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AN INTRODUCTION TO THE USE OF THE SCATTERING MATRIX IN NETWORK THEORY

by

HERBERT J. CARLIN

Report R-366-54, PIB-300

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MRI

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Title Page
Acknowledgment
Abstract
Table of Contents
56 Pages of Text
1 Page of References

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ABSTRACT

The general algebraic properties of scattering parameters are discussed with respect to their application to lumped, linear, reciprocal networks. A basic definition is given for the reflection factor of a two terminal network and this is then extended to n-terminal pair networks where the relevant descriptive tool is the scattering matrix. The normalization of the scattering matrix is presented and power transfer through n-ports, and the properties of maximum output and biconjugate networks are shown to be simply described in terms of this normalized scattering matrix.

A final section discusses the basis for applying the scattering formalism to network synthesis in the frequency domain by giving fundamental network realizability theorems in terms of the scattering matrix.

TABLE OF CONTENTS

I. Application of Scattering Parameters

Either the short or open circuit parameters of an n-port (2n terminal pair) (1) completely describe the properties of the network. However in many instances it is convenient to choose one or the other set of quantities depending on the type of problem being considered. For example if the voltage is required at an open-circuited i^{th} pair of terminals with current specified at the j^{th} pair of terminals one immediately has $V_i = Z_{ij} I_j$ if all terminal pairs are open circuited. The required voltage is very simply found because the network is operating under conditions which precisely correspond to those under which the open circuit impedance matrix may be defined. Clearly the short circuit admittances could have been used to write the required relation i.e. $V_i = \Delta_Y / \Delta_{Y-ji} I_j$ where Δ_Y is the determinant of the Y matrix, and Δ_{Y-ji} is the ji th cofactor of that determinant. However this would be a poor choice and would needlessly complicate the formal algebra. The admittance or impedance quantities can in fact be used to compute the performance of a network under any sort of terminating conditions, but as in the above example, if the network functions in a special fashion one or another of these coefficients provides a particularly simple description of the network operation.

The choice of a simple formulation is particularly important in theoretical studies where important properties of the network may be obscured by complicated algebraic equations. The scattering parameters of a network are a set of quantities which can describe the performance of a network under any specified terminating conditions just as the impedance or admittance quantities, but whereas the scattering coefficients may not be particularly convenient for short or open circuit computations they may be applied in a relatively simple fashion when the network is terminated in a real load impedance. Thus questions of power transfer from a finite impedance generator to a resistive load are frequently best handled by scattering relations, and it is possible to deduce important general theorems for this kind of operation which are hidden when impedance or admittance quantities are used. Practical applications occur in problems concerning filters, equalizers, matching networks, and balanced networks. (The Wheatstone bridge is a simple example of the latter type of network). The scattering coefficients were first mentioned by Campbell and Foster[2], in 1922 and were more completely exploited recently in connection with transmission line problems including the non-reciprocal case.[3][4][5] This is because the relation between reflected and incident wave amplitudes on a transmission line is a linear set of equations whose coefficients are scattering parameters.

It was, in fact, the stimulus of the waveguide investigators which has lead to a renewed interest in the application of scattering coefficients to lumped circuits.(6)(7)(10)(11)

One final property of the scattering parameters should be mentioned. Many networks have no impedance matrix (for example a four-pole consisting of a series impedance and a ground wire) or no admittance matrix (a four pole consisting of an element with two terminal pairs connected in parallel across it) or neither (an ideal transformer). However every physical network has a scattering matrix. This fact is useful in stating general theorems which apply to all networks with no special exceptions.

II. Scattering Relations for a One Terminal Pair Network (One Port)

The scattering coefficients express the relations between incident and reflected voltages or currents. These quantities in turn may be defined in terms of the conventional voltages and currents at the accessible terminal pairs of a network. In Fig.-1 the voltage and current at the one accessible terminal pair of a 1-port (one terminal pair or driving point impedance) is indicated

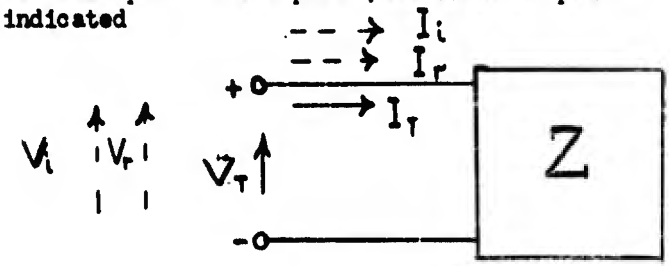


Fig.-1 Definition of scattering quantities

The total voltage V_T and current I_T are separated into incident and reflected components polarized as shown in the figure. The subscript i is for "incident", and r for "reflected". In terms of these components the voltage and current are given as:

$$V_T = V_i + V_r \qquad (1$$

$$I_T = I_i + I_r \qquad (2$$

These two equations only partially define the components. In addition one needs relations between V_i and I_i, and V_r and I_r. It is convenient to make incident and reflected currents proportional respectively to incident and reflected voltages. Thus a useful definition is

$$V_i = I_i Z_o \qquad (3$$

Here the incident current into the network (this is the positive sense of I_i polarized as in Fig.-1) is multiplied by a proportionality constant Z_o, which may be real or complex to give V_i. Z_o is termed the "normalization number". There is usually some preferred value of Z_o dictated by the problem at hand but as far as the general definitions are concerned Z_o is completely arbitrary though it must be chosen in advance.

The relation between reflected quantities is defined as:

$$V_r = - I_r Z_o \qquad (4$$

Here, in line with the physical sense of the word "reflected", V_r is made proportional to the incident current polarized outward from the network. Figure 1 indicates that the positive sense of I_r is into the network; hence the minus sign in equation 4 for outward I_r.

Equations 1 to 4 completely define the incident and reflected components in terms of total voltages and currents, since they constitute an independent set of 4 equations in the 4 unknowns, V_i, I_i, V_r, I_r. To solve for the voltage components substitute equations 3 and 4 into 2, and add after multiplying by Z_o.

$$V_T + Z_o I_T = V_i + V_r + V_i - V_r = 2 V_i$$

$$V_i = \frac{1}{2} \left[V_T + Z_o I_T \right] \qquad (5$$

If the equations are subtracted one obtains:

$$V_r = \frac{1}{2} \left[V_T - Z_o I_T \right] \qquad (6$$

Applying 3 and 4 to equations 5 and 6 leads to

$$I_i = \frac{V_i}{Z_o} = \frac{1}{2Z_o} \left[V_T + Z_o I_T \right] \qquad (7$$

$$I_r = -\frac{V_r}{Z_o} = -\frac{1}{2Z_o}\left[V_T - Z_o I_T\right] \qquad (8$$

Figures 2 and 3 shows a method for measuring the amplitudes of incident and reflected voltage components determined directly from equations 5 and 6.

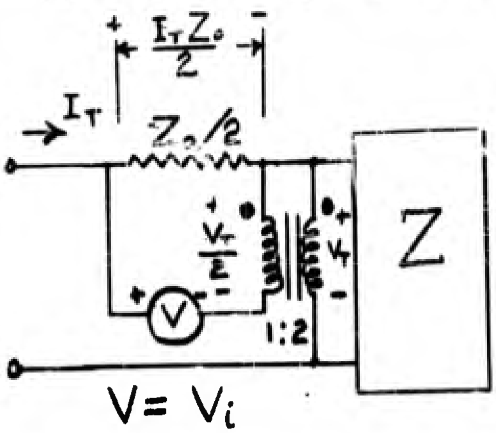


Fig.-2

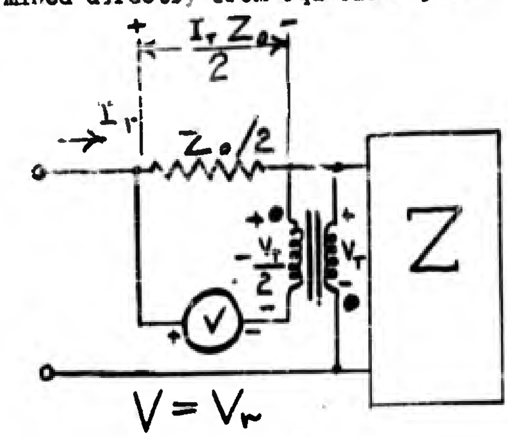


Fig.-3

Measurement of incident and reflected voltages.

The scattering coefficients define the relations between incident and reflected quantities. The voltage scattering coefficient for the network of Fig.-1 is obtained by dividing equation 6 by equation 5:

$$\frac{V_r}{V_i} = S^V = \frac{V_T - Z_o I_T}{V_T + Z_o I_T}$$

If numerator and denominator are divided by I_T, the total current

$$S^V = \frac{\frac{V_T}{I_T} - Z_o}{\frac{V_T}{I_T} + Z_o} = \frac{Z - Z_o}{Z + Z_o} \qquad (9a$$

Inverting this

$$Z = Z_o \frac{1 + S^V}{1 - S^V} \qquad (9b$$

where $Z = \frac{V_T}{I_T}$ is the input impedance.

The current scattering coefficient is

$$S^I = \frac{I_r}{I_i} = \frac{-Z_o V_r}{Z_o V_i} = - S^V = \frac{Z_o - Z}{Z_o + Z} \qquad (10a$$

and

$$Z = Z_o \frac{1 - S^I}{1 + S^I} \qquad (10b$$

The quantities S^V and S^I are also known as voltage and current reflection factors respectively.

Observe from equations 9 and 10 that if Z_o is real and Z is purely imaginary

$$S^V = \frac{jX - Z_o}{jX + Z_o} = \varepsilon^{j2\tan^{-1}\frac{-Z_o}{X}} \qquad (11$$

and

$$S^I = - \varepsilon^{j2\tan^{-1}\frac{Z_o}{X}} \qquad (12$$

The amplitude of the reflection factor in this case is unity. On the other hand if $Z = Z_o$ the reflection factors are zero. This latter case is known as matched operation, and it is clear that the reflection factor is a quantitative measure of the deviation of the impedance Z from matched conditions.

The incident and reflected quantities delivered by a generator with an arbitrary internal impedance into an arbitrary two-terminal network (the network may contain active elements) will now be computed. Fig.-4 shows the equivalent impedance and Thèvenin generator of the two terminal network.

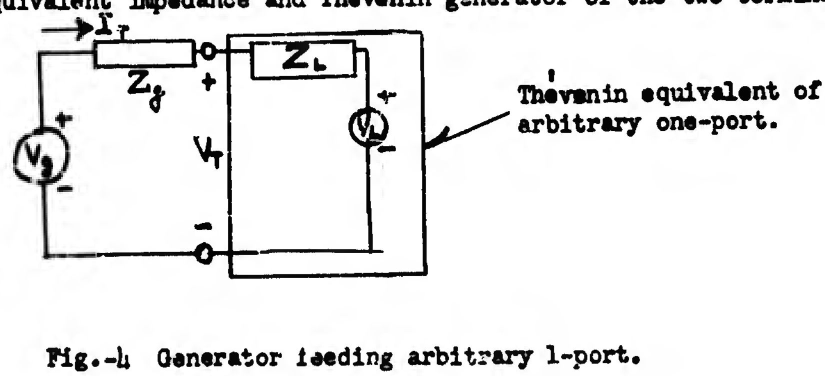


Fig.-4 Generator feeding arbitrary 1-port.

From an inspection of Fig.-4.

$$I_T = \frac{V_g - V_L}{Z_g + Z_L}$$

$$V_T = \frac{(V_g - V_L)\, Z_L}{Z_g + Z_L} + V_L = \frac{V_g Z_L + V_L Z_g}{Z_g + Z_L}.$$

Applying equations 5 and 6:

$$V_i = \frac{1}{2}\left[\frac{V_g Z_L + V_L Z_g}{Z_g + Z_L} + \frac{(V_g - V_L)\, Z_o}{Z_g + Z_L}\right]$$

$$= \frac{1}{2}\left[\frac{V_g(Z_L + Z_o) + V_L\,(Z_g - Z_o)}{Z_g + Z_L}\right] \qquad (13$$

$$V_r = \frac{1}{2}\left[\frac{V_g(Z_L - Z_o) + V_L\,(Z_g + Z_o)}{Z_g + Z_L}\right] \qquad (14$$

Both V_i and V_r are functions of the generator impedance, the load, and the Thevinin voltage. If, the load is passive $V_L = 0$, and the ratio $\frac{V_r}{V_i}$ is precisely the value S^V given by equation 9, a function of the load impedance and Z_o.

