THE FORCES AND MOMENTS EXPERIENCED BY A TRAILING SHIP MOVING IN THE WAKE OF A LEADING SHIP

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T. O. # 4

Report No. 16

Dear Sir:

This is the final comprehensive report on "The Forces and Moment Experienced by a Trailing Ship Moving in the Wake of a Leading Ship". It summarizes to date the work which has been completed on this phase of Contract No. N7onr-43904, T. O. # 4.

Very sincerely,

Adolf G. Strandhagen

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ABSTRACT

A ship following in the wake of another ship of comparable size experiences a change in hydrodynamic force and moment due to the tubulence and waves generated by the leading ship. For two identical wall-sided ships, of finite draft, in tandem; the force and moment due to waves, experienced by the trailing ship, are derived by using Lagally's theorem and the usual assumptions used in the theory of wave resistance. The force is taken in two components, horizontal and vertical; and the moment is found about a transverse axis through the center of mass of the trailing ship.

Procedures applicable to the evaluation of all integrals involved are indicated and the results for the force components and significant moment components are listed. Numerical values are calculated for some significant force and moment components and curves are plotted showing the trend of the various components.

I. Introduction

In 1950, the authors, presented a paper, "The Dynamic Stability on Course of a Towed Ship", before the Society of Naval Architects and Marine Engineers (1). It was an introductory investigation on the stability of a towed ship in which only the motion in the horizontal plane was considered. Due to the complexity of the problem and to determine the effect of quantities of major importance, many refinements were idealized or neglected.

One effect which was neglected was the hydrodynamic influence of the towboat on the towed ship. As was pointed out by B. H. Peters, in the discussion which followed the afore mentioned paper, this effect is very significant in the analysis of the towing problem, especially for the case of a short tow. This fact was experimentally verified by E. G. Barrillon (2) who found that the hydrodynamic influence on the trailing ship was noticeable even when the distance between the towing and towed ship was of the order of five ship lengths.

To obtain a physical picture of the problem, imagine an unpowered ship moving with steady velocity on a calm sea. The ship experiences a total resistance, the sum of frictional and residual resistance, of which wave resistance is the chief part of the latter. If now another unpowered ship moving with the same steady velocity is introduced directly ahead of the existing ship, the trailing ship will experience a change in resistance. The Change in resistance is largely due to the turbulence and waves generated by the leading ship. The turbulence and additional waves affect both the frictional and wave resistance encountered by the trailing ship. In addition, the presence of the leading ship near the trailing ship will give rise to forces of mutual action which may be classed as due to local disturbances.

In 1936, T. H. Havelock wrote a paper on the mutual action between two bodies (3). The general action between two bodies are given and are applied to several cases of a pair of doublets which are oriented in different positions. Only the horizontal component force is dealt with and the vertical component and moment experienced by the trailing body are not discussed.

In a later paper (2), Havelock applied one part of the equations developed in his earlier paper to a wall-sided vessel of infinite draft. Equations for the horizontal component forces were given but curves showing the trend were not shown. As a first approximation, the use of an infinite draft is quite adequate since those parts of the ship-form which are situated very deep below the fluid surface do not affect the wave resistance seriously. For a closer approximation, however, Havelock does show (4) that the effect of decreasing draft becomes appreciable especially for small draft-length ratios at high velocities.

The present investigation is an extension of the work done by Havelock and deals with only that part of the hydrodynamic force and moment experienced by the trailing ship which is due to waves and the presence of the leading ship. The finite draft which was neglected by Havelock will also be considered. The effect of friction on the force and moment, which is complicated by the turbulence created by the leading ship, will not be discussed.

II. Assumptions and Methods in Wave Resistance

Any surface ship in motion encounters a total resistance which may be classified into several mutually dependent components. They may be classified as (a) frictional resistance which is assumed to be the resistance of a flat plate, of the same roughness, length, and surface area as the hull of the ship; (b) pressure resistance caused by pressure changes due to viscosity, especially in the after body; and (c) wave resistance which follow behind a ship. Another type of resistance is spray resistance. However, this is an independent physical effect and is only important for high speed ships.

In the present state of development the theoretical analysis of wave resistance is, at best, an approximation. To facilitate practical analysis many assumptions have to be made and thus a complete solution cannot be obtained even for the simplest of ship-shaped forms. Perhaps the greatest violation of physical fact is the use of an ideal fluid and the assumption that the wave resistance is independent of the other forms of resistance. These assumptions are justifiable only because of the need for systematically simplifying a complex problem.

Further assumptions are as follows:

(1) The velocity deviations due to the presence of the ship's hull are small relative to the main stream velocity.

(2) The height of waves generated by the ship's hull is small relative to their length.

(3) Trim and sinkage do not affect the wave resistance.

(4) Interference between the ship's hull and wave system does not affect the wave resistance.

(5) The inclination of the tangent plane to the median plane at any point on the surface of the ship is small.

(6) The fluid is non-viscous, of infinite depth and unbounded on the free surface.

(7) Surface tension on the free surface is negligible.

(8) The condition of atmospheric pressure on the free surface is satisfied at the undisturbed free surface and not at the disturbed free surface.

Taking all these assumptions into consideration, the actual physical picture is far removed from the theoretical picture with its many assumptions. With all these assumptions, the general agreement which exists between experimental and theoretical results is very surprising indeed.

There are several methods of calculating the effect of waves on the hull of a ship. To mention a few, the pressure changes due to wave motion may be integrated over the surface of the hull; the wave energy left behind the ship may be calculated; the energy dissipation using Rayleigh's dissipation coefficient may be calculated; and forces experienced by hydrodynamic singularities (sources and sinks) may be calculated. The method which seems best suited to the two body problem of towing is the last one, the calculation of forces experienced by the hydrodynamic singularities which together with the free stream represent the bodies.

Lagally's theorem (5) is indispensable in this method. It enables the calculation of forces and moments experienced by the hydrodynamic singularities. The resultant forces $\overline{F_1}$ and resultant moments $\overline{M_1}$ acting on a body whose surface is a closed stream surface of the fluid motion, are given by the vectors

$$\left.\begin{aligned} \overline{F_i} &= -4\pi\rho\, m_i \overline{q_i} \\ \overline{M_i} &= -\overline{r_i} \times 4\pi\rho\, m_i \overline{q_i} \end{aligned}\right\} \qquad (2.1)$$

where m_i is the strength of a source internal to the stream surface, $\overline{q_i}$ is the resultant fluid velocity vector at the location of the source due to all other sources, ρ is the mass density of the fluid, and $\overline{r_i}$ is the displacement of the source m_i from the moment center. The forces $\overline{F_i}$ have the direction of $-\overline{q_i}$ and the lines of action pass through the point at which the source is located. The vectors $\overline{M_i}$ which represent the resultant moments is perpendicular to the plane formed by $\overline{r_i}$ and $\overline{q_i}$.

The motion of the ideal fluid produced by the hydrodynamic singularities and the free stream is described by a velocity potential ϕ (x, y, z, t) which must satisfy Laplace's equation $\nabla^2\phi = 0$ and conditions at the boundaries of the fluid, at the surface of the hull, and at the free surface of the liquid. It must also satisfy the equations of wave motion. Of all the conditions to be satisfied, the conditions at the free surface cause the most difficulty.

III. Free Surface Condition

In elementary hydrodynamics problems, any body moving uniformly in an unbounded ideal fluid at rest at infinity, experiences, by D'Alembert's paradox, no resistance. It may, due to its orientation with the free stream, experience a moment. Unlike an unbounded fluid, in the practical application of theoretical hydrodynamics to problems of naval architecture, a free surface must always be considered. For this case, despite the use of an idealized fluid in the analysis, a body moving uniformly on or near a free surface encounters resistance due to the energy required in creating gravity waves which travel with the body.

To obtain the free surface condition, consider a stationary right handed systen of xyz axes in the free surface of a stream flowing with steady velocity c in the negative x-direction. The origin and the x and y axes are placed in the undisturbed free surface and the z-axis is vertically upward.

Now consider two points on the free surface of the stream. One point is taken near a disturbance and the other point is taken at infinity. The disturbance is of such a nature that its effect at infinity is negligible. For these two points, assuming that any deviation of a fluid particle is resisted by a force proportional to the fluid velocity, the pressure equation may be written as

$$\frac{p}{\rho}+\frac{1}{2}\left[(c+\phi_x)^2+\phi_y^2+\phi_z^2\right]+g(z+\eta)-\mu'\phi=\frac{p_o}{\rho}+\frac{c^2}{2}+gz \qquad (3.1)$$

where ρ is the mass density of the fluid, p and p_o are the pressures at the disturbance and at infinity respectively and η is the surface elevation from the undisturbed free surface. The quantities $-(c+\phi_x)$, $-\phi_y$, $-\phi_z$ are the absolute velocities of the fluid at the disturbance. The quantity $\mu'\phi$ is the force potential which introduces the hypothesis of a frictional force proportional to the deviation of the fluid velocity from the uniform flow c. This hypothesis is adopted to keep various integrals convergent in the subsequent analysis (6), (7). However, since wave resistance is calculated for a frictionless fluid the final result will be simplified by letting the frictional coefficient μ' tend to zero.

The quantities $-\phi_x$, $-\phi_y$, $-\phi_z$ are the deviations, at the disturbance, of the fluid velocity from the uniform flow c in the x, y and z directions. By assuming that the vector sum of these deviations is small, i.e., that $(\phi_x^2 + \phi_y^2 + \phi_z^2)$ is negligible, the pressure equation becomes,

$$\frac{p}{\rho} + c\phi_x + g\eta - \mu'\phi = \frac{p_0}{\rho} \qquad (3.2)$$

Now by assuming that the pressure on the free surface at the disturbance is equal to the pressure at the undisturbed surface at infinity, i.e., $p = p_0$,

$$c\phi_x + g\eta - \mu'\phi = 0$$

or

$$\eta = \frac{1}{g}(\mu'\phi - c\phi_x) \qquad (3.3)$$

This equation gives the elevation of the free surface at the disturbance.

The general boundary condition which must be satisfied at the free surface is that there shall be no flow across the free surface, i.e., the velocity, relative to the free surface, of all particles lying in it must be tangent to the surface. If the free surface is given by the equation $F(x,y,z,t) = 0$, this boundary condition may be expressed as

$$\frac{dF}{dt} = F_t + uF_x + vF_y + wF_z = 0 \qquad (3.4)$$

where $u = -(c + \phi_x)$, $v = -\phi_y$, $w = -\phi_z$.

In the present problem, the equation of the free surface is given by

$$z = \eta(x, y)$$

or

$$F(x,y,z,t) = z - \eta(x,y). \tag{3.5}$$

The Eq. (3.4) becomes

$$(c+\phi_x)\eta_x + \phi_y\eta_y - \phi_z(1-\eta_z) = 0 \tag{3.6}$$

Assuming that the wave height is small in comparison to the wave length, the slope of the waves become small quantities of first. Then by neglecting small quantities of second order, Eq. (3.6) becomes,

$$\eta_x = \frac{1}{c}\phi_z \tag{3.7}$$

Combining Eq. (3.3) and Eq. (3.7) the result is the free surface condition

$$\phi_{xx} + K_0\phi_z - \mu\phi_x = 0 \tag{3.8}$$

at $z = 0$, and where $K_o = g/c^2$ and $\mu = \mu'/c$. The velocity potential which describes the motion of the ideal fluid and the distribution of sources and sinks, which together with the free stream represents the bodies, must satisfy this free surface condition.

IV. Velocity Potential

Using the same axes as in the previous section, consider a simple source of strength m at $(0,0,-f)$, traveling with uniform velocity c in the positive x-direction. In the absence of a free surface and in a fluid of infinite extent in all directions, the velocity potential is

$$\phi = m/r_1 \tag{4.1}$$

where $(r_1)^2 = x^2 + y^2 + (z + f)^2$. In the presence of a free surface, the velocity potential may be written as (8)

$$\phi = m/r_1 + \sum_i m_i/r_i \tag{4.2}$$

where $(r_i)^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2$. The summation $\sum_i m_i/r_i$ represents an image system, a distribution of hydrodynamic singularities, in an unbounded fluid, which together with the given source will produce a free surface.

Making use of the known integral (8)

$$\left.\begin{aligned}&\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\int_0^\infty e^{K(i\bar{\omega}-z)}dK\\&\text{for}\quad z>0\quad\text{and}\\&\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\int_0^\infty e^{K(i\bar{\omega}+z)}dK\\&\text{for}\quad z<0\end{aligned}\right\}\qquad(4.3)$$

the velocity potential is assumed to be of the form

$$\left.\begin{aligned}\phi=\frac{m}{2\pi}&\int_{-\pi}^{\pi}d\theta\int_0^\infty e^{K[i\bar{\omega}-(z+f)]}dK\\&+\int_{-\pi}^{\pi}d\theta\int_0^\infty F(\theta,K)\,e^{K(z+i\bar{\omega})}dK\end{aligned}\right\}\qquad(4.4)$$

where $\bar{\omega} = x\cos\theta + y\sin\theta$, and only the real part is to be used.

Substituting ϕ into the free surface condition, Eq. (3.8), it can be shown that

$$F(\theta,K)=\frac{m}{2\pi}\left[1-2\left(\frac{K+i\mu\sec\theta}{K-K_0\sec^2\theta+i\mu\sec\theta}\right)\right].\qquad(4.5)$$

The velocity potential then becomes, after some reduction,

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} - \frac{K_0 m}{\pi}\int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^\infty \frac{e^{K[(z-f)+i\bar{\omega}]}}{K - K_0\sec^2\theta + i\mu\sec\theta}\, dK \tag{4.6}$$

where

$$r_1^2 = x^2 + y^2 + (z+f)^2$$
$$r_2^2 = x^2 + y^2 + (z-f)^2$$
$$\bar{\omega} = x\cos\theta + y\sin\theta.$$

The limiting value of Eq. (4.6) is to be taken for $\mu \to 0$.

The first term represents the given source at (0,0-f); the second term represents a sink at (0,0,f), and the last term represents a continuous distribution of sources and sinks, in the plane $z = f$, trailing in the negative x-direction to infinity. The last two terms form the image system which is required to produce the free surface (9).

Instead of an isolated source, for a general continuous distribution of sources below the free surface, the source strength m is replaced by $\int \sigma \, dS$ where σ is the surface density of a source on a surface dS at a point (h,k,-f) within the fluid. Applying the principle of inverse flow, the velocity potential for a general continuous distribution of sources in an ideal fluid with a free surface is

$$\left.\begin{aligned} \phi &= cx + \int \left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sigma\, dS \\ &\quad - \frac{K_0}{\pi}\int \sigma\, dS \int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^{\infty} \frac{e^{K[(z-f)+i\bar{\omega}]}}{K - K_0\sec^2\theta + i\mu\sec\theta}\, dK \end{aligned}\right\} \quad (4.7)$$

where c is the speed of the free stream in the negative x-direction and

$$r_1^2 = (x-h)^2 + (y-k)^2 + (z+f)^2$$
$$r_2^2 = (x-h)^2 + (y-k)^2 + (z-f)^2$$
$$\bar{\omega} = (x-h)\cos\theta + (y-k)\sin\theta.$$

V. Source Distribution

A complete determination of the distribution of sources and sinks which will satisfy all the boundary conditions is a very difficult problem. The problem is somewhat simplified by considering a distribution of sources and sinks taken over the plane $y = 0$ instead of a distribution over a volume. This is only permissible for a narrow ship's hull. If at the point $(h,0,-f)$, σ is the source density, ~~the total strength over a small~~ area dS is σdS and the corresponding flux is $4\pi\sigma\, dS$. Considering the entire source distribution, since it is continuous within its limits, the velocity flows outward normally on both sides of the plane $y = 0$. The velocity normal to the plane $y = 0$ is thus $4\pi\sigma\, dS/\,2dS = \pm 2\pi\sigma$.

