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RADIATION CHARACTERISTICS OF RECTANGULAR SURFACE SOURCES

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U. S. NAVAL ORDNANCE LABORATORY
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ABSTRACT: In this report, a theoretical study of the radiation characteristics of rectangular surface sources has been made when these sources radiate uniformly over their surfaces and obey Lambert's Cosine Law. This study includes a presentation of the history of the development of such formulae up to the present. Very general equations of two forms have been derived giving the total flux on an elementary receiving area, when this elementary receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines. The classical method of surface integration introduced by Lambert has been used. These equations are very general in form, and it is shown how each of the equations found in the literature become special cases of these more general equations in which the coordinates of the elementary receiving area or the direction cosines of its surface normal have particular values. A simple translation and rotation of coordinates makes it possible to also consider the equally important problem in which the source is permitted to have arbitrary coordinates and a surface normal with arbitrary direction cosines. Some numerical examples of the radiation field of a square surface source are given and possible military applications are cited.

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RADIATION CHARACTERISTICS OF RECTANGULAR SURFACE SOURCES

I. INTRODUCTION

1. Often it would be quite useful if it were possible to predict the total flux from a source of radiation of arbitrary geometry falling on a receiver when the source and receiver are at arbitrary distances from each other and have arbitrary orientations. This flux might be of any form such as electromagnetic flux, acoustic flux, or it may consist of a flow of particles. This problem, in addition to having many everyday applications, also is one of the problems facing the military when devices are constructed to detect potential targets, for certainly one of the most important problems in such considerations which must be faced is that of determining the total flux from the target falling on the receiver. The receiver, of course, must also convert this flux into a signal having a sufficiently high level above the noise, but such considerations form another chapter which lies beyond the scope of the present report. We shall be concerned here only with methods for determining the total flux falling on the receiver and will not be concerned with the manner in which the receiver utilizes this flux.

2. Calculations of the flux from sources of arbitrary geometry have not been attempted because they are usually regarded as too complex. To be sure, they can be carried out when the source and receiving element are separated by distances which are large compared with the dimensions of the source and receiving element and the absorption and scattering of the intervening medium are neglected, for in this case the inverse square law can be applied. There is then no problem in determining the total flux falling on the receiver for any position and orientation if it is known for one position and orientation. Frequently however, receivers are called upon to detect sources at close range so that the geometries of both the source and the receiver influence very markedly the total flux falling on the receiver. Under these conditions, the inverse square law fails, and there is then no simple way of calculating the total flux falling on the receiver. The practical solution to such a problem is usually the engineering one in which a receiver is chosen and measurements made. Such measurements, however, are useful only under the conditions under which they are taken and become of very little value when changes are made or new variables appear.

3. To predict the total flux falling on a receiver of given geometry for any position and orientation in space, one must first make detailed studies of the radiation fields of the sources being considered. A radiation field of a source of flux of some form can be described in various ways, usually depending upon how it is to be detected. Since we will be interested here in calculating the total flux falling on a small plane receiving element (thermocouple, photoconducting cell, etc.), we will consider the radiation field of such a source to be defined by the total flux falling on a small elementary receiving element (small compared to other dimensions) placed at different points in the surrounding region and having different orientations. To determine the forms of the radiation fields of various sources, surface and volume integrations become necessary. Many of these integrations have already been carried out for sources of simple geometries, providing certain simplifying assumptions are made concerning the properties of the source and flux, and the elementary receiving area is placed in some particular position or plane and is oriented in some particular direction. Similar problems involving "action at a distance" have also appeared in other fields of physics in which surface and volume integrations have been necessary.

4. In the case of surface sources, which will concern us in this report, some assumptions are usually made concerning the nature of the flux emitted from the surface in question. Surface integrations of the type previously mentioned mean that the contributions from each of the surface elements are simply added. In the case of radiant energy, if the contributions of these surface elements are to be added, then the wave lengths of the flux must be both random in phase across the surface and small compared to the size of the surface elements involved. This means then that the problem of evaluating the radiation field of such a surface source emitting radiant energy is reduced to one in geometrical optics rather than in physical optics. Similarly in the case of particles, the particles must be small compared to these surface elements and they must not interact with each other if these surface integrations are to be of value. Assuming that these conditions are fulfilled, two additional assumptions are usually made concerning the nature of the radiating surface. The first is that the surface radiates uniformly over its surface and the second that it obeys Lambert's Cosine Law (reference (a)). This of course represents a further idealization of the problem, and such a source is commonly referred to as a "perfect diffuse emitter". Many sources will be found which do not conform to such an idealization (reference (b)). Even though precautions are taken to be sure that they radiate uniformly over their surfaces, deviations still are found because Lambert's Cosine Law is not satisfied. Heated polished surfaces for example always radiate more flux at oblique emergence than would be expected from Lambert's Cosine Law. Occasionally, a cosine squared dependence with angle of emergence is found for

particular materials with particular surface conditions. Fortunately, there are many sources which approximately obey Lambert's Cosine Law (e.g. magnesium oxide, finely ground glass surfaces, thermal radiation from molten silver and unpolished platinum), particularly if the surface is somewhat rough, and the range in angle of emergence is not too large. We will therefore assume that the surface sources being considered here radiate uniformly over their surface and obey Lambert's Cosine Law.

5. In this report, we will confine our attention to the rectangular emitting surface. The primary reason for this is that many sources are rectangular in shape. However, there are also sources which do not possess simple forms and perhaps do not radiate uniformly over their surfaces or obey Lambert's Cosine Law. In these cases, the source can usually be divided up into a series of smaller sources which do approximately fulfill these conditions of uniformity and satisfy Lambert's Cosine Law. The rectangular source element is one of the most useful forms for such calculations, and the total flux falling on the elementary receiving area from such a complex source will simply be the sum of the individual contributions of each of these smaller rectangular sources. The general problem which will concern us here is then more specifically that of calculating the total flux from the rectangular surface source (assuming the rectangular source radiates uniformly over its surface and obeys Lambert's Cosine Law) falling on the elementary receiving area when it has arbitrary coordinates and a surface normal with arbitrary direction cosines.

II. History

6. If one assumes that the rectangular source radiates uniformly over its surface and obeys Lambert's Cosine Law, then the rectangular emitting surface is, of course, a rather old problem (reference (c)). The origin of calculations on the field characteristics of emitting rectangular sources seems to be in the papers of E. P. Hyde and K. Norden (reference (d)) published in 1907 and 1908. Both of these authors were concerned with calculating the flux density on an elementary receiving area from a mercury arc of tubular form, which could be approximated by a cylinder or an infinitely long strip. Apparently the first calculations for rectangular surfaces of finite dimensions were those made by Bordoni in Italy in 1908. The two cases in which he was interested are shown schematically in Fig. 1 and Fig. 2. Bordoni calculated the flux density on an elementary receiving area located on the surface normal through the midpoint of one side of the rectangle when the receiving area was parallel to the rectangular surface. As illustrated by Fig. 2, he also concerned himself with the case in which the surface normal to the elementary receiving area was rotated until it was parallel to one pair of the sides of the rectangle. Both of these equations are correct and in

agreement with the often repeated calculations for these problems appearing in later years.

7. Another of the early workers on the radiation properties of rectangular sources was B. Jones (reference (f)) in the United States who published three papers during the period from 1909-1911. These papers described some additional radiation characteristics of rectangular sources as well as other surface sources. The particular problems involving rectangular sources in which he was interested are those shown in Figs. 3, 4, 5, and 6. Figs. 3 and 4 will be recognized as modifications of one of the problems considered earlier by Bordoni. The Jones formula for the flux density for the case represented by Fig. 3 was written down incorrectly in his first paper but was corrected in the following paper in 1910. The formula for the case represented by Fig. 4 appears correctly in the paper of 1910. The problems suggested by Figs. 5 and 6 represent the first problems in which the elementary receiving area was located on surface normals at arbitrary distances from the corner or center of the rectangle and was oriented in directions other than parallel or perpendicular to its edges. In the 1909 paper, the formula for the case represented by Fig. 5 was given incorrectly. The problem suggested by Fig. 6 was solved correctly in Jones' 1911 paper. Unfortunately Jones has made several mistakes so that his papers have been severely criticized by later writers. These errors on the rectangular source will be discussed in detail later. The interest in the rectangular source seemed then to subside until 1923. At that time P. J. Waldram (reference (g)) in England published an approximate formula for the flux density on a horizontal surface when the rectangular source was tilted at an arbitrary angle with respect to the horizontal. The particular problem in which he was interested is shown in Fig. 7. He assumed that the rectangle could, without appreciable error, be placed on the surface of a sphere and then considered the flux density on a small horizontal receiving element placed at the center of the sphere. This formula is then valid only when the distance from the rectangular source is large compared with its dimensions.

8. The next real advance in predicting the radiation fields of rectangular sources was made in Japan by Z. Yamanouti (reference (b)) in 1924. Yamanouti calculated the flux density on an elementary receiving area when it had arbitrary coordinates (X_o, Y_o, Z_o) and was oriented in the three directions of the coordinates. For these calculations, the rectangular source was confined to one of the coordinate planes at the origin of the coordinates as suggested by Fig. 8. The three directions of the surface normals of the elementary receiving area are indicated in the figure. Yamanouti's considerations represent an extension of the work of Jones in which two arbitrary coordinates were possible for one particular orientation of the receiving element (see Fig. 5). This work, however, still

had more general applications, for the three formulae obtained were looked upon as the three components of a "light vector". To obtain an equation for the flux density on an elementary receiving area having arbitrary coordinates and a surface normal with arbitrary direction cosines, Yamanouti suggested that a scalar product could be formed with the unit vector along the direction of the surface normal of the elementary receiving area, but this equation was never actually written because of the author's interest in the simpler cases. The three formulae obtained by Yamanouti are considerably more complex than the previous formulae derived by others, and some mistakes in signs appear which will be mentioned later.

9. The next contribution to the subject was that of H. H. Higbie and his coworkers (reference (i)) in this country in 1925 and 1926. Higbie gave a formula for the flux density on an elementary receiving area with arbitrary coordinates, when it was oriented perpendicularly to the rectangular surface. The particular problem and the terminology used in setting up this formula is that indicated in Fig. 9. Clearly this is equivalent to one of the special cases treated earlier by Yamanouti, and Yamanouti's corresponding equation can be reduced to Higbie's equation when both are expressed in the same terms (assuming that the previously mentioned errors in signs in Yamanouti's equations have been corrected). Higbie also discussed two other special cases which are of interest and are shown schematically in Figs. 10 and 11. The problem suggested by Fig. 10 is also a special case of Yamanouti's equations and can (assuming that the previously mentioned errors in sign have been corrected) be brought into agreement with Yamanouti's corresponding equation when both equations are expressed in the same terms. Fig. 11 is another modification of the Jones problem (see Fig. 6) and is a special case of the [illegible] general equation in the present paper. These equations will be discussed more in detail later. In 1928 O. Seibert (reference (j)) in Germany published a paper in which he again derived the two earlier equations of Bordoni. In this paper, however, he used spherical trigonometry and projection methods rather than the usual method of surface integration. Wilhelm Nusselt (reference (k)) in Germany during the same year also published a paper showing how projection methods on a unit sphere were equivalent to surface integration.

10. P. Moon (reference (l)) and D. E. Spencer (reference (m)) in this country have also carried out extensive calculations on rectangular sources using spherical trigonometry and vector methods. Like Yamanouti they have been concerned with evaluating the flux density on an elementary receiving area, when it has arbitrary coordinates and is oriented in the directions of the three coordinates (see Fig. 8). The equations obtained are in slightly different forms from those of Yamanouti, but can, after a slight transformation, be shown to be equivalent (except for the errors in sign in Yamanouti's equations).

11. The most recent papers on this subject are those of G. Bethe's (reference (n)) in Germany. These papers present a very general equation for the rectangular source when the source itself is confined to one of the coordinate planes and the elementary receiving area is located on one of the coordinate axes and has arbitrary orientations. This equation is very long and requires over half of an ordinary size journal page to record it. This equation, however, represents a further generalization of the problem beyond the equations of Yamanouti and will, after a translation of coordinates, be found to be equivalent to the more condensed equation in the present paper when the two are reduced to the same terminology. The author then presents two special cases of this equation which are new and will be discussed later.