If the normalization number Z_o is chosen equal to Z_g, the incident voltage from equation 13 becomes

$$V_i = \frac{V}{2}\,[Z_o - Z_g] \qquad (15$$

and in this case is not a function of either the load impedance, the generator impedance, or the active elements in the arbitrary 1-port. The generator may then be termed a "matched generator". If the generator and load in the passive case ($V_L = 0$) are both matched ($Z_g = Z_L = Z_0$), the reflected voltage is zero and the impressed voltage V_T across the load is precisely equal to the incident voltage $V/2 = V_i$.

It is sometimes useful to consider a passive load with current flowing in it as a generator of incident voltage. In Fig.-5 such a load is shown. To compute this as an equivalent generator the total flow of current out of the load is used. This is $-I_T = I_T'$

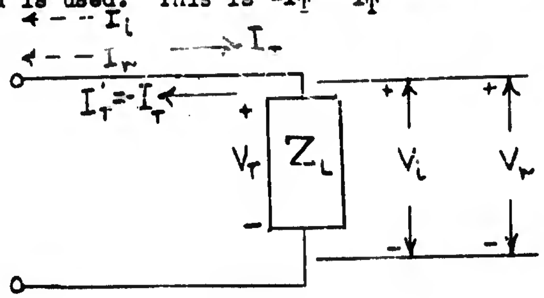


Fig.-5 Load as an equivalent generator.

Then applying equations 5 and 6

$$V_i = \frac{1}{2}\left[V_T + Z_o I_T'\right]$$

$$= \frac{V_T}{2}\left[1 - \frac{Z_o}{Z_L}\right] \qquad (16$$

and

$$V_r = \frac{V_T}{2}\left[1 + \frac{Z_o}{Z_L}\right] \qquad (17$$

From 3 and 4, I_i and I_r out of the load are given by

$$I_i = \frac{V_T}{2R_o}\left[1 - \frac{Z_o}{Z_L}\right] \qquad (18$$

$$I_r = -\frac{V_T}{2R_o}\left[1 + \frac{Z_o}{Z_L}\right] \qquad (19$$

There is no generation of V_i or I_i if the load impedance Z_i is matched to Z_o. For the mismatched case the equivalent generation of incident quantities is given by equations 16 and 19. This concept will be found useful in the discussion of networks with a number of terminated parts.

It was pointed out that the scattering relations find useful application in the study of power transfer. Accordingly it is important to compute the power in a load in terms of incident and reflected quantities.

The complex power delivered to the load of Fig.-1 is

$$W = V_T\, I_T^* = (V_i + V_r)\,(I_i^* + I_r^*)$$

Substituting equations 3 and 4

$$W = \frac{1}{Z_o^*}\left[V_i + V_r\right]\left[V_i^* - V_r^*\right]$$

$$= \frac{1}{Z_o^*}\left[|V_i|^2 + |V_r|^2 + j2I_m(V_r V_i^*)\right]$$

with Im = "Imaginary part of"

If $\frac{1}{Z_o} = G_o + j\,B_o = Y_o$

Then

$$W = G_o\left[|V_i|^2 - |V_r|^2\right] + 2B_o \mathrm{Im}\,(V_r V_i^*)$$

$$+ j\left[2G_o \mathrm{Im}\,(V_r V_i^*) - B_o\,(|V_L|^2 + |V_r|^2)\right] \qquad (20$$

In terms of the current components

$$W = R_o \left[|I_i|^2 - |I_r|^2 \right] - 2X_o I_m \left[I_r^* I_i \right]$$

$$+ j \left[2 R_o I_m (I_r^* I_i) + X_o (|I_i|^2 - |I_r|^2) \right] \qquad (21$$

with $Z_o = R_o + jX_o$

The real power takes on a particularly simple form if the normalizing number Z_o is real i.e. $X_o = B_o = 0$.

The real power is then

$$P = \text{Re}\, W = \frac{1}{R_o} \left[|V_i|^2 - |V_r|^2 \right] = R_o \left[|I_i|^2 - |I_r|^2 \right] \qquad (22$$

If the incident and reflected powers are defined as

$$P_i = |I_i|^2 R_o = |V_i|^2 G_o \qquad (23$$

$$P_r = |I_i|^2 R_o = |V_i|^2 G_o \qquad (24$$

Then referring to equation 22, <u>if and only if</u> Z_o is real,

$$P = P_i - P_r \qquad (25$$

It is clear that this simple relation will not apply to the general case of equations 20 and 21, with P_i and P_r defined as in equations 23, 24.

If $I_r = S^I I_i$, and $V_r = S^V V_i$ are substituted in equation 24, and then into equation 25 one has (Z_o real):

$$P = \left[P_i - |S^I|^2 \; |I_i|^2 \; R_o \right] = \left[P_i - |S^V|^2 \; |V_i|^2 \; G_o \right]$$

or

$$P = P_i \left[1 - |S^I|^2 \right] = P_i \left[1 - |S^V|^2 \right] \qquad (26$$

Again it must be emphasized that this is only true if Z_o is real. In this case the maximum value of P occurs when $|S^I| = |S|^V = 0$ i.e. the matched loadcase (See eq. 26) and then

$$P_{MAX} = P_i \qquad (27$$

If the generator supplying the load in Fig.-4 has a real internal impedance and the normalization number R_o is chosen to equal this then if equation 23 is substituted in equation 27 with V_i given by equation 15 as $\frac{V}{2}$

$$P_{MAX} = \frac{V^2}{4R_o} \qquad (28$$

and this occurs when the load is matched to R_o.

P_i is therefore the power delivered from a matched generator into a matched load (the so-called "available generator power"), and equation 26 directly measures the deviation from this value of power in terms of the squared amplitude of the load reflection factor.

One final point should be made in connection with the scattering properties of a one port. It is convenient to normalise all impedance quantities to the number Z_o. This will give a set of equations whose form is the same as would occur with $Z_o = 1$. Thus the equation for the voltage reflection factor (19a) could be written

$$S^V = \frac{\frac{Z}{Z_o} - 1}{\frac{Z}{Z_o} + 1} = \frac{\mathfrak{z} - 1}{\mathfrak{z} + 1} \qquad (29a$$

where $\mathfrak{z} = \frac{Z}{Z_o}$ is the load impedance normalized to Z_o. Similarly equation 9b becomes

$$\mathfrak{z} = \frac{1 + S^V}{1 - S^V} \tag{29b}$$

To find the true load impedance from eq. 29b one must multiply $\mathfrak{z}$ by Z_o.

As an illustration of scattering computations for a 1-port suppose it is desired to construct an impedance which provides a variable phase angle with constant power absorbing properties. From equation 26 this requires that $|S^V| = K$ a constant.

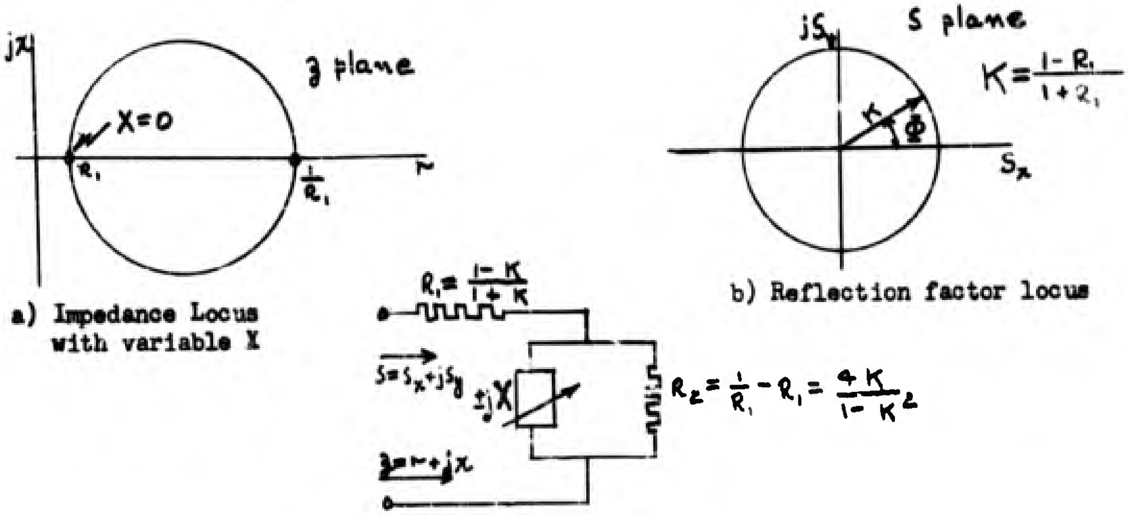
The equation for the normalized impedance is given by using eq. 29b with $S^V = K\,\varepsilon^{j\Phi}$ where Φ is variable.

$$\mathfrak{z} = \frac{1 + K\,\varepsilon^{j\Phi}}{1 - K\,\varepsilon^{j\Phi}}$$

When $\Phi = 0$, $\mathfrak{z} = \frac{1 + K}{1 - K}$. When $\Phi = \pi$, $\mathfrak{z}$ is $\frac{1 - K}{1 + K}$ the reciprocal of the value for $\Phi = 0$. The locus of $\mathfrak{z}$ is in fact a circle as shown in Fig.-6a. Figure 6b shows the corresponding reflection factor locus circle. A network with this locus is shown in Fig.-6c.

$$R_1 = \frac{1 - K}{1 + K}, \quad R_1 + R_2 = \frac{1}{R_1} = \frac{1 + K}{1 - K},$$

so that $R_2 = \frac{4K}{1 - K^2}$.



a) Impedance Locus with variable X

b) Reflection factor locus

c) Network for loci of a) and b)

Fig.-6 Constant power absorbtion, variable phase network.

III. Scattering Relations in an N-Terminal Pair Network (N-port)*

The scattering relationships derived in section II may be carried over in very similar form to the analysis of n-ports. The basic change is the replacement of scalars by matrices.

Consider the passive network N of Fig.-7.

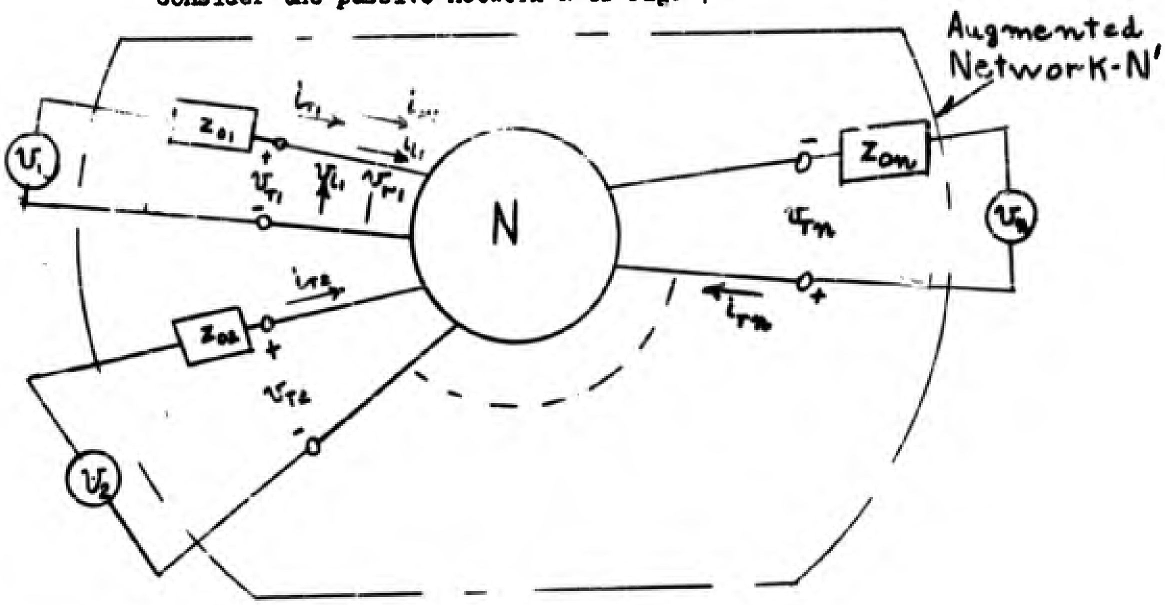


Fig.-7 Relations in general N-Port.

It is presumed that each accessible terminal pair can be energized with voltage and current $[v_{T1}, v_{T2} \ldots, i_{T1}, i_{T2}, \ldots]$ and hence at each terminal pair the incident and reflected components can be found. Thus for the k^{th} pair

$$v_{Tk} = v_{ki} + v_{kr} \quad (30a$$

$$i_{Tk} = i_{ki} + i_{kr} \quad (30b$$

$$v_{ki} = z_{oR} \; i_{ki} \quad (30c$$

$$v_{kr} = - z_{ok} \; i_{kr} \quad (30d$$

* In the remainder of the text capitals generally represent matrices, and lower case scalar elements.