This normal velocity is related to the form of the ship by considering the condition which must be satisfied at the surface of the ship's hull. The boundary condition is that the normal velocity of the ship's hull and the fluid in contace with it must be the same. The equation which expresses this boundary condition is the same as Eq. (3.4).

$$\frac{dF}{dt} = F_t + uF_x + vF_y + wF_z = 0$$

If $y = F(x,z)$ is the equation of the surface of the ship, then this boundary condition is expressed by

$$(c - u)\, F_x + v - w\, F_z = 0 \tag{5.1}$$

It is assumed here that the inclination of the tangent plane to the median plane at any point on the surface of the ship is small. This condition is not satisfied at the bottom of modern ships. However, since most of the wave resistance is due to the upper part of the immersed portion of the hull, it is a reasonable assumption. With this assumption F_x and F_z will be small, and since $u = -\phi_x$, $v = -\phi_y$ and $w = -\phi_z$ were considered small; by neglecting small quantities of second order, Eq. (5.1) becomes

$$v = -c\, F_x. \tag{5.2}$$

Now by assuming that the velocity in the y-direction at the hull is equal to the velocity normal to the plane $y = 0$,

$$\sigma = -\frac{c}{2\pi} F_x = -\frac{c}{2\pi} \frac{\partial y}{\partial x} \tag{5.3}$$

where $\partial y/\partial x$ is obtained from $y = F(x,z)$ the equation describing the surface of the hull. Eq. (5.3) is then the usual approximation for the density of the distribution of sources and sinks over the median plane of the ship's hull.

VI. Forces on the Towed Ship

Consider two identical wall-sided ships A and B, one right behind the other, placed in a stream which flows with constant speed c in the negative x-direction. The ships are spaced such that the distance between their centers of masses is equal to L. The ships are of length $2l$, beam 2b, and draft d. Ship A represents the towed ship and ship B represents the towing ship.

A right handed system of body axes x, y, z with the origin in the undisturbed free surface is fixed in ship A as shown in Fig. (6.1). The equation which describes the water plane section of ship A is

$$y = b\left[1 - (x/l)^2\right] \qquad (6.1)$$

where $-l \leq x \leq l$. For the ship B the equation is

$$y = b\left\{1 - \left[(x - L)/l\right]^2\right\} \qquad (6.2)$$

where $(L - l) \leq x \leq (L + l)$.

Each ship will now be replaced by a continuous distribution of sources and sinks over the vertical median plane as described in the last section. Following

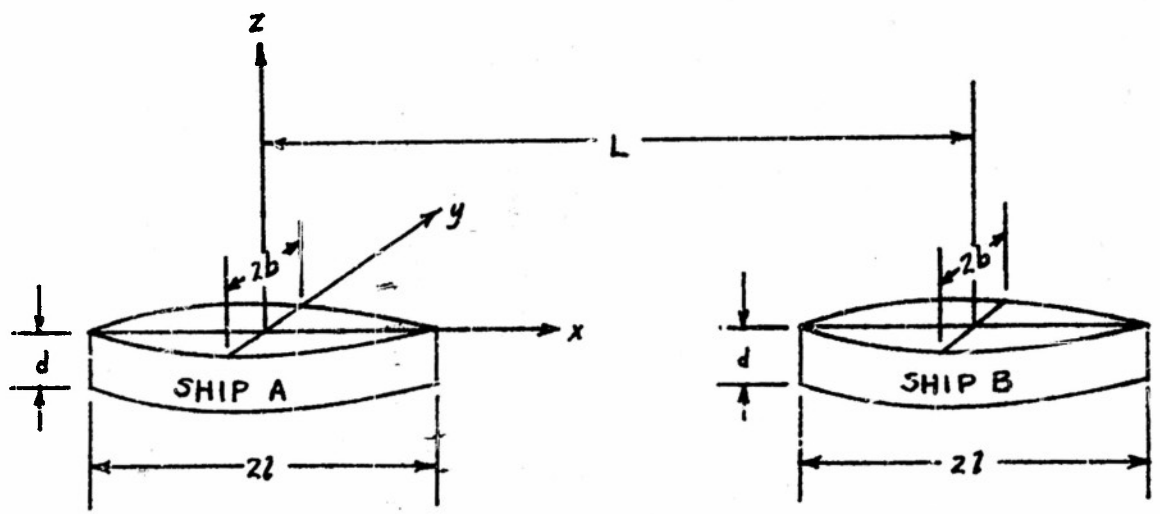

FIG. 6.1 ORIENTATION OF SHIPS A AND B

Eq. (5.3) and referring all quantities to the axes fixed in ship A, the source densities, σ_A and σ_B, for the ships A and B are

$$\sigma_A = (bcx)/(\pi l^2)$$

where $-l \leq x \leq l$, and

$$\sigma_B = \left[bc(x - L)\right]/(\pi l^2)$$

where $(L - l) \leq x \leq (L + l)$.

From Eq. (4.7) and again referring all quantities to the axes fixed in ship A, the velocity potential of the above source distribution in the uniform stream will be,

$$\left.\begin{aligned} \phi = ch &+ \left[\int_0^d\int_{-l}^{l}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\sigma_A\,dh\,df + \int_0^d\int_{L-l}^{L+l}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\sigma_B\,dh\,df\right] \\ &- \frac{K_0}{\pi}\left[\int_0^d\int_{-l}^{l}\sigma_A\,dh\,df + \int_0^d\int_{L-l}^{L+l}\sigma_B\,dh\,df\right]\int_{-\pi}^{\pi}\sec^2\theta\,d\theta \\ &\times\int_0^\infty \frac{e^{-K(f-z)+iK\varpi}}{K-K_0\sec^2\theta+i\mu\sec\theta}\,dK \end{aligned}\right\} \quad (6.4)$$

where u and w are the magnitudes of the components of $\overline{q}$ in the x and z directions respectively. The resultant forces in the x and z directions on the distribution A will then be

$$\left.\begin{aligned} R_x &= -4\pi\rho\int_0^d\int_{-l}^{l} u\,\sigma_A\,dh'\,df', \\ R_z &= -4\pi\rho\int_0^d\int_{-l}^{l} w\,\sigma_B\,dh'\,df'. \end{aligned}\right\} \qquad (6.7)$$

The velocity components u and w in terms of the velocity potential ϕ are

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial h}, \\ w &= -\frac{\partial\phi}{\partial f}. \end{aligned}\right\} \qquad (6.8)$$

where the term $(1/r_1)\,\sigma_A\,dhdf$, the velocity potential for a source at any point (x,y,z), has been eliminated from ϕ. Then

$$u(h',0,-f') = -c - \int_0^d\int_{-l}^{l} \alpha_1 \sigma_A \, dh\,df - \int_0^d\int_{L-l}^{L+l} (\alpha_1 - \alpha_2)\sigma_B \, dh\,df$$
$$+ \frac{iK_0}{\pi}\left[\int_0^d\int_{-l}^{l} \sigma_A \, dh\,df + \int_0^d\int_{L-l}^{L+l} \sigma_B \, dh\,df\right]$$
$$\times \int_{-\pi}^{\pi} \sec\theta \, d\theta \int_0^{\infty} \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K - K_0\sec^2\theta + i\mu\sec\theta} K\,dK \qquad (6.9)$$

and

$$w(h',0,-f') = \int_0^d\int_{-l}^{l} \beta_1 \sigma_A \, dh\,df + \int_0^d\int_{L-l}^{L+l} (\beta_1 + \beta_2)\sigma_B \, dh\,df$$
$$+ \frac{K_0}{\pi}\left[\int_0^d\int_{-l}^{l} \sigma_A \, dh\,df + \int_0^d\int_{L-l}^{L+l} \sigma_B \, dh\,df\right]$$
$$\times \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^{\infty} \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K - K_0\sec^2\theta + i\mu\sec\theta} K\,dK \qquad (6.10)$$

where

$$\alpha_1 = \frac{h' - h}{\left[(h' - h)^2 + (f + f')^2\right]^{3/2}},$$
$$\alpha_2 = \frac{h' - h}{\left[(h' - h)^2 + (f - f')^2\right]^{3/2}}, \qquad (6.11)$$

$$\left.\begin{aligned} \beta_1 &= \frac{f + f'}{\left[(h' - h)^2 + (f + f')^2\right]^{3/2}}, \\ \beta_2 &= \frac{f - f'}{\left[(h' - h)^2 + (f - f')^2\right]^{3/2}}. \end{aligned}\right\} \tag{6.11}$$

Substituting Eq. (6.9) and Eq. (6.10) into Eq. (6.7)

$$R_x = X_1 + X_2 + X_3 + X_4 + X_5 + X_6, \tag{6.12}$$

where

$$X_1 = 4\pi\rho c \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df',$$

$$X_2 = 4\pi\rho \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{-l}^{l} \alpha_1 \sigma_A \, dh df,$$

$$X_3 = 4\pi\rho \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \alpha_1 \sigma_B \, dh df,$$

$$X_4 = 4\pi\rho \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \alpha_2 \sigma_B \, dh df,$$

$$X_5 = -4 i \rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{-l}^{l} \sigma_A \, dh df$$

$$\times \int_{-\pi}^{\pi} \sec\theta \, d\theta \int_0^\infty \frac{e^{-K(f+f') + iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu \sec\theta} \, K dK ,$$

$$X_6 = -4 i \rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \sigma_B \, dh df$$

$$\times \int_{-\pi}^{\pi} \sec\theta \, d\theta \int_0^\infty \frac{e^{-K(f+f') + iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu \sec\theta} \, K dK .$$

The result for the force R_z is

$$R_z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 , \tag{6.13}$$

where

$$Z_1 = -4 \pi \rho \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{-l}^{l} \beta_1 \sigma_A \, dh df ,$$

$$Z_2 = -4 \pi \rho \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \beta_1 \sigma_B \, dh df ,$$

$$Z_3 = -4\pi\rho \int_0^d \int_{-l}^{l} \sigma_A' dh' df' \int_0^d \int_{L-l}^{L+l} \beta_2 \sigma_B \, dh \, df ,$$

$$Z_4 = -4\rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' dh' df' \int_0^d \int_{-l}^{l} \sigma_A \, dh \, df$$

$$\times \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^\infty \frac{e^{-K(f+f') + iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu\sec\theta} K dK ,$$

$$Z_5 = -4\rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' dh' df' \int_0^d \int_{L-l}^{L+l} \sigma_B \, dh \, df$$

$$\times \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^\infty \frac{e^{-K(f+f') + iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu\sec\theta} K dK .$$

VII. Interpretation of Force Components

The different components of the forces in the x and z directions represent the forces between the various parts of the source distribution. To describe the various force and moment components, the interpretation of the various parts of the velocity potential which describes the flow due to the source distribution, which is given in Section IV, must be known.

The first component of R_x,

$$X_1 = 4\pi\rho c\int_0^d\int_{-l}^{l}\sigma_A' dh' df'$$

represents the force on the distribution A due to the uniform stream. However, since the total source strength over the distribution is zero, it is equal to zero.

For the second component

$$X_2 = 4\pi\rho\int_0^d\int_{-l}^{l}\sigma_A'' dh' df'\int_0^d\int_{-l}^{l}\alpha_1\sigma_A dh\, df$$

consider two sources of strength m and m' at the

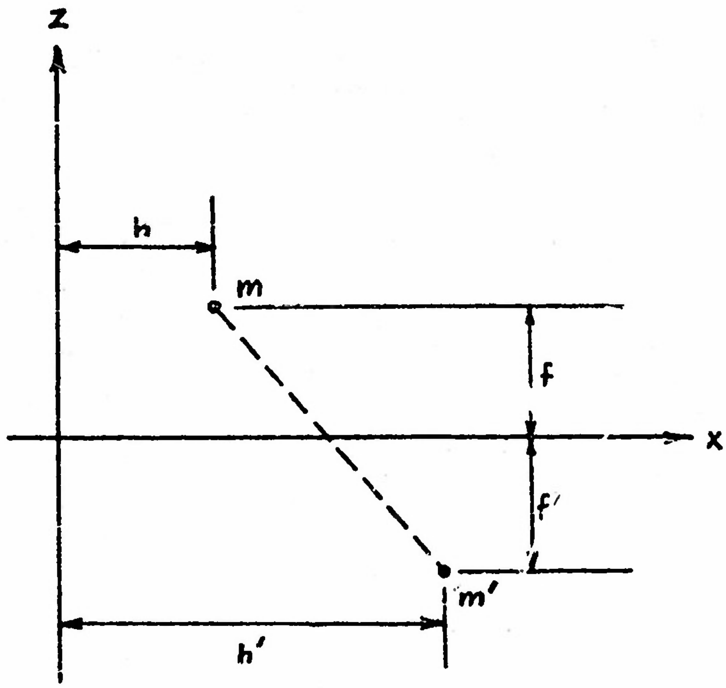

FIG. 7.1 SOURCES m AND m' FOR INTERPRETATION OF X_2

points $(h,0,f)$ and $(h',0,-f')$ respectively, as shown in Fig. (7.1). The magnitude of the force between m and m' is given by

$$\frac{4\pi\rho m m'}{r^2}$$

where r is the distance between m and m'. The force acts along the line drawn between m and m' and is an attraction when m and m' are of like sign. When m and m' are of unlike sign the force is a repulsion.

Referring to Fig. (7.1)

$$r^2 = (h'-h)^2 + (f+f')^2.$$

Then the force between m and m' becomes

$$\frac{4\pi\rho m m'}{(h'-h)^2 + (f+f')^2}$$

The horizontal component of this force is

$$\frac{4\pi\rho m m' (h'-h)}{\left[(h'-h)^2+(f+f')^2\right]^{3/2}}.$$

If m represents a certain distribution over an area A in the plane $y = 0$, we may write

$$m = \iint_A \sigma_A \, dh \, df.$$

Similarly if m' represents an identical distribution except possibly for the sign of m', we may write

$$m' = \iint_A \sigma_A' \, dh' \, df'.$$

Then for the given distribution of sources, the magnitude of the resultant force in the x-direction is

$$4\pi\rho \iint_A \sigma_A' \, dh' \, df' \iint_A \frac{(h'-h)}{\left[(h'-h)^2 + (f+f')^2\right]^{3/2}} \sigma_A \, dh \, df$$

Comparing this equation with the component X_2, it is clear that the form and magnitude are the same. Hence X_2 represents the resultant force in the x-direction, on the distribution A, due to the forces which exist between the sources and sinks in the distribution A and its image system, excluding that part of the image system which trails aft to infinity.

By following the same reasoning, it can be shown that the component X_3 is the resultant force in the x-direction, on the distribution A, due to the forces between the sources and sinks in the distribution A and the image of the sources and sinks in the distribution B, again excluding that part of the image system of the distribution B which trails aft to infinity. The component X_4 represents the resultant force in the x-direction, on the distribution A, due to the forces between the sources and sinks in the distribution A and B.

For the remaining distribution which trails aft to infinity for both distribution A and B, the velocity potential is

$$-\frac{K_0}{\pi}\left[\int_0^d\int_{-l}^{l}\sigma_A\,dh\,df+\int_0^d\int_{L-l}^{L+l}\sigma_B\,dh\,df\right]\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_0^\infty\frac{e^{-K(f+f')+iK\bar{\omega}}}{K-K_0\sec^2\theta+i\mu\sec\theta}\,K\,dK$$

as given by Eq. (6.4). The first part gives the trailing system for the distribution A and the second part gives the trailing system for the distribution B.

The resultant force in the x-direction on the distribution A for this case cannot be shown as simply as was done in the previous cases. However, a little reflection on Lagally's theorem and this part of the velocity potential will show that X_5 represents the resultant force in the x-direction, on the distribution A, due to the forces between the trailing system for the distribution A and the distribution A itself. This expression is the usual form of the wave resistance for a ship alone on a calm sea.

The component X_6 is the resultant force in the x-direction due to the forces between the distribution A and the trailing system for the distribution B. Physically this term represents the wave resistance on the distribution A due to the wave interference caused by the waves left by the distribution B, or the towing ship.