12. In view of the somewhat unsatisfactory state in which the rectangular surface source has been left, it is felt that it might be worthwhile to solve this problem independently to see if the equations of others can be verified and perhaps add new and simpler formulae. It was also felt that there was a genuine need to sort of coordinate or bring together all these apparently very different formulae, so that it would be possible to see wherein they differ, or perhaps are similar, and also whether or not they are correct. Another motivating force for this report has been the desire to calculate the radiation fields of irregular surface sources which could be approximated by a series of smaller irregularly oriented rectangular emitting surfaces. To do these three things it became clear that a very general approach would be preferable in which a very general equation was found which described all these cases. We therefore undertook to solve the problem of calculating the total flux from a rectangular emitting surface falling on an elementary receiving area having arbitrary coordinates and a surface normal with arbitrary direction cosines. This problem was solved before we found the papers of Bethe. It will be recognized that Bethe's problem is very close to the problem proposed here, for he has limited his rectangular source to one of the coordinate planes and permitted his elementary receiving area to have arbitrary orientations along one of the coordinate axes, while we are suggesting that the rectangular source be limited to one of the coordinate planes at the center of the coordinates and the elementary receiving area permitted to have arbitrary coordinates and a surface normal with arbitrary direction cosines. Thus, although the integral equations for the two problems are different, the resulting equations should be reducible to each other when the two equations are expressed in the same terms, for in going from one problem to the other, a simple translation of coordinates is necessary. This is fortunate, since each of the two sets of rather detailed calculations gives added weight to the other. We should like then in the following to give a rather brief derivation of our formula for the rectangular source, when the elementary receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines.

We will show how all the previously mentioned equations become special cases of this general formula and point out wherein we think errors exist. Some examples will then be given of radiation fields of a square surface source. The more complex calculations were made possible by utilizing the mechanized computing facilities of the Naval Ordnance Laboratory.

III. General Equations for Radiation Field of Rectangular Surface Source

13. To describe in more detail the problem in which we shall be interested, it will be necessary to refer to Fig. 12. In the (Y,Z) plane of the coordinate system, we have indicated schematically a plane rectangular surface whose center is at the origin of the coordinate system. We will assume that this rectangular surface acts as a source of flux and will not be concerned with the mechanism by which this flux arises. It is also unimportant as to the form which this flux takes. As suggested earlier, it may be either a wave disturbance or it may consist of a flow of particles. If the flux is in the form of radiant energy, we shall not be concerned with its frequency distribution or its plane of polarization for these properties become important only when the response of the receiver to the incident flux is considered. At point (X_o, Y_o, Z_o) there is indicated an elementary receiving area ds' which is small compared with all the dimensions considered and has a surface normal N' with direction cosines (α, β, γ). Both the coordinates of the elementary receiving area and the direction cosines of its surface normal are completely arbitrary. We shall assume that each element of area of the rectangular source emits the same flux* per unit solid angle in a direction normal to its surface, and that it obeys Lambert's Cosine Law. If, in addition, we assume that the intervening space between the source and receiver is homogeneous and neither absorbs nor scatters the flux passing through it, it is possible to calculate rigorously the total flux falling on the elementary area ds'. In performing these calculations, we shall neglect the contributions of multiple reflections, multiple absorptions and reemissions between the surfaces which might contribute to the total flux falling on ds', for ds' will always be assumed to be so small or such an efficient energy converter that such effects will not make appreciable contributions.

14. The radiation characteristics of the rectangular source must now be expressed in mathematical terms. We postulate that each unit area of the rectangular source radiates the same total

*The units of flux in the c.g.s. system are ergs/sec; in the m.k.s. system watts.

energy** B per second per unit solid angle in the direction of its normal. On a microscopic time basis, B, of course, must fluctuate because for particles the flux cannot be considered to be distributed uniformly throughout space and for waves the energy flow must be periodic. However, if the times of observation are made large compared to these microscopic disturbances, then B will, if the source continues to radiate in the same manner, establish an average value. In all the following remarks it will be assumed that the times of observation are sufficiently large that such an average value for B has significance. If we consider a small elementary area ds of the rectangular source, the total flux within the element of solid angle $d\omega$ must then be of amount $B\,ds\,d\omega$. We have suggested earlier that each elementary area of the rectangular source must obey Lambert's Cosine Law (reference (a)). This law simply states that the total flux within the solid angle $d\omega$ being radiated by a small area element decreases with the cosine of the angle between the surface normal and the direction of emergence of the radiation. Since, for such small areas radiating within a small element of solid angle $d\omega$ the effective cross section area of the source also decreases with the cosine of this same angle, B for a surface obeying Lambert's Cosine Law must be independent of the angle of emergence of the flux. A source whose total flux falls off with the cosine of the angle between the surface normal and the direction of emergence of the flux is as previously stated known as a "perfect diffuse emitter". Lambert's Cosine Law is, of course, only an approximation for many sources do not obey this law, but we shall, for want of a better description of such sources, assume it to be valid. The total flux radiated by the area element ds in a direction θ with its normal within the solid angle $d\omega$ would, with the assumption that it obeys Lambert's Cosine Law, be expected to be of the amount

$$d^2F = B \cos\theta \, ds \, d\omega. \qquad (1)$$

**By total energy we mean all the energy entering within the unit solid angle. Thus, if the flux is in the form of particles, it must then be the total energy contained by the particles which enter within this unit solid angle. If the radiation is periodic, it must be integrated over all frequencies; if it is transverse, it must be integrated over all planes of polarization. The units of energy in the c.g.s. system are ergs; in the m.k.s. system joules.

15. A similar equation can also be written for the total energy emerging in a direction Θ in the time dt by simply inserting a dt on the end of the previous equation. We will not, however, be concerned with energy in this paper for most receivers only integrate over the time in the short interval in which they are first exposed to the radiation. After this brief initial exposure to the radiation, an equilibrium is established in which the total flux falling on the receiver is equal to the flux expended in maintaining a certain signal level plus the flux lost by reflection, heat, radiation, noise, and other such losses inherent in detectors.

16. If, as suggested by Fig. 12, this flux is allowed to be incident on the receiving area ds' at an arbitrary angle Θ' with its normal N', we must, in order to determine the total flux falling on ds', limit the solid angle dω to that intercepted at ds by the area ds'. From the definition of the solid angle, we can write

$$d\omega = \frac{ds' \cos \theta'}{a^2} \qquad (2)$$

a being the distance between ds and ds'. If now this value for dω is inserted in Eq. 1, the total flux which the elementary area ds radiates to the elementary area ds' is then

$$d^2F = \frac{B ds \cos \theta \; ds' \cos \theta'}{a^2} \qquad (3)$$

This equation is one of the old and universally accepted laws of photometry (reference (a)) and is commonly referred to as Lambert's Photometric Law. It is also valid for small sources which do not obey Lambert's Cosine Law, providing B is defined as the total energy radiated per second per unit area per unit solid angle in the direction of Θ.

17. This equation provides the starting point for quantitative calculations in this report as has been the case for others working on these problems. All one has to do is to perform the indicated integration over the surface of the rectangle in order to obtain the total flux falling on the elementary receiving area ds'. To adapt this equation to the problem at hand, it is necessary to refer to Fig. 12. The radiating area element ds can be expressed in terms of an element of area of the rectangle, which obviously will be dY dZ. The angle Θ between the normal N and the direction of a will be

given by

$$\cos \theta = X_0/a \tag{4}$$

To fix the magnitude of the angle θ' between N' and a, we use the fact that the cosine of the angle between two lines whose direction cosines are known is given by

$$\cos \theta' = \alpha \cos(a, X) + \beta \cos(a, Y) + \gamma \cos(a, Z), \tag{5}$$

where α, β, γ are the direction cosines of N'.
From Figure 12 it is quite a straightforward task to write down the direction cosines of a. Assuming a is positive when going from ds to ds', we find its direction cosines are

$$\cos(a, X) = X_0/a, \quad \cos(a, Y) = \frac{Y_0 - Y}{a}, \quad \cos(a, Z) = \frac{Z_0 - Z}{a}. \tag{6}$$

The magnitude of a is also easily determined. It is

$$a = \sqrt{X_0^2 + (Y_0 - Y)^2 + (Z_0 - Z)^2}. \tag{7}$$

If now we substitute for ds, cos θ, cos θ' and a in Equation 3, the integral equation which must be solved in order to determine the total flux falling on ds' from the rectangular source must then be

$$dF = B\,ds'\,X_0 \int_{-w}^{+w} \int_{-\ell}^{+\ell} \frac{\left[\alpha X_0 + \beta (Y_0 - Y) + \gamma (Z_0 - Z)\right] dY\,dZ}{\left[X_0^2 + (Y_0 - Y)^2 + (Z_0 - Z)^2\right]^2}. \tag{8}$$

It should perhaps be noted that the direction cosines of the surface normal N' have been chosen to be positive in the same sense in which a is positive. Thus N' will always be directed in the general direction of the incident radiation and, hence, will be the surface normal for the surface on which the radiation is not incident. This same convention for N^6 has been used for all the figures in this paper. The integration over the Z direction is readily carried out. This integral is an integral of the form

$$\int \frac{(A-\gamma Z)\,dZ}{(B-CZ+Z^2)^2},$$

and is equal to the sum of two integrals each of which are of standard forms. When the integration was carried out, the following result was found.

$$\int \frac{(A-\gamma Z)\,dZ}{(B-CZ+Z^2)^2} = \frac{(2A-\gamma Z)(2Z-C)}{2(4B-C^2)(B-CZ+Z^2)} + \frac{\gamma}{2(B-CZ+Z^2)}$$

$$+ \frac{4A-2\gamma C}{(4B-C^2)^{3/2}} \tan^{-1} \frac{2Z-C}{\sqrt{4B-C^2}}.$$

This equation must of course be evaluated at the limits -w and +w. When this is done, we find that

$$\int_{-w}^{+w} \frac{(A-\gamma Z)\,dZ}{(B-CZ+Z^2)^2} = \frac{(2A-\gamma w)(2w-C)}{2(4B-C^2)(B-Cw+w^2)} + \frac{\gamma}{2(B-Cw+w^2)}$$

$$+ \frac{4A-2\gamma C}{(4B-C^2)^{3/2}} \tan^{-1} \frac{2w-C}{\sqrt{4B-C^2}} + \frac{(2A-\gamma w)(2w+C)}{2(4B-C^2)(B+Cw+w^2)} \qquad (9)$$

$$- \frac{\gamma}{2(B+Cw+w^2)} + \frac{(4A-2\gamma C)}{(4B-C^2)^{3/2}} \tan^{-1} \frac{2w+C}{\sqrt{4B-C^2}}.$$

18. To complete the problem, each of these terms must be integrated over the Y direction within the limits of $-\ell$ to $+\ell$. This looks rather discouraging since from Eq. 8 we see also that A and B are functions of Y and must be inserted before integration is attempted. Some of the terms in the previous equation will then be found to divide up into other terms, each of which must be integrated separately. It is, of course, impossible to carry out such a detailed integration here, and about all that can be done is to give some general suggestions which might be useful to anyone who cares to repeat the calculations, and then simply give the final result. To reduce each of the integrals to a standard form, a change of variables is necessary. For all the integrals, the same change of variables is utilized. This is quite fortunate for all the integrations can be carried out in terms of this new variable and the various terms collected and simplified before worrying about the limits. The change of variables utilized was

$$v = Y - Y_0, \quad dv = dY \tag{10}$$

With this new variable, the limits on the integral must be changed to $-(Y_0+\ell)$ to $-(Y_0-\ell)$. In carrying out these integrations, the following well-known integral forms were encountered.

$$\int \frac{dv}{a+v^2}, \quad \int \frac{dv}{(a+v^2)(b+v^2)}, \quad \int \frac{v\,dv}{(a+v^2)(b+v^2)}, \quad \int \frac{v^2\,dv}{(a+v^2)(b+v^2)}$$

Terms involving arc tangents were always integrated by parts. After these integrations were carried out, the terms in Eq. 9 were found to give thirty-eight separate terms without inserting the limits for v. Fortunately, a very large number of these terms cancel out and the resultant terms simplify considerably. The resulting value for this integral was then the following.

