. Reversing every sign in Table I will give the data for the other half of our (m_{kj}) matrix.

The choice of $\epsilon = 1$ is justified as follows: Our ϵ equals Karpov's e_t, see [2] page 8, where

(11) $e_t^2 = e_{t_1}^2 + e_{t_2}^2$

and e_{t_1}, and e_{t_2} are time and equivalent distance errors respectively.

The least count of our counters is 0.625 μ sec. and the estimated error in any z determination is .001 ft. v. [1]. If we let $e_{t_1} = 0.625$ μ sec. with $e_t = \epsilon = 1$ then $e_{t_2} = .78$ which corresponds to a distance error of .0025 ft. at a velocity of 3000 feet per second. It seems unlikely that our distance error is this large; therefore the $\epsilon = 1$ value might reasonably be somewhat diminished. For convenience it was not.

In the 5 atmosphere case, the maximum percentage error in K_D, .91% occurs for distribution (15) + − + − − +. Hence all percentage errors will be contained in the band ± .91% about the true curve. If all distributions are considered equally likely, this extreme value occurs in 1/16 of the cases. Two other distributions causing curvature change, (7) + + − − + and (13) + − − + +, have percentage errors of .74% and .63% respectively. All others (i. e. 13 remaining distributions) have absolute percentage errors of .46% or less.

Similarly, the error band for the 1 atmosphere case is found to be ± 4.60% or five times the width of the 5 atmosphere case. 13 distributions out of 16 have absolute percentage errors of 2.20% or less.

If the scatter observed in the experimental data is due to the assumed random errors of this idealized problem, it will be bounded by the above limits. If, however, points fall outside these limits, then 1) the estimated ϵ is too small, 2) errors other than random timing errors contribute to the scatter or 3) the analysis of the idealized problem does not apply. We show later that the bounds on error in K_D for any distributions such that $|m_{kj}| \leq 1$ differ only slightly, if at all, from those already obtained. Thus the idealized problem furnishes reasonable bounds.

The scatter observed in the K_D vs. M curves in Bergdolt's report [1] for 1 and 5 atmospheres is bounded by the limits mentioned earlier, if a few extreme points are disregarded. There are several valid reasons for disregarding this small number of extreme points. At the time the rounds represented in Figures 9, 10 and 11 of [1] were fired, there was no control of the humidity of the air in the range and no system of checks in the operation of the counters. Later, it was also discovered that the calibration of the pressure gauge was in error. Any of these three types of error might cause points to fall outside the limits stated above; however subsequent experience with the counters indicates them as the likely source.

Since these three difficulties have recently been minimized by the use of dry air, auxiliary check counters and recalibrated pressure gauges, we shall expect more recent data to fall within the bounds specified. In fact, returning to Table I, it should be noted that 13/16's of the distributions have percentage errors falling within the bounds ± 1/2 maximum percentage error i. e. for 1 atmosphere ± 2.20% and for 5 atmosphere ± .46%. Hence we may conjecture that in the practical case, 80% of the points will cluster within the central half of the maximum band width.

The combined counting and distance errors may in actuality take on all values between 0 and $\pm \epsilon$; thus the m_{kj}'s need only lie in the range $|m_{kj}| \leq 1$.

We prove in Appendix 1 that all distributions with m_{kj}'s which fall within this range have errors in K_D bounded by limits only slightly larger than those just obtained for $m_{kj} = \pm 1$. It seems likely that these latter are also true bounds for any distribution such that $|m_{kj}| \leq 1$. While a proof of this conjecture has not been obtained, no exception has yet been found. Its truth or falsity is not of much importance at present since the limits actually obtained in Appendix 1 only differ by a small positive quantity from those for $m_{kj} = \pm 1$. (See Table II)

Thus we may conclude that the scatter band-width indicated by our idealized calculation for the 32 distributions obtained above gives an accurate estimate of the band-width to be expected in practical experience with the five station range. It is to be understood that suitable adjustments in the assumed ϵ must be made as changes occur in the fundamental accuracy with which time and distance can be measured.

F. D. Bennett
F. D. BENNETT

J. M. Bartos
J. M. BARTOS

APPENDIX 1

The computations discussed earlier are limited to an ϵ of 1 and m_{kj} of either $\pm$ 1. Now we shall examine other possibilities for which $-1 \leq m_{kj} \leq +1$.