With the foregoing interpretation, the force R_x acting on the distribution A, may be classed into three parts; (1) the wave resistance of the towed ship as if existing alone on a calm sea, (2) the mutual action between the two systems which may be classed as due to local disturbances, and (3) the wave interference acting on the towed ship (3).

Following the same procedure, the interpretation of the various components in R_z may be given. The first component Z_1 represents the resultant vertical force on the distribution A due to the forces between the distribution A and its image system excluding that part of the image system which trails aft to infinity. The component Z_2 represents the resultant vertical force on the distribution A due to the forces between the distribution A and the image system of the distribution B, again excluding that part of the image system of the distribution B which trails aft to infinity. The component Z_3 represents the resultant vertical force due to the forces between the distribution A and the distribution B. Finally, Z_4 and Z_5 represent the resultant vertical force on the distribution A due to the forces between the trailing system for both distribution A and B and the distribution A itself.

VIII. Moment on Towed Ship

To obtain the expression for the moment, consider again the source of strength $\sigma_A' dh' df'$ at the point $(h', 0, -f')$ in the distribution A. By Lagally's theorem for moments, the moment about the center of mass of the distribution A due to the force on this source is given by

$$d\overline{L} = -\overline{r} \times 4\pi\rho(\sigma_A' dh' df')\,\overline{q} \tag{8.1}$$

where $\overline{r}$ is the displacement vector of the source $\sigma_A' dh' df'$ from the center of mass of the towed ship as shown in Fig. (8.1).

If the source distribution is taken in the vertical median plane of the towed ship

$$\overline{r} = h'\,\overline{i} + (f' - \epsilon)\,\overline{k} \tag{8.2}$$

where $\overline{i}$ and $\overline{k}$ are unit vectors in the positive x and z directions, and ϵ is the vertical distance of the center of gravity of the towed ship from the origin of the coordinate system. The resultant velocity at the point $(h', 0, -f')$ may be written as

$$\overline{q} = u\,\overline{i} + v\,\overline{j} + w\,\overline{k} \tag{8.3}$$

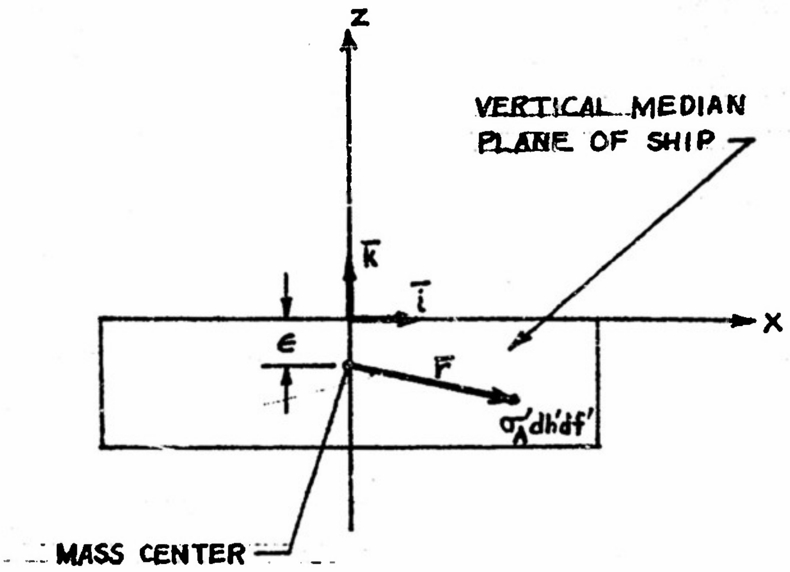

FIG. 8.1 MASS CENTER AND DISPLACEMENT VECTOR $\bar{r}$ FOR MOMENT

where $\bar{j}$ is a unit vector in the y-direction. However, for a distribution of sources taken in the vertical median plane, the resultant velocity v at this point will be equal to zero. Then

$$\bar{r} \times \bar{q} = \bar{j}\left[(f' - \epsilon)\, u - h'\, w\right] \tag{8.4}$$

and

$$\overline{dL} = -\, 4\pi\rho(\sigma_A' dh' df')\left[(f' - \epsilon)u - h'w\right]\bar{j} \tag{8.5}$$

The total resultant moment on the distribution A about an axis through the center of gravity and parallel to the y-axis will then be

$$L_y = -\, 4\pi\rho \int_0^d\!\!\int_{-l}^{l} \sigma_A'\left[(f' - \epsilon)u - h'w\right] dh' df'. \tag{8.6}$$

The quantities u and w are given by Eq. (6.9) and Eq. (6.10) respectively. Substituting into Eq. (8.6) the moment then becomes

$$L_y = M_1 + M_2 + M_3 + \cdots + M_{11} \tag{8.7}$$

where

$$M_1 = 4\pi\rho\int_0^d\int_{-l}^{l} c\,(f'-\epsilon)\,\sigma_A'\,dh'\,df' \,,$$

$$M_2 = 4\pi\rho\int_0^d\int_{-l}^{l}(f'-\epsilon)\,\sigma_A'\,dh'\,df'\int_0^d\int_{-l}^{l}\alpha_1\sigma_A\,dh\,df \,,$$

$$M_3 = -4\pi\rho\int_0^d\int_{-l}^{l}(f'-\epsilon)\,\sigma_A'\,dh'\,df'\int_0^d\int_{L-l}^{L+l}\alpha_1\sigma_B\,dh\,df \,,$$

$$M_4 = -4\pi\rho\int_0^d\int_{-l}^{l}(f'-\epsilon)\,\sigma_A'\,dh'\,df'\int_0^d\int_{L-l}^{L+l}\alpha_2\sigma_B\,dh\,df \,,$$

$$M_5 = -4i\rho K_0\int_0^d\int_{-l}^{l}(f'-\epsilon)\,\sigma_A'\,dh'\,df'\int_0^d\int_{-l}^{l}\sigma_A\,dh\,df$$

$$\times\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_0^\infty\frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K-K_0\sec^2\theta+i\mu\sec\theta}\,K\,dK \,,$$

$$M_6 = -4i\rho K_0\int_0^d\int_{-l}^{l}(f'-\epsilon)\,\sigma_A'\,dh'\,df'\int_0^d\int_{L-l}^{L+l}\sigma_B\,dh\,df$$

$$\times\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_0^\infty\frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K-K_0\sec^2\theta+i\mu\sec\theta}\,K\,dK \,,$$

$$M_7 = 4\pi\rho\int_0^d\int_{-l}^{l} h'\sigma_A' dh'df' \int_0^d\int_{-l}^{l} \beta_1\sigma_A dh\,df ,$$

$$M_8 = 4\pi\rho\int_0^d\int_{-l}^{l} h'\sigma_A' dh'df' \int_0^d\int_{L-l}^{L+l} \beta_1\sigma_B dh\,df ,$$

$$M_9 = 4\pi\rho\int_0^d\int_{-l}^{l} h'\sigma_A' dh'df' \int_0^d\int_{L-l}^{L+l} \beta_2\sigma_B dh\,df ,$$

$$M_{10} = 4\rho K_0\int_0^d\int_{-l}^{l} h'\sigma_A' dh'df' \int_0^d\int_{-l}^{l} \sigma_A dh\,df$$

$$\times\int_{-\pi}^{\pi}\sec^2\theta\, d\theta\int_0^\infty \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K-K_0\sec^2\theta+i\mu\sec\theta}K\,dK ,$$

$$M_{11} = 4\rho K_0\int_0^d\int_{-l}^{l} h'\sigma_A' dh'df' \int_0^d\int_{L-l}^{L+l} \sigma_B dh\,df$$

$$\times\int_{-\pi}^{\pi}\sec^2\theta\, d\theta\int_0^\infty \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K-K_0\sec^2\theta+i\mu\sec\theta}K\,dK .$$

As was done for the forces R_x and R_z the interpretation for the various components of L_y is as follows. M_1 represents the moment on the distribution A about the center of mass due to the uniform stream. The sum of the components M_2 through M_6 represents the moment of the distribution A about the center of mass due to the horizontal components of the forces between the distribution A and (a) the image system of the distribution A excluding the trailing system of sources and sinks, (b) the distribution B, (c) the image system of sources and sinks, (d) the trailing system of the distribution A, and (e) the trailing system of the distribution B. In like manner the sum of the components M_7 through M_{11} represents the moment due to the vertical component of the forces between the various parts of the source distribution.

IX. Reduction of Some Integrals

Many of the integrals which appear in the expressions for R_x, R_z and L_y may be rewritten in other forms more suitable for further integration. In all of these expressions we have the integrals

$$N_1 = \int_{-\pi}^{\pi} \sec\theta \, d\theta \int_0^{\infty} \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu \sec\theta} \, K dK \qquad (9.1)$$

and

$$N_2 = \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^{\infty} \frac{e^{-K(f+f')+iK(h'-h)\cos\theta}}{K - K_0 \sec^2\theta + i\mu \sec\theta} \, K dK \, . \qquad (9.2)$$

These integrals may be evaluated by considering K as a complex variable and using a suitable contour. Both integrals will have simple poles within the contours and the position of the simple poles will depend upon the sign of $\sec\theta$.

Consider, first, the integral N_1. To fix the pole of the integral, the range of integration with respect to θ is reduced from $-\pi, \pi$ to $0, \pi/2$. Then the integration in θ and K may be written as

$$N_1 = 2\int_0^{\frac{\pi}{2}} \sec\theta\, d\theta \int_0^{\infty} \left[\psi_1(\theta,K) - \psi_2(\theta,K)\right] e^{-K(f+f')} K\, dK \qquad (9.3)$$

where

$$\left.\begin{aligned} \psi_1(\theta,K) &= \frac{e^{iK(h'-h)\cos\theta}}{K - K_0\sec^2\theta + i\mu\sec\theta}, \\ \psi_2(\theta,K) &= \frac{e^{-iK(h'-h)\cos\theta}}{K - K_0\sec^2\theta + i\mu\sec\theta}. \end{aligned}\right\} \qquad (9.4)$$

Choosing a contour bounded by the positive half of the real axis and the negative half of the imaginary axis, it can be shown that

$$\left.\begin{aligned} \lim_{\mu\to 0} \int_0^{\infty} \psi_1(\theta,K)\, e^{-K(f+f')} K\, dK &= \\ = 2\pi i \left\{K_0\sec^2\theta \left[e^{iK_0(h'-h)\sec\theta - K_0(f+f')\sec^2\theta}\right]\right\} & \\ + \int_0^{\infty} \xi_1(m)\, e^{m(h'-h)\cos\theta}\, m\, dm & \\ + i\int_0^{\infty} \xi_2(m)\, e^{m(h'-h)\cos\theta}\, m\, dm & \end{aligned}\right\} \qquad (9.5)$$

where $(h' - h) < 0$. Here, it is to be noted that $\mu \to 0$. The expressions for $\xi_1(m)$ and $\xi_2(m)$ are as follows:

$$\left.\begin{aligned} \xi_1(m) &= \frac{K_0 \sec^2\theta \left[\cos m(f+f')\right] + m \sin m(f+f')}{m^2 + K_0^2 \sec^4\theta}, \\ \xi_2(m) &= \frac{K_0 \sec^2\theta \left[\sin m(f+f')\right] - m \cos m(f+f')}{m^2 + K_0^2 \sec^4\theta}, \end{aligned}\right\} \quad (9.6)$$

For $(h' - h) > 0$, a contour bounded by the positive half of both the real and imaginary axes is chosen. Then

$$\left.\begin{aligned} \lim_{\mu \to 0} \int_0^\infty \psi_1(\theta, K)\, e^{-K(f+f')} K\, dK = \\ = \int_0^\infty \xi_1(m)\, e^{-m(h'-h)\cos\theta}\, m\, dm \\ - i \int_0^\infty \xi_2(m)\, e^{-m(h'-h)\cos\theta}\, m\, dm \end{aligned}\right\} \quad (9.7)$$

where again, $\mu \to 0$.

Following similar procedures, it may be shown that

$$\left.\begin{aligned}
&\lim_{\mu\to 0}\int_0^\infty \psi_2(\theta,K)\, e^{-K(f+f')} K\,dK = \\
&\quad = 2\pi i \left\{ K_0 \sec^2\theta \left[e^{-iK_0(h'-h)\sec\theta - \nu_0(f+f')\sec^2\theta} \right] \right\} \\
&\qquad + \int_0^\infty \xi_1(m)\, e^{m(h'-h)\cos\theta}\, m\,dm \\
&\qquad\quad - i \int_0^\infty \xi_2(m)\, e^{m(h'-h)\cos\theta}\, m\,dm
\end{aligned}\right\} \qquad (9.8)$$

where $(h' - h) < 0$, and

$$\left.\begin{aligned}
&\lim_{\mu\to 0}\int_0^\infty \psi_2(\theta,K)\, e^{-K(f+f')} K\,dK = \\
&\quad = \int_0^\infty \xi_1(m)\, e^{-m(h'-h)\cos\theta}\, m\,dm \\
&\qquad + i \int_0^\infty \xi_2(m)\, e^{-m(h'-h)\cos\theta}\, m\,dm
\end{aligned}\right\} \qquad (9.9)$$

where $(h' - h) > 0$.

Substituting these equations into Eq. (9.3) the result is

$$
\left.\begin{aligned}
N_1 = 2\int_0^{\frac{\pi}{2}} \sec\theta\, d\theta \Big\{ & -4\pi K_0 i \sec^2\theta\, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] \\
& + 2i\int_0^{\infty} \xi_2(m)\, e^{m(h'-h)\cos\theta}\, m\,dm \Big\}
\end{aligned}\right\} \quad (9.10)
$$

for $(h' - h) < 0$, and

$$
N_1 = -2\int_0^{\frac{\pi}{2}} \sec\theta\, d\theta \left\{ 2i\int_0^{\infty} \xi_2(m)\, e^{-m(h'-h)\cos\theta}\, m\,dm \right\} \quad (9.11)
$$

for $(h' - h) > 0$.

For N_2 the result is

$$
\left.\begin{aligned}
N_2 = 2\int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \Big\{ & 4\pi K_0 \sec^2\theta\, e^{-K_0(f+f')\sec^2\theta} \sin\left[K_0(h'-h)\sec\theta\right] \\
& + 2\int_0^{\infty} \xi_2(m)\, e^{m(h'-h)\cos\theta}\, m\,dm \Big\}
\end{aligned}\right\} \quad (9.12)
$$

for $(h' - h) < 0$, and

$$N_2 = 2\int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \int_0^{\infty} 2\,\xi_1(m)\, e^{-m(h'-h)\cos\theta}\, m\, dm \tag{9.13}$$

for $(h' - h) > 0$.