$$\begin{aligned}\iint_{-w}^{+w} \frac{(A-\gamma Z)\,dY\,dZ}{(B-CZ+Z^2)^2} = {} & \frac{\gamma X_0 - \alpha(Z_0-w)}{2X_0\sqrt{X_0^2+(Z_0-w)^2}} \tan^{-1} \frac{v}{\sqrt{X_0^2+(Z_0-w)^2}} \\ & - \frac{\gamma X_0 - \alpha(Z_0+w)}{2X_0\sqrt{X_0^2+(Z_0+w)^2}} \tan^{-1} \frac{v}{\sqrt{X_0^2+(Z_0+w)^2}} \\ & + \frac{\beta X_0 + \alpha v}{2X_0\sqrt{X_0^2+v^2}} \tan^{-1} \frac{Z_0+w}{\sqrt{X_0^2+v^2}} \\ & - \frac{\beta X_0 + \alpha v}{2X_0\sqrt{X_0^2+v^2}} \tan^{-1} \frac{Z_0-w}{\sqrt{X_0^2+v^2}} \; .\end{aligned} \tag{11}$$

If now we insert the proper limits on the variable v as suggested earlier, we obtain the following equation from Eq. 8 for the total flux falling on the elementary area ds' from the rectangular source.

$$
\begin{aligned}
dF = \frac{B\,ds'}{2}\Bigg[& \frac{\gamma X_0 - \alpha(Z_0 - w)}{\sqrt{X_0^2 + (Z_0 - w)^2}} \left\{ \tan^{-1} \frac{Y_0 + \ell}{\sqrt{X_0^2 + (Z_0 - w)^2}} - \tan^{-1} \frac{Y_0 - \ell}{\sqrt{X_0^2 + (Z_0 - w)^2}} \right\} \\
& - \frac{\gamma X_0 - \alpha(Z_0 + w)}{\sqrt{X_0^2 + (Z_0 + w)^2}} \left\{ \tan^{-1} \frac{Y_0 + \ell}{\sqrt{X_0^2 + (Z_0 + w)^2}} - \tan^{-1} \frac{Y_0 - \ell}{\sqrt{X_0^2 + (Z_0 + w)^2}} \right\} \\
& + \frac{\beta X_0 - \alpha(Y_0 - \ell)}{\sqrt{X_0^2 + (Y_0 - \ell)^2}} \left\{ \tan^{-1} \frac{Z_0 + w}{\sqrt{X_0^2 + (Y_0 - \ell)^2}} - \tan^{-1} \frac{Z_0 - w}{\sqrt{X_0^2 + (Y_0 - \ell)^2}} \right\} \\
& - \frac{\beta X_0 - \alpha(Y_0 + \ell)}{\sqrt{X_0^2 + (Y_0 + \ell)^2}} \left\{ \tan^{-1} \frac{Z_0 + w}{\sqrt{X_0^2 + (Y_0 + \ell)^2}} - \tan^{-1} \frac{Z_0 - w}{\sqrt{X_0^2 + (Y_0 + \ell)^2}} \right\} \Bigg].
\end{aligned}
\tag{12}
$$

An alternative and somewhat simpler form of this equation can be obtained by expressing each of these differences in angles in terms of a single angle. dF may then also be written in the following alternate form.

$$
\begin{aligned}
dF = \frac{B\,ds'}{2}\Bigg[& \frac{\gamma X_0 - \alpha(Z_0 - w)}{\sqrt{X_0^2 + (Z_0 - w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_0^2 + (Z_0 - w)^2}}{X_0^2 + (Z_0 - w)^2 + Y_0^2 - \ell^2} \\
& - \frac{\gamma X_0 - \alpha(Z_0 + w)}{\sqrt{X_0^2 + (Z_0 + w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_0^2 + (Z_0 + w)^2}}{X_0^2 + (Z_0 + w)^2 + Y_0^2 - \ell^2} \\
& + \frac{\beta X_0 - \alpha(Y_0 - \ell)}{\sqrt{X_0^2 + (Y_0 - \ell)^2}} \tan^{-1} \frac{2w\sqrt{X_0^2 + (Y_0 - \ell)^2}}{X_0^2 + (Y_0 - \ell)^2 + Z_0^2 - w^2} \\
& - \frac{\beta X_0 - \alpha(Y_0 + \ell)}{\sqrt{X_0^2 + (Y_0 + \ell)^2}} \tan^{-1} \frac{2w\sqrt{X_0^2 + (Y_0 + \ell)^2}}{X_0^2 + (Y_0 + \ell)^2 + Z_0^2 - w^2} \Bigg].
\end{aligned}
\tag{13}
$$

19. These last two equations become quite powerful tools in radiation problems, for the user can now calculate the total flux from a rectangular source falling on an elementary receiving area having arbitrary coordinates (X_0,Y_0,Z_0) and a surface normal with arbitrary direction cosines (α, β, γ). These equations are very general in form so that all similar equations for which the rectangular source is confined to one of the coordinate planes at the center of the coordinates become special cases of these equations in which either the coordinates or the direction cosines of the surface normal of the elementary receiving area have particular values. After a simple translation and rotation of axes, these equations may also be used for the equally important case in which the elementary receiving area is fixed in one of the coordinate planes at the center of the coordinates, and the rectangular source permitted to have arbitrary coordinates and a surface normal with arbitrary direction cosines. Using the equations in this form, it is then possible to calculate the total flux falling on an elementary receiving area in the presence of an array of such rectangular sources. If the source is a surface radiator of irregular shape or does not radiate uniformly over its surface or obey Lambert's Cosine Law, it is usually possible to divide this surface up into a series of smaller rectangular sources which radiate approximately uniformly over their surfaces and approximately obey Lambert's Cosine Law within the limits of the angles intercepted at the source by the elementary receiving element. The radiation fields of more complex surface sources can then also be approximated by using these equations. The only precaution necessary in using these equations is that of being certain that the plane containing the elementary receiving area does not intersect the rectangular source, for if it does, these equations (except in very special cases) no longer apply. When this plane intersects the rectangular radiating surface in a line parallel to one of the pairs of sides, corrections in the limits can be made to take care of the reduction in effective radiating surface so as to obtain the total flux falling one one side of the elementary receiving element. Care must be taken, however, to shield the opposite side of the elementary receiving area from radiation so as to obtain the calculated flux. When the plane containing the elementary receiving area intersects the rectangular source in a line which is not parallel to a pair of its sides, triangular sources have to be considered and these were considered to be outside the scope of the present report.

20. One of the most immediate questions which confronted us was how we could be sure that these equations are correct in view of the errors which others have made on simpler aspects of the problem. In order to establish their correctness, we immediately started a search of the literature for all types of equations for rectangular sources and also worked out independently a series of problems in which (X_0, Y_0, Z_0) and (α, β, γ) had certain restricted values. This literature search was responsible for the history of the subject presented earlier. The results obtained by others and from these

independent calculations were then compared with the restricted forms obtained from Eqs. 12 or 13 when particular values for (X_o, Y_o, Z_o) and (α, β, γ) were inserted. The most direct verification of the correctness of these equations was that obtained from Bethe's equation. After a translation of coordinates, insertion of direction cosines in place of angles, and a change in sign of one of these direction cosines, all the various terms could be collected and Eq. 12 was found to be the result. This is fortunate for the solutions of two closely related complex problems involving different integrations serve as checks on each other. It should also be stated that in all other cases in which independent calculations on simpler problems done by others or the author were correct, the restricted form of the general equation was always in agreement.

IV. Radiation Equations for Special Simplified Cases

21. Since the literature on the rectangular emitting surface is rather meager, widely distributed both in periodicals and time, and often in error, it may be of interest to consider some special cases of the above formulae. In considering these special cases, we will, insofar as possible, cite the originator of each of these formulae for many of these formulae have often been rederived in the literature. We will also point out wherein we believe errors have been made when we believe they are present. In cases in which we believe errors to be present, we have not relied solely on the general equation of the present paper to establish these errors, but have also carried out independent calculations starting with the original problem suggested by each equation. In presenting these special equations, they will be considered in the order of decreasing complexity.

A. Bethe's First Formula

22. One of the special cases which immediately suggests itself is that in which the surface normal is not limited in its orientation, but the coordinates of the receiving element are limited to one of the coordinate planes. Such a formula was published by Bethe (reference (n)) in 1951. If in Fig. 12 we confine the receiving element, for example, to the (X,Y) plane, we must then in Eq. 13 set $Z_o = 0$. Doing this we find that

$$dF = \frac{B\,ds'}{2}\left[\frac{2\alpha w}{\sqrt{X_o^2+w^2}}\tan^{-1}\frac{2\ell\sqrt{X_o^2+w^2}}{X_o^2+w^2+Y_o^2-\ell^2} + \frac{\beta X_o-\alpha(Y_o-\ell)}{\sqrt{X_o^2+(Y_o-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o-\ell)^2}}{X_o^2+(Y_o-\ell)^2-w^2} - \frac{\beta X_o-\alpha(Y_o+\ell)}{\sqrt{X_o^2+(Y_o+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o+\ell)^2}}{X_o^2+(Y_o+\ell)^2-w^2}\right]. \qquad (14)$$

This equation is a very much condensed version of Bethe's equation and will be found equivalent to it.

B. Equations for which α, β, and γ are zero

23. Another of the simpler cases in the order of decreasing complexity is that in which the elementary receiving area is restricted so that its surface normal rotates in a plane parallel to one of the coordinate planes. Equations for these three cases can be arrived at from Eq. 13 by simply setting $\alpha = 0$, $\beta = 0$, and $\gamma = 0$. When this is done, the following three equations were found.

$$
\begin{aligned}
dF_{y,z} = \frac{B\,ds'}{2}\Bigg[& \frac{\gamma X_o}{\sqrt{X_o^2+(Z_o-w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_o^2+(Z_o-w)^2}}{X_o^2+(Z_o-w)^2+Y_o^2-\ell^2} \\
& - \frac{\gamma X_o}{\sqrt{X_o^2+(Z_o+w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_o^2+(Z_o+w)^2}}{X_o^2+(Z_o+w)^2+Y_o^2-\ell^2} \\
(\alpha = 0) \qquad & + \frac{\beta X_o}{\sqrt{X_o^2+(Y_o-\ell)^2}} \tan^{-1} \frac{2w\sqrt{X_o^2+(Y_o-\ell)^2}}{X_o^2+(Y_o-\ell)^2+Z_o^2-w^2} \\
& - \frac{\beta X_o}{\sqrt{X_o^2+(Y_o+\ell)^2}} \tan^{-1} \frac{2w\sqrt{X_o^2+(Y_o+\ell)^2}}{X_o^2+(Y_o+\ell)^2+Z_o^2-w^2} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
dF_{x,z} = \frac{B\,ds'}{2}\Bigg[& \frac{\gamma X_o-\alpha(Z_o-w)}{\sqrt{X_o^2+(Z_o-w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_o^2+(Z_o-w)^2}}{X_o^2+(Z_o-w)^2+Y_o^2-\ell^2} \\
& - \frac{\gamma X_o-\alpha(Z_o+w)}{\sqrt{X_o^2+(Z_o+w)^2}} \tan^{-1} \frac{2\ell\sqrt{X_o^2+(Z_o+w)^2}}{X_o^2+(Z_o+w)^2+Y_o^2-\ell^2} \qquad (15) \\
(\beta = 0) \qquad & - \frac{\alpha(Y_o-\ell)}{\sqrt{X_o^2+(Y_o-\ell)^2}} \tan^{-1} \frac{2w\sqrt{X_o^2+(Y_o-\ell)^2}}{X_o^2+(Y_o-\ell)^2+Z_o^2-w^2} \\
& + \frac{\alpha(Y_o+\ell)}{\sqrt{X_o^2+(Y_o+\ell)^2}} \tan^{-1} \frac{2w\sqrt{X_o^2+(Y_o+\ell)^2}}{X_o^2+(Y_o+\ell)^2+Z_o^2-w^2} \Bigg],
\end{aligned}
$$

$$dF_{X,Y} = \frac{Bds'}{2}\left[-\frac{\alpha(Z_0-w)}{\sqrt{X_0^2+(Z_0-w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2+(Z_0-w)^2}}{X_0^2+(Z_0-w)^2+Y_0^2-\ell^2}\right.$$

$$+\frac{\alpha(Z_0+w)}{\sqrt{X_0^2+(Z_0+w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2+(Z_0+w)^2}}{X_0^2+(Z_0+w)^2+Y_0^2-\ell^2}$$

$$(\gamma=0)\qquad +\frac{\beta X_0-\alpha(Y_0-\ell)}{\sqrt{X_0^2+(Y_0-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0-\ell)^2}}{X_0^2+(Y_0-\ell)^2+Z_0^2-w^2}$$

$$\left.-\frac{\beta X_0-\alpha(Y_0+\ell)}{\sqrt{X_0^2+(Y_0+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0+\ell)^2}}{X_0^2+(Y_0+\ell)^2+Z_0^2-w^2}\right].$$

These equations, although rigorous and apparently new, still represent examples of cases, which are considered too complicated to be evaluated.