The normalizing number z_{ok} may be different for each port.

Each reflected voltage will be found as a linear combination of all the incident voltages. A convenient way to set up the complete system of all the linear equations for the v_{kr} is to make use of equation 15 which shows that if a matched generator is used (i.e. internal generator impedance equal to the normalising number) the incident voltage is one half the open circuit voltage of the generator. In Fig.-7 therefore the network N is shown excited at its various ports by a system of generators with voltages $v_1, v_2, \ldots$ and internal impedances $z_{o1}, z_{o2}, \ldots$ equal to the arbitrarily assigned normalizing numbers.

At the typical k^{th} port the situation is precisely the same as that in Fig.-4, with the excitations at all ports except the k^{th} combining to give an equivalent Thévinin circuit at the k^{th} port. The incident voltage at this port due to the generator V_k is therefore by equation 15:

$$v_{ik} = v_k/2 \tag{31}$$

Suppose now the network N' shown in Fig.-7 is considered. This is formed by connecting the internal generator impedances to the network N as shown in the figure. The network N' will be termed "the augumented network". Suppose the admittance matrix of this augumented network is $\hat{Y}$. Then

$$I_T = \hat{Y}\, V \tag{32}$$

where I_T is the matrix of currents into the ports, and V is the matrix of generator voltages.

$$I_T = \begin{bmatrix} i_{T1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ i_{Tk} \end{bmatrix} \qquad V = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{bmatrix}$$

At each port

$$v_{Tk} = v_k - z_{ok}\, i_{Tk}$$

or in matrix notation

$$V_T = V - Z_o\, I_T \tag{33}$$

where Z_o is a diagonal matrix of normalization numbers:

$$Z_o = \begin{bmatrix} z_{o1} & 0 & 0 & \cdots \\ 0 & z_{o2} & 0 & \cdots \\ 0 & 0 & z_{o3} & \cdots \\ \vdots & & & \end{bmatrix} \qquad (34$$

Substitute eq. 32 into 33

$$V_T = V - Z_o \hat{Y} V \qquad (35$$

Now note that the matrix form of equation 31 is

$$V_i = \frac{1}{2} V \qquad (36$$

and replace V in equation 35.

$$V_T = 2V_i - 2Z_o \hat{Y} V_i \qquad (37$$

The matrix form of equation 30a is

$$V_T = V_i + V_r$$

or

$$V_r = V_T - V_i$$

Replace V_T by equation 37

$$V_r = 2V_i - 2Z_o \hat{Y} V_i - V_i = [I - 2Z_o \hat{Y}] V_i \qquad (38$$

where I is the identity matrix.

Equation 38 is the required relation between reflected and incident voltages, and the voltage scattering matrix is given by:

$$S^V = I - 2 Z_o \hat{Y} \tag{39}$$

So that the general scattering equation for an n-port takes the form:

$$V_r = S^V V_i \tag{40}$$

The matrix form of eqs. 30c and 30d may be substituted into eq. 40 to give:

$$- Z_o I_r = S^V Z_o I_i$$

$$I_r = - Z_o^{-1} S^V Z_o I_i \tag{41}$$

or

$$I_r = S^I I_i \tag{42}$$

with

$$S^I = - Z_o^{-1} S^V Z_o \tag{43}$$

In general the matrices S^I and S^V <u>are not</u> the negatives of each other but if the normalising numbers for all the ports are identical and equal to the scalar z_o then:

$$S^I = - \frac{1}{z_o} I S^V I z_o = - S^V \Big|_{z_{o1}=z_{o2}=\dots=z_o} \tag{44}$$

Equation 40 represents the set of linear equations:

$$\begin{aligned}
v_{r1} &= s_{11} v_{i1} + s_{12} v_{i2} + \dots + s_{1n} v_{in} \\
v_{r2} &= s_{21} v_{i1} + s_{22} v_{i2} + \dots + s_{2n} v_{in} \\
&\vdots \\
v_{rn} &= s_{n1} v_{i1} + s_{n2} v_{i2} + \dots + s_{nn} v_{in}
\end{aligned} \tag{45}$$

The individual coefficients may be determined from equation 39. The diagonal terms are of the form

$$s_{kk} = 1 - 2\, z_{oK}\, \hat{y}_{kk} \tag{46}$$

The off diagonal terms are given by

$$s_{jk} = -\, 2\, z_{oj}\, \hat{y}_{jk} \tag{47}$$

$\hat{y}_{kk}$ and $\hat{y}_{jk}$ are the short circuit driving point and transfer admittances of the augmented network N'. It is clear from equation 47 that s_{jk} does not generally equal s_{kj} so that the S matrix is not symmetrical,

$\frac{s_{jk}}{s_{kj}} = \frac{z_{oj}}{z_{ok}}$. However if all the z_{oj} are equal the S matrix is symmetrical.

(A normalization technique is described in section VIII which always leads to a symmetrical scattering matrix for reciprocal networks even when all port normalization numbers are different.)

It is possible to derive the S matrix directly from the impedance or admittance matrix of the network N if these arrays exist. If the network N has an impedance matrix Z, then

$$V_T = Z\, I_T$$

and substituting the matrix forms of eqs. 30a, 30b, 30c, 30d.:

$$V_i + V_r = Z \left[Z_o^{-1}\; V_i - Z_o^{-1}\; V_r \right]$$

$$Z\, Z_o^{-1}\, V_r\, V_r = Z\, Z_o^{-1}\, V_i - V_i$$

$$V_r = \left[Z\, Z_o^{-1} + I \right]^{-1} \left[Z\, Z_o^{-1} - I \right] V_i$$

$$S^V = (Z\, Z_o^{-1} + I)^{-1}\, (Z\, Z_o^{-1} - I) \tag{48}$$

Another form for this equation is obtained if $Y_o = Z_o^{-1}$, $Y = Z^{-1}$ are used, then from equation 48

$$S^V = (Z\,Y_o + I)\,ZZ^{-1}\,(Z\,Y_o - I)$$

$$= (Z^{-1}\,Z\,Y_o + Z^{-1})^{-1}\,(Z^{-1}\,Z\,Y_o - Z^{-1})$$

$$= (Y_o + Y)^{-1}\,(Y_o - Y) \qquad (49$$

Still another form can be obtained by observing that $\hat{Y} = [Z + Z_o]^{-1}$. Then from equation 39

$$S^V = (I - 2\,Z_o\,\hat{Y}) = I - 2\,Z_o\,(Z + Z_o)^{-1}$$

then

$$S^V\,(Z + Z_o) = (Z + Z_o) - 2\,Z_o = Z - Z_o$$

or

$$S^V = (Z - Z_o)\,(Z + Z_o)^{-1} \qquad (50$$

The corresponding equations for S^I are:

$$S^I = (Y\,Y_o^{-1} + I)^{-1}\,(Y\,Y_o^{-1} + I) \qquad (51$$

$$S^I = (Z_o + Z)^{-1}\,(Z_o - Z) \qquad (52$$

$$S^I = (Y - Y_o)\,(Y + Y_o)^{-1} \qquad (53$$

The Z and Y matrices may be formed from the scattering matrices by inverting equations 49, 50, 52, 53.

$$Z = (I + S^I)^{-1} Z_o (I - S^I) \quad (54$$

$$Z = (I - S^V)^{-1} (I + S^V) Z_o \quad (55$$

$$Y = (I + S^V)^{-1} Z_o (I - S^V) \quad (56$$

$$Y = (I - S^I)^{-1} (I + S^I) \quad (57$$

As an example of a computation using scattering coefficients consider the network of Fig.-8a. This has no impedance matrix so that to compute the voltage scattering coefficients, the form of equation 49 is used.

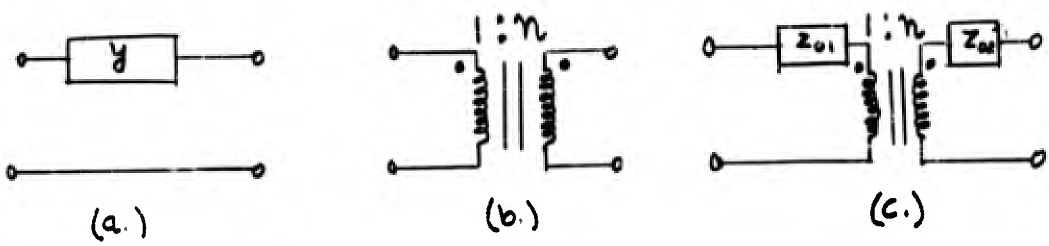


Fig.-8 Two-ports used for scattering computation.

In this case let

$$Y_o = \begin{bmatrix} y_o & 0 \\ 0 & y_o \end{bmatrix} = y_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

That is the same normalizing number for both ports. Then since the admittance matrix for Fig.-8a is

$$Y = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix}$$

The scattering matrix is found directly from equation 49

$$S^Y = \begin{bmatrix} y+y_o & -y \\ -y & y+y_o \end{bmatrix}^{-1} \begin{bmatrix} y_o-y & y \\ y & y_o-y \end{bmatrix}$$

$$= \frac{1}{y_o^2 + 2\,y_o\,y} \begin{bmatrix} y_o+y & y \\ y & y_o+y \end{bmatrix} \begin{bmatrix} y_o-y & y \\ y & y_o-y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{y_o}{y_o + 2y} & \frac{2y}{y_o + 2y} \\ \frac{2y}{y_o + 2y} & \frac{y_o}{y_o + 2y} \end{bmatrix}$$

The reader should check this by equations 46 and 47.

The network of Fig.-8b, has neither a Z or Y matrix, so that the use of equations 46 and 47 is required. (Note that even if Z or Y exists equations 46 and 47 may be the most convenient.)

Suppose that in this case the normalizing numbers for the two ports are z_{o1}, z_{o2}. It is necessary to find the admittance matrix of Fig. 8c. This may be done by any conventional technique, but the direct computation of voltage current ratios is most convenient. Then

$$\hat{y}_{11} = \frac{1}{z_{o1} + \frac{z_{o2}}{n^2}} = \frac{n^2}{z_{o2} + n^2 z_{o1}}$$

$$\hat{y}_{22} = \frac{1}{z_{o2} + n^2\, z_{o1}}$$

$$\hat{y}_{12} = \frac{-n}{z_{o2} + n^2 z_{o1}}$$

From eq. 46

$$s^v_{11} = 1 - \frac{2n^2 z_{o1}}{z_{o2} + n^2 z_{o1}} = \frac{z_{o2} - n^2 z_{o1}}{z_{o2} + n^2 z_{o1}}$$

$$s^v_{22} = 1 - \frac{2\, z_{o2}}{z_{o2} + n^2 z_{o1}} = \frac{-(z_{o2} - n^2 z_{o1})}{z_{o2} + n^2 z_{o1}}$$

$$s^v_{12} = \frac{2n\, z_{o1}}{z_{o2} + n^2 z_{o1}}$$

$$s^v_{21} = \frac{2n\, z_{o2}}{z_{o2} + n^2 z_{o1}}$$

It is of interest to note that $|S_{11}| = |S_{22}|$. In general, it will later be shown that this is true of all lossless 2-ports. Certain special cases of the above equations may also be considered. If $z_{o1} = z_{o2}$, the normalizing numbers cancel from all the equations and they become only a function of the turns ratio. On the other hand if

$$z_{o2} = n^2 z_{o1}$$

then

$$s_{11} = s_{22} = 0$$

$$s_{12} = \frac{1}{n}$$

$$s_{21} = n$$

If this transformer is terminated in the normalizing impedances, it acts as an impedance matching transformer, that is input and output reflections are zero.

IV. Scattering Computations in Terminated Networks

It was shown in the last section that the scattering parameters are readily calculated by exciting a network with generators whose internal impedances equal the normalizing numbers. The resultant array of coefficients is of course applicable to any set of excitations, but provides the simplest description of network operation when matched conditions prevail (i.e. network terminated in the normalizing impedances).

Consider the n-port network of Fig.-9.

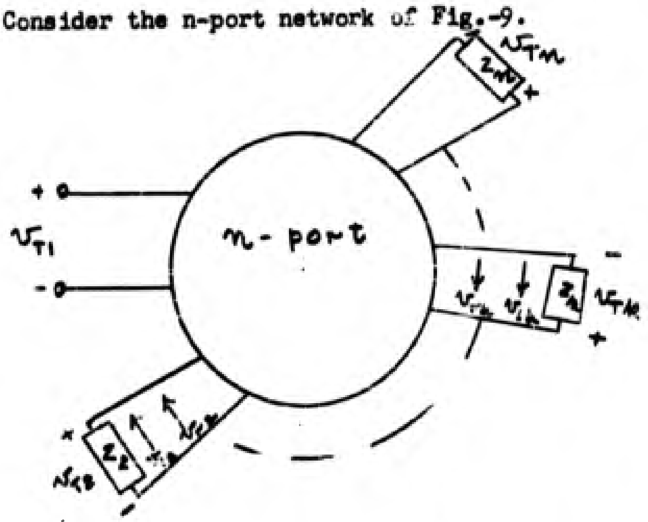


Fig.-9 Network with arbitrary terminations.