Returning to the expressions for X_5 and X_6 , it is to be noted that they are imaginary. By substituting the imaginary part of Eq. (9.10) and Eq. (9.11) into X_5 and X_6 , real forces are obtained. The result is,

$$X_5 + X_6 =$$

$$\left.\begin{aligned}
&= -16\rho K_0 \int_0^d\!\!\int_{-l}^{l} \sigma_A' dh'df' \int_0^d\!\!\int_{-l}^{h'} \sigma_A dh df \int_0^{\frac{\pi}{2}} \sec\theta\, d\theta \int_0^{\infty} \xi_2(m) e^{-m(h'-h)\cos\theta}\, m dm \\
&\quad + 16\rho K_0 \int_0^d\!\!\int_{-l}^{l} \sigma_A' dh'df' \int_0^d\!\!\int_{h'}^{l} \sigma_A dh df\, \sec\theta\, d\theta \int_0^{\infty} \xi_2(m) e^{m(h'-h)\cos\theta}\, m dm \\
&\quad + 32\pi\rho K_0^2 \int_0^d\!\!\int_{-l}^{l} \sigma_A' dh'df' \int_0^d\!\!\int_{h'}^{l} \sigma_A dh df \int_0^{\frac{\pi}{2}} \sec^3\theta\, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] d\theta \\
&\quad + 16\rho K_0 \int_0^d\!\!\int_{-l}^{l} \sigma_A' dh'df' \int_0^d\!\!\int_{L-l}^{L+l} \sigma_B dh df\, \sec\theta\, d\theta \int_0^{\infty} \xi_2(m) e^{m(h'-h)\cos\theta}\, m dm \\
&\quad - 32\pi\rho K_0^2 \int_0^d\!\!\int_{-l}^{l} \sigma_A' dh'df' \int_0^d\!\!\int_{L-l}^{L+l} \sigma_B dh df \int_0^{\frac{\pi}{2}} \sec^3\theta\, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] d\theta .
\end{aligned}\right\} \tag{9.14}$$

Further reduction on this equation is possible. Consider the first two terms in the right member. In the first term, the integration with respect to the variables h and h' is over the area shown in Fig. (9.1). By changing the order of integration, the integration in h and h' becomes

$$\int_{-l}^{l} \sigma_A dh \int_{h}^{l} \sigma_A' dh' \; . \; . \; . \; . \; e^{-m(h'-h)\cos\theta} \; . \; . \; . \; . \tag{9.15}$$

By interchanging h and h', the value of the integral is not changed. Hence the integration in h and h' may be written as

$$\int_{-l}^{l} \sigma_A' dh' \int_{h'}^{l} \sigma_A dh \; . \; . \; . \; . \; e^{m(h'-h)\cos\theta} \; . \; . \; . \; . \tag{9.16}$$

In expression (9.15), there is a limitation on $(h'-h)$, i.e., $(h'-h) > 0$. By the process of interchanging h and h', the limitation on $(h'-h)$ in expression (9.16) becomes $(h'-h) < 0$.

Now by comparing the integration in h and

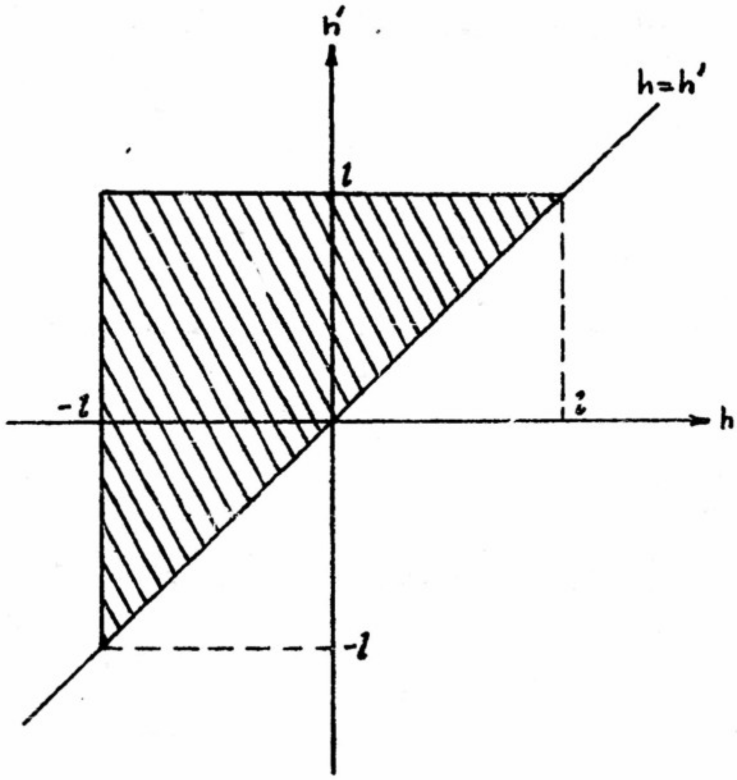

FIG. 9.1 AREA OF INTEGRATION FOR EQUATION (9.15)

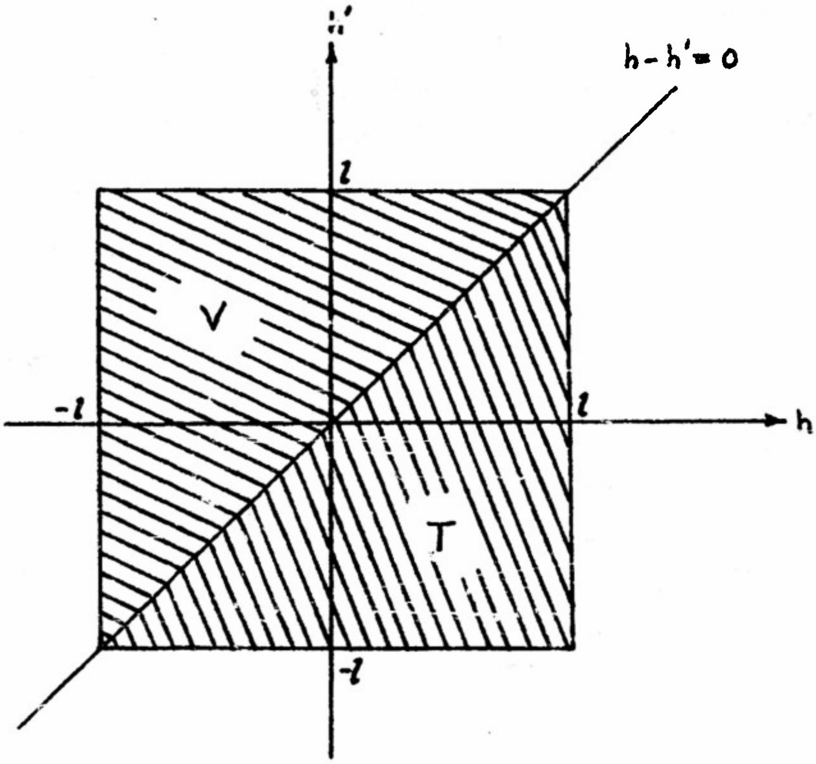

FIG. 9.2 AREAS OF INTEGRATION FOR EQUATIONS (9.17) AND (9.18)

h' in the second term of the right member of Eq. (9.14) with expression (9.16), it is to be noted that they are identical except for their sign. Thus the sum of the first two terms in the right member of Eq. (9.14) is equal to zero.

By a similar procedure, the third term in the right member of Eq. (9.14) may be rewritten in another form. For the integration in h and h',

$$\int_{-l}^{l} \sigma_A' dh' \int_{h'}^{l} \sigma_A \cos\left[K_0 (h'-h) \sec\theta\right] dh\,, \tag{9.17}$$

by changing the order of integration and then interchanging h and h', the result is

$$\int_{-l}^{l} \sigma_A' dh' \int_{-l}^{h'} \sigma_A \cos\left[K_0 (h'-h) \sec\theta\right] dh. \tag{9.18}$$

Referring to Fig. (9.2), expression (9.17) represents area T and expression (9.18) represents area V. Since $\cos\left[K_0 (h'-h) \sec\theta\right]$ is symmetric or even about $(h'-h) = 0$, the total area may be written as

$$\int_{-l}^{l} \sigma_A' dh' \int_{-l}^{l} \sigma_A \cos\left[K_o(h'-h)\sec\theta\right] dh. \tag{9.19}$$

Then

$$\int_{-l}^{l} \sigma_A' dh' \int_{h'}^{l} \sigma_A \cos\left[K_o(h'-h)\sec\theta\right] dh = \frac{1}{2}\int_{-l}^{l} \sigma_A' dh' \int_{-l}^{l} \sigma_A \cos\left[K_o(h'-h)\sec\theta\right] dh \tag{9.20}$$

where the limitation on $(h' - h)$ has been removed.

Thus the third term in the right member of Eq. (9.14) may be written in the form

$$-16\rho\pi K_o^2 \int_0^d \int_{-l}^{l} \sigma_A' dh' df' \int_0^d \int_{-l}^{l} \sigma_A \, dh \, df \int_0^{\frac{\pi}{2}} \sec^3\theta \, e^{-K_o(f+f')\sec^2\theta} \cos\left[K_o(h'-h)\sec\theta\right] d\theta \tag{9.21}$$

With these reductions, the force R_x becomes

$$R_x = X_1 + X_2 + X_3 + X_4 + X_5' + X_6' + X_5'' \tag{9.22}$$

where X_1, X_2, X_3, X_4, are given by the equations following Eq. (9.12), and where

$$X_5' = -16\pi\rho K_0^2 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{-l}^{l} \sigma_A \, dh df \int_0^{\frac{\pi}{2}} \sec^3\theta \, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] d\theta ,$$

$$X_6' = 16 \rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \sigma_B \, dh df \int_0^{\frac{\pi}{2}} \sec\theta \, d\theta \int_0^{\infty} \xi_2(m) e^{m(h'-h)\cos\theta} m \, dm ;$$

$$X_6'' = -32\pi\rho K_0^2 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{L-l}^{L+l} \sigma_B \, dh df \int_0^{\frac{\pi}{2}} \sec^3\theta \, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] d\theta .$$

For the force R_z, as expressed in Eq. (6.13), all terms are real. Hence when substituting N_2 into Eq. (6.13), the real parts are to be taken. The final result is

$$R_z = Z_1 + Z_2 + Z_3 + Z_4' + Z_4'' + Z_4''' + Z_5' + Z_5'' \qquad (9.23)$$

where Z_1, Z_2, Z_3 are given by the equations following Eq. (6.13) and where

$$Z_4' = -16\rho K_0 \int_0^d \int_{-l}^{l} \sigma_A' \, dh' df' \int_0^d \int_{-l}^{l} \sigma_A \, dh df \int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta \int_0^{\infty} \xi_1(m) e^{-m(h'-h)\cos\theta} m \, dm ,$$

$$Z_4'' = -16\rho K_0 \int_0^d\int_{-l}^{l} \sigma_A' \, dh'df' \int_0^d\int_{h'}^{l} \sigma_A \, dh\,df \int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta \int_0^\infty \xi_1(m)\, e^{m(h'-h)\cos\theta} \, m\,dm ,$$

$$Z_4''' = -32\pi\rho K_0^2 \int_0^d\int_{-l}^{l} \sigma_A' \, dh'df' \int_0^d\int_{h'}^{l} \sigma_A \, dh\,df \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-K_0(f+f')\sec^2\theta} \sin\left[K_0(h'-h)\sec\theta\right] d\theta ,$$

$$Z_5' = -16\rho K_0 \int_0^d\int_{-l}^{l} \sigma_A' \, dh'df' \int_0^d\int_{L-l}^{L+l} \sigma_B \, dh\,df \int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta \int_0^\infty \xi_1(m)\, e^{m(h'-h)\cos\theta} \, m\,dm ,$$

$$Z_5'' = -32\pi\rho K_0^2 \int_0^d\int_{-l}^{l} \sigma_A' \, dh'df' \int_0^d\int_{L-l}^{L+l} \sigma_B \, dh\,df \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-K_0(f+f')\sec^2\theta} \sin\left[K_0(h'-h)\sec\theta\right] d\theta .$$

There is no corresponding cancellation of Z_4' and Z_4'' as was the case for R_x. Also, Z_4''' cannot be reduced to the integral over the entire distribution A because the factor $\sin\left[K_0(h'-h)\sec\theta\right]$ is not symmetric about $(h'-h) = 0$

By following a similar procedure, the final for form of the moment L_y is

$$\begin{aligned} L_y = M_1 + M_2 + M_3 + M_4 + M_5' + M_6' + M_6'' \\ + M_7 + M_8 + M_9 + M_{10}' + M_{10}'' + M_{10}''' \\ + M_{11}' + M_{11}'' \end{aligned}$$

where M_1, M_2, M_3, M_4 and M_7, M_8, M_9 are given by the equations follwoing Eq. (8.7) and where

$$M_5' = -16\pi\rho K_0^2 \int_0^d\int_{-l}^{l}(f'-\epsilon)\sigma_A' dh' df' \int_0^d\int_{-l}^{l}\sigma_A dh df \int_0^{\frac{\pi}{2}} \sec^3\theta\, e^{-K_0(f+f')\sec^2\theta} \cos[K_0(h'-h)\sec\theta]\, d\theta,$$

$$M_6' = 16\rho K_0 \int_0^d\int_{-l}^{l}(f'-\epsilon)\sigma_A' dh' df' \int_0^d\int_{L-l}^{L+l}\sigma_B dh df \int_0^{\frac{\pi}{2}} \sec\theta\, d\theta \int_0^{\infty} \xi_2(m)\, e^{m(h'-h)\cos\theta}\, m\, dm,$$

$$M_6'' = -32\pi\rho K_0^2 \int_0^d\int_{-l}^{l}(f'-\epsilon)\sigma_A' dh' df' \int_0^d\int_{L-l}^{L+l}\sigma_B dh df \int_0^{\frac{\pi}{2}} \sec^3\theta\, e^{-K_0(f+f')\sec^2\theta} \cos[K_0(h'-h)\sec\theta]\, d\theta,$$

$$M_{10}' = 16\rho K_0 \int_0^d\int_{-l}^{l}h'\sigma_A' dh' df' \int_0^d\int_{-l}^{h'}\sigma_A dh df \int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \int_0^{\infty} \xi_1(m)\, e^{-m(h'-h)\cos\theta}\, m\, dm,$$

$$M_{10}'' = 16\rho K_0 \int_0^d\int_{-l}^{l}h'\sigma_A' dh' df' \int_0^d\int_{h'}^{l}\sigma_A dh df \int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \int_0^{\infty} \xi_1(m)\, e^{m(h'-h)\cos\theta}\, m\, dm,$$

$$M_{10}''' = 32\pi\rho K_0^2 \int_0^d\int_{-l}^{l}h'\sigma_A' dh' df' \int_0^d\int_{h'}^{l}\sigma_A dh df \int_0^{\frac{\pi}{2}} \sec^4\theta\, e^{-K_0(f+f')\sec^2\theta} \sin[K_0(h'-h)\sec\theta]\, d\theta,$$

$$M_{11}' = 16\rho K_0 \int_0^d\int_{-l}^{l}h'\sigma_A' dh' df' \int_0^d\int_{L-l}^{L+l}\sigma_B dh df \int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \int_0^{\infty} \xi_1(m)\, e^{m(h'-h)\cos\theta}\, m\, dm,$$

$$M_{11}'' = 32\pi\rho K_0^2 \int_0^d\int_{-l}^{l}h'\sigma_A' dh' df' \int_0^d\int_{L-l}^{L+l}\sigma_B dh df \int_0^{\frac{\pi}{2}} \sec^4\theta\, e^{-K_0(f+f')\sec^2\theta} \sin[K_0(h'-h)\sec\theta]\, d\theta.$$

X. Evaluation of Components of R_x

In the foregoing discussion, the expressions for the forces R_x and R_z, and the moment L_y acting on the towed ship have been formulated. To determine the variation of these forces and moments on the towed ship, the various integrals involved must now be evaluated. The evaluation may be by direct numerical integration for a particular case or the integrals may be evaluated analytically for a general case. In either case the amount of labor involved is considerable. In this and the following sections, the analytical approach will be used so that later numerical calculations will be applicable to any particular case within the limits of the problem.

The present section deals with the various components of the force R_x. By using Eq. (6.3), the equations for the source densities of the well-sided ships A and B, the first component of R_x becomes

$$X_1 = \frac{4\rho bc^2}{l^2} \int_0^d \int_{-l}^{l} h' \, dh' \, df' . \qquad (10.1)$$

Since X_1 is an odd function in h' with limits of integration $-l$ to l, $X_1 = 0$. This is to be expected because, the total source strength is zero. It is in agreement with D'Alembert's Principle which states that, any body moving uniformly in an unbounded ideal fluid at rest at infinity, experiences no resistance.