C. Equation for which γ and Z_o are zero

24. One of the simpler variations of the previous formulae is the case in which the receiver is restricted to the (X,Y) plane in such a way that its surface normal also rotates in this plane. The equation describing the total flux falling on the receiver in this case can be obtained by setting both γ and Z_o in Eq. 13 equal to zero, or by simply setting $Z_o = 0$ in the last of the previous equations. In this case dF_{xy} reduces to

$$dF_{xy} = \frac{B\,ds'}{2}\left[\frac{2\alpha w}{\sqrt{X_o^2+w^2}}\tan^{-1}\frac{2\ell\sqrt{X_o^2+w^2}}{X_o^2+w^2+Y_o^2-\ell^2} + \frac{\beta X_o-\alpha(Y_o-\ell)}{\sqrt{X_o^2+(Y_o-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o-\ell)^2}}{X_o^2+(Y_o-\ell)^2-w^2} - \frac{\beta X_o-\alpha(Y_o+\ell)}{\sqrt{X_o^2+(Y_o+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o+\ell)^2}}{X_o^2+(Y_o+\ell)^2-w^2}\right]. \tag{16}$$

This formula also, to the best of our knowledge, has not appeared in the literature.

D. Equation for which α and Z_o are zero

25. Another interesting case in that shown in Fig. 13. In this case the receiver is still confined to the (X,Y) plane but its surface normal is restricted to rotation in the (Y,Z) plane. Clearly from Fig. 12, we would expect both α and Z_o to be zero for this case. This time we will substitute these values in Eq. 12 because the result is simpler than the corresponding result from Eq. 13. Thus we find that

$$dF_{y,z} = B\,ds'\left[\frac{\beta X_0}{\sqrt{X_0^2+(Y_0-l)^2}}\tan^{-1}\frac{w}{\sqrt{X_0^2+(Y_0-l)^2}}\right. \qquad (17)$$

$$\left. - \frac{\beta X_0}{\sqrt{X_0^2+(Y_0+l)^2}}\tan^{-1}\frac{w}{\sqrt{X_0^2+(Y_0+l)^2}}\right].$$

For this equation to be valid, the plane containing ds' must not intersect the rectangular source. It is particularly interesting to note that when Y_0 also is zero that $dF_{y,z}$ vanishes. This is to be expected since the same total flux would fall on both sides of the receiver regardless of its orientation with the result that the total flux of a given sign falling on the receiver must be zero.

E. Jones' Formula for Figure 6

26. Some of the simpler formulae in the literature which are closely related to the previous formulae are the Jones formula for the case suggested in Figure 6 and the Higbie formula for that in Figure 11. To obtain the Jones formula, the normal to the receiving element must be directed along the radius R drawn from the center of the rectangle to the receiving element. Clearly then from Figures 6 and 12, the following substitutions should be made.

$$\alpha = \frac{X_0}{R}, \quad \beta = \frac{Y_0}{R}, \quad \gamma = 0, \quad Z_0 = 0.$$

To obtain the equation in the form in which Jones originally wrote it, it is necessary to substitute in Equation 12 rather than Equations 13 and 14. The result is

$$dF_{x,y} = \frac{Bds'}{R} X_o \left[\frac{w}{\sqrt{X_o^2+w^2}} \left\{ \tan^{-1} \frac{Y_o+l}{\sqrt{X_o^2+w^2}} - \tan^{-1} \frac{Y_o-l}{\sqrt{X_o^2+w^2}} \right\} \right.$$

$$\left. + \frac{l}{\sqrt{X_o^2+(Y_o-l)^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2+(Y_o-l)^2}} + \frac{l}{\sqrt{X_o^2+(Y_o+l)^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2+(Y_o+l)^2}} \right]. \quad (18)$$

This is the Jones formula (reference (f)) which was published correctly for the first time in 1911. Another form of this equation can be obtained by substituting in Equation 14 rather than Equation 12, but this additional equation, although somewhat simpler, does not have any particular advantages over the form in which it was first published.

F. Higbie's Formula for Figure 11

27. The Higbie Levine formula for the case described by Figure 11 is a special case of dF_{xy} in Equation 15. To reduce dF_{xy} to the Higbie Levine formula, the following substitutions in Figure 12 will be necessary.

$$X_o = a \sin \phi \qquad \alpha = \cos \phi \qquad 2l = f$$

$$Y_o = a \cos \phi - l \qquad \beta = -\sin \phi \qquad 2w = m$$

$$Z_o = w \qquad \gamma = 0$$

When these substitutions are made in dF_{xy} of Equation 15, the following result was found

$$dF_{XY} = \frac{B\,ds'}{2}\left[\tan^{-1}\frac{m}{a} + \frac{f\cos\phi - a}{\sqrt{a^2+f^2-2af\cos\phi}}\tan^{-1}\frac{m}{\sqrt{a^2+f^2-2af\cos\phi}}\right.$$

$$\left. + \frac{m\cos\phi}{\sqrt{a^2\sin^2\phi+m^2}}\tan^{-1}\frac{f\sqrt{a^2\sin^2\phi+m^2}}{a^2+m^2-af\cos\phi}\right]. \qquad (19)$$

This equation is in the form in which it was first published (reference (i)) and is, of course correct.

G. Yamanouti's Formulae for Figure 8

28. All other formulae found in the literature were for cases in which the surface normal to the receiving element was parallel to one of the three directions of the coordinates. This is clearly the case discussed first by Yamanouti (reference (h)) which is described by Figure 8. To arrive at the equations of Yamanouti, it is necessary to set α, β and γ equal to unity. We can make these substitutions in either Equation 12 or Equation 13. To obtain the equations in their original forms, it is necessary to use Equation 12. Thus

$$dF_X = \frac{B\,ds'}{2}\left[-\frac{Z_0-w}{\sqrt{X_0^2+(Z_0-w)^2}}\left\{\tan^{-1}\frac{Y_0+l}{\sqrt{X_0^2+(Z_0-w)^2}} - \tan^{-1}\frac{Y_0-l}{\sqrt{X_0^2+(Z_0-w)^2}}\right\}\right.$$

$$+\frac{Z_0+w}{\sqrt{X_0^2+(Z_0+w)^2}}\left\{\tan^{-1}\frac{Y_0+l}{\sqrt{X_0^2+(Z_0+w)^2}} - \tan^{-1}\frac{Y_0-l}{\sqrt{X_0^2+(Z_0+w)^2}}\right\} \qquad (20)$$

$$(\alpha=1)\quad -\frac{Y_0-l}{\sqrt{X_0^2+(Y_0-l)^2}}\left\{\tan^{-1}\frac{Z_0+w}{\sqrt{X_0^2+(Y_0-l)^2}} - \tan^{-1}\frac{Z_0-w}{\sqrt{X_0^2+(Y_0-l)^2}}\right\}$$

$$\left.+\frac{Y_0+l}{\sqrt{X_0^2+(Y_0+l)^2}}\left\{\tan^{-1}\frac{Z_0+w}{\sqrt{X_0^2+(Y_0+l)^2}} - \tan^{-1}\frac{Z_0-w}{\sqrt{X_0^2+(Y_0+l)^2}}\right\}\right],$$

$$dF_Y = \frac{\beta ds'}{2}\left[\frac{X_0}{\sqrt{X_0^2+(Y_0-\ell)^2}}\left\{\tan^{-1}\frac{Z_0+w}{\sqrt{X_0^2+(Y_0-\ell)^2}} - \tan^{-1}\frac{Z_0-w}{\sqrt{X_0^2+(Y_0-\ell)^2}}\right\}\right.$$

$$(\beta=1) \quad \left. - \frac{X_0}{\sqrt{X_0^2+(Y_0+\ell)^2}}\left\{\tan^{-1}\frac{Z_0+w}{\sqrt{X_0^2+(Y_0+\ell)^2}} - \tan^{-1}\frac{Z_0-w}{\sqrt{X_0^2+(Y_0+\ell)^2}}\right\}\right],$$

$$dF_Z = \frac{\beta ds'}{2}\left[\frac{X_0}{\sqrt{X_0^2+(Z_0-w)^2}}\left\{\tan^{-1}\frac{Y_0+\ell}{\sqrt{X_0^2+(Z_0-w)^2}} - \tan^{-1}\frac{Y_0-\ell}{\sqrt{X_0^2+(Z_0-w)^2}}\right\}\right. \tag{20}$$

$$(\gamma=1) \quad \left. - \frac{X_0}{\sqrt{X_0^2+(Z_0+w)^2}}\left\{\tan^{-1}\frac{Y_0+\ell}{\sqrt{X_0^2+(Z_0+w)^2}} - \tan^{-1}\frac{Y_0-\ell}{\sqrt{X_0^2+(Z_0+w)^2}}\right\}\right].$$

These equations are in agreement with those of Yamanouti with the exception of a change in sign of the w in the numerator in the coefficient of the second term in dF_X, and a change in sign in the ℓ under the radical in the denominator of the coefficient of the second term in dF_Y. Both of these, we believe, are misprints and we have written to Yamanouti concerning this.

H. Condensed Form of Yamanouti's Equations

29. It should also be mentioned that a condensed version of these equations can be obtained by making substitutions in Equation 13. These equations are

$$dF_X = \frac{\beta ds'}{2}\left[-\frac{Z_0-w}{\sqrt{X_0^2+(Z_0-w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2(Z_0-w)^2}}{X_0^2+(Z_0-w)^2+Y_0^2-\ell^2}\right.$$

$$+\frac{Z_0+w}{\sqrt{X_0^2+(Z_0+w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2+(Z_0+w)^2}}{X_0^2+(Z_0+w)^2+Y_0^2-\ell^2}$$

$$(\alpha=1) \quad -\frac{Y_0-\ell}{\sqrt{X_0^2+(Y_0-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0-\ell)^2}}{X_0^2+(Y_0-\ell)^2+Z_0^2-w^2} \tag{21}$$

$$\left.+\frac{Y_0+\ell}{\sqrt{X_0^2+(Y_0+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0+\ell)^2}}{X_0^2+(Y_0+\ell)^2+Z_0^2-w^2}\right],$$

$$dF_Y = \frac{\beta ds'}{2}\left[\frac{X_0}{\sqrt{X_0^2+(Y_0-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0-\ell)^2}}{X_0^2+(Y_0-\ell)^2+Z_0^2-w^2}\right.$$

$$(\beta=1)\qquad\left.-\frac{X_0}{\sqrt{X_0^2+(Y_0+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_0^2+(Y_0+\ell)^2}}{X_0^2+(Y_0+\ell)^2+Z_0^2-w^2}\right],$$

$$dF_Z = \frac{\beta ds'}{2}\left[\frac{X_0}{\sqrt{X_0^2+(Z_0-w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2+(Z_0-w)^2}}{X_0^2+(Z_0-w)^2+Y_0^2-\ell^2}\right. \qquad (21)$$

$$(\gamma=1)\qquad\left.-\frac{X_0}{\sqrt{X_0^2+(Z_0+w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_0^2+(Z_0+w)^2}}{X_0^2+(Z_0+w)^2+Y_0^2-\ell^2}\right].$$