This is driven at port 1 and is arbitrarily terminated at its various ports. Each of the terminating loads may be replaced by an equivalent generator if equation 16 and 17 are used. Thus the equivalent components of voltage incident and reflected on the network at port k is given by,

$$v_{ik} = \frac{v_{Tk}}{2} \left[1 - \frac{z_{ok}}{z_k} \right] \qquad (58$$

$$v_{rk} = \frac{v_{Tk}}{2}\left[1 + \frac{z_{ok}}{z_k}\right] \tag{59}$$

Observe that the ratio

$$\frac{v_{ik}}{v_{rk}} = \frac{z_k - z_{ok}}{z_k + z_{ok}} = s_L^V \tag{60}$$

is precisely the voltage reflection factor of the termination itself. The reader must be warned that the numerator of eq. 60 is v_{ik} (not v_{rk}) the incident voltage on the network at port k. If $z_k = z_{ok}$ then $v_{ik}/v_{rk} = 0$, and the equivalent generated incident voltage $v_{ik} = 0$.

Suppose that in Fig.-9 the network is match terminated and excited from port 1. The system of equations 45 applies to the relations between the incident and reflected voltages, but by the discussion in the previous paragraph all equivalent incident voltages $v_{i2}, v_{i3} \ldots, v_{in}$ are zero. Therefore

$$\begin{aligned}
v_{r1} &= s_{11}\, v_{i1} \\
v_{r2} &= s_{21}\, v_{i1} \\
&\vdots \\
v_{rk} &= s_{k1}\, v_{i1} \\
&\vdots \\
v_{rn} &= s_{kn}\, v_{in}
\end{aligned}$$

But at each terminal pair

$$v_{TK} = v_{ik} + v_{rk} = v_{rk} \quad (v_{ik} = 0)$$

Further if the network is driven by a matched generator $v_{i1} = \frac{v_1}{2}$.

Hence

$$\left.\begin{aligned}
\frac{v_{T2}}{v_1} &= \frac{s_{21}}{2} \\
\frac{v_{Tk}}{v_1} &= \frac{s_{k1}}{2}
\end{aligned}\right\} \quad \text{Match terminated case} \tag{61}$$

and equations 61 show that the scattering coefficients give directly the voltage transfer functions of a match terminated network. Thus in the matching transformer ($z_{o1} = n^2 z_{o2}$) of the preceding section, $s_{12} = 1/n$, $s_{21} = n$, and by equations 61

$$\frac{v_{T2}}{v_1} = \frac{n}{2}$$

$$\frac{v_{T1}}{v_2} = \frac{1}{2n}$$

Since in this case the input impedance is z_{o1}, equal to the generator impedance $v_1 = 2\, v_{T1}$, and similarly $v_2 = 2\, v_{T2}$, where v_{T1} and v_{T2} are the primary and secondary voltages of the transformer. Then

$$\frac{v_{T2}}{v_{T1}} = n$$

which is the usual voltage relationship in the ideal transformer.

If the network has mismatched terminations equation 60 may be inserted into the general scattering equations (eq. 45) to determine network operation. As an example of this consider Fig.-10.

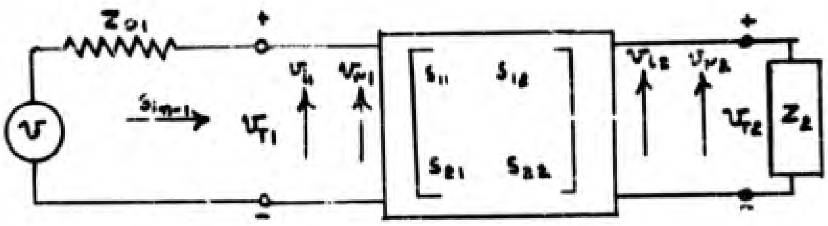


Fig.-10 - Two-port with mismatched termination.

The network is driven by a matched ($z_g = z_{o1}$) generator and it is required to find the ratio v_{T2}/v , and the input reflection factor when the load impedance does not equal the normalizing impedance for port 2, e.g. $z_2 \neq z_{o2}$.

The voltage scattering equations are

$$v_{r1} = s_{11} v_{i1} + s_{12} v_{i2}$$

$$v_{r2} = s_{21} v_{i1} + s_{22} v_{i2}$$

The voltage reflection factor of the load is

$$s_2 = \frac{z_2 - z_{o2}}{z_2 + z_{o2}}$$

so that according to eq. 60

$$v_{i2} = s_2 v_{r2}$$

Inserting this into the second scattering equation and solving for V_{r2}

$$v_{r2} = \frac{s_{21} v_{i1}}{1 - s_2 s_{22}}$$

The first equation is then

$$v_{r1} = s_{11} v_{i1} + \frac{s_{12} s_{21} s_2 v_{i1}}{1 - s_2 s_{22}}$$

Therefore the input reflection factor at port 1 is:

$$s_{in-1} = \frac{v_{r1}}{v_{i1}} = s_{11} + \frac{s_{12} s_{21} s_2}{1 - s_2 s_{22}} \qquad (62$$

The voltage $v_{T2} = v_{i2} + v_{r2}$

Therefore

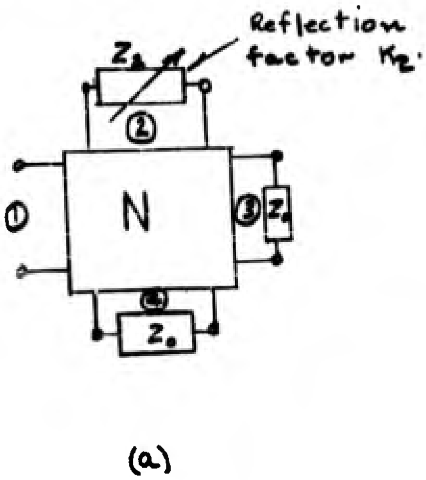
$$v_{T2} = v_{r2}(1 + s_2) = \frac{s_{21}(1 + s_2)\, v_{i1}}{1 - s_2\, s_{22}}$$

Since the generator is matched $v_{i1} = \frac{v}{2}$, so that finally:

$$\frac{v_{T2}}{v} = \frac{s_{21}(1 + s_2)}{2(1 - s_2\, s_{22})} \qquad (63$$

If $s_2 = 0$ (matched case), eq. 63 becomes the form given by equation 61, and the input reflection factor of equation 62 reduces to s_{11}.

As another example of the use of the scattering coefficients in a mismatched case, consider the general 4-port of Fig.-11a.



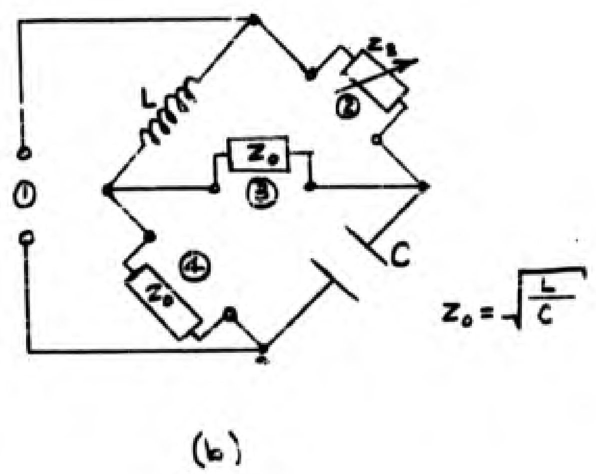


Fig.-11 Operation of a four-port under mismatched conditions.

The impedance at port 2 is adjustable, but ports 3 and 4 are matched. The network is energized from port 1. It is required to determine the conditions on the network N to produce a null voltage at port 3. Suppose the voltage scattering matrix of the 4-port (all normalizing impedances equal to z_0 so that the matrix is symmetrical) is known. The conditions at the ports are given as $v_{i3} = v_{i4} = 0$, $v_{i2} = K_2 v_{r2}$. The reflected voltages at ports 2 and 3 are

$$v_{r2} = s_{12} v_{i1} + s_{22} v_{i2} = s_{12} v_{i1} + K_2 s_{22} v_{r2}$$

$$v_{r3} = s_{13} v_{i1} + s_{23} v_{i2} = s_{13} v_{i1} + K_2 s_{23} v_{r2}$$

From these

$$v_{r3} = s_{13} v_{i1} + \frac{K_2 s_{23} s_{12} v_{i1}}{1 - K_2 s_{22}}$$

or since $v_{r3} = v_{T3}$ (i.e. since $v_{i3} = 0$)

$$v_{T3} = v_{i1} \left[s_{13} + \frac{K_2 s_{23} s_{12}}{1 - K_2 s_{22}} \right]$$

For $v_{T3} = 0$, the bracketed term is zero and,

$$K_2 = \frac{s_{13}}{-s_{23} s_{12} + s_{13} s_{22}}$$

If z_0 is real, $|K_2|$ can not be greater than unity, and the above equation can therefore only be satisfied when

$$\left| s_{22} + \frac{s_{12}}{s_{13}} s_{23} \right| \geq 1$$

In many bridge circuits, for example that of Fig.-11b, $s_{22} \simeq 0$ (in the case of Fig.-11b, this equation is exactly satisfied, but at higher frequencies when parasitic elements are not-negligible the approximate equality will be true), and in this case the requirement becomes:

$$|s_{13}| \leq |s_{12} s_{23}|$$

Thus in this case without any knowledge of the internal structure of the network, it is possible by a simple measurement of voltage transfer function amplitude (see eq. 45) under matched conditions to ascertain the possibility of obtaining a voltage null with an arbitrary (but physical) impedance at port 2.

V. Power Relationships in an n-Terminal Pair Network (n-port) in Terms of Scattering

The formulation of the power equations for an n-port is of basic importance in the application of the scattering matrix to network theory. The case where all the normalizing numbers are real (but not necessarily equal) is of particular interest and this will now be considered. Consider an arbitrary n-port; by equations 25 and 26, the total power entering the typical k^{th} port is

$$P_k = P_{ik} - P_{rk}$$

This may be written as

$$P_k = \frac{|v_{ik}|^2}{R_{ok}} - \frac{|v_{rk}|^2}{R_{ok}} = G_{ok}\,(v_{ik}\,v_{ik}^* - v_{rk}\,v_{rk}^*)$$

The total real power entering the network and dissipated therein is:

$$P = \Sigma\, P_k = \Sigma\, G_{ok}\left[v_{ik}\,v_{ik}^* - v_{rk}\,v_{rk}^*\right] \qquad (64$$

In matrix notation this is

$$P = V_i^{*T}\;G_o\,V_i - V_r^{*T}\;G_o\,V_r \qquad (65$$

(The matrices in eq. 65 were defined in section III)

Then since

$$V_r = S^V\,V_i$$

$$P = \left[V_i^{*T} \ G_o \ V_i - V_i^{*T} \ S^{V*T} \ G_o \ S^V \ V_i \right]$$

$$P = V_i^{*T} \left[G_o - S^{V*T} \ G_o \ S^V \right] V_i \qquad (66$$

Let Q be the bracketed term. Q has the following interesting properties. First since G_o is a diagonal matrix:

$$Q^T = G_o - S^{VT} \ G_o \ S^{V*}$$

and secondly since G_o has only real elements

$$Q^* = G_o - S^{VT} \ G_o \ S^{V*}$$

or

$$Q^T = Q^* \qquad (67$$

A matrix satisfying the above condition is known as a Hermitian[(8)] matrix. The elements of such a matrix satisfy

$$\left.\begin{array}{l} q_{ij} = q_{ji}^* \\ q_{jj} \text{ is real} \end{array}\right\} \qquad (68$$

Equation 66 is known as a Hermitian form. Clearly the derivation of eq. 66 indicates that such a form must be a real scalar when it is expanded (it is the power), and in fact this form is exactly analogous to the quadratic form.[(8)]

Algebraically the real nature of the Hermitian form can be seen from the expansion:

$$P = V_i^{*T} \; Q \; V_i$$

$$= \begin{bmatrix} v_{i1}^* & v_{i2}^* & \dots \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \dots \\ q_{12}^* & q_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \end{bmatrix}$$

$$= \sum_{j,k} v_{ij}^* \, v_{ik} \, q_{jk} \qquad (69$$

and since terms of eq. 69 occur in conjugate pairs

$$v_{ij}^* \, v_{ik} \, q_{jk} + v_{ik}^* \, v_{ij} \, q_{kj} = v_{ij}^* \, v_{ik} \, q_{jk} + v_{ij} \, v_{ik}^* \, q_{jk}^*$$

$$= 2 \, \mathrm{Re} \left[v_{ij}^* \, v_{ik} \, q_{jk} \right]$$

The total sum is a matrix consisting of a single real element i.e. the real power dissipated in the network.