The second component of R_x is

$$X_2 = \frac{4\rho b^2c^2}{l^2}\int_0^d\int_{-l}^{l} h'dh'df' \int_0^d\int_{-l}^{l} h\left\{\frac{(h'-h)}{\left[(h'-h)^2+(f+f')^2\right]^{3/2}}\right\}dh\,df. \qquad (10.2)$$

It represents the resultant horizontal force on the distribution A due to its own image system excluding that part of the image system which trails aft to infinity. To show that this component is also equal to zero, consider the function

$$f(h',h) = \frac{h'h\,(h'-h)}{\left[(h'-h)^2+(f+f')^2\right]^{3/2}}. \qquad (10.3)$$

By interchanging h' and h

$$f(h,h') = \frac{hh'\,(h - h')}{\left[(h - h')^2 + (f + f')^2\right]^{3/2}}$$

or

$$f(h,h') = -\frac{h'h\,(h' - h)}{\left[(h' - h)^2 + (f + f')^2\right]^{3/2}} \qquad (10.4)$$

Thus

$$f(h',h) = -\,f(h,h') \qquad (10.5)$$

Hence $f(h',h)$ is skew symmetric about $(h' - h) = 0$ and for the limits $-l$ to l for both h' and h,

$$X_2 = 0 \qquad (10.6)$$

The third and fourth components of R_x represent the resultant horizontal force on the distribution A due to the distribution B and its image system, excluding the trailing distribution. Because of the presence of some common terms in the process of integration, it is convenient to evaluate these two components together. With the definitions of the source densities given by Eq. (6.3)

$$X_3 + X_4 = \frac{4\rho b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} h' dh' df' \int_0^d \int_{L-l}^{L+l} (h-L)(\alpha_1 - \alpha_2) dh df \qquad (10.7)$$

where α_1 and α_2 are given by Eq. (6.11). Integrating with respect to f, f' and h', the resulting expression after some reduction is

$$X_3 + X_4 = \frac{4\rho b^2 c^2}{\pi l^4} \int_{L-l}^{L+l} (h-L)(I_1 + I_2) dh \qquad (10.8)$$

where

$$I_1 = 4h\left\{\left[(h-l)^2+d^2\right]^{\frac{1}{2}} - \left[(h+l)^2+d^2\right]^{\frac{1}{2}} - d\ln\left|\frac{d+\left[(h-l)^2+d^2\right]^{\frac{1}{2}}}{(h-l)}\right|\right.$$

$$\left. + d\ln\left|\frac{d+\left[(h+l)^2+d^2\right]^{\frac{1}{2}}}{(h+l)}\right|\right\}$$

$$+4\left\{\frac{(h-l)\left[(h-l)^2+d^2\right]^{\frac{1}{2}}}{2} - \frac{(h+l)\left[(h+l)^2+d^2\right]^{\frac{1}{2}}}{2}\right.$$

$$\left. + \frac{d^2}{2}\ln\left|\frac{(h-l)+\left[(h-l)^2+d^2\right]^{\frac{1}{2}}}{(h+l)+\left[(h+l)^2+d^2\right]^{\frac{1}{2}}}\right|\right\}$$

and

$$I_2 = h\left\{\left[(h-l)^2+4d^2\right]^{\frac{1}{2}} - \left[(h+l)^2+4d^2\right]^{\frac{1}{2}}\right.$$

$$\left. - 2d\ln\left|\frac{2d+\left[(h-l)^2+4d^2\right]^{\frac{1}{2}}}{(h-l)}\right| + 2d\ln\left|\frac{2d+\left[(h+l)^2+4d^2\right]^{\frac{1}{2}}}{(h+l)}\right|\right\}$$

$$-\left\{\frac{(h-l)\left[(h-l)^2+4d^2\right]^{\frac{1}{2}}}{2} - \frac{(h+l)\left[(h+l)^2+4d^2\right]^{\frac{1}{2}}}{2}\right.$$

$$\left. + 2d^2\ln\left|\frac{(h-l)+\left[(h-l)^2+4d^2\right]^{\frac{1}{2}}}{(h+l)+\left[(h+l)^2+4d^2\right]^{\frac{1}{2}}}\right|\right\}.$$

The exact integration of Eq. (10.8) with respect to h is a long and tedious process, the final result being very lengthy. A process of numerical integration also involves a great amount of labor but is more suitable because the result is more concise. For numerical integration Eq. (10.8) may be written as

$$\left.\begin{aligned} X_3 + X_4 = \frac{16\rho b^2 c^2}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta \Big[& -8(\alpha+\eta)(A_1+A_2+A_3+A_4) \\ & +4(A_5+A_6+A_7) + 2(\alpha+\eta)(A_8+A_9+A_{10}+A_{11}) \\ & -(A_{12}+A_{13}+A_{14})\Big]\, d\eta \end{aligned}\right\} \quad (10.9)$$

where $\eta = (h - L)/2l$ and $\alpha = L/2l$. The quantity α is the ratio of the distance between the centers of mass of the two ships and the ship length. With the definition $\beta = d/2l$, the ratio of the draft to ship length, the terms in the integrand are defined as

$$A_1 = \left\{\left[2(\alpha+\eta)-1\right]^2 + 4\beta^2\right\}^{\frac{1}{2}},$$

$$A_2 = -\left\{\left[2(\alpha+\eta)+1\right]^2 + 4\beta^2\right\}^{\frac{1}{2}},$$

$$A_3 = -2\beta \ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)-1} \right| ,$$

$$A_4 = 2\beta \ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)+1} \right| ,$$

$$A_5 = \frac{1}{2} \left[2(\alpha+\eta)-1 \right] \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_6 = -\frac{1}{2} \left[2(\alpha+\eta)+1 \right] \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_7 = 2\beta^2 \ln \left| \frac{\left[2(\alpha+\eta)-1 \right] + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta)+1 \right] + \left\{ \left[2(\alpha+\eta)+1 \right] + 4\beta^2 \right\}^{\frac{1}{2}}} \right| ,$$

$$A_8 = \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_9 = -\left\{ \left[2(\alpha+\eta)+1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_{10} = -4\beta \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)-1} \right| ,$$

$$A_{11} = 4\beta \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)+1} \right| ,$$

$$A_{12} = \frac{1}{2}\left[2(\alpha+\eta)-1 \right] \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_{13} = -\frac{1}{2}\left[2(\alpha+\eta)+1 \right] \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$A_{14} = 8\beta^2 \ln \left| \frac{\left[2(\alpha+\eta)-1 \right] + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta)+1 \right] + \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}} \right| .$$

By choosing values for α and β, Eq. (10.9) may now be evaluated. The procedure is to obtain the ordinates of the integrand for several values of η within the interval $-\frac{1}{2}$ to $+\frac{1}{2}$. Then by Simpson's rule the approximate value of the integral may be found.

This force is analogous to the mutual attraction between two bodies. As is to be expected, preliminary numerical calculations show that $X_3 + X_4$ is a very small quantity. Individually, however, X_3 and X_4 are not of the same order of magnitude as their sum. They are just about equal to each other in magnitude. One is positive while the other is negative. Hence their difference is very small.

The component X_5' which represents the wave resistance of ship A as if alone has been evaluated by T. H. Havelock for a distribution of doublets instead of sources and sinks. It may be written as

$$X_5' = -16\pi\rho K_0^2 \int_0^{\frac{\pi}{2}} (P^2 + Q^2) \sec^5\theta \, d\theta \qquad (10.10)$$

where

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \iint \sigma \begin{matrix} \cos \\ \sin \end{matrix} \left[K_0 (h' - h) \sec\theta \right] e^{-K_0 (f + f') \sec^2\theta} \, dh \, df \qquad (10.11)$$

The results of the integration of this component for the wall-sided ship described in section VI are summarized in the Proc. Roy. Soc., Ser. A, Vol. 108, pp 582 - 591.

The component X_6' gives the resultant horizontal force on the distribution A due to the forces between the distribution A and the trailing system of sources and sinks for the distribution B. With the substitution of the source densities, it becomes

$$\left.\begin{aligned} X_6' = \frac{16\rho K_0 b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} h' dh' df' \int_0^d \int_{L-l}^{L+l} (h-L) dh df \int_0^{\frac{\pi}{2}} \sec\theta \, d\theta \\ \times \int_0^{\infty} \xi_2(m) e^{m(h'-h)\cos\theta} m dm \end{aligned}\right\} \qquad (10.12)$$

where

$$\xi_2(m) = \frac{K_0 \sec^2\theta \left[\sin m(f+f')\right] - m \cos m(f+f')}{m^2 + K_0^2 \sec^4\theta} \qquad (10.13)$$

and $(h' - h) < 0$.

The integral

$$\int_0^{\frac{\pi}{2}} \sec\theta \, d\theta \int_0^{\infty} \frac{K_0 \sec^2\theta \left[\sin mf\right] - m \cos mf}{m^2 + K_0^2 \sec^4\theta} e^{-mx\cos\theta} m dm \qquad (10.14)$$

where $x > 0$ has been evaluated in terms of the two variables x and f by the National Physical Laboratory (Teddington, Middlesex, England) and has been tabulated (10). A numerical method of evaluating Eq. (10.12) is possible by the use of such tabulated results. However, it will not be pursued here as the range of tabulated values is not sufficient for application in Eq. (10.12).

Using the substitutions $\nu = m/K_0$, $K_0(h - h') = H$, $K_0(f \quad f') = D$ and integrating with respect to f', Eq. (10.12) becomes

$$\chi_S' = \frac{16 \rho K_0 b^2 c^2}{\pi^2 l^4} \int_{-l}^{l} h' dh' \int_{L-l}^{L+l} (h-L)\, dh \int_0^d \left[I_1 + I_2 \right] df \qquad (10.15)$$

where

$$I_1 + I_2 = - \int_0^{\frac{\pi}{2}} \sec\theta \left[M \sec^2\theta - \frac{dM}{dD} \right] d\theta \Bigg|_{K_0 f}^{K_0(f+d)} , \qquad (10.16)$$

$$M = \int_0^{\infty} \frac{\cos \nu D}{\nu^2 + \cos^4\theta} \; e^{-\nu H \cos\theta} \, d\nu \qquad (10.17)$$

The limits of integration $K_0 f$ and $K_0(f + d)$

are to be substituted for the variable D. By using the Laplace Transform of $\cos \nu\zeta$ Eq. (10.17) may now be written as

$$\left.\begin{aligned} M &= \frac{1}{\sec^2\theta}\int_0^\infty \left[\cos \nu D \cos \nu\zeta\right] e^{-\nu H\cos\theta} d\nu \\ &\times \int_0^\infty (\cos \nu\zeta)\, e^{-\zeta\sec^2\theta} d\zeta \,. \end{aligned}\right\} \qquad (10.18)$$

By writing

$$\cos \nu D \cos \nu\zeta = \frac{1}{2}\left[\cos \nu(\zeta+D) + \cos \nu(\zeta-D)\right]$$

and taking the Laplace Transform with respect to

$$M = \frac{1}{2\sec^2\theta}\int_0^\infty \left[\frac{H\cos\theta}{H^2\cos^2\theta+(\zeta+D)^2} + \frac{H\cos\theta}{H^2\cos^2\theta+(\zeta-D)^2}\right] e^{-\zeta\sec^2\theta} d\zeta \,. \qquad (10.19)$$

The derivative of Eq. (10.19) with respect to D is

$$\frac{dM}{dD} = \frac{H}{2\sec^3\theta}\int_0^\infty \frac{d}{dD}\left[\frac{1}{H^2\cos^2\theta+(\zeta+D)^2} + \frac{1}{H^2\cos^2\theta+(\zeta-D)^2}\right] e^{-\zeta\sec^2\theta} d\zeta$$

or

$$\frac{dM}{dD} = \frac{H}{2\sec^3\theta}\int_0^\infty \frac{d}{d\zeta}\left[\frac{1}{H^2\cos^2\theta+(\zeta+D)^2} - \frac{1}{H^2\cos^2\theta+(\zeta-D)^2}\right]e^{-\zeta\sec^2\theta}\,d\zeta. \tag{10.20}$$

Integrating Eq. (10.20) by parts

$$\frac{dM}{dD} = \frac{H}{2\sec\theta}\int_0^\infty \left[\frac{1}{H^2\cos^2\theta+(\zeta+D)^2} - \frac{1}{H^2\cos^2\theta+(\zeta-D)^2}\right]e^{-\zeta\sec^2\theta}\,d\zeta\ . \tag{10.21}$$

Then by substituting Eq. (10.19) and Eq. (10.21) into eq. (10.16)

$$M\sec^2\theta - \frac{dM}{dD} = -\frac{H}{\sec\theta}\int_0^\infty \frac{1}{H^2\cos^2\theta+(\zeta-D)^2}\,e^{-\zeta\sec^2\theta}\,d\zeta\ . \tag{10.22}$$

Using the transformation $\lambda = \zeta - D$, Eq. (10.22) becomes

$$M\sec^2\theta - \frac{dM}{dD} = \frac{H}{\sec\theta}\,e^{-D\sec^2\theta}\int_{-D}^\infty \frac{e^{-\lambda\sec^2\theta}}{H^2\cos^2\theta+\lambda^2}\,d\lambda\ . \tag{10.23}$$

Then from Eq. (10.16)

$$I_1 + I_2 = -H\int_0^{\frac{\pi}{2}} e^{-D\sec^2\theta}\left[\int_D^{\infty}\frac{e^{-\lambda\sec^2\theta}}{H^2\cos^2\theta+\lambda^2}\,d\lambda\right]_{K,\theta} \tag{10.24}$$

Now by using the transformation $\tan\theta = x$ and integrating by parts

$$\left.\begin{aligned} I_1 + I_2 = &-\int_0^{\frac{\pi}{2}}\frac{H\,dx}{(1+x^2)\left[H^2+(1+x^2)D^2\right]} \\ &+ 2\int_0^{\infty} e^{-D(1+x^2)}dx\int_D^{\infty}\frac{H\lambda\, e^{-\lambda(1+x^2)}}{\left[H^2+(1+x^2)\lambda^2\right]^2}\,d\lambda \end{aligned}\right\} \tag{10.25}$$

By letting x be a complex variable and choosing a contour consisting of the upper half plane, the first term in Eq. (10.25) may be evaluated as

$$\int_0^{\infty}\frac{H\,dx}{(1+x^2)\left[H^2+(1+x^2)D^2\right]} = \frac{\pi}{2H}\left[1-\frac{D}{(H^2+D^2)^{\frac{1}{2}}}\right]. \tag{10.26}$$

It can be shown that the second term in

Eq. (10.25) is of the order $1/H^3$. For practical purposes the magnitude of H is such that this term may be neglected. Then from Eq. (10.15)

$$X_6' \approx \frac{8\rho K_0 b^2 c^2}{\pi l^4}\int_{-l}^{l} h'dh' \int_{L-l}^{L+l}(h-L)\,dh \int_0^{d}\left[\frac{1}{H} - \frac{D}{H(H^2+D^2)^{\frac{1}{2}}}\right]df\,\Bigg|_{K_1 f}^{K_1(f+d)} \qquad (10.27)$$

Now by completing the integration with respect to h', f and f' and using $\eta = (h - L)/2l$, $\alpha = L/2l$ and $\beta = d/2l$

$$\left.\begin{aligned} X_6' \approx -\frac{32\rho b^2 c^2}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta\Big[&(B_1+B_2)-\tfrac{1}{2}(B_3+B_4)+4B_5-8B_6 \\ &-4(\alpha+\eta)(B_7+B_8)+2(\alpha+\eta)(B_9+B_{10}) \\ &+8(\alpha+\eta)(B_{11}+B_{12})-8(\alpha+\eta)(B_{13}+B_{14})\Big]\,d\eta \end{aligned}\right\} \qquad (10.28)$$

where

$$B_1 = \left[2(\alpha+\eta)-1\right]\left\{\left[2(\alpha+\eta)-1\right]^2+4\beta^2\right\}^{\frac{1}{2}},$$

$$B_2 = -\left[2(\alpha+\eta)+1\right]\left\{\left[2(\alpha+\eta)+1\right]^2+4\beta^2\right\}^{\frac{1}{2}},$$

$$B_3 = \left[2(\alpha+\eta)-1\right]\left\{\left[2(\alpha+\eta)-1\right]^2+16\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_4 = -\left[2(\alpha+\eta)+1\right]\left\{\left[2(\alpha+\eta)+1\right]^2+16\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_5 = \beta^2 \ln\left|\frac{\left[2(\alpha+\eta)-1\right]+\left\{\left[2(\alpha+\eta)-1\right]^2+4\beta^2\right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta)+1\right]+\left\{\left[2(\alpha+\eta)+1\right]^2+4\beta^2\right\}^{\frac{1}{2}}}\right| ,$$

$$B_6 = \beta^2 \ln\left|\frac{\left[2(\alpha+\eta)-1\right]+\left\{\left[2(\alpha+\eta)-1\right]^2+16\beta^2\right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta)+1\right]+\left\{\left[2(\alpha+\eta)+1\right]^2+16\beta^2\right\}^{\frac{1}{2}}}\right| ,$$

$$B_7 = \left\{\left[2(\alpha+\eta)-1\right]^2+4\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_8 = \left\{\left[2(\alpha+\eta)+1\right]^2+4\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_9 = \left\{\left[2(\alpha+\eta)-1\right]^2+16\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_{10} = \left\{\left[2(\alpha+\eta)+1\right]^2+16\beta^2\right\}^{\frac{1}{2}} ,$$

$$B_{11} = \beta \ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)-1} \right| ,$$

$$B_{12} = -\beta \ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)+1} \right| ,$$

$$B_{13} = \beta \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta)-1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)-1} \right| ,$$

$$B_{14} = -\beta \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta)+1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta)+1} \right| .$$

Then by following the same procedure as for the component X_3 X_4, X_6' may be evaluated.