I. Equations of Moon and Spencer

30. The closest equations to those of Yamanouti are those of Moon and Spencer (reference (m)). In fact, the problem discussed by Moon and Spencer is identical with that of Yamanouti, but the manner in which the equations are set up is a little different. Referring to Figure 14, Moon and Spencer determined the coordinates of the receiving element from one of the corners of the rectangle. To transform Yamanouti's equations to those of Moon and Spencer, a translation of coordinates is necessary. Referring to Figure 14, the translation must be such that

$$Y_0 = \ell + u, \quad Z_0 = w + v.$$

Introducing this translation of coordinates into Yamanouti's equations, we find that

$$dF_X = \frac{B\,ds'}{2}\Bigg[-\frac{v}{\sqrt{X_0^2+v^2}}\left\{\tan^{-1}\frac{2l+u}{\sqrt{X_0^2+v^2}}-\tan^{-1}\frac{u}{\sqrt{X_0^2+v^2}}\right\}$$

$$+\frac{2w+v}{\sqrt{X_0^2+(2w+v)^2}}\left\{\tan^{-1}\frac{2l+u}{\sqrt{X_0^2+(2w+v)^2}}-\tan^{-1}\frac{u}{\sqrt{X_0^2+(2w+v)^2}}\right\}$$

$$-\frac{u}{\sqrt{X_0^2+u^2}}\left\{\tan^{-1}\frac{2w+v}{\sqrt{X_0^2+u^2}}-\tan^{-1}\frac{v}{\sqrt{X_0^2+u^2}}\right\}$$

$$+\frac{2l+u}{\sqrt{X_0^2+(2l+u)^2}}\left\{\tan^{-1}\frac{2w+v}{\sqrt{X_0^2+(2l+u)^2}}-\tan^{-1}\frac{v}{\sqrt{X_0^2+(2l+u)^2}}\right\}\Bigg],$$

$$dF_Y = \frac{B\,ds'}{2}\Bigg[\frac{X_0}{\sqrt{X_0^2+u^2}}\left\{\tan^{-1}\frac{2w+v}{\sqrt{X_0^2+u^2}}-\tan^{-1}\frac{v}{\sqrt{X_0^2+u^2}}\right\} \tag{22}$$

$$-\frac{X_0}{\sqrt{X_0^2+(2l+u)^2}}\left\{\tan^{-1}\frac{2w+v}{\sqrt{X_0^2+(2l+u)^2}}-\tan^{-1}\frac{v}{\sqrt{X_0^2+(2l+u)^2}}\right\}\Bigg],$$

$$dF_Z = \frac{B\,ds'}{2}\Bigg[\frac{X_0}{\sqrt{X_0^2+v^2}}\left\{\tan^{-1}\frac{2l+u}{\sqrt{X_0^2+v^2}}-\tan^{-1}\frac{u}{\sqrt{X_0^2+v^2}}\right\}$$

$$-\frac{X_0}{\sqrt{X_0^2+(2w+v)^2}}\left\{\tan^{-1}\frac{2l+u}{\sqrt{X_0^2+(2w+v)^2}}-\tan^{-1}\frac{u}{\sqrt{X_0^2+(2w+v)^2}}\right\}\Bigg].$$

These equations are the equations of Moon and Spencer (reference (m)) with the exception of the B term which they define in a different manner.

J. Higbie's Formula for Figure 9

31. An equation which is equivalent to one of the Yamanouti equations is that derived by Higbie (reference (i)) suggested in Figure 9. To transform Equation 12 to the case suggested by Figure 9, the following substitutions are necessary.

$$\alpha = 0 \qquad Y_o + l = -g \qquad Z_o + w = n$$
$$\beta = -1 \qquad Y_o - l = -f \qquad Z_o - w = -m.$$
$$\gamma = 0$$

Making these substitutions in Equation 12,

$$dF_y = \frac{B\,ds'}{2}\left[\frac{X_o}{\sqrt{X_o^2+g^2}}\left\{\tan^{-1}\frac{n}{\sqrt{X_o^2+g^2}} + \tan^{-1}\frac{m}{\sqrt{X_o^2+g^2}}\right\} - \frac{X_o}{\sqrt{X_o^2+f^2}}\left\{\tan^{-1}\frac{n}{\sqrt{X_o^2+f^2}} + \tan^{-1}\frac{m}{\sqrt{X_o^2+f^2}}\right\}\right] \qquad (23)$$

This is Higbie's equation published in 1925 and it is, of course, correct. This equation is equivalent to dF_y in Yamanouti's equations.

K. Bethe's Formula for Figure 15

32. Another recent variation of Yamanouti's equations is that given by Bethe (reference (n)) shown schematically in Figure 15. In this case, the surface of the receiving element is parallel to the rectangular emitting source and its motion is confined to a plane determined by the X_o axis and the line $Y_o = Z_o$. This is a special case of the first of Yamanouti's equations (that is, with $\alpha = 1$) in which Y_o is simply set equal to Z_o. In order to write this equation in the final form in which it appears, it is preferable to make these substitutions directly in Equation 13. Doing this, dF_x has the following value.

$$dF_x = \frac{B\,ds'}{2}\left[\frac{Y_o+w}{\sqrt{X_o^2+(Y_o+w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_o^2+(Y_o+w)^2}}{X_o^2+(Y_o+w)^2+Y_o^2-\ell^2}\right.$$

$$-\frac{Y_o-w}{\sqrt{X_o^2+(Y_o-w)^2}}\tan^{-1}\frac{2\ell\sqrt{X_o^2+(Y_o-w)^2}}{X_o^2+(Y_o-w)^2+Y_o^2-\ell^2}$$

$$+\frac{Y_o+\ell}{\sqrt{X_o^2+(Y_o+\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o+\ell)^2}}{X_o^2+(Y_o+\ell)^2+Y_o^2-w^2} \qquad (24)$$

$$\left.-\frac{Y_o-\ell}{\sqrt{X_o^2+(Y_o-\ell)^2}}\tan^{-1}\frac{2w\sqrt{X_o^2+(Y_o-\ell)^2}}{X_o^2+(Y_o-\ell)^2+Y_o^2-w^2}\right].$$

This is Bethe's equation in the form in which it was written. It is more useful in connection with square sources since in this case the flux dF_x would be that in a plane normal to the square surface and through its diagonal.

L. Jones' Formula for Figure 5

33. One of the simpler special cases of the Yamanouti equations is that suggested by Figure 5 in connection with Jones' contributions. In this case, the receiving element is confined to a plane parallel to the (X, Y) plane but displaced a distance $Z_o = w$ and the surface normal lies in a direction parallel to the Y axis. Obviously we must then make the following substitutions.

$$\beta = 1, \quad Y_o = \ell + a, \quad Z_o = w .$$

Making these substitutions in dF_y of Equations 20, we find that

$$dF_Y = \frac{\beta ds'}{2}\left[\frac{X_o}{\sqrt{X_o^2 + a^2}} \tan^{-1}\frac{2w}{\sqrt{X_o^2+a^2}} - \frac{X_o}{\sqrt{X_o^2+(2\ell+a)^2}} \tan^{-1}\frac{2w}{\sqrt{X_o^2+(2\ell+a)^2}}\right] . \tag{25}$$

Except for the constant factor outside which was neglected but could be inferred from previous equations and the misprint of an exponent as a subscript, this is the Jones' formula (reference (f)). It has often been rederived in the literature.

M. Higbie's Formula for Figure 10

34. Another special case of these general equations is that of Higbie and Levine represented schematically by Figure 10. In this case, the receiving element is located at a distance a in the Z direction from a normal through one corner of the rectangle and is oriented with its surface parallel to that of the rectangle. In order then to transform these general equations to Higbie's formula, the following substitutions must be made.

$$\alpha = 1, \quad Y_o = -\ell, \quad Z_o = a - w, \quad 2\ell = f, \quad 2w = m .$$

The most direct method of arriving at Higbie's equation is that of using Equation 13. Making these substitutions in this equation,

$$dF_X = \frac{\beta ds'}{2}\left[\frac{a}{\sqrt{X_0^2+a^2}}\tan^{-1}\frac{f}{\sqrt{X_0^2+a^2}} + \frac{f}{\sqrt{X_0^2+f^2}}\tan^{-1}\frac{m\sqrt{X_0^2+f^2}}{X_0^2+f^2+a^2-am}\right.$$

$$\left. + \frac{m-a}{\sqrt{X_0^2+(m-a)^2}}\tan^{-1}\frac{f}{\sqrt{X_0^2+(m-a)^2}}\right]. \tag{26}$$

This is the Higbie Levine formula (reference (i)) which appeared first in 1926 and it is, of course, correct.

N. Jones' Formulae for Fig. 3 and Fig. 4

35. Jones also discussed two additional cases which are represented by Figs. 3 and 4. Obviously, if we are to transform the general case in Fig. 12 to that suggested by Fig. 3, the following substitutions must be made.

$$\alpha = 1, \quad Y_0 = -l, \quad Z_0 = w.$$

These substitutions can be made in any one of the Eqs. 12, 13, 20, 21 with the same result.

$$dF_X = \frac{\beta ds'}{2}\left[\frac{2w}{\sqrt{X_0^2+(2w)^2}}\tan^{-1}\frac{2l}{\sqrt{X_0^2+(2w)^2}}\right.$$

$$\left. + \frac{2l}{\sqrt{X_0^2+(2l)^2}}\tan^{-1}\frac{2w}{\sqrt{X_0^2+(2l)^2}}\right] \tag{27}$$

This equation was first published (reference (f)) incorrectly in 1909 but was corrected in a following paper in 1910. For Fig. 4, the receiving element is parallel to the rectangular source and is located on a surface normal through the center of the rectangle. This again is a special case of the previously mentioned equations in this paragraph in which $\alpha = 1$ and Y_0 and Z_0 are both zero. Making these substitutions,

$$dF_x = 2\beta ds' \left[\frac{w}{\sqrt{X_o^2+w^2}} \tan^{-1} \frac{\ell}{\sqrt{X_o^2+w^2}} + \frac{\ell}{\sqrt{X_o^2+\ell^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2+\ell^2}} \right] \quad (28)$$

This equation is again correct and appears in Jones' paper (reference (f)) in 1911.

O. Bordoni's Formulae For Figure 1 and Figure 2

36. The last two equations which we shall mention are those of Bordoni (reference (e)) represented by Figures 1 and 2. These are slight modifications of the cases already discussed in connection with Jones' papers. Thus to reduce our general equations to those represented in Figure 2, the following substitutions must be made in Equation 12.

$$\beta = -1, \quad Y_o = -\ell, \quad Z_o = 0$$

Doing this,

$$dF_y = \beta ds' \left[\tan^{-1} \frac{w}{X_o} - \frac{Y_o}{\sqrt{X_o^2+(2\ell)^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2+(2\ell)^2}} \right]. \quad (29)$$

To obtain the equation representing Figure 1, the following substitutions must be made.

$$\alpha = 1, \quad Y_o = -\ell, \quad Z_o = 0.$$

Thus Bordoni's equation for the case in Figure 1 is

$$dF_x = \beta ds' \left[\frac{w}{\sqrt{X_o^2+w^2}} \tan^{-1} \frac{2\ell}{\sqrt{X_o^2+w^2}} + \frac{2\ell}{\sqrt{X_o^2+(2\ell)^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2+(2\ell)^2}} \right]. \quad (30)$$

As suggested earlier, both of these equations are correct and apparently represent the first contribution to the radiation fields of rectangular sources of finite dimensions.

P. Special Photometric Equations

37. There are two other well-known simple photometric laws which can be used also as a check on these equations. Since Lambert's Photometric Equation given earlier as Eq. 3 was used in deriving these equations, it also ought to be contained in these equations as a special case. To establish that this is the case, it is necessary simply to assume that X_o is much larger than Y_o, Z_o, ℓ, and w. If we make this assumption in Eq. 13 and expand the arctangent terms in series retaining only the first terms, we find that

$$dF = \frac{B\,ds'\,(4\ell w)\alpha}{X_o^2}, \qquad (31)$$

which is in accordance with Lambert's Photometric Law. Finally if we make ℓ and w very large and of the same order of magnitude, then for increasingly smaller values of X_o, Y_o, and Z_o, Eq. 13 reduces to

$$dF = \pi B\,ds'\,\alpha, \qquad (32)$$

again a well-known result.

38. It should perhaps be mentioned again that caution must be exercised in the use of these equations. If the plane containing the elementary receiving area intersects the rectangular source, these equations are then no longer applicable except in very limited cases.

V. Radiation Field of a Square Surface Source

A. Deviations from Inverse Square Law

39. As a particular example of these equations, a few calculations were carried out for a square surface source. One of the simplest and most often calculated cases for these problems is that in which the elementary receiving area is placed on the X_o axis parallel to the source ($\alpha = 1$, $Y_o = Z_o = 0$). For such a case, our general equation goes over into the following simple form:

$$dF_x = B\,ds'\left[\frac{4w}{\sqrt{X_o^2 + w^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2 + w^2}}\right], \qquad (33)$$

w being the width of the square. In Fig. 16 we have plotted some data arrived at from this equation. In this figure, the ordinate is $dF_x/B\ ds'$ and the abscissa is X_o in units of the width w. The full line curve in this figure gives the dependence of $dF_x/B\ ds'$ on X_o as would be expected from this equation.