From § III we have

$$K_D = \frac{2m}{\rho\, d^2} \left(\frac{a_2}{a_1} \right) . \qquad (8)$$

By logarithmic differentiation we get

$$\frac{\Delta K_D}{K_D} = \left[\frac{\Delta a_2}{a_2} - \frac{\Delta a_1}{a_1} \right] . \qquad (A-1)$$

It is easily shown that $\bar{a}_1$ and $\bar{a}_2$ are both positive quantities. Therefore

$$\frac{\Delta K_D}{K_D} \leq \frac{|\Delta a_2|}{a_2} + \frac{|\Delta a_1|}{a_1} . \qquad (A-2)$$

Now

$$\begin{aligned} \Delta a_1 &\leq |\Delta a_1|_{max} \\ \Delta a_2 &\leq |\Delta a_2|_{max} . \end{aligned} \qquad (A-3)$$

Furthermore

$$\begin{aligned} a_1 = \bar{a}_1 + \Delta a_1 &\geq \bar{a}_1 - |\Delta a_1|_{max} \\ a_2 = \bar{a}_2 + \Delta a_2 &\geq \bar{a}_2 - |\Delta a_2|_{max} . \end{aligned} \qquad (A-4)$$

Thus,

$$\left| \frac{\Delta K_D}{K_D} \right|_{max} \leq \frac{|\Delta a_2|_{max}}{\bar{a}_2 - |\Delta a_2|_{max}} + \frac{|\Delta a_1|_{max}}{\bar{a}_1 - |\Delta a_1|_{max}} \quad \text{or} \qquad (A-5)$$

$$\left| \frac{\Delta K_D}{K_D} \right|_{max} \leq A_2 + A_1 . \qquad (A-5a)$$

In order to find the bounds on $\frac{\Delta K_D}{K_D}$, it is now necessary to find values for $|\Delta a_1|_{max}$ and $|\Delta a_2|_{max}$.

From § III we have

$$(7) \qquad Z(a - \bar{a}) = \epsilon \begin{pmatrix} \sum m_{kj} z_j^0 \\ \sum m_{kj} z_j^1 \\ \sum m_{kj} z_j^2 \\ \sum m_{kj} z_j^3 \end{pmatrix}.$$

We solve for the individual Δa_i's and obtain the following expressions

$$(A\text{-}6) \qquad \Delta a_1 = \frac{\epsilon}{\det Z} \begin{Bmatrix} -M_{12} \Sigma_0 + M_{22} \Sigma_1 \\ -M_{32} \Sigma_2 + M_{42} \Sigma_3 \end{Bmatrix}$$

and

$$(A\text{-}7) \qquad \Delta a_2 = \frac{\epsilon}{\det Z} \begin{Bmatrix} +M_{13} \Sigma_0 - M_{23} \Sigma_1 \\ +M_{33} \Sigma_2 - M_{43} \Sigma_3 \end{Bmatrix}$$

where $\Sigma_i = \sum m_{kj} z_j^i$ and M_{ij} is the signed minor or cofactor of the i j$\underline{th}$ element of the Z matrix.

For our particular Z matrix, all M_{ij}'s for which $(i + j) = 2n + 1$ are equal to zero. M_{ij}'s for which $(i + j) = 2n$ are constants which we can evaluate. Hence we may simplify (A-6) and (A-7) as follows

$$(A\text{-}6a) \qquad \Delta a_1 = \frac{\epsilon}{\det Z} \left[M_{22} \Sigma_1 + M_{42} \Sigma_3 \right]$$

$$(A\text{-}7a) \qquad \Delta a_2 = \frac{\epsilon}{\det Z} \left[M_{13} \Sigma_0 + M_{33} \Sigma_2 \right].$$

To find $|\Delta a_1|_{max}$ and $|\Delta a_2|_{max}$ we examine the behavior of the bracketed functions as follows.