Finally, the component X_6'' which represents part of the force acting on ship A due to the wave interference caused by ship B may be written as

$$\left.\begin{aligned} X_6'' = & -\frac{32\rho K_0^2 b^2 c^2}{\pi l^4} \int_0^{d}\int_{-l}^{l} h' dh' df' \int_0^{d}\int_{L-l}^{L+l} (h-L)\, dh\, df \\ & \times \int_0^{\frac{\pi}{2}} \sec^3\theta \, e^{-K_0(f+f')\sec^2\theta} \cos\left[K_0(h'-h)\sec\theta\right] d\theta \end{aligned}\right\} \quad (10.29)$$

Integrating with respect to h, f, h' and f'

$$
\left.
\begin{aligned}
X_6'' = -\frac{32\rho b^2c^2}{\pi K_0^4 l^4}\int_0^{\frac{\pi}{2}}\left[1-e^{-K_0 d\sec^2\theta}\right]^2\Big[&2(K_0^2 l^2\sec^2\theta + \cos^6\theta)\cos(K_0 L\sec\theta) \\
&+(K_0^2 l^2\cos^4\theta - \cos^6\theta)\cos(K_0 L_1\sec\theta) + 2K_0 l\cos^4\theta\sin(K_0 L_1\sec\theta) \\
&+(K_0^2 l^2\cos^4\theta - \cos^6\theta)\cos(K_0 L_2\sec\theta) - 2K_0 l\cos^4\theta\sin(K_0 L_2\sec\theta)\Big]d\theta
\end{aligned}
\right\} \quad (10.30)
$$

where $L_1 = L - 2l$ and $L_2 = L + 2l$. In complex notation X_6'' may be written as

$$
\left.
\begin{aligned}
X_6'' = -\frac{32\rho b^2c^2}{\pi K_0^4 l^4}\left[1-e^{-K_0 d\sec^2\theta}\right]^2\Big\{&2(K_0^2 l^2\cos^4\theta + \cos^6\theta)\cos(K_0 L\sec\theta) \\
&+ \text{Real part of } \left[\cos^6\theta\,(K_0 l\sec\theta + i)^2 e^{-iK_0 L_1\sec\theta}\right] \\
&+ \text{Real part of } \left[\cos^6\theta\,(K_0 l\sec\theta + i)^2 e^{iK_0 L_2\sec\theta}\right]\Big\}\,d\theta\,.
\end{aligned}
\right\} \quad (10.31)
$$

To evaluate X_8'', consider the first integral,

$$J_1 = \int_0^{\frac{\pi}{2}} \left[1-e^{-K_0 d \sec^2\theta}\right]^2 \left(K_0^2 l^2 \cos^3\theta + \cos^5\theta\right) \cos\left(K_0 L \sec\theta\right) d\theta . \qquad (10.32)$$

Letting $p = 2K_0 l$, $K_0 d = \beta p$ where $\beta = d/2l$ and using the transformation

$$\sec\theta = 1 + t/p \qquad (10.33)$$

with

$$d\theta = \frac{dt}{(2pt)^{\frac{1}{2}}\left(1+\frac{t}{p}\right)\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}} \qquad (10.34)$$

the integral J_1 may be written as

$$\left.\begin{aligned} J_1 = \text{Real part of } \frac{p^2 i\alpha p}{4(2p)^{\frac{1}{2}}} \int_0^{\infty} \frac{1-2e^{-\beta p-(2\beta-i\alpha)t+\frac{\beta t^2}{p}}-e^{-2\beta p-(4\beta-i\alpha)t-\frac{2\beta t^2}{p}}}{t^{\frac{1}{2}}\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}} \\ \times\left[\frac{1}{\left(1+\frac{t}{p}\right)^4}+\frac{4}{p^2\left(1+\frac{t}{p}\right)^6}\right] dt \end{aligned}\right\} \qquad (10.35)$$

where $\alpha = L/2l$. Each term in the expanded integrand is of the form

$$J_1' = e^{-\frac{\gamma p}{2}} \int_0^\infty \frac{e^{-(\gamma - i\alpha)t - \frac{\gamma t^2}{2p}}}{t^{\frac{1}{2}}\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}} \left[\frac{1}{\left(1+\frac{t}{p}\right)^4} + \frac{4}{p^2\left(1+\frac{t}{p}\right)^6}\right] dt \qquad (10.36)$$

where γ takes on the values 0, 2β , 4β .

A close approximation to eq. (10.36) may be obtained by expanding

$$\left(1+\frac{t}{2p}\right)^{-\frac{1}{2}} \left[\frac{1}{\left(1+\frac{t}{p}\right)^4} + \frac{4}{p^2\left(1+\frac{t}{p}\right)^6}\right]$$

in ascending powers of $1/p$ and integrating term by term. The result is

$$\left.\begin{aligned} &1 + \left(-\frac{\gamma t^2}{2} - \frac{17t}{4}\right)\frac{1}{p} + \left(\frac{\gamma^2 t^4}{2} + \frac{17\gamma t^3}{8} + \frac{355t^2}{2} + 4\right)\frac{1}{p^2} \\ &+ \left(-\frac{\gamma^3 t^6}{48} - \frac{17\gamma^2 t^5}{32} - \frac{355\gamma t^4}{64} - \frac{8799t^3}{384} - 2\gamma t^2 - 25t\right)\frac{1}{p^3} + \cdots \end{aligned}\right\} \qquad (10.37)$$

For practical values of $p = 2K_c l$, the expansion to

$1/p^3$ is sufficient. With this expansion, each term in Eq. (10.36) is of the form

$$t^{r-\frac{1}{2}} e^{-(\gamma - i\alpha)t} \tag{10.38}$$

where $r = 0, 1, 2, 3, \ldots$.

To integrate these terms, the known relationship (11)

$$\int_0^\infty e^{-st} t^{z-1} dt = s^{-z} \Gamma(z) \tag{10.39}$$

where the real parts of x and z are positive and $\Gamma(z)$ is the gamma function, is used. In the present notation Eq. (10.39) may be written as

$$\int_0^\infty t^{r-\frac{1}{2}} e^{-(\gamma \pm i\alpha)t} dt = \Gamma(r+\tfrac{1}{2})\, \delta^{r+\frac{1}{2}} e^{\mp(r+\frac{1}{2})i\theta} \tag{10.40}$$

where

$$\left.\begin{aligned} \delta &= (\gamma^2+\alpha^2)^{-\frac{1}{2}} \, , \\ \theta &= \arctan\frac{\alpha}{\gamma} \, . \end{aligned}\right\} \tag{10.41}$$

Using Eq. (10.40) and the expansion given by expression (10.37), Eq. (10.36) now becomes

$$J_1 = \text{Real part of } \frac{r_e^{2} e^{i\alpha p}}{4(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \left[Q_1(0) - 2e^{-\beta p} Q_1(2\beta) + e^{-2\beta p} Q_1(4\beta) \right] \qquad (10.42)$$

where

$$\left.\begin{aligned} Q_1(\gamma) = {} & \delta^{\frac{1}{2}} e^{\frac{1}{2} i\theta} - \left[\frac{3}{8} \gamma \delta^{\frac{5}{2}} e^{\frac{5}{2} i\theta} + \frac{17}{8} \delta^{\frac{3}{2}} e^{\frac{3}{2} i\theta} \right] \frac{1}{p} \\ & + \left[\frac{105}{128} \gamma^2 \delta^{\frac{9}{2}} e^{\frac{9}{2} i\theta} + \frac{225}{64} \delta^{\frac{7}{2}} e^{\frac{7}{2} i\theta} \right. \\ & \left. + \frac{1065}{128} \delta^{\frac{5}{2}} e^{\frac{5}{2} i\theta} + 4 \delta^{\frac{3}{2}} e^{\frac{3}{2} i\theta} \right] \frac{1}{p^2} \\ & - \left[\frac{10395}{3072} \gamma^3 \delta^{\frac{13}{2}} e^{\frac{13}{2} i\theta} + \frac{16065}{1024} \gamma^2 \delta^{\frac{11}{2}} e^{\frac{11}{2} i\theta} \right. \\ & + \frac{37275}{1024} \gamma \delta^{\frac{9}{2}} e^{\frac{9}{2} i\theta} + \frac{131985}{3072} \delta^{\frac{7}{2}} e^{\frac{7}{2} i\theta} \\ & \left. + \frac{3}{2} \gamma \delta^{\frac{5}{2}} e^{\frac{5}{2} i\theta} + \frac{25}{2} \delta^{\frac{3}{2}} e^{\frac{3}{2} i\theta} \right] \frac{1}{p^3} + \cdots \end{aligned}\right\} \qquad (10.43)$$

with $\beta = 0$, 2β, 4β in turn, and δ and θ being defined by Eq. (10.41)

Now consider the second integral in Eq. (10.31) and let it be denoted by J_2.

$$\left.\begin{aligned} J_2 = \text{Real part of} \int_0^{\frac{\pi}{2}} \left[1 - e^{K_0 d \sec^2\theta}\right]^2 \\ \times \left[\cos^5\theta \, (K_0 l \sec\theta + i)^2 e^{-iK_0 L_1 \sec\theta}\right] d\theta \end{aligned}\right\} \qquad (10.44)$$

Using the same substitutions as were used for J_1,

$$\left.\begin{aligned} J_2 = \text{Real part of } \frac{p^2 e^{-i\alpha' p}}{4(2p)^{\frac{1}{2}}} \\ \times \int_0^{\frac{\pi}{2}} \frac{e^{i\alpha' t}\left[1 - e^{-\beta p\left(1+\frac{t}{p}\right)^2}\right]^2 \left[\left(1+\frac{t}{p}\right) + \frac{2i}{p}\right]}{t^{\frac{1}{2}}\left(1+\frac{t}{p}\right)^6 \left(1+\frac{t}{2p}\right)^{\frac{1}{2}}} \, dt \end{aligned}\right\} \qquad (10.45)$$

where $\alpha' = L_1/2l$. Following the same procedure

$$J_2 = \text{Real part of } \frac{p^2 e^{-i\alpha' p}}{4(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \left[Q_2(0) - 2e^{-\beta p} Q_2(2\beta) + e^{-2\beta p} Q_2(4\beta)\right] \qquad (10.46)$$

where

$$
\left.\begin{aligned}
Q_2(\gamma) = {} & \delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta} + \left[-\frac{3}{8}\gamma\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta} - \frac{17}{8}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta} + 4i\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta}\right]\frac{1}{p} \\
& + \left[\frac{105}{128}\gamma^2\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta} + \frac{225}{64}\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} + \left(\frac{1065}{128} - \frac{3}{2}i\gamma\right)\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta}\right. \\
& \left. - \frac{21}{2}i\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta} - 4\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta}\right]\frac{1}{p^2} + \left[-\frac{10395}{3072}\gamma^3\delta^{\frac{13}{2}}e^{-\frac{13}{2}i\theta}\right. \\
& - \frac{16065}{1024}\gamma^2\delta^{\frac{11}{2}}e^{-\frac{11}{2}i\theta} + \left(\frac{105}{32}i\gamma^2 - \frac{57275}{1024}\gamma\right)\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta} \\
& + \left(\frac{315}{16}i\gamma - \frac{131985}{3072}\right)\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} + \left(\frac{3}{2}\gamma + \frac{1089}{32}i\right)\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta} \\
& \left. + \frac{25}{2}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta}\right]\frac{1}{p^3} + \cdots\cdots
\end{aligned}\right\} \quad (10.47)
$$

with $\gamma = 0,\ 2\beta\ ,\ 4\beta$ in turn and
$\theta = \arctan \alpha'/\gamma.$

Similarly for the third integral in Eq. (10.31) denoted by J_3 ,

$$
\left.\begin{aligned}
J_3 = {} & \text{Real part of} \int_0^{\frac{\pi}{2}} \left[1 - e^{-K_0 d \sec^2\theta}\right]^2 \\
& \times \left[\cos^6\theta\,(K_0 l \sec\theta + i)^2 e^{iK_0 L_2 \sec\theta}\right] d\theta
\end{aligned}\right\} \quad (10.48)
$$

the result is

$$J_3 = \text{Real part of } \frac{r_o^{2}\,{}^{i\alpha'' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_3(0) - 2e^{-\beta p}Q_3(2p) + e^{-2\beta p}Q_3(4p)\right] \qquad (10.49)$$

where $\alpha'' = L_2/2l$, and $Q_3(\gamma)$ is the same as $Q_2(\gamma)$ with the negative sign on the exponential factor changed to a positive sign. The definitions for δ and θ are $\delta = [\gamma^2 + (\alpha'')^2]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha''/\gamma$.