40. Often in problems of this sort, the inverse square law is applied without being sufficiently careful as to the magnitude of the error introduced by its use. To indicate the type of agreement to be found between the exact Eq. 33 for the square and that to be expected on the basis of an inverse square law, we have included a second dotted curve in Fig. 16 showing the values of $dF_x/B\ ds'$ found from the inverse square law. If we set $\alpha = 1$ in Eq. 31, the equation describing the dependence of dF_x on X_o, assuming an inverse square law, must then be

$$dF_x = B\,ds' \frac{(4w^2)}{X_o^2} \; . \tag{34}$$

For small values of X_o, the inverse square law gives larger values than the exact equation, the divergence between the two values being larger as X_o decreases. For small values of X_o compared with w, the ordinate for the full line curve approaches π while that for the dotted curve goes to infinity.

B. Special Case for which $\gamma = Y_o = Z_o = 0$

41. Another quite simple example is that for which the elementary receiving area is still on the X_o axis, but its surface normal is free to rotate in the (X, Y) plane. For such a case in our general equations, we must set $\gamma = 0$ and $Y_o = Z_o = 0$. Doing this we find that

$$dF = B\,ds' \left[\frac{4\alpha w}{\sqrt{X_o^2 + w^2}} \tan^{-1} \frac{w}{\sqrt{X_o^2 + w^2}} \right] . \tag{35}$$

This equation is identical in form with Eq. 33 except for the presence of the α. If we plot $dF/B\ ds'$ as a function of X_o and θ', we can then represent these values by polar plots of the form shown in Fig. 17. For a fixed value of X_o, $dF/B\ ds'$ falls off simply with the cosine of the angle θ'. In this figure, the lengths of the vectors at each of the indicated values X_o (X_o = w, 2w, 3w) contained in the circles of diameters

$$\frac{4}{\sqrt{K^2+1}} \tan^{-1} \frac{1}{\sqrt{K^2+1}}$$

are the magnitudes of dF/B ds', K having the values 1, 2, and 3 for the indicated values of X_o. Below each of the circles is indicated the maximum value of dF/B ds'. In each case, the elementary receiving area itself is normal to these vectors. For each value of X_o, values of dF/B ds' are significant only up to the angle θ' at which the plane of the elementary receiving surface itself begins to intersect the square surface (for X_o = w, θ' = 45°; X_o = 2w, θ' = 60°). Beyond this angle, these calculations of course have no value.

C. Isophotopic Plots for (Y, Z) Plane with $\alpha = 1$

42. In the last series of figures, we have plotted field data of a more detailed type. For each of these figures, the elementary receiving area was confined to a given (Y, Z) plane (that is, having a fixed X_o value) and had a surface normal parallel to the X_o axis (that is $\alpha = 1$). In carrying out these calculations, polar coordinates were introduced in the (Y, Z) plane and calculations were made for a fixed angle ϕ with the Z_o axis and at various distances from the X_o axis. If in the (Y, Z) plane the radius r and the distance X_o are expressed in units of w, Eq. 24 can, for purposes of calculation, be written in the form

$$dF_x = \frac{B\,ds'}{2}\left[\frac{\bar{r}\cos\phi - 1}{\sqrt{\bar{X}_o^2 + (\bar{r}\cos\phi - 1)^2}}\tan^{-1}\frac{2\sqrt{\bar{X}_o^2 + (\bar{r}\cos\phi - 1)^2}}{\bar{X}_o^2 + \bar{r}^2 - 2\bar{r}\cos\phi}\right.$$

$$+ \frac{\bar{r}\cos\phi + 1}{\sqrt{\bar{X}_o^2 + (\bar{r}\cos\phi + 1)^2}}\tan^{-1}\frac{2\sqrt{\bar{X}_o^2 + (\bar{r}\cos\phi + 1)^2}}{\bar{X}_o^2 + \bar{r}^2 + 2\bar{r}\cos\phi}$$

$$- \frac{\bar{r}\sin\phi - 1}{\sqrt{\bar{X}_o^2 + (\bar{r}\sin\phi - 1)^2}}\tan^{-1}\frac{2\sqrt{\bar{X}_o^2 + (\bar{r}\sin\phi - 1)^2}}{\bar{X}_o^2 + \bar{r}^2 - 2\bar{r}\sin\phi}$$

$$\left. + \frac{\bar{r}\sin\phi + 1}{\sqrt{\bar{X}_o^2 + (\bar{r}\sin\phi + 1)^2}}\tan^{-1}\frac{2\sqrt{\bar{X}_o^2 + (\bar{r}\sin\phi + 1)^2}}{\bar{X}_o^2 + \bar{r}^2 + 2\bar{r}\sin\phi}\right].$$

In arriving at numerical values from this equation, values for $dF_x/B\ ds'$ were calculated along fixed directions (that is, constant φ and variable $\overline{r}$) in each of six (Y, Z) planes for which X_o had the values 1, 2, 4, 6, 8, and 10. Intervals of 10° in φ were used in each of these planes. From this data, curves were plotted in each plane along each of the chosen directions (that is, for fixed φ and variable $\overline{r}$) and values of $\overline{r}$ chosen for which the quantity $dF_x/B\ ds'$ had certain constant values. These data were then plotted on sheets of coordinate paper corresponding to each of these six planes, and all the points along each of the directions for which $dF_x/B\ ds'$ had the same value were joined together. The results of these detailed calculations and plotting are given in Figs. 18, 19, 20, 21, 22, and 23. In each of these figures, the values of $dF_x/B\ ds'$ along each of these isophotopic lines is indicated directly on the figure. The square itself is plotted to scale at the center of each of these figures. Most of these isophotopic lines, except for Fig. 18, can be approximated quite well by circles if the calculated and plotted values are accurate to three figures.

43. Figures of this type can be quite useful in making rapid determinations of the total flux or flux density on an elementary receiving area placed in various planes and having surface normals with various orientations. All one needs to know are the coordinates of the point in question (which are determined from the figure directly), the magnitudes of B and ds' and then either dF or dF/ds' can be determined directly from such curves.

VI. CONCLUSIONS AND ACKNOWLEDGMENTS

44. In this report, we have presented a rather detailed investigation of the subject of the radiation characteristics of rectangular sources, when these sources radiate uniformly over their surfaces and obey Lambert's Cosine Law. This has been done from a very general viewpoint so that the elementary receiving area may have arbitrary coordinates and a surface normal with arbitrary direction cosines. After a simple translation and rotation of axes, these equations may also be used for the equally important case in which the elementary receiving area is confined to one of the coordinate planes at the center of the coordinates and the rectangular source permitted to have arbitrary coordinates and a surface normal with arbitrary direction cosines. Using the equations in this form, it is then possible to calculate the total flux falling on an elementary receiving area in the presence of an array of rectangular sources of various sizes having arbitrary coordinates and surface normals with arbitrary direction cosines, for the total flux will simply be equal the sum of the individual contributions of each of these sources.

45. These equations are also directly applicable to problems for which the previously designated elementary receiving area becomes the source of the flux and the rectangular surface the receiving surface. Thus one can, by utilizing these equations, calculate the total flux entering a rectangular aperture from a small elementary source area when the source or the receiving area has arbitrary coordinates and a surface normal with arbitrary direction cosines. Assuming that the source in each case radiates uniformly over its surface and obeys Lambert's Cosine Law, the only limitation in the use of these equations is that of being certain in each case that the plane containing the elementary source or receiving area does not intersect the rectangle, for if it does, these equations (except in very special cases) are no longer applicable.

46. A search of the literature has alsobeen made which resulted in the finding of seventeen formulae describing various aspects of the radiation fields of rectangular sources. We have pointed out that all these formulae are apparently very different, and that there are a few errors both in transcribing and arriving at these formulae. We have shown, however, that each of these formulae (except Bethe's) can be arrived at from the more general formulae presented in this report by simply substituting particular values for X_o, Y_o, Z_o, α, β, and γ. When discrepancies appeared between the formula arrived at from the general equation presented in this report and the corresponding equation published by others, we have shown where this discrepancy exists and have to our satisfaction established these discrepancies as errors in the equations of others. As a result of this investigation, we believe that all of these formulae, with the exception of Jones' formula for Fig. 5 and the formulae of Yamanouti for Fig. 8, are correct when they are written in the final form in which the authors intended. The errors appearing in Jones' formula are readily found from preceding equations, but the errors in signs made in Yamanouti's equations cannot be inferred from his paper because they involve complex integrations which cannot, because of their length, be discussed in detail. These errors in sign may, however, be the result of a misprint or due to an error in copying. It perhaps should also be mentioned again that the Jones' formula for Fig. 3 appeared incorrectly in his first publication but was corrected in a following publication. Waldram's formula was the only one which was not discussed, for his formula, by his own admission, was only an approximate one.

47. The equations presented in this report for the rectangular source appear to be new. The first form of the equation, although not specifically written by Yamanouti, can nevertheless be obtained from Yamanouti's components of the "light vector" if the light vector is written for an arbitrary position and orientation and certain definitions are brought into agreement.

As mentioned before, however, a discrepancy in two signs appears between the two equations. The first form of the equation presented in this report will also be found to agree with Bethe's much longer equation after a translation of coordinates insertion of direction cosines, change in sign of one of these direction cosines, and a collection of the various terms. The alternate form of the equation seems, however, to be new.

48. In the last section of the paper we have presented a few simple examples of calculations for the radiation field of a square surface source and also some more detailed plots of the radiations field when the elementary receiving area is parallel to the square source and is confined to planes normal to its axis. Calculations of the latter type are very long and require mechanized equipment to carry them out. Curves of this type are, however, very useful for rapid evaluations of the total flux or the flux density on the elementary receiving area. The number of isophotopic curves which one plots would depend on the accuracy and variability in the position of the elementary receiving area which could only be anticipated by the user.

49. The examples which we have cited do not illustrate the full versatility of these equations, for they can be used to determine the total flux or the flux density on an elementary receiving area which is restricted to any type of surface. All one needs to solve the particular problem are the coordinates X_o, Y_o, Z_o and the direction cosines of the surface normal over the complex surface, and then isophotopic lines can be plotted over this surface in much the same way as was done in Figs. 18 to 23. Radiation fields of targets having complex geometries can also, with sufficient patience, be calculated with these equations, for there are many targets which can be approximated by a series of rectangular sources. Mechanized computing machines will be quite helpful in carrying out these calculations, for, without them, these equations become quite tedious to evaluate even for single isolated points.

50. This report may be subject to criticism because it is completely theoretical and has no experimental data to justify these equations. Further objections may also be raised because some surfaces may not radiate uniformly over their surfaces and may not obey Lambert's Cosine Law. To the first objection we can only say that detailed experimental measurements on such problems require a considerable amount of time and this time simply was not available. Even after measurements are taken, they may not, because of the second objection, describe adequately the theoretical calculations. In cases of this sort the measurements are further complicated because one must also determine how B depends on the coordinates and on the angle of emergence. Knowing the dependence of B on the coordinates and on the angle of emergence, additional integrations must be attempted to solve the problem rigorously. If these cannot be done, the radiating surface must then be broken up into a

series of smaller surfaces for which these conditions of uniformity and satisfying of Lambert's Cosine Law are valid.

51. The author is indebted to Gerald Fine of this Laboratory who expressed a considerable interest in this problem and carried out detailed calculations which were necessary for plotting Figs. 18, 19, 20, 21, 22, and 23. These calculations were very long and tedious and were done with the mechanized computing facilities of the Naval Ordnance Laboratory. Harold Feldman of this Laboratory plotted these data and is responsible for the drawing of these figures. The author would also like to acknowledge his interest in this work.