In any physical passive network the real power can never be negative. The conservation of energy therefore requires that

$$P \geqslant 0 \qquad (70$$

Algebraically the Hermitian form for P must have non-negative values for all complex values of the variables v_{ij}. Just as a real quadratic form (which has real variables) satisfying this requirement is known as a Positive Definite (always positive), or Positive Semi-Definite (positive or zero) (abbreviated PD and PSD), form, so the form with complex variables is known as a Positive Definite or Positive Semi-Definite Hermitian form.

The necessary and sufficient conditions that a Hermitian form Q be PD or PSD is that all the principle minors of Q be non-negative.(9) In the important case that all the normalizing numbers are equal to the scalar g_0 (See also the normalization discussion in section VII),

$$G_o = g_o I \tag{71}$$

and

$$Q = I - S^{V*T} S^V \tag{72}$$

If the network is purely reactive, there is no dissipation so that $P = 0$ for all V_i. This can only occur when

$$Q = 0 = I - S^{V*T} S^V \tag{73}$$

or

$$S^{V*T} S^V = I \tag{74a}$$

A matrix which satisfies equation 74a is a unitary matrix. The special restrictions under which equation 74a were derived are applicable to reactive networks with the normalizing numbers at every port real and equal, or when the normalization method described in the next section is used. For such a network (containing only reciprocal elements) the scattering matrix is symmetrical with $S^T = S$, so that equation 74a becomes

$$S^{V*} S^V = I \tag{74b}$$

This relation clearly applies to the current scattering matrix as well, and so may be written without any superscripts as

$$S^* S = I \tag{74c}$$

If this equation is expanded according to the usual rules of matrix multiplication the following equation in terms of the elements of S are obtained:

$$\Sigma\, s_{ir}\, s^*_{ji} = \left\{ \begin{matrix} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{matrix} \right\} = \delta_{ij} \tag{74d}$$

VI. Scattering Matrix Normalization

The scattering equations for power dissipation, and equations relating impedance and admittance coefficients to the scattering parameters have an awkward form when the normalizing numbers for the various ports differ from each other. Further under these conditions the scattering matrix is asymmetric even though the network only contains reciprocal elements. It is useful, therefore, to introduce an appropriate normalization which gives a uniform set of scattering equations and for reciprocal networks makes the normalized scattering matrix symmetric about its main diagonal.

The clue to the normalization procedure is contained in equation 66 of section VI. This expresses the power in terms of incident variables and a hermitian matrix Q which is a property of the network and the port normalization numbers

$$Q = G_o - S^{*T} \; G_o \; S \qquad (75^{\Delta}$$

In a physical network Q is the matrix of Positive Definite or Semi Definite hermitian form. Suppose that a diagonal matrix D_o is defined as

$$D_o = (G_o)^{\frac{1}{2}} = (R_o^{-1})^{\frac{1}{2}} , \quad D_o^{-1} = (G_o^{-1})^{\frac{1}{2}} = (R_o)^{\frac{1}{2}} \qquad (76$$

where the exponential operator 1/2 on the diagonal matrices* G_o, R_o, etc means the square root of each element in the matrix. The Q matrix may be written as:

$$Q = D_o \, (I - D_o^{-1} \, S^{*T} \, D_o \, . \, D_o \, S \, D_o^{-1}) \, D_o$$

or

$$\bar{Q} = D_o^{-1} \, Q \, D_o^{-1} = (I - \bar{S}^{*T} \, \bar{S}) \qquad (77a$$

where

$$\bar{S} = D_o \, S \, D_o^{-1}, \quad S = D_o^{-1} \, \bar{S} \, D_o \qquad (78$$

and

$$Q = D_o \, \bar{Q} \, D_o \qquad (79$$

[Δ] See footnote on following page.

Equation 77a is the normalized equivalent of equation 75.[Δ] Power in terms of $\bar{Q}$ may be written as

$$P = V_i^{*T} \; Q \, V_i = V_i^{*T} \; D_o \, \bar{Q} \, D_o \, V_i \qquad (80$$

In terms of $\bar{Q}$, the necessary and sufficient conditions that the power P be non-negative is that $\bar{Q}$ be the matrix of a Positive Definite or Semi Definite hermitian form is:

$$P = \bar{V}_i^{*T} \; \bar{Q} \; \bar{V}_i \qquad (81a$$

where

$$\bar{V}_i = D_o \, V_i, \; V_i = D_o^{-1} \; \bar{V}_i \qquad (82$$

If the network is lossless, the power P = 0, and the normalized scattering matrix is unitary:

$$\bar{S}^{*T} \; \bar{S} = I \quad \text{(Lossless case)} \qquad (77b$$

Algebraically we say that equation 77 defines a non-singular hermitian transformation (D_o^{-1} has an inverse D_o, and all its non zero elements are on the main diagonal and purely real hence it is non-singular and hermitian) and this transformation does not change the rank nor definiteness character of the matrix Q. Equation 82 gives the transformation to the new normalised incident voltage variables.

If it is recalled that a typical j^{th} element of the diagonal matrix D_o is $\frac{1}{\sqrt{R_{oj}}}$, then from equation 78:

$$\bar{s}_{jk} = s_{jk} \sqrt{\frac{R_{ok}}{R_{oj}}} \qquad (83a$$

$$\bar{s}_{jj} = s_{jj} \qquad (83b$$

[Δ] In the following text, no superscript is used for S. In general S^V is implied; where it is necessary to distinguish between voltage and current scattering matrices the superscript notation is employed.

but by equation 47

$$s_{jk} = -2 R_{oj} \hat{y}_{jk}$$

so that

$$\bar{s}_{jk} = -2 \hat{y}_{jk} \sqrt{R_{ok} R_{oj}} = \frac{-2 \hat{y}_{jk}}{\sqrt{G_{oj} G_{ok}}} \quad (j \neq k) \qquad (84$$

Since $\hat{y}_{jk} = \hat{y}_{kj}$, it follows that

$$\bar{s}_{jk} = \bar{s}_{kj} \qquad (85$$

i.e. the normalized scattering matrix $\bar{S}$ is symmetric.

Equation 84 may be written in terms of the elements of a normalized augumented admittance matrix $\tilde{Y}$

$$\tilde{Y} = D_o^{-1} \hat{Y} D_o^{-1} \qquad (86a$$

or

$$\tilde{y}_{jk} = \frac{\hat{y}_{jk}}{\sqrt{G_{oj} G_{ok}}} \qquad (86b$$

Hence

$$\bar{s}_{jk} = -2 \tilde{y}_{jk} \quad (j \neq k) \qquad (87a$$

$$\bar{s}_{jj} = 1 - 2 \tilde{y}_{jj} \qquad (87b$$

or

$$\bar{S} = I - 2 \tilde{Y} \qquad (87c$$

It is interesting to observe that this algebraic process of normalization corresponds to a transformation of the original network in which ideal transformers are placed at the various ports. If each of the ports of the original network has placed in series with it a resistor of value equal to the port normalization number the result is the augumented network $\hat{Y}$ for the originally specified circuit. Equations 86a and b state that $\tilde{Y}$ is formed from $\hat{Y}$ by placing an ideal transformer of primary to secondary ratio $1/\sqrt{R_{oj}}$ across each port of the augumented network. The result is shown in Fig.-12a. If the transformers are moved so that the augumenting resistors appear in series with the primaries the result is as shown in Fig.-12b. and all the normalizing resistors have unit value.

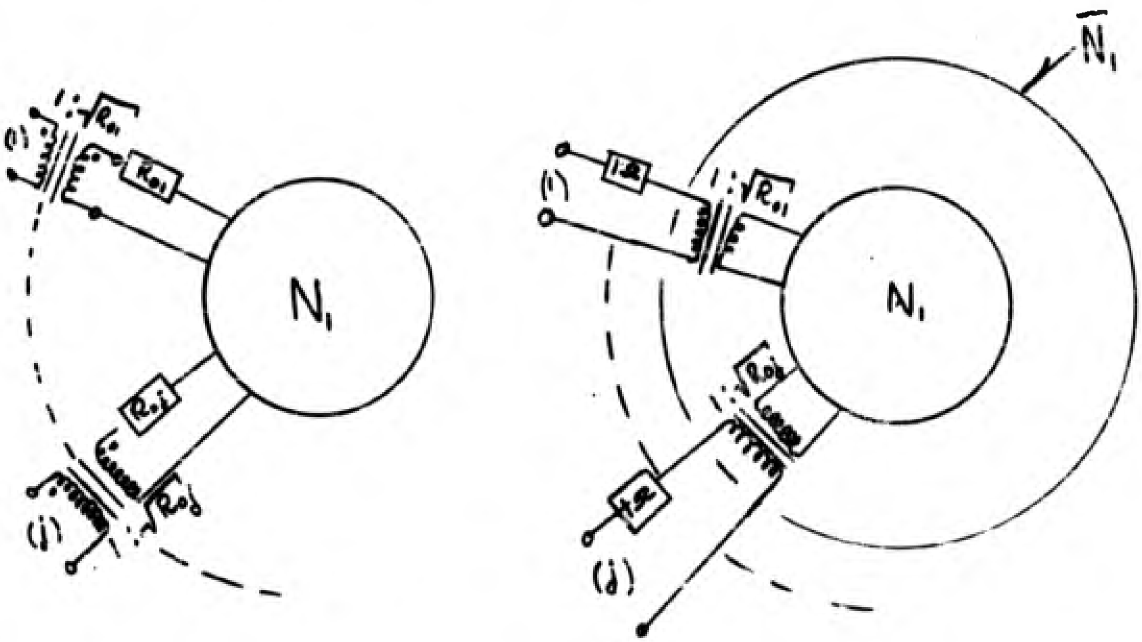


Fig.-12 Network normalization with ideal transformers.

The network of 12b is entirely equivalent to that of Fig.-12a with respect to the accessible ports. In other words the normalized augumented network may be formed using unit resistors at each port if the original network N_1 is transformed to $\bar{N}_1$ by the transformers of ratio $1/\sqrt{R_{oj}}$ as shown in Fig.-12b. The network $\bar{N}_1$ is the normalized transformation of N_1, and this process may always be performed whether or not N_1 has an open circuit impedance and/or short circuit admittance matrix. If the open circuit impedance matrix of N_1 exists and is Z, then the impedance matrix $\bar{Z}$ of the normalized network $\bar{N}_1$ from Fig.-12b is:

$$\bar{Z} = D_o Z D_o, \quad \bar{z}_{ij} = \frac{z_{ij}}{\sqrt{R_{oi} R_{oj}}} \tag{88a}$$

Similarly the normalized short circuit admittance matrix of N_1 is

$$\bar{Y} = D_o^{-1} Y D_o^{-1}, \quad \bar{y}_{ij} = \frac{y_{ij}}{\sqrt{G_{oi} G_{oj}}} \tag{88b}$$

Corresponding to this the normalized total current and voltage variables are:

$$\bar{V} = D_o V, \quad \bar{I} = D_o I \tag{88c}$$

In terms of the normalized admittance and impedance matrices the scattering matrix equations are given as:

$$\bar{S}^V = (\bar{Z} - I)(\bar{Z} + I)^{-1} = (\bar{Z} + I)^{-1} (\bar{Z} - I) \tag{89a}$$

$$\bar{S}^V = (I + \bar{Y})^{-1} (I - \bar{Y}) = (I - \bar{Y})(I + \bar{Y})^{-1} \tag{89b}$$

$$\bar{S}^I = (I + \bar{Z})^{-1} (I - \bar{Z}) = (I - \bar{Z}) (I + \bar{Z})^{-1} \tag{90a}$$

$$\bar{S}^I = (\bar{Y} - I) (\bar{Y} + I)^{-1} = (\bar{Y} + I)^{-1} (\bar{Y} - I) \tag{90b}$$

$$\bar{Z} = (I - \bar{S}^V)^{-1} (I + \bar{S}^V) = (I + \bar{S}^V) (I - \bar{S}^V)^{-1} \tag{91a}$$

$$\bar{Z} = (I + \bar{S}^I)^{-1} (I - \bar{S}^I) = (I - \bar{S}^I) (I + \bar{S}^I)^{-1} \tag{91b}$$

$$\bar{Y} = (I + \bar{S}^V)^{-1} (I - \bar{S}^V) = (I - \bar{S}^V) (I + \bar{S}^V)^{-1} \tag{92a}$$

$$\bar{Y} = (I - \bar{S}^I)^{-1} (I + \bar{S}^I) = (I + \bar{S}^I) (I - \bar{S}^I)^{-1} \tag{92b}$$

Observe that in normalized coordinates

$$\bar{S}^V = - \bar{S}^I \qquad (93$$

The above group of equations should be compared with the corresponding non-normalized set which are given in section III.