Finally by substituting J_1, J_2, and J_3 into Eq. (10.31)

$$X_6'' = -\frac{32\rho b^2 c^2}{\pi l^4 K_o^4}\left[2J_1 + J_2 + J_3\right]. \qquad (10.50)$$

XI. Evaluation of Components of R_z

The first vertical component Z_1, the resultant force on the distribution A due to the forces between the distribution A and its image system, excluding the trailing system of sources and sinks, may be written as

$$Z_1 = -\frac{4\rho b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} h' dh' df' \int_0^d \int_{-l}^{l} h \beta_1 \, dh \, df \qquad (11.1)$$

where

$$\beta_1 = \frac{f + f'}{\left[(h' - h)^2 + (f + f')^2\right]^{3/2}}.$$

Integrating with respect to f', f and h'

$$Z_1 = -\frac{4\rho b^2 c^2}{\pi l^4} \int_{-l}^{l} h \, C(h) \, dh \qquad (11.2)$$

where

$$
\left.\begin{aligned}
C(h) = d\Big\{&\left[(h-l)^2+d^2\right]^{\frac{1}{2}}-\left[(h+l)^2+d^2\right]^{\frac{1}{2}}-\left[(h-l)^2+4d^2\right]^{\frac{1}{2}}\\
&+\left[(h+l)^2+4d^2\right]^{\frac{1}{2}}\Big\}+(l^2-h^2)\Big\{\operatorname{csch}^{-1}\left(\frac{h-l}{d}\right)\\
&-\operatorname{csch}^{-1}\left(\frac{h+l}{d}\right)-\frac{1}{2}\operatorname{csch}^{-1}\left(\frac{h-l}{2d}\right)+\frac{1}{2}\operatorname{csch}^{-1}\left(\frac{h+l}{2d}\right)\Big\}\\
&+2dh\Big\{-\sinh^{-1}\left(\frac{h-l}{d}\right)+\sinh^{-1}\left(\frac{h+l}{d}\right)+\sinh^{-1}\left(\frac{h-l}{2d}\right)\\
&-\sinh^{-1}\left(\frac{h+l}{2d}\right)\Big\}.
\end{aligned}\right\} \quad (11.3)
$$

As was done previously for the component $X_3 + X_4$, the integration of Eq. (11.2) with respect to h may conveniently be done by numerical integration. By letting $\tau = h/2l$ with $\beta = d/2l$, Eq. (11.2) may be written as

$$Z_1 = -\frac{16\rho b^2 c^2}{\pi} \left\{ 2\beta\tau\left[D_1 + D_2 + D_3 + D_4\right] + \tau(1+4\tau^2)\left[D_5 + D_6 + D_7 + D_8\right] + 8\beta\tau^2\left[D_9 + D_{10} + D_{11} + D_{12}\right]\right\} d\tau \qquad (11.4)$$

where

$$D_1 = \left[(2\tau-1)^2 + 4\beta^2\right]^{\frac{1}{2}},$$

$$D_2 = -\left[(2\tau+1)^2 + 4\beta^2\right]^{\frac{1}{2}},$$

$$D_3 = -\left[(2\tau-1)^2 + 16\beta^2\right]^{\frac{1}{2}},$$

$$D_4 = \left[(2\tau+1)^2 + 16\beta^2\right]^{\frac{1}{2}},$$

$$D_5 = \ln \frac{2\beta + \left[(2\tau-1)^2 + 4\beta^2\right]^{\frac{1}{2}}}{(2\tau-1)},$$

$$D_6 = -\ln \frac{2\beta + [(2T+1)^2 + 4\beta^2]^{\frac{1}{2}}}{(2T+1)} ,$$

$$D_7 = -\frac{1}{2} \ln \frac{4\beta + [(2T-1)^2 + 16\beta^2]^{\frac{1}{2}}}{(2T-1)} ,$$

$$D_8 = \frac{1}{2} \ln \frac{4\beta + [(2T+1)^2 + 16\beta^2]^{\frac{1}{2}}}{(2T+1)} ,$$

$$D_9 = -\ln \left\{ (2T-1) + [(2T-1)^2 + 4\beta^2]^{\frac{1}{2}} \right\} ,$$

$$D_{10} = \ln \left\{ (2T+1) + [(2T+1)^2 + 4\beta^2]^{\frac{1}{2}} \right\} ,$$

$$D_{11} = \ln \left\{ (2T-1) + [(2T-1)^2 + 16\beta^2]^{\frac{1}{2}} \right\} ,$$

$$D_{12} = \ln \left\{ (2T+1) + [(2T+1)^2 + 16\beta^2]^{\frac{1}{2}} \right\} .$$

By choosing values for β and using Simpson's rule, Z_1 may now be evaluated.

The second vertical component, Z_2, the resultant vertical force on the distribution A due to the forces between the distribution A and the image system, excluding the trailing system, for the distribution B, may be written as

$$Z_2 = -\frac{4\rho b^2 c^2}{\pi l^4}\int_0^d\int_{-l}^{l} h' \, dh' \, df' \int_0^d\int_{L-l}^{L+l} (h-L)\beta_1 \, dh \, df \qquad (11.5)$$

where

$$\beta_1 = \frac{f + f'}{\left[(h' - h)^2 + (f + f')^2\right]^{\frac{3}{2}}}$$

This component may also be evaluated in the same manner as the horizontal component $X_3 + X_4$. Integrating with respect to f', f and h'

$$Z_2 = -\frac{4\rho b^2 c^2}{\pi l^4}\int_{L-l}^{L+l} (h-L)\, C(h)\, dh \qquad (11.6)$$

where $C(h)$ is given by Eq. (11.3). Then again, with

$$\eta = (h - L)/2l \quad , \quad \alpha = L/2l \quad , \quad \beta = d/2l$$

$$\left. \begin{aligned} Z_2 = -\frac{16\rho\omega^2 c^2}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta \Big\{ & 2\beta\,(E_1 + E_2 + E_3 + E_4) \\ & + \left[1 - 4(\alpha+\eta)^2\right](E_5 + E_6 + E_7 + E_8) + 8(\alpha+\eta)(E_9 + E_{10}) \Big\}\, d\eta \end{aligned} \right\} \tag{11.7}$$

where

$$E_1 = \left\{ \left[2(\alpha+\eta) - 1\right]^2 + 4\beta^2 \right\}^{\frac{1}{2}} ,$$

$$E_2 = -\left\{ \left[2(\alpha+\eta) + 1\right]^2 + 4\beta^2 \right\}^{\frac{1}{2}} ,$$

$$E_3 = -\left\{ \left[2(\alpha+\eta) - 1\right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$E_4 = \left\{ \left[2(\alpha+\eta) + 1\right]^2 + 16\beta^2 \right\}^{\frac{1}{2}} ,$$

$$E_5 = \ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta) - 1\right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta) - 1} \right| ,$$

$$E_6 = -\ln \left| \frac{2\beta + \left\{ \left[2(\alpha+\eta) + 1\right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta) + 1} \right| ,$$

$$E_7 = \frac{1}{2} \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta) - 1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta) - 1} \right| ,$$

$$E_8 = \frac{1}{2} \ln \left| \frac{4\beta + \left\{ \left[2(\alpha+\eta) + 1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{2(\alpha+\eta) + 1} \right| ,$$

$$E_9 = -\ln \left| \frac{\left[2(\alpha+\eta) - 1 \right] + \left\{ \left[2(\alpha+\eta) - 1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta) + 1 \right] + \left\{ \left[2(\alpha+\eta) + 1 \right]^2 + 4\beta^2 \right\}^{\frac{1}{2}}} \right| ,$$

$$E_{10} = \ln \left| \frac{\left[2(\alpha+\eta) - 1 \right] + \left\{ \left[2(\alpha+\eta) - 1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}}{\left[2(\alpha+\eta) + 1 \right] + \left\{ \left[2(\alpha+\eta) + 1 \right]^2 + 16\beta^2 \right\}^{\frac{1}{2}}} \right| .$$

Again, by choosing values for α and β, and by using Simpson's rule, the approximate value of the integral may be found.

The third vertical component, the resultant vertical force on the distribution A due to the forces between the sources and sinks in the distribution A and B, is

$$Z_3 = -\frac{4\rho b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} h' dh' df' \int_0^d \int_{L-l}^{L+l} (h-L)\, \beta_2 \, dh \, df \qquad (11.8)$$

where

$$\beta_2 = \frac{f - f'}{\left[(h' - h)^2 + (f + f')^2\right]^{3/2}}$$

That this component is equal to zero is not evident from Eq. (11.8). By integrating with respect to f' and f, Z_3 may be shown equal to zero. The first integration with respect to f' gives

$$Z_3 = -\frac{4\rho b^2 c^2}{\pi l^4}\int_{-l}^{l} h' dh' \left[\int_{0}^{d}\int_{L-l}^{L+l} (h-L)\, F\, dh\, df\right] \qquad (11.9)$$

where

$$F = \frac{1}{\left[(H')^2 + (f-d)^2\right]^{\frac{1}{2}}} - \frac{1}{\left[(H')^2 + f^2\right]^{\frac{1}{2}}} \qquad (11.10)$$

with $H' = (h - h')$.

Then by integrating with respect to f, it may be shown that

$$Z_3 = 0. \qquad (11.11)$$

The components Z_4', Z_4'', Z_5' may be evaluated in the same manner as X_6'. By following the same procedure, it can be shown that these components are of the order $1/H^3$. As in the case of the horizontal component X_6', these small terms will be neglected.

The components Z_4''' and Z_5'' can be evaluated in the same manner as X_6''. The component Z_4''' represents one part of the resultant vertical force on the distribution A due to the forces between the distribution A and the trailing system of sources and sinks behind the distribution A. Substituting the definitions for the source densities it may be written as

$$\left.\begin{aligned} Z_4''' &= -\frac{32\rho K_0^2 b^2 c^2}{\pi l^4}\int_0^d\int_{-l}^{l} h' dh' df' \int_0^d\int_{h'}^{l} h dh df \\ &\quad \times \int_0^{\frac{\pi}{2}} \sec^2\theta \, e^{-K_0(f+f')\sec\theta} \sin\left[K_0(h'-h)\sec\theta\right] d\theta \end{aligned}\right\} \qquad (11.12)$$

Integrating with respect to f, f', and h'

$$
\left.
\begin{aligned}
Z_4''' = -\frac{128\rho b^2c^2}{\pi p^2}\int_0^{\frac{\pi}{2}} & \left[1-e^{-\beta p\sec^2\theta}\right]\left\{\left[\cos^2\theta-\frac{4}{p^2}\cos^4\theta\right]\sin(p\sec\theta)\right. \\
& \left. +\frac{4}{p}\cos^3\theta\cos(p\sec\theta)+\frac{p}{3}\cos\theta\right]d\theta
\end{aligned}
\right\} \quad (11.13)
$$

where $p = 2K_0 l$ and $\beta p = K_0 d$.

In complex notation, Eq. (11.13) may be written as

$$
\left.
\begin{aligned}
Z_4''' = -\frac{128\rho b^2c^2}{\pi p^2}\int_0^{\frac{\pi}{2}} & \left[1-e^{-\beta p\sec^2\theta}\right]^2 \\
\times & \left[\!\left[\text{Imag. part of } \left\{\cos^4\theta\left[\sec\theta+\frac{2i}{p}\right]e^{ip\sec\theta}\right. \right.\right. \\
& \left.\left. +\frac{p}{3}\cos\theta\right]\!\right] d\theta
\end{aligned}
\right\} \quad (11.14)
$$

Using the transformation (10.33), Eq. (11.14) becomes

$$
Z_4''' = -\frac{128\rho b^2c^2}{\pi p^2}\left[\text{Imag. part of } \frac{e^{ip}}{(2p)^{\frac{1}{2}}} I_3 + \frac{p}{3} I_4\right] \quad (11.15)
$$

where

$$I_3 = \int_0^\infty \frac{e^{it}\left[\left(1+\frac{t}{p}\right)+\frac{2i}{p}\right]^2\left[1-2e^{-\beta p\left(1+\frac{t}{p}\right)^2}+e^{-2\beta p\left(1+\frac{t}{p}\right)^2}\right]}{t^{\frac{1}{2}}\left(1+\frac{t}{p}\right)^5\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}}\,dt \tag{11.16}$$

When expanded, each term in Eq. (11.16) is of the form

$$e^{-\frac{\gamma p}{2}}\int_0^\infty \frac{e^{-(\gamma-i)t}\,e^{-\frac{\gamma t^2}{2p}}\left[\left(1+\frac{t}{p}\right)+\frac{2i}{p}\right]^2}{t^{\frac{1}{2}}\left(1+\frac{t}{p}\right)^5\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}}\,dt$$

where γ takes on the value $0, 2\beta, 4\beta$.

Now by expanding

$$\frac{e^{-\frac{\gamma t^2}{2p}}\left[\left(1+\frac{t}{p}\right)+\frac{2i}{p}\right]^2}{\left(1+\frac{t}{p}\right)^5\left(1+\frac{t}{2p}\right)^{\frac{1}{2}}}$$

in ascending powers of $(1/p)$ and using Eq. (10.40). Eq. (11.16) may be written as

$$I_3 = \pi^{\frac{1}{2}}\left[Q_1(0)-2e^{-\beta p}Q_1(2\beta)+e^{-2\beta p}Q_1(4\beta)\right] \tag{11.17}$$

where

$$Q_4(\gamma) = \delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} + \left[-\frac{3}{8}\gamma\delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} - \frac{13}{8}\delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} + 4i\delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \right] \frac{1}{p}$$

$$+ \left[\frac{105}{128}\gamma^2\delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} + \frac{145}{64}\delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta} + \left(\frac{657}{128} - \frac{3}{2}i\gamma\right)\delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} \right.$$

$$\left. - \frac{17}{2}i\delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} - 4\delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \right] \frac{1}{p^2}$$

$$+ \left[-\frac{10395}{3072}\gamma^3\delta^{\frac{13}{2}} e^{\frac{13}{2}i\theta} - \frac{12285}{1024}\delta^{\frac{11}{2}} e^{\frac{11}{2}i\theta} \right.$$

$$+ \left(\frac{105}{32}i\gamma^2 - \frac{22995}{1224}\gamma\right)\delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} + \left(\frac{385}{16}i\gamma - \frac{68085}{3072}\right)\delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta}$$

$$\left. + \left(\frac{3}{2}\gamma + \frac{1065}{32}i\right)\delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} + \frac{21}{2}\delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} \right] \frac{1}{p^3} + \cdots$$

with

$$\delta = (\gamma^2 + 1)^{-\frac{1}{2}} ,$$

$$\theta = \arctan \frac{1}{\gamma} .$$

Now returning to Eq. (11.15),

$$I_4 = \int_0^{\frac{\pi}{2}} \left[1 - e^{-\beta p \sec^2\theta}\right]^2 \cos\theta \, d\theta \, . \tag{11.18}$$

By expanding Eq. (11.18)

$$\left.\begin{aligned} I_4 = \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta - 2\int_0^{\frac{\pi}{2}} e^{-K_0 d \sec^2\theta} \cos\theta \, d\theta \\ + \int_0^{\frac{\pi}{2}} e^{-2K_0 d \sec^2\theta} \cos\theta \, d\theta \, . \end{aligned}\right\} \tag{11.19}$$

The first term is equal to one. The last two terms are of the form

$$I_4' = \int_0^{\frac{\pi}{2}} e^{-\xi \sec^2\theta} \cos\theta \, d\theta$$

where ξ equals $K_0 d$ and $2 K_0 d$.

By using the transformation $\sec^2\theta = 1 + t^2$

$$I_4' = e^{-\xi} \int_0^{\infty} (1+t^2)^{-\frac{3}{2}} e^{-\xi t^2} dt \, . \tag{11.21}$$

Eq. (11.21) is in a form such that the confluent hypergeometric function (12)

$$W_{k,m}(\xi) = \frac{e^{-\frac{1}{2}\xi}\xi^{\kappa}}{\Gamma(\frac{1}{2}-\kappa+m)} (\xi t^2)^{-\kappa-\frac{1}{2}+m} (1+t^2)^{\kappa-\frac{1}{2}+m} e^{-\xi t^2} (2t)\xi dt \qquad (11.22)$$

may be used. When $k = m = -\frac{1}{2}$

$$W_{-\frac{1}{2},-\frac{1}{2}}(\xi) = \frac{e^{-\xi}}{\Gamma(\frac{1}{2})} (1+t^2)^{-\frac{3}{2}} e^{-\xi t^2} dt. \qquad (11.23)$$

Hence

$$I_4' = e^{-\xi} (1+t^2)^{-\frac{3}{2}} e^{-\xi t^2} dt = W_{-\frac{1}{2},-\frac{1}{2}}(\xi)\pi^{\frac{1}{2}}. \qquad (11.24)$$

Then

$$I_4 = 1 - 2W_{-\frac{1}{2},-\frac{1}{2}}(K_0 d)\pi^{\frac{1}{2}} + W_{-\frac{1}{2},-\frac{1}{2}}(2K_0 d)\pi^{\frac{1}{2}}. \qquad (11.25)$$

For large values of $|\xi|$ when $|\arg \xi| < \pi - \psi < \pi$, the asymptotic expansion of $W_{k,m}(\xi)$ is given by the formula (12)

$$W_{k,m}(\xi) \sim e^{-\frac{1}{2}\xi}\xi^{k}\left\{1+\sum_{n=1}^{\infty}\frac{[m^2-(k-\frac{1}{2})^2][m^2-(k-\frac{3}{2})^2]\cdots[m^2-(k-n+\frac{1}{2})^2]}{n!\,\xi^n}\right\} \tag{11.26}$$

When $k = m = -\frac{1}{2}$, Eq. (11.26) reduces to

$$W_{-\frac{1}{2},-\frac{1}{2}}(\xi) \sim e^{-\frac{1}{2}\xi}\xi^{-\frac{1}{2}}\left[1-\frac{3}{4\xi}+\frac{45}{32\xi^2}-\frac{1575}{384\xi^3}+\frac{99225}{6144\xi^4}-\cdots\right]. \tag{11.27}$$

The expansion of the confluent hypergeometric function for all values of ξ such that $|\arg\xi| < \pi$; and, for values of $\arg\xi$ such that $\pi \leq |\arg\xi| < \frac{3}{2}\pi$ is (12)

$$W_{k,m}(\xi) = \frac{e^{-\frac{1}{2}\xi}\xi^{k}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\Gamma(-s-k-m+\frac{1}{2})\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\Gamma(-k+m+\frac{1}{2})}\xi^{s}\,ds \tag{11.28}$$

When $k = m = -\frac{1}{2}$

$$W_{-\frac{1}{2},-\frac{1}{2}}(\xi) = \frac{e^{-\frac{1}{2}\xi}\xi^{-\frac{1}{2}}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\Gamma(-s+\frac{3}{2})\Gamma(-s+\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}\xi^{s}\,ds \tag{11.29}$$

The integral may be evaluated by calculating the residues at the simple pole when $s = \frac{1}{2}$, and at the double poles when $s = 3/2, 5/2, 7/2, \ldots$. The calculation of the residue at the simple pole is simple enough but the double poles pose a more difficult problem because the residues involve logarithmic terms. By using a known result, Eq. (11.20) may be evaluated without calculating the required residues.