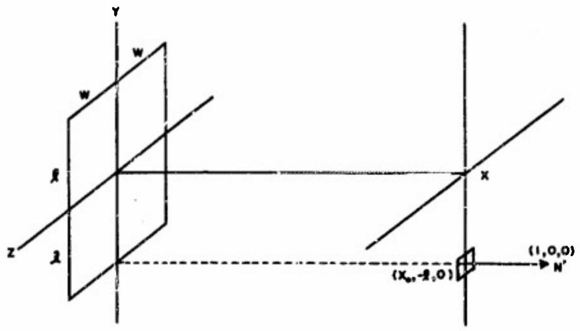

FIG. 1
BORDONI'S FIRST PROBLEM

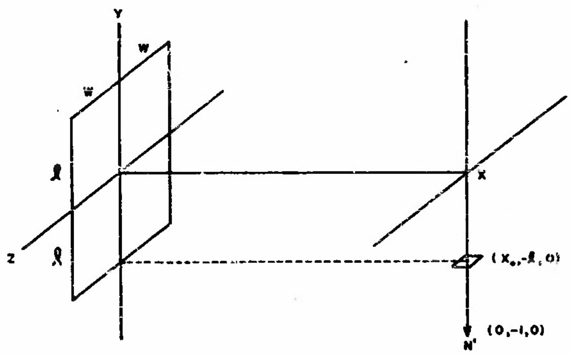

FIG. 2
BORDONI'S SECOND PROBLEM

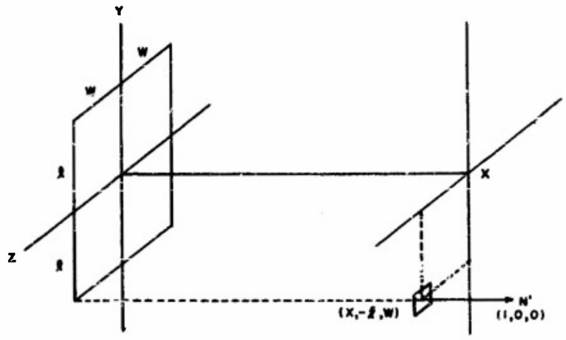

FIG. 3
JONES' FIRST PROBLEM

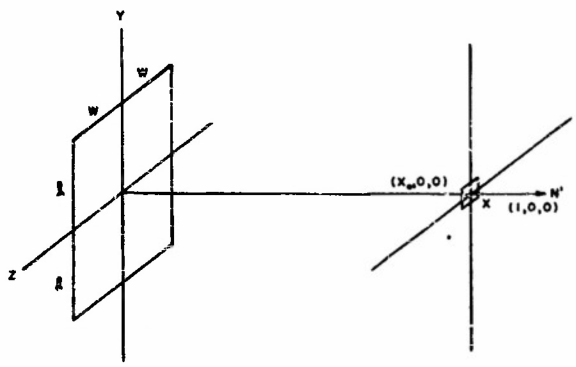

FIG. 4
JONES' SECOND PROBLEM

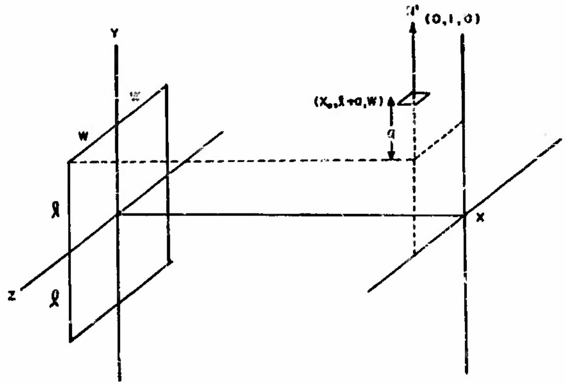

FIG. 5
JONES' THIRD PROBLEM

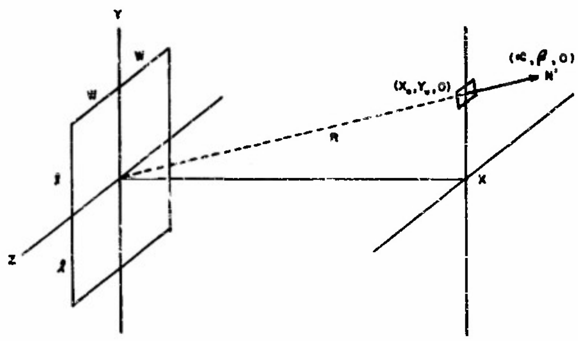

FIG. 6
JONES' FOURTH PROBLEM

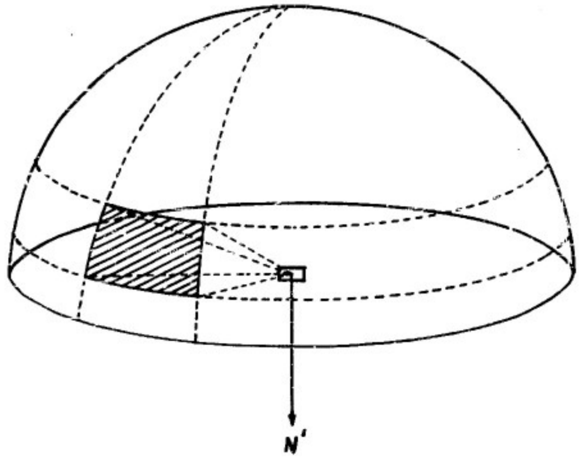

FIG. 7
WALDRAM'S PROBLEM

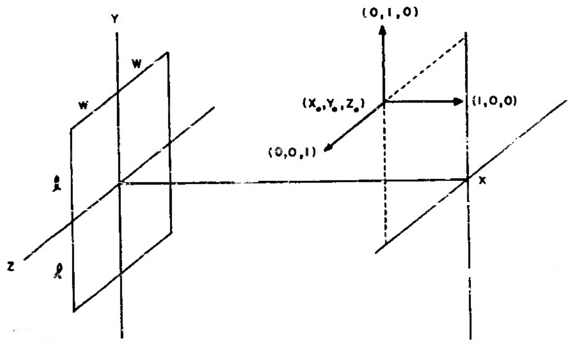

FIG. 8
YAMANOUTI'S COMPONENTS
OF THE "LIGHT VECTOR"