It is interesting to note that with impedance and scattering matrices in normalized form, it is possible to perform a type of linear transformation on $\bar{Z}$ which produces the identical transformation on $\bar{S}^V$.

Suppose $\bar{Z}$ is transformed to $\bar{Z}^1$ by

$$\bar{Z}^1 = C^T \bar{Z} C \qquad (94$$

with C square real and non-singular and further with:

$$C^T C = I, \qquad (95$$

Such a transformation is termed an <u>orthogonal transformation</u>. If $\bar{Z}$ is purely real or imaginary, it is always possible to find an orthogonal transformation which makes the matrix $\bar{Z}^1$ a purely diagonal array. The elements of this diagonal array are known as the characteristic roots, or eigenvalues of $\bar{Z}$.

Substituting $\bar{Z}^1$ into equation 89a one obtains a transformed scattering matrix $\bar{S}^{V1}$:

$$\bar{S}^{V1} = (C^T \bar{Z} C - I)(C^T \bar{Z} C + I)^{-1}$$

$$= C^T(\bar{Z} - (C^T)^{-1} C^{-1}) C C^{-1} (\bar{Z} + (C^T)^{-1} C^{-1}) (C^T)^{-1}$$

and using equations 95

$$\bar{S}^{V1} = C^T (\bar{Z} - I)(\bar{Z} + I) C$$

$$= C^T \bar{S} C = C^{-1} \bar{S} C \qquad (96$$

Thus a real orthogonal transformation on $\bar{Z}$ produces the same transformation on $\bar{S}^V$. Further if $\bar{Z}$ is purely imaginary (lossless case) or real the orthogonal transformation which diagonalizes $\bar{Z}$ simultaneously diagonalizes $\bar{S}$. If a given network is transformed as defined by equations 94 and 95 it is a simple matter to find the scattering matrix of the new network by 96. After this slight digression let us return to a further consideration of the properties of normalized scattering parameters.

The relation between normalized incident and reflected variables is easily established. The non-normalized voltage relations are:

$$V_r = S^V V_i$$

Substituting equations 78 and 82

$$V_r = D_o^{-1} \bar{S}^V D_o D_o^{-1} \bar{V}_i$$

$$= D_o^{-1} \bar{S}^V \bar{V}_i$$

or

$$D_o V_r = \bar{S}^V \bar{V}_i$$

Hence we have

$$\bar{V}_r = \bar{S}^V \bar{V}_i \qquad (94$$

where

$$\bar{V}_r = D_o V_r, \; V_r = D_o^{-1} \bar{V}_r \qquad (95$$

In addition the normalized current scattering relations are:

$$\bar{I}_r = \bar{S}^I \bar{I}_i$$

with

$$\bar{I}_r = D_o^{-1} I_r$$

$$\bar{I}_i = D_o^{-1} I_i$$

In other words a problem of analysis may be treated by taking a given set of incident excitations, normalizing these by equation 82, applying the result to the normalized network of scattering matrix $\bar{S}$, and then solving for the normalized reflected quantities by equation 94. The non-normalized set of reflected voltages V_r may then be found by equation 95.

As a simple example of the use of normalization consider the ideal transformer intended for operation between real impedances R_{01} and R_{02} as shown in Fig.-13a. The normalized network is shown in Fig.-13b with the two additional normalizing transformers. These are combined with the given transformer in Fig.-13c.

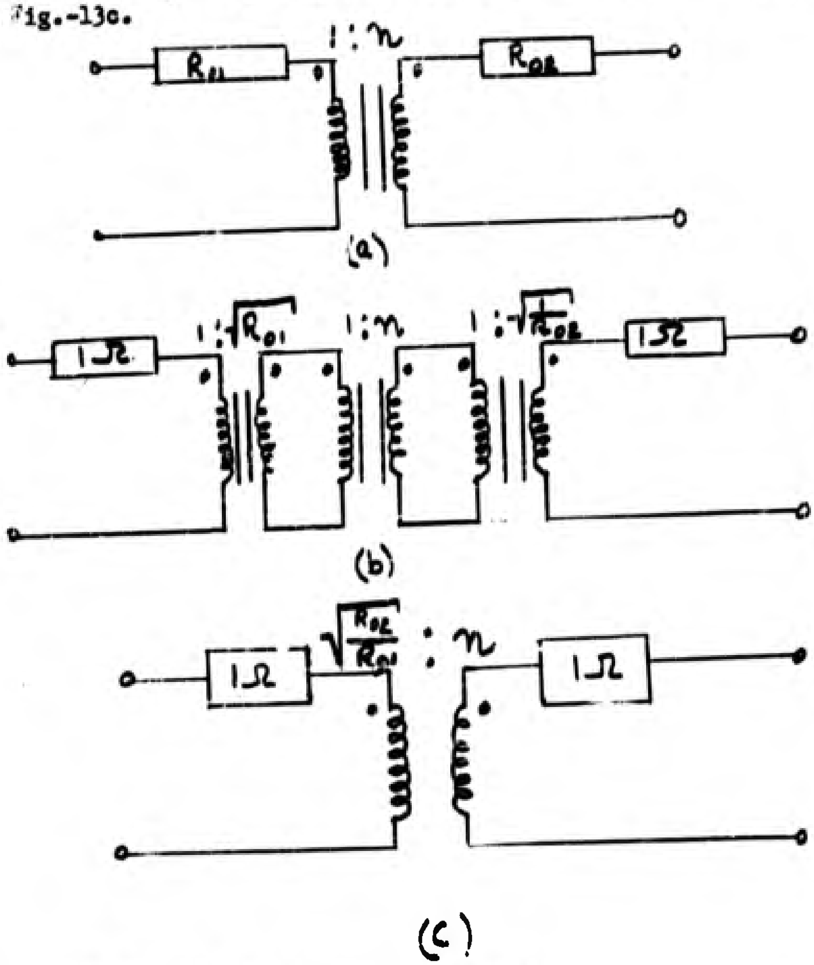


Fig.-13 Normalization of ideal transformer.

The normalized scattering coefficients determined directly from Fig.-13c are

$$\bar{s}^V_{11} = \frac{1 - n'^2}{1 + n'^2}$$

$$\bar{s}^V_{22} = \frac{n'^2 - 1}{n'^2 + 1}$$

$$\bar{s}^V_{12} = \bar{s}_{21} = \frac{2n'}{1 + n'^2}$$

where $n' = \sqrt{\frac{R_{o1}}{R_{o2}}}$.

The transformer in the normalized network produces a match when $n' = 1$ or $n = \sqrt{\frac{R_{o2}}{R_{o1}}}$, which is the condition for match in the non-normalized network. In this case $\bar{s}^V_{11} = \bar{s}^V_{22} = 0$ $\bar{s}^V_{12} = \bar{s}^V_{21} = 1$.

This example indicates that any zeros present in a non-normalized scattering matrix will also be present in the normalized matrix. This is easily seen to be generally true by inspection of equation 83. Further the squared amplitude of coefficients in the normalized scattering matrix measure the ratio of power delivered to a port with respect to maximum available power i.e. available power gain. To see this observe that if a generator of internal resistance R_{oj} and voltage v_j excites a network terminated in the port normalising resistances, the power delivered to R_{ok} is

$$P_k = |v_j|^2 \, |\hat{y}_{jk}|^2 \, R_{ok}$$

and available power (the incident power) at port j is

$$P_{i-j} = \frac{|v_j|^2}{4R_{oj}}$$

so that the available gain is

$$\frac{P_k}{P_{i-j}} = 4\, R_{oj}\, R_{ok}\, |\hat{y}_{jk}|^2$$

and thus by equation 84

$$\frac{P_k}{P_{i-j}} = |\bar{s}_{jk}|^2 \quad (j \neq k) \tag{96*}$$

Note that in the example of Fig.-13 the matching transformer in normalized coordinates produced a gain of 1 ($\bar{s}_{12}^2 = 1$), whereas in non-normalized coordinates (see the example in section III) $|s_{12}|^2 = \frac{1}{n^2}$, $|s_{21}|^2 = n^2$, i.e. the amplitude of non-normalized scattering coefficients are not equal to available gain between various network ports. It is therefore only the amplitude of normalized scattering coefficients whose upper bound is unit amplitude at real frequencies.

This section is concluded by a summary of the more important properties of normalized scattering coefficients:

a) The necessary and sufficient conditions that the power dissipated in a network be non-negative is that the matrix $\bar{Q} = I - \bar{S}^{*T}\,\bar{S}$ be the matrix of a positive or semidefinite hermitian form.

b) If the network is lossless its normalized scattering matrix must be unitary i.e. $\bar{S}^{*T}\,\bar{S} = I$.

c) The power dissipated in a network is invariant with the change to normalized coordinates.

d) If the non-normalized scattering matrix of a network has any zero elements these appear in the same location in the normalized scattering array.

---

* When j = k:

$$\frac{P_k}{P_{i-k}} = 1 - |\bar{s}_{kk}|^2$$

e) If a network is terminated in resistances equal to the port normalizing numbers, then the squared amplitude of the normalized scattering coefficient $|\bar{s}_{ij}|^2$, $(i \neq j)$ is equal to the available power gain of the non-normalized and normalized networks between ports i and j.

f) The normalized scattering matrix of a reciprocal network is symmetrical about the main diagonal.

g) An orthogonal transformation applied to a normalized impedance matrix produces the same orthogonal transformation on the normalized scattering matrix.

h) Normalization of the scattering matrix corresponds to the introduction of ideal transformers at the various network ports to produce a normalized network.

VII. Power Transfer in Lossless 2-port Networks

The unitary requirement on the normalized* scattering matrix of lossless networks (equation 77b) may be used to establish, in a relatively simple fashion, some important terminal properties of lossless networks. Consider a lossless two-port with voltage scattering matrix:

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Since normalized coordinates are used the matrix is symmetrical. If the unitary character of S is applied by the use of equation 77d the following equations are obtained

$$|s_{11}|^2 + |s_{12}|^2 = 1 \qquad (97a$$

$$|s_{22}|^2 + |s_{12}|^2 = 1 \qquad (97b$$

$$s_{11} s_{21}^* + s_{12} s_{22}^* = 0 \qquad (97c$$

* In future discussion, unless otherwise indicated, all scattering matrices will be presumed normalized according to equation 78. The bar ($\bar{s}$) will not be used. Further, unless otherwise indicated the voltage scattering matrix will be used and the "v" superscript will be omitted.

It is clear from equations 97a and 97b that

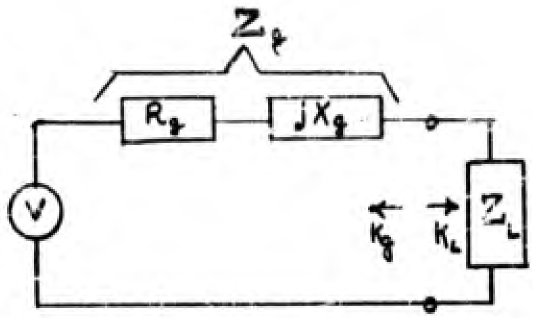
$$|s_{11}| = |s_{22}| \tag{98}$$

Thus in a lossless two port network the magnitude of the reflection factor measured under match terminated conditions is the same at either port of the network.

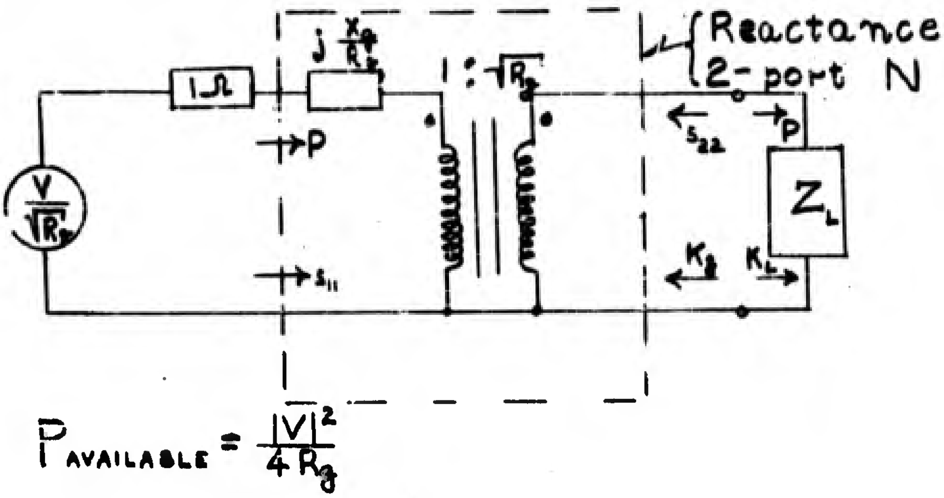
The following problem is another illustration of the application of unitary constraints to lossless networks. A load of fixed reflection factor amplitude but arbitrary phase terminates a generator with complex internal impedance. The limits of power transfer to the load are to be determined in terms of the generator and load reflection factor amplitudes. Power may lie anywhere between these bounds since the phase angle between load and generator impedances is not fixed.