For Δa_1:

$$M_{22} \Sigma_1 + M_{42} \Sigma_3 = M_{22} \Sigma m_{kj} z_j^1 + M_{42} \Sigma m_{kj} z_j^3$$

$$= \Sigma m_{kj} \left[M_{22} z_j + M_{42} z_j^3 \right]$$

$$= \Sigma m_{kj} \psi_j ; \qquad (\psi_j = M_{22} z_j + M_{42} z_j^3). \quad *$$

Analysis of ψ_j for $j = \pm 2, \pm 1$ and 0, summarized in Table III, shows that the maximum $|\Sigma m_{kj} \psi_j|$ occurs only for distributions +1, -1, $\pm$1, +1, -1 and -1, +1, $\pm$1, -1, +1. All values of $|m_{kj}| \leq 1$ must yield a smaller $|\Sigma m_{kj} \psi_j|$.

Similarly for Δa_2:

$$M_{13} \Sigma_0 + M_{33} \Sigma_2 = M_{13} \Sigma m_{kj} z_j^0 + M_{33} \Sigma m_{kj} z_j^2$$

$$= \Sigma m_{kj} \left[M_{13} z_j^0 + M_{33} z_j^2 \right]$$

$$= \Sigma m_{kj} \phi_j ; \quad (\phi_j = M_{13} z_j^0 + M_{33} z_j^2).$$

Distributions giving maximum $|\Sigma m_{kj} \phi_j|$ are +1, -1, -1, -1, +1 and -1, +1, +1, +1, -1, (see Table III). This analysis is verified by the computations summarized in Table I. It may be noted that $|\Delta a_2|_{max}$ and $|\Delta a_1|_{max}$ do not occur simultaneously. Hence, we conjecture that for a given ϵ value, $\frac{\Delta K_D}{K_D}$ is bounded by $\pm a_2$. In practice the contribution of the a_1 term to $\frac{\Delta K_D}{K_D}$ is small; and may be neglected so that the preceding expression gives a good estimate of the bound.

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*For actual computation, any common numerical factors in ψ_j were factored out and combined with $\frac{\epsilon}{\det Z}$.

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TABLE I

d_k	Δa_0	Δa_1	Δa_2	Δa_3	% ERROR IN K_D 5 ATM.	1 ATM.
(1)+++++	1.00	0	0	0	0	0
(2)++++-	1.17	.03	-.011	-.0013	-.46	-2.20
(3)+++-+	.31	-.27	.006	.0027	.28	1.25
(4)+++--	.49	-.23	-.006	.0013	-.18	-1.16
(5)++-++	.03	0	.011	0	.46	2.20
(6)++-+-	.20	.03	0	-.0013	-.01	-.01
(7)++--+	-.66	-.27	.017	.0027	.74	3.45
(8)++---	-.49	-.23	.006	.0013	.28	1.24
(9)+-+++	.31	.27	.006	-.0027	.18	1.14
(10)+-++-	.49	.30	-.006	-.0040	-.29	-1.26
(11)+-+-+	-.37	0	.011	0	.46	2.20
(12)+-+--	-.20	.03	0	-.0013	-.01	-.01
(13)+--++	-.66	.27	.017	-.0027	.63	3.34
(14)+--+-	-.49	.30	.006	-.0040	.17	1.14
(15)+---+	-1.34	0	.023	0	.91	4.60
(16)+----	-1.17	.03	.011	-.0013	.45	2.19

5 Atmosphere True Curve

$$t = 5000 + 500\,z + 2.5z^2 + .02z^3$$

1 Atmosphere True Curve

$$t = 3800 + 520\,z + .5z^2 + .01\,z^3$$

TABLE II

	5 ATM	1 ATM
A_1	.0006	.0006
A_2	.0092	.0478
$A_2 + A_1$	.0098	.0484
$100\ (A_2 + A_1)$	.98%	4.84%
Max. % Error in K_D From Table I	.91%	4.60%

(A-5)
$$\left|\frac{\Delta K_D}{K_D}\right|_{Max.} \leq A_2 + A_1$$

Where

$$A_2 = \frac{|\Delta a_2|_{Max.}}{\bar{a}_2 - |\Delta a_2|_{Max.}}$$

$$A_1 = \frac{|\Delta a_1|_{Max.}}{\bar{a}_1 - |\Delta a_1|_{Max.}}$$

TABLE III

j	ψ_j^*	ϕ_j^*
-2	+ 750	+ 50
-1	- 6000	- 25
0	0	- 50
1	+ 6000	- 25
2	- 750	+ 50

Note:

$$\psi_j = 7 \times 2^2 \times 5^9\ \psi_j^*$$

$$\phi_j = 9 \times 2^4 \times 5^9\ \phi_j^*$$

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