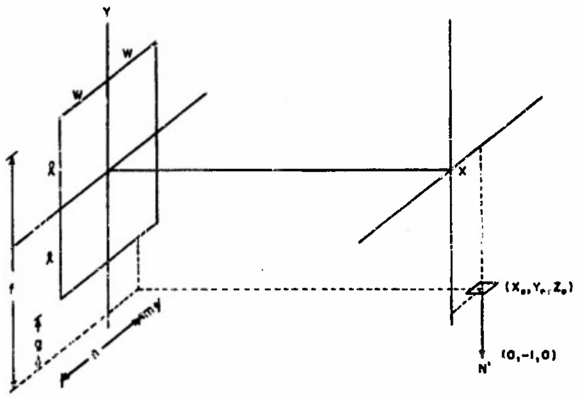

FIG. 9
HIGBIE'S FIRST PROBLEM

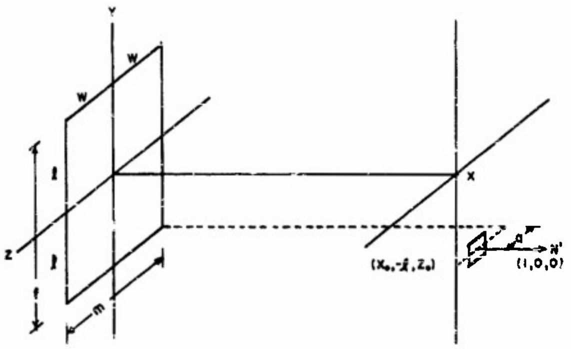

FIG. 10
HIGBIE'S SECOND PROBLEM

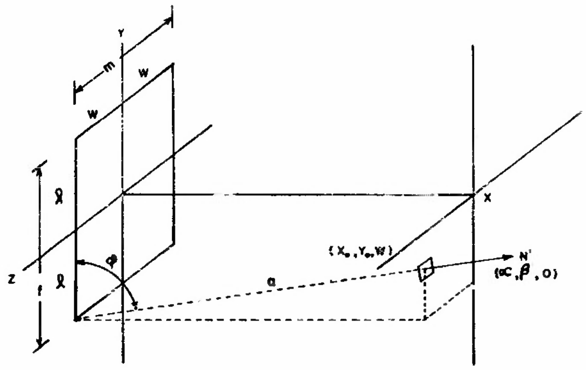

FIG. 11
HIGBIE'S THIRD PROBLEM

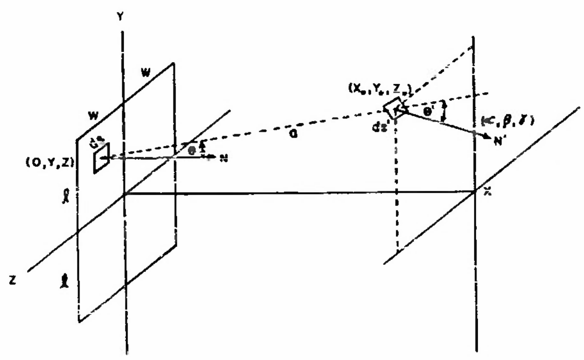

FIG. 12
GENERAL PROBLEM SOLVED
IN THIS REPORT

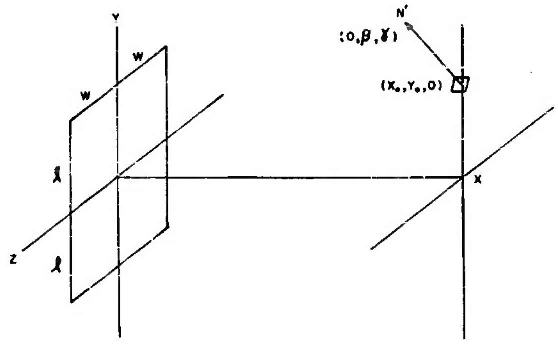

FIG. 13
ELEMENTARY RECEIVING AREA CONFINED TO (X,Y) PLANE WITH SURFACE NORMAL, FREE TO ROTATE IN (Y,Z) PLANE

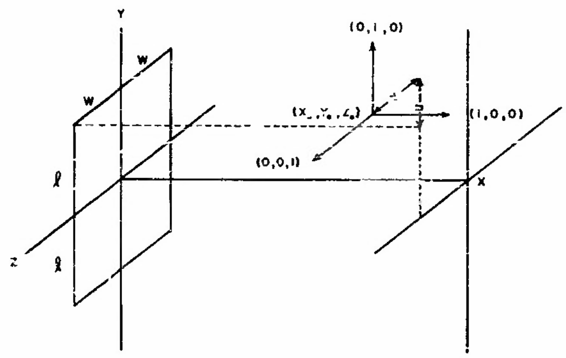

FIG. 14
SPENCER'S AND MOON'S MODIFICATION OF THE GENERAL PROBLEM PRESENTED BY YAMANOUTI

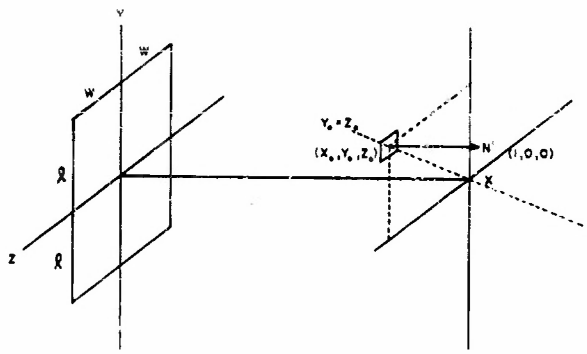

FIG. 15
ONE OF BETHE'S SPECIAL PROBLEMS

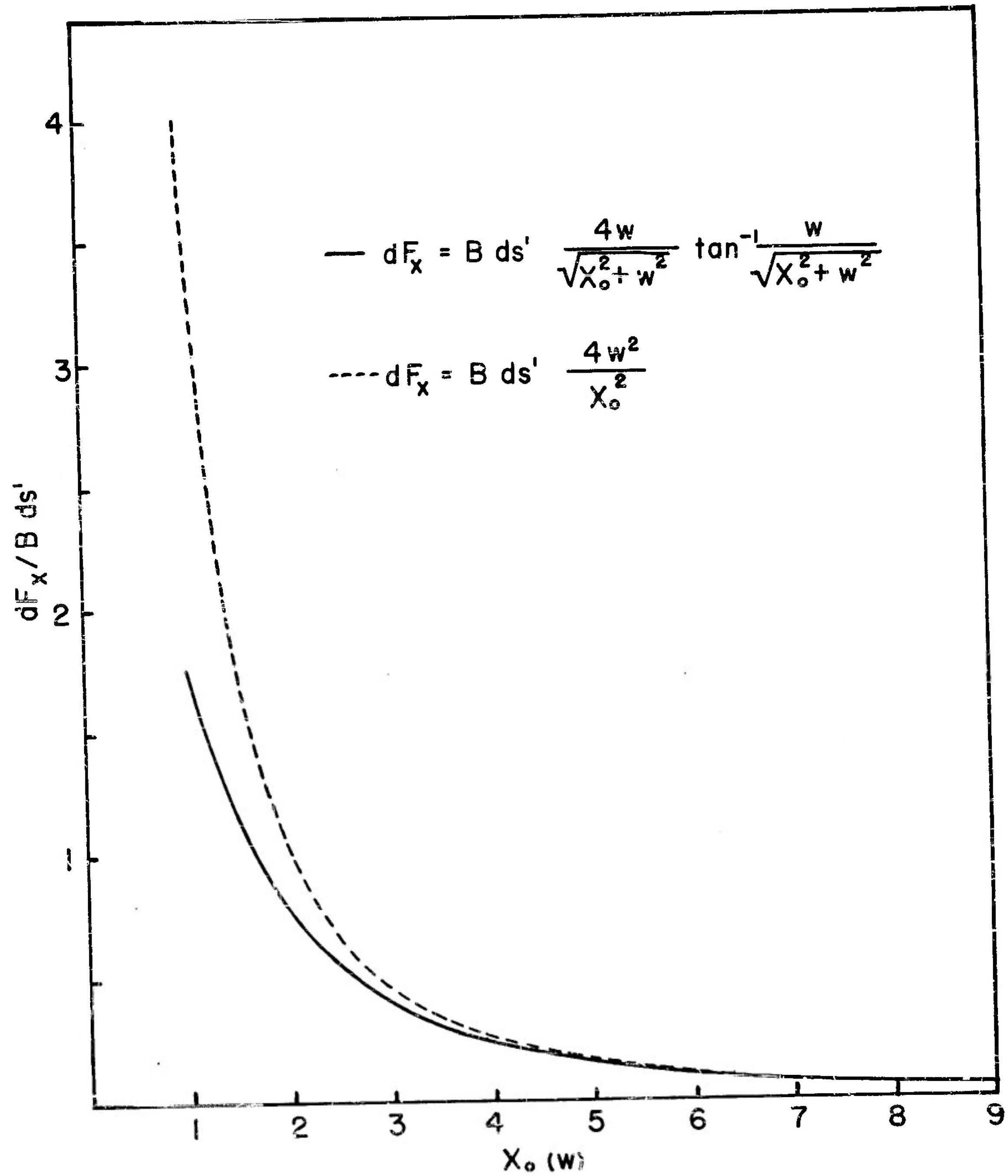

FIG. 16
DEVIATIONS FROM INVERSE SQUARE LAW

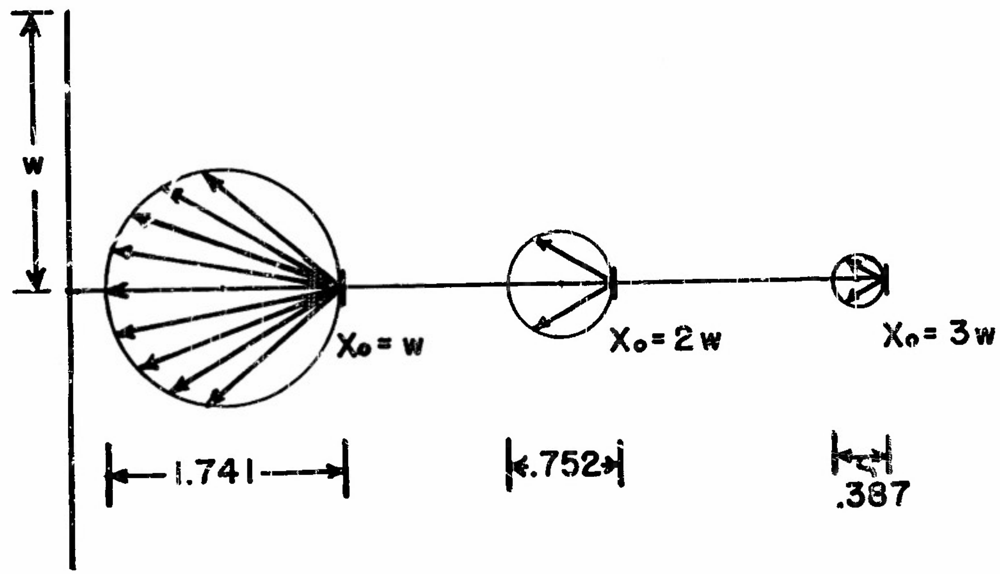

FIG. 17

TOTAL FLUX ON ELEMENTARY RECEIVING AREA FOR GIVEN VALUE OF X_o WHEN SURFACE NORMAL IS ROTATED IN (X,Y) PLANE

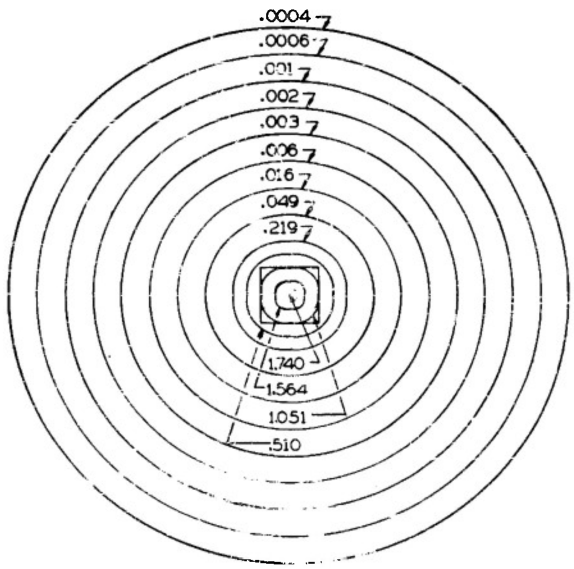

FIG. 18
RADIATION FIELD OF A SQUARE SOURCE
FOR $\alpha = 1$ AND $X_0 = w$

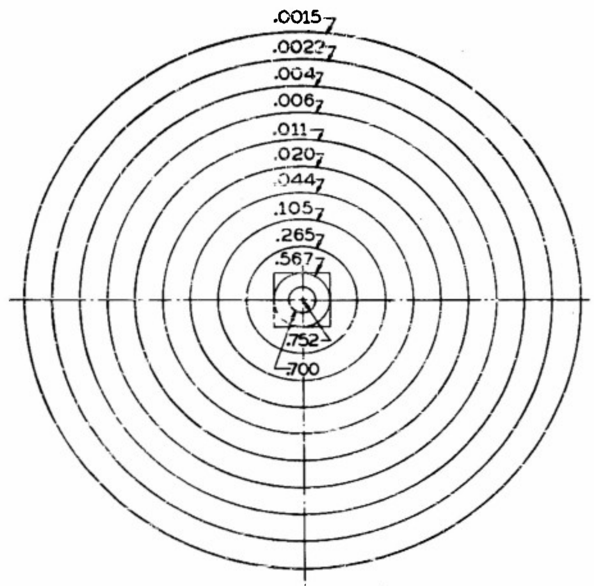

FIG. 19

RADIATION FIELD OF A SQUARE SOURCE FOR $\alpha = 1$ AND $X_0 = 2w$

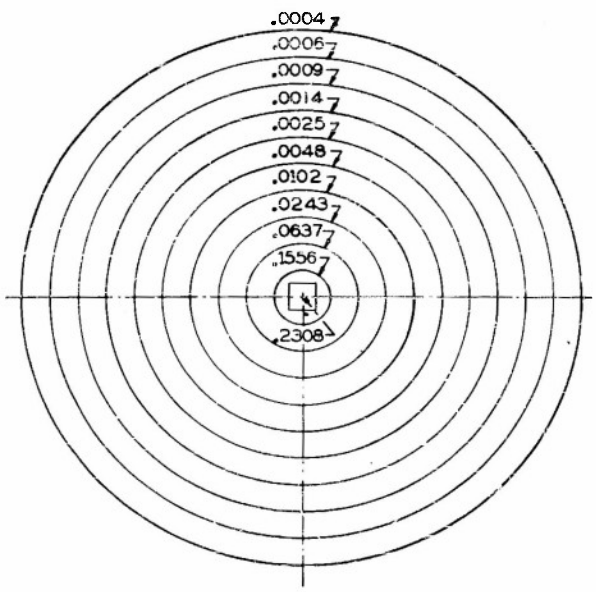

FIG. 20
RADIATION FIELD OF A SQUARE SOURCE
FOR $\alpha = 1$ AND $X_0 = 4w$

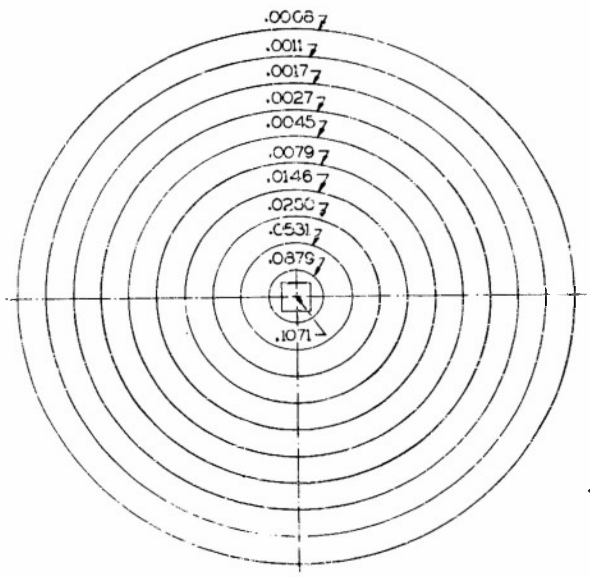

FIG. 21
RADIATION FIELD OF A SQUARE SOURCE
FOR $\alpha = 1$ AND $X_0 = 6w$

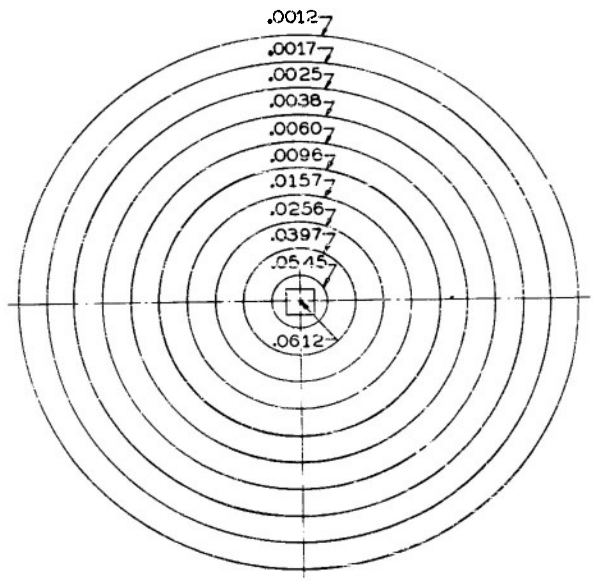

FIG. 22
RADIATION FIELD OF A SQUARE SOURCE
FOR $\alpha = 1$ AND $X_0 = 8w$

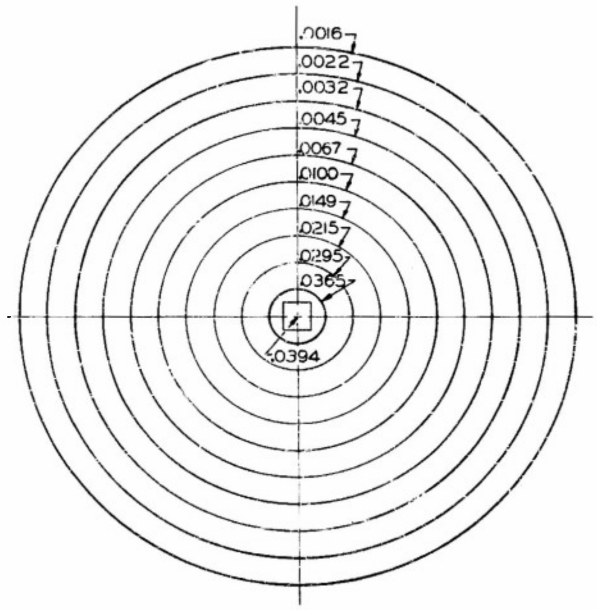

FIG. 23
RADIATION FIELD OF A SQUARE SOURCE
FOR $\alpha = 1$ AND $X_0 = 10w$

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