This problem is most readily handled by representing the circuits in a rather special way.* In Fig.-14a the actual arrangement is shown with a generator of reflection factor amplitude $|K_g|$ (measured with internal voltage shorted) feeding the load of reflection factor $|K_L|$. In Fig.-14b the generator is represented as a voltage source with one ohm impedance plus a lossless 2-port N. The power P at the output of the 2-port is precisely the same as that delivered to the load in Fig.-14a. Since the 2-port is lossless the power delivered to its input is also P. As the relative phase between K_g and K_L is varied (amplitudes remaining fixed) the power delivered to the load in both 14a and 14b varies in the same manner. Accordingly in finding the range of power variation the power delivered to the reactance 2-port N of Fig.-14b will be studied.

* The circuit representation is similar to one suggested by J.W.E. Griemsmann.



(a)



(b)

Fig.-14 Power to an arbitrary load.

The scattering coefficient amplitudes of the lossless network N are easily written in terms of $|K_g|$. Referring to Fig.-14b

$$|s_{22}| = |K_g| \qquad (99a$$

and by equation 98 since the network is lossless

$$|s_{22}| = |s_{11}| = |K_g| \qquad (99b$$

$$|s_{12}|^2 = 1 - |s_{11}|^2 = 1 - |K_g|^2 \qquad (99c$$

The net input reflection factor to the network N when terminated in Z_L is determined by the relative phase of K_g and K_L. It is the excursion of this input reflection magnitude which determines the range of power transferred to the load. Suppose the maximum reflection amplitude at the input is $|K|_{MAX}$ and the minimum $|K|_{MIN}$. The maximum and minimum power transfer ratios (normalized to the available generator power $|V|^2/4R_g$) are then

$$G_{MAX} = 1 - |K|^2_{MIN} \qquad (100a)$$

$$G_{MIN} = 1 - |K|^2_{MAX} \qquad (100b)$$

The power transfer problem is therefore reduced to the determination of $|K|_{MAX}$ and $|K|_{MIN}$.

The 2-port N has a net input reflection factor K given by equation 62.

$$K = s_{11} + \frac{s_{12}^2 K_L}{1 - K_L s_{22}} = \frac{s_{11} - K_L(s_{11}s_{22} - s_{12}^2)}{1 - K_L s_{22}} \qquad (101)$$

Assuming a reference phase of zero for s_{11}, the unitary equation 97c for N gives

$$|s_{11}|\,|s_{12}|\,\epsilon^{j\Phi_{12}} + |s_{12}|\,|s_{22}|\,\epsilon^{j(\Phi_{22}-\Phi_{12})} = 0$$

or

$$\Phi_{22} = 2\,\Phi_{12} + \pi \qquad (102)$$

Substituting equations 99b, c and 102 into equation 101 gives K in terms of Φ_{22}, K_g, and $K_L = |K_L|\,\epsilon^{j\Phi}$.

$$K = \frac{|K_g| - |K_L|\,\epsilon^{j(\Phi_{22}+\Phi)}}{1 - K_L|K_g|\,\epsilon^{j\Phi_{22}}} = \frac{\left|\frac{K_g}{K_L}\right| - \epsilon^{j\Phi^1}}{|K_g|\left[\frac{1}{|K_g K_L|} + \epsilon^{j\Phi^1}\right]}$$

where

$$\Phi^1 = \Phi + \Phi_{22} + \pi$$

Thus the value of $|K|^2$ is:

$$|K|^2 = \frac{\left(\left|\frac{K_g}{K_L}\right| + \left|\frac{K_L}{K_g}\right| + 2 \cos \Phi^1 \right)}{\left(\frac{1}{|K_g K_L|} + |K_g K_L| + 2 \cos \Phi^1 \right)} \qquad (103$$

In equation 103, the terms involving $|K_g|$ and $|K_L|$ in both numerator and denominator have the algebraic form $x + 1/x$, $0 \leq x \leq 1$, and this has a minimum value of 2 over this range of x. Hence equation 103 is positive for all values of Φ^1. The extreme values of $|K|^2$ in equation 103 occur for $\cos \Phi^1 = \pm 1$, and therefore

$$|K|^2_{MAX} = \left(\frac{|K_g| + |K_L|}{1 + |K_g||K_L|} \right)^2 \qquad (104a$$

$$|K|^2_{MIN} = \left(\frac{|K_g| - |K_L|}{1 - |K_g||K_L|} \right)^2 \qquad (104b$$

The possible range of power is obtained by substituting these equations into equations 100.

A more compact expression is obtained by defining a new quantity, the "voltage standing wave ratio". This has the physical significance of the ratio of maximum to minimum voltage amplitude when a standing wave is present on a transmission line, but for the present problem may be defined as

$$\rho = \frac{1 + |K|}{1 - |K|} \geq 1 \qquad (105a$$

or in terms of ρ

$$|K| = \frac{\rho - 1}{\rho + 1} \qquad (105b)$$

If ρ_K, and ρ_L are obtained by substituting $|K_g|$ and $|K_L|$ in equation 105a, then corresponding to $|K|_{MAX}$ and $|K|_{MIN}$ (equations 104), the maximum and minimum input voltage standing wave ratios are:

$$\rho_{MAX} = \rho_g \rho_L \qquad (106a)$$

$$\rho_{MIN} = \left|\frac{\rho_g}{\rho_L}\right|^{\delta} \quad \begin{cases} \delta = 1 & \text{if } \rho_g > \rho_L \\ \delta = 1 & \text{if } \rho_g < \rho_L \end{cases} \qquad (106b)$$

In terms of voltage standing wave ratios of load and generator the values of G_{MIN} and G_{MAX} are (see equations 100)

$$G_{MIN} = \frac{4 \rho_g \rho_L}{(\rho_g \rho_L + 1)^2} \qquad (107a)$$

$$G_{MAX} = \frac{4 \left|\frac{\rho_g}{\rho_L}\right|^{\delta}}{\left(\left(\frac{\rho_g}{\rho_L}\right)^{\delta} + 1\right)^2} \qquad (107b)$$

In summary, the previous discussion has arrived at two important results.

1) If a generator of voltage standing wave ratio ρ_g, feeds a load of voltage standing wave ratio ρ_L, the ratio of extremal values of power delivered to the load is $\frac{G_{MAX}}{G_{MIN}}$ and can be computed from equations 107.

2) If a lossless 2-port has a standing wave ratio ρ_g when terminated in a unit normalized load, and is actually terminated in a load of standing wave ratio ρ_L (phase unknown), the maximum possible standing wave ratio at the input of the 2-port is $\rho_g \rho_L$, and the minimum possible standing wave ratio is $\left(\frac{\rho_g}{\rho_L}\right)^{\delta}$. (See equations 106).

VIII. Biconjugate and Maximum Output Networks

A biconjugate network is one in which, under prescribed resistive terminating conditions, each port is decoupled from at least one other port. The normalized scattering matrix of a biconjugate network (the normalization is with respect to the various prescribed port terminating resistors) has at least one zero element in every row and column. A biconjugate network is not necessarily lossless.

A maximum output or matched network is a lossless n-port in which, under prescribed resistive terminating conditions, each port presents an input impedance equal to the port termination. Since the port inputs are matched to the prescribed terminations, a generator with Thevinin impedance equal to the prescribed terminating resistance feeds its maximum power to the network.

The unitary relations yield many interesting properties of maximum output networks. For example, any maximum output 4-port is a biconjugate network; also a lossless network with three ports cannot be constructed as a maximum output network.

To illustrate first how the tools of algebra may be used to derive general network properties of maximum output networks consider the following theorem.

Theorem A. A lossless network with real scattering coefficients and n-pairs of ports can not be physically realized as a maximum output network unless n, the number of ports is even.

A network with ideal transformers interconnected in an arbitrary fashion is an example of a lossless structure with real scattering coefficients.

This theorem may be proved in a general fashion[10] by using certain special algebraic properties of a unitary symmetric matrix. Recall first (see equation 96) that a unitary symmetric scattering matrix can be diagonalized by an orthogonal transformation. If λ is the diagonal matrix

$$\lambda = C^{-1} S C \qquad (108$$

$$(\lambda_{ij}) = 0 \text{ for } i \neq j$$

Such an orthogonal transformation leaves certain important properties of S invariant. First the resulting matrix λ is still unitary. This means that

$$|\lambda_{ii}|^2 = 1 \qquad (109a$$

since there are only diagonal elements in the λ matrix. Since the orthogonal matrix C is real, and S by hypothesis has only real elements, λ_{ii} in the transformed diagonal matrix must be real or

$$\lambda_{ii} = \pm 1 \qquad (109b$$

The second invariant which is useful here is that the sum of the diagonal elements of the original S matrix (the "trace" or "spur") is equal to the sum of the diagonal elements in any orthogonal transformation of S. Thus

$$\sum_i s_{ii} = \Sigma \lambda_{ii} \qquad (110$$

Suppose it is assumed that the original matrix S is that of a maximum output network. Then

$$s_{ii} = 0$$

or

$$\sum_i \lambda_{ii} = 0 \qquad (111$$

But since each λ_{ii} is ± 1, the number of ports must be even to satisfy equation 111, and the theorem is proved.

A group of useful theorems can be proved concering lossless matched networks with 3 and 4 ports by direct recourse to the unitary equations without the need of abstract algebra of the type used in theorem A. For example a property earlier mentioned is easily proved.

Theorem B. A lossless network with 3 ports is not physically realizable as a maximum output network. (In this case the restriction of real elements used in Theorem A is not part of the hypothesis).

To prove this consider just the diagonal elements in the unitary equation

$$S^* S = I$$

Thus with $S_{11} = S_{22} = S_{33} = 0$ (the matched network case)

$$|s_{12}|^2 + |s_{13}|^2 = 1$$

$$|s_{12}|^2 + |s_{23}|^2 = 1$$

$$|s_{13}|^2 + |s_{23}|^2 = 1$$

The only possible solution of these equations is

$$|s_{12}| = |s_{13}| = |s_{23}| = \frac{1}{\sqrt{2}}$$

An off diagonal element of the unitary relations ($\Sigma s_{1r} s_{2r}^* = 0$) gives

$$s_{11} s_{12}^* + s_{12} s_{22}^* + s_{13} s_{23}^* = 0 = s_{13} s_{23}^*$$

or either s_{13} or s_{23} is zero which contradicts the requirement $|s_{13}| = |s_{23}| = \frac{1}{\sqrt{2}}$.
Thus a matched 3 port is inconsistent with the unitary equations, and is not physically realizable.

In as much as a maximum output 3-port is not physically possible it might be interesting to determine the best power transfer possibilities with a lossless 3-port.

Theorem C. If a lossless 3 port is to have the same power transfer ratio between prescribed resistive terminations located at each port the best power transfer that can be achieved at each port is to transmit 2/3 of the available power from a generator to the network.

To prove this observe that $T = |s_{12}| = |s_{13}| = |s_{23}|$ if power transfer between each pair of ports is the same. Then from the unitary relations

$$|s_{11}|^2 + 2\,T^2 = |s_{22}|^2 + 2\,T^2 = |s_{33}|^2 + 2\,T^2 = 1$$

or $|s_{11}| = |s_{22}| = |s_{33}| = K$. Hence

$$K^2 + 2\,T^2 = 1$$

The off diagonal relation used in the previous theorem gives

$$(s_{11}s_{12}^* + s_{12}s_{22}^* + s_{13}s_{23}^* = KT\left\{e^{j(\Phi_{11}-\Phi_{12})} + e^{j(\Phi_{12}-\Phi_{22})}\right\} + T^2\,e^{j(\Phi_{13}-\Phi_{23})} = 0$$

or $T = K$

Hence

$$3\,T^2 = 1$$

$$T = K = \frac{1}{\sqrt{3}}$$

The ratio of power delivered to the network to the available power is

$$1 - K^2 = \frac{2}{3}$$

and the theorem is proved.

Many other theorems may be proved by similar means and a few of these will be listed here without proof.

Theorem D. A lossless reciprocal 4-port which has one pair of ports matched and decoupled under prescribed resistive terminating conditions is a maximum output biconjugate network.

Theorem E. A lossless reciprocal 4-port which under prescribed resistive terminating conditions has ports 1 and 2 matched, with 1 decoupled from 3 and 2 decoupled from 4, is a maximum output biconjugate network.

Theorem F. A maximum output 4-port is a biconjugate network.

Theorem G. A maximum output 4-port with port 1 decoupled from 3, 2 from 4 has

$$|s_{14}| = |s_{41}| = |s_{23}| = |s_{32}| \text{ and } |s_{12}| = |s_{21}| = |s_{34}| = |s_{43}|.$$

A simple example of a maximum output network is shown in Fig.-15.

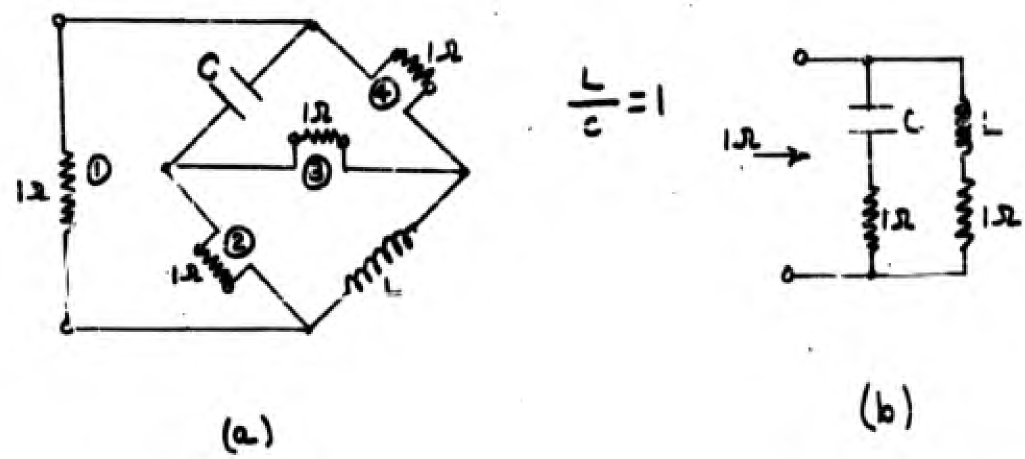


Fig.-15 Example of Maximum Output Network

The network is shown with normalized unit terminations at the 4-ports. It is clear that for $L/C = 1$ this is a balanced bridge with ports 1 and 3 decoupled. Thus for excitation at port 1 no current flows in 3 and the network presents the constant resistance combination shown in Fig.-15b to ports 1 and 3. Hence Theorem D applies and the network is of maximum output type. At all frequencies total available power from generators at each port is transmitted to the network, but the power division to the various ports is a function of frequency. When $\omega L = 1/\omega C = 1$, 50 % of the available power in at port 1 is transferred to port 2, and 50 % to port 4.

Another example of a maximum output 4-port is shown in Fig.-16. The three winding transformer is the so called hybrid coil. The additional ideal transformer shown at port 4 is merely to effect the normalization to a one ohm port terminating resistance.

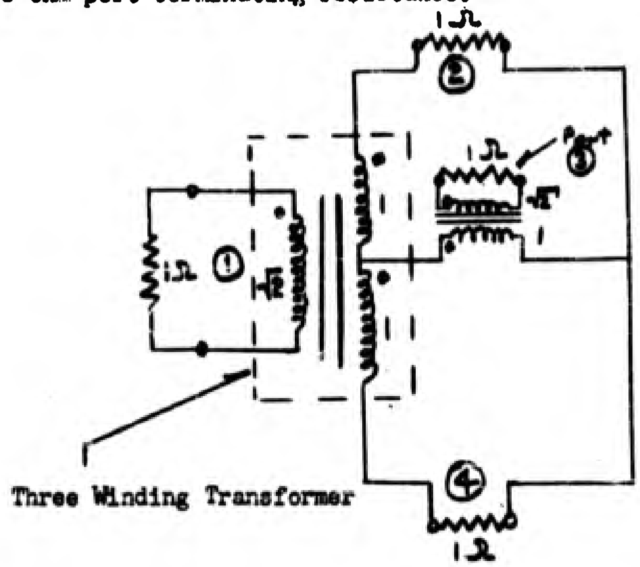


Fig.- 16 Hybrid coil maximum output network.

From symmetry(5) considerations it is not difficult to show that port 1 is decoupled from 3 and further that the impedance at ports 1 and 3 looking into the terminated network is 1Ω. Hence Theorem D indicates that this is a maximum output network. Thus it is immediately established that ports 2 and 4 are decoupled and matched. Further this network transfers 50% of the available power at one port to each of the two remaining coupled ports, and this factor is independent of frequency. A complete tabulation of maximum output networks similar to Fig.-16 is given in reference 2.

IX. Frequency Dependence of The Scattering Matrix

The preceding sections have considered the properties of the scattering matrix at any real fixed frequency. The most general physical constraint on the scattering matrix was shown to be the positive definite or semi definite character of the hermitian form whose matrix is (normalized):

$$Q = I - S^* S$$

When the network is lossless S is unitary or

$$S^* S = I$$

Even when the scattering matrix has elements which are functions of a complex frequency variable $p = \sigma + j\omega$ (ω is real angular frequency) rather than just complex constants (as has been the case heretofore), the general realizability requirements are quite similar to those which apply to the matrix of constants.

The following theorem(10) is basic to the consideration of a scattering matrix which applies over the entire frequency spectrum to a network of linear, passive reciprocal elements.

Theorem A. The necessary and sufficient conditions that a normalized scattering matrix S(p), whose elements are functions of the complex frequency $p = \sigma + j\omega$, corresponds to a physically realizable lumped passive linear reciprocal network is that

a) $Q(j\omega) = I - S^*(j\omega)\, S(j\omega)$ (112
be the matrix of a positive definite or semi-definite hermitian form for all real ω and

b) The elements of S(p) be rational functions of p with real coefficients (i.e. ratio of two polynomials in p) which are analytic for Re $p > 0$, (i.e. the denominator polynomials of the various elements of S must contain no roots in the right half of the $p = \sigma + j\omega$ plane).

An equivalent form for the above theorem is

Theorem B. The necessary and sufficient conditions that a normalized scattering matrix $S(p)$, whose elements are functions of the complex frequency $p = \sigma + j\omega$, correspond to a physically realizable lumped passive linear reciprocal network is that

$$Q(p) = I - S^{*}(p)\ S(p) \quad p = \sigma + j\omega \qquad (113)$$

$$\sigma \geqslant 0.$$

be the matrix of a positive definite or semi-definite hermitian form for all p restricted as in equation 113.

Theorem B will be proved for the case where a network possesses an impedance matrix. This was originally carried out by Belevitch[11] in 1948, and the proof for the case of a network with no impedance matrix was later carried out by him[10] in 1951. For the proof of this latter case the reader is referred to section IX of reference (10).

Referring to equation 91a

$$Z = (I + S)\ (I - S)^{-1}$$

or

$$Z(I - S) - S = I$$

$$Z(I - S) + (I - S) = 2I$$

$$Z = 2(I - S)^{-1} - I \qquad (114)$$

In equation 114 all matrix elements are functions of the complex frequency $p = \sigma + j\omega$. The necessary and sufficient conditions that an impedance matrix be physically realizable are[12] that the matrix $R(\sigma, \omega) = \text{Re}\ Z(p) = 1/2\ [Z(p) + Z^{*}(p)]$ be the matrix of a positive definite or semi-definite quadratic form for $\text{Re}\ p \geqslant 0$. This will be translated into the appropriate requirement on S. Use equation 114 to find R. Thus:

$$R(\sigma,\omega) = \frac{1}{2}\left[Z + Z^{*}\right] = \left[I - S\right]^{-1} - I + \left[I - S^{*}\right]^{-1}$$

Premultiply the above equation by $I - S^*$ and post multiply it by $I - S$. When terms are collected the result is

$$(I - S)^* R(\sigma,\omega) (I - S) - I - S^* S = Q(\sigma,\omega) \qquad (115$$

Now $R(\sigma,\omega)\ \sigma \geq 0$ is the matrix of a PD or PSD quadratic form, which is merely a special case of an hermitian form in that all the elements are real so that $R(\sigma,\omega)$ is both real and symmetric. It is a mathematical fact that the transformation given in equation 115 changes $R(\sigma,\omega)$ into a new hermitian form $Q(\sigma,\omega)$ with complex elements, no matter what S may be, provided I - S is non-singular and symmetric which of course is true. (The non-singular property follows from the hypothesized existence of an impedance matrix; see equation 115). Further a non singular hermitian transformation (specifically the one given in equation 115) preserved the PD or PSD character of the original matrix. Hence $Q(\sigma,\omega)$, $\sigma \geq 0$ is a PD or PSD hermitian form. Further if equation 115 is inverted to give

$$((I - S)^*)^{-1}\ Q(\sigma,\omega)\ (I - S)^{-1} = R(\sigma,\omega) \qquad (116$$

it follows by the same reasoning that if $Q(\sigma,\omega)\ \sigma \geq 0$ is PD or PSD then $R(\sigma,\omega)\ \sigma \geq 0$ is PD or PSD. Thus theorem B is proved for the case where Z exists. To complete the general proof, it is shown in reference (11) that when $Q(\sigma,\omega)$, $\sigma \geq 0$ is PD or PSD but Z does not exist i.e. (I - S) is singular, the network can always be represented as the interconnection of two networks one with ideal transformers the other possessing an impedance matrix.

Theorem B is therefore true for all cases.

Belevitch[10] shows that theorem A is sufficient to give theorem B by considering the hermitian form

$$P = V^{*T}\ Q(j\omega)\ V = V^{*T}\ V - V^{*T}\ S^*\ .\ SV$$

which is PD or PSD for all ω by the hypothesis of theorem A. Further if the elements of S are analytic in the right half p plane so are the elements of SV for any complex values of the components of V. The quantity $V^{*T}\ S^*\ .\ SV$ is a hermitian form of the variable p, and is a generalization of $s^*_{ij}(p) \cdot s_{ij}(p)$ which as the squared amplitude of a complex function has its maximum value on the $j\omega$ boundary for all $p = \sigma + j\omega, \sigma \geq 0$ if s_{ij} is analytic in the right half plane (i.e. $\sigma \geq 0$). Similarly the hermitian form $V^{*T}\ S^*\ .\ SV$ has its maximum value on $j\omega$ if all elements of SV are analytic in the right half of the p plane.

In equation 116 $V^{*T}V$ is not a function of p and is a positive number for any column matrix V. Further $V^{*T} S \cdot SV$ has its maximum value on the boundary, hence P has its minimum value on jω. But $P \geq 0$ on jω by hypothesis, hence $P \geq 0$ for all $p = \sigma + j\omega$, $\sigma \geq 0$. Thus Theorem B follows from A.

The converse is true as well. Note that if as hypothesised in theorem B, $P(\sigma, \omega) \geq 0$ for $\sigma \geq 0$, then $Q(\sigma, \omega)$ must have non-negative principle minors, hence in particular its diagonal terms (the first order principle minors) satisfy

$$1 - |s_{ij}(p)|^2 \geq 0 \qquad p = \sigma + j\omega$$
$$\sigma \geq 0$$

Therefore $|s_{ij}(p)|^2$ is bounded everywhere in the right half plane i.e.

$$|s_{ij}(p)| \leq 1$$

so that $s_{ij}(p)$ must be analytic for p in the right half plane. In fact any physical $s_{ij}(p)$ takes the right half of the p plane into unit circle in the s_{ij} plane. Thus Theorem A follows from B.

An important special case of these two theorems applies to lossless networks and is given as

Theorem C. The necessary and sufficient conditions that S(p) be the normalized scattering matrix of a linear lossless reciprocal network is that

a) S(p) be unitary for $p = j\omega$, i.e. $S(j\omega)\, S^*(j\omega) = I$

and

b) The elements of S be analytic in the right half of the p plane.

These theorems have been applied recently to various problems of network design such as the synthesis of reactance 2-ports(13), the design of matching networks(6)(7), and the synthesis of power and voltage equalizers(14). Space does not permit any discussion of these applications here and the interested reader must consult these references.

In conclusion it should be restated that the scattering formulation of network equations appears to be a most powerful tool for many problems of analysis and synthesis. This formulation appeals very strongly as a unified approach to network theory, for every physical network without exception possesses a scattering matrix.

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