When $k = m = -3/2$ Havelock gives the result (4)

$$
\left.
\begin{aligned}
W_{-\frac{3}{2},-\frac{3}{2}}(\xi) = \frac{8}{15}\pi^{\frac{1}{2}}\xi^{-\frac{3}{2}}e^{-\frac{1}{2}\xi}\Big\{ & -2\xi^{\frac{1}{2}} + \frac{1}{2}\xi^{\frac{3}{2}} - \frac{3}{8}\xi^{\frac{5}{2}} \\
& - \sum_{n=3}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(n-2)}\xi^{n+\frac{1}{2}}\Big(\ln\tfrac{1}{4}\gamma\xi + 2\sum_{p=1}^{n}\frac{1}{2p-1} - \sum_{1}^{n}\frac{1}{p} - \sum_{1}^{n-3}\frac{1}{p}\Big)
\end{aligned}
\right\} \qquad (11.30)
$$

where γ is known as Euler's or Mascheroni's constant and is equal to 0.5772157 .... . (As a point of interest, the value of γ has been calculated by J. C. Adams to 260 places of decimals.) (12)

From Eq. (11.30) Havelock obtains the result

$$\left.\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^5\theta\, e^{-\xi\sec^2\theta}\, d\theta = \frac{8}{15} e^{-\xi}\Big\{ & 1 - \frac{1}{4}\xi + \frac{3}{16}\xi^2 + \frac{37}{192}\xi^3 + \frac{113}{3072}\xi^4 \\ & - \frac{263}{2048}\xi^5 + \cdots + \left(\frac{5}{32}\xi^3 + \frac{35}{256}\xi^4 + \frac{63}{1024}\xi^5 + \cdots\right) \ln \frac{1}{4}\gamma\xi \Big\} \,.\end{aligned}\right\} \quad (11.31)$$

By differentiating Eq. (11.31) twice with respect to ξ, the required integral, Eq. (11.20), may be obtained without calculating the residues at the simple and double poles. The result is

$$\left.\begin{aligned}\int_0^{\frac{\pi}{2}} \cos\theta\, e^{-\xi\sec^2\theta}\, d\theta = e^{-\xi}\Big\{ & 1 + \frac{1}{2}\xi + \frac{1}{16}\xi^2 - \frac{7}{192}\xi^3 - \frac{79}{3072}\xi^4 + \cdots \\ & + \left(\frac{1}{2}\xi + \frac{3}{8}\xi^2 + \frac{17}{96}\xi^3 + \frac{31}{418}\xi^4 + \cdots\right) \ln \frac{1}{4}\gamma\xi \Big\} \,.\end{aligned}\right\} \quad (11.32)$$

By substituting the appropriate expression into Eq. (11.19), I_4 may now be evaluated.

Another method of evaluating Eq. (11.21) is possible. Integrals of this form can be evaluated in terms of Bessel functions of the second kind of orders zero and one. Consider the integral (13)

$$\left.\begin{aligned} L_{2n+1} &= \int_0^\infty e^{-\xi t^2}(1+t^2)^{\frac{2n+1}{2}}\,dt \\ &= \frac{n+\xi}{\xi}\int_0^\infty e^{-\xi t^2}(1+t^2)^{\frac{2n-1}{2}}\,dt - \frac{2n-1}{2\xi}\int_0^\infty e^{-\xi t^2}(1+t^2)^{\frac{2n-3}{2}}\,dt \end{aligned}\right\} \tag{11.33}$$

where n is an interger.

This may be reduced to the general form

$$L_{2n+1} = \frac{n+\xi}{\xi} L_{2n-1} - \frac{2n-1}{2\xi} L_{2n-3} \tag{11.34}$$

Starting with $n = 0$, it can be seen that if L_1 and L_{-1} are known, all the other L's may be determined. The definitions for L_1 and L_{-1} are

$$L_1 = \int_0^\infty e^{-\xi t^2}(1+t^2)^{\frac{1}{2}}\,dt \tag{11.35}$$

$$L_{-1} = \int_0^\infty e^{-\xi t^2}(1+t^2)^{-\frac{1}{2}}\,dt \tag{11.36}$$

To evaluate Eq. (11.31), from Eq. (11.34) it may be written that

$$\left.\begin{aligned} L_{-3} &= \int_0^\infty e^{-\xi t^2}(1+t^2)^{-\frac{3}{2}}\,dt \\ &= 2\xi\left[L_1 - L_{-1}\right] \;. \end{aligned}\right\} \qquad (11.37)$$

By applying the transformation t sinh to the integrals L_1 and L_{-1}, they can be transformed into known expressions for modified Bessel Functions of the second kind K_0 and K_1 (14).

$$\left.\begin{aligned} L_{-1} &= \frac{1}{2}e^{\frac{\xi}{2}}K_0\left(\frac{\xi}{2}\right) \\ L_1 &= \frac{1}{4}e^{\frac{\xi}{2}}\left[K_0\left(\frac{\xi}{2}\right)+K_1\left(\frac{\xi}{2}\right)\right] \;. \end{aligned}\right\} \qquad (11.38)$$

Since the functions K_0 and K_1 are tabulated (14), L_{-3} may now be found from Eq. (11.37). Hence

$$\left.\begin{aligned} I_4' &= e^{-\xi}\int_0^{\infty}(1+t^2)^{-\frac{1}{2}}e^{-\xi t^2}\,dt \\ &= \frac{\xi}{2}e^{-\frac{\xi}{2}}\left[K_1\left(\frac{\xi}{2}\right)-K_0\left(\frac{\xi}{2}\right)\right]. \end{aligned}\right\} \tag{11.39}$$

The latter method of evaluating I_4' appears more concise and will be pursued here. When $\xi = K_0 d$

$$I_4' = \frac{1}{2}K_0 d\, e^{-\frac{K_0 d}{2}}\left[K_1\left(\frac{K_0 d}{2}\right)-K_0\left(\frac{K_0 d}{2}\right)\right]. \tag{11.40}$$

When $\xi = 2K_0 d$

$$I_4' = K_0 d\, e^{-K_0 d}\left[K_1(K_0 d)-K_0(K_0 d)\right]. \tag{11.41}$$

Thus, from Eq. (11.19)

$$\left.\begin{aligned} I_4 = 1 &- K_0 d\, e^{-\frac{K_0 d}{2}}\left[K_1\left(\frac{K_0 d}{2}\right)-K_0\left(\frac{K_0 d}{2}\right)\right] \\ &+ K_0 d\, e^{-K_0 d}\left[K_1(K_0 d)-K_0(K_0 d)\right]. \end{aligned}\right\} \tag{11.42}$$

Now by substituting I_3 and I_4, Eq. (11.17) and Eq. (11.32) respectively, into Eq. (11.15) the component Z_4''' may be evaluated.

The vertical component, Z_5'', one part of the resultant vertical force on the distribution A due to the forces between the trailing system of sources and sinks behind the distribution B and the distribution A itself, may be written as

$$\left. \begin{aligned} Z_5'' = -\frac{32\rho K_0^2 b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} h' \, dh' \, df' \int_0^d \int_{L-l}^{L+l} (h-L) \, dh \, df \\ \times \int_0^{\frac{\pi}{2}} \sec^4\theta \left\{ \sin\left[K_0 (h'-h) \sec\theta\right] \right\} e^{-K_0 (f+f') \sec^2\theta} \, d\theta \end{aligned} \right\} \quad (11.43)$$

,

Integrating with respect to f', f, h' and h and writing in complex form

$$\left. \begin{aligned} Z_5'' = -\frac{32\rho b^2 c^2}{\pi K_0^4 l^4} \int_0^{\frac{\pi}{2}} \left[1 - e^{-\beta p \sec^2\theta}\right]^2 \Big\{ 2\left[K_0^2 l^2 \cos^2\theta + \cos^4\theta\right] \sin(K_0 L \sec\theta) \\ + \text{Imag. part of} \left[\cos^4\theta \, (K_0 l \sec\theta + i)^2 \left(-e^{-iK_0 L_1 \sec\theta}\right)\right] \\ + \text{Imag. part of} \left[\cos^4\theta \, (K_0 l \sec\theta + i)^2 \left(e^{iK_0 L_2 \sec\theta}\right)\right] \Big\} \, d\theta \end{aligned} \right\} \quad (11.44)$$

where $p = 2K_0 l$, $\beta p = K_0 d$, $L_1 = L - 2l$ and $L_2 = L + 2l$

Now by using the transformation given by Eq. (10.33) and using Eq. (10.40)

$$Z_5'' = -\frac{32\rho b^2 a^2}{\pi K_0^4 l^4}\left[2J_5 + J_6 + J_7\right] \tag{11.45}$$

with the following definitions for J_5, J_6 and J_7.

$$J_5 = \text{Imag. part of } \frac{p^2 e^{i\alpha p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_5(0) - 2e^{-\beta p}Q_5(2\beta) + e^{-2\beta p}Q_5(4\beta)\right] \tag{11.46}$$

where

$$Q_5(\gamma) = \delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \left[\frac{3}{8}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \frac{13}{8}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p}$$

$$+ \left[\frac{105}{128}\gamma^2\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \frac{195}{64}\gamma\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \frac{657}{128}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + 4\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta}\right]\frac{1}{p^2}$$

$$- \left[\frac{10395}{3072}\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta} + \frac{12285}{1024}\gamma^2\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta} + \frac{22995}{1024}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta}\right.$$

$$\left. + \frac{68075}{3072}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \frac{5}{2}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \frac{11}{2}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p^3} + \cdots$$

with $\delta = (\gamma^2 + \alpha^2)^{-\frac{1}{2}}$ and $\theta = \arctan \alpha/\gamma$.

$$J_6 = \text{Imag. part of } -\frac{p\, e^{-i\alpha' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_6(0) - 2e^{-\beta p}Q_6(2\beta) + e^{-2\beta p}Q_6(4\beta)\right] \tag{11.47}$$

where

$$Q_6(\gamma) = \delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta} - \left[\frac{1}{8}\gamma\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta} + \frac{15}{8}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta} - 4i\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta}\right]\frac{1}{p}$$

$$+\left[\frac{105}{128}\gamma^2\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta} + \frac{195}{64}\gamma\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} + \left(\frac{457}{128} - \frac{3}{2}i\gamma\right)\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta}\right.$$

$$\left. - \frac{17}{2}i\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta} - 4\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta}\right]\frac{1}{p^2} - \left[\frac{10395}{3072}\gamma^3\delta^{\frac{13}{2}}e^{-\frac{13}{2}i\theta}\right.$$

$$+ \frac{12285}{1024}\gamma^2\delta^{\frac{11}{2}}e^{-\frac{11}{2}i\theta} - \left(\frac{105}{32}i\gamma^2 - \frac{22995}{1024}\gamma\right)\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta}$$

$$-\left(\frac{225}{16}i\gamma - \frac{68075}{3072}\right)\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} - \left(\frac{3}{2}\gamma + \frac{1065}{32}i\right)\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta}$$

$$\left. - \frac{21}{2}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta}\right]\frac{1}{p^3} + \cdots\cdot$$

with $\delta = \left[\gamma^2 + (\alpha')^2\right]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha'/\gamma$.

$$J_7 = \text{Imag. part of } \frac{p^2 e^{i\alpha' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_7(0) - 2e^{-\beta p}Q_7(2\beta) + e^{-2\beta p}Q_7(4\beta)\right] \tag{11.48}$$

where $Q_7(\gamma)$ is the same as $Q_6(\gamma)$ except that the sign of the exponent of the exponential factors is positive instead of negative; and where $\delta = [\gamma^2 + (\alpha'')^2]^{-\frac{1}{2}}$ θ = arctan α''/γ .

XII. Evaluation of Components of L_y

Since the integration of the various moment components follows the same procedures as were used for the force components, the detailed steps will not be indicated in this section. Only the final results of the significant components, M_5', M_6'', M_{10}''' and M_{11}'', will be listed. The other components may be evaluated by the procedures indicated in the previous sections. The components M_5' and M_6'' represent moments due to horizontal forces while M_{10}''' and M_{11}'' are moments due to vertical forces.

From Section IX, the component M_5' may be written as

$$\left.\begin{aligned} M_5' &= -\frac{16\rho K_0^2 b^2 c^2}{\pi l^4}\int_0^d\int_{-l}^{l}(f'-e)h'\,dh'\,df'\int_0^d\int_{-l}^{l}h\,dh\,df \\ &\quad \times\int_0^{\frac{\pi}{2}}\sec^5\theta\, e^{-K_0(f+f')\sec^2\theta}\cos\left[K_0(h'-h)\sec\theta\right]d\theta \end{aligned}\right\} \qquad (12.1)$$

By completing the integration with respect to f, f', h, h' and θ, M_5' may be written as the sum of three parts.

$$M_5' = M_{51}' + M_{52}' + M_{53}' \quad . \tag{12.2}$$

The expressions for M_{51}', M_{52}', and M_{53}' are

$$\left.\begin{aligned}
M_{51}' &= -\frac{16\rho b^2 e^2}{\pi K_0^6 l^4}\Big\{\frac{4p^2}{15} + \frac{96}{105} - p^2 e^{-\beta p} L_{-7}(\beta p) - 4e^{-\beta p} L_{-9}(\beta p) \\
&\quad + \frac{p^2}{2} e^{-2\beta p} L_{-7}(2\beta p) + 2e^{-2\beta p} L_{-9}(2\beta p) \\
&\quad + \text{Real part of } \frac{p^2 e^{ip}}{2(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \Big[Q_8(0) - 2e^{-\beta p} Q_8(2\beta) + e^{-2\beta p} Q_8(4\beta)\Big]\Big\}, \\
M_{52}' &= \frac{16\rho b^2 c^2 e}{\pi K_0^4 l^4}\Big\{\frac{p^2}{3} + \frac{16}{15} - p^2 e^{-\beta p} L_{-5}(\beta p) - 4e^{-\beta p} L_{-7}(\beta p) \\
&\quad + \frac{p^2}{2} e^{-2\beta p} L_{-5}(2\beta p) + 2e^{-2\beta p} L_{-7}(2\beta p) \\
&\quad + \text{Real part of } \frac{p^2 e^{ip}}{2(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \Big[Q_6(0) - 2e^{-\beta p} Q_6(2\beta) + e^{-2\beta p} Q_6(4\beta)\Big]\Big\}, \\
M_{53}' &= \frac{16\rho b^2 c^2 d}{\pi K_0^4 l^4}\Big\{\frac{p^2}{2} e^{-\beta p} L_{-5}(\beta p) + 2e^{-\beta p} L_{-7}(\beta p) \\
&\quad - \frac{p^2}{2} e^{-2\beta p} L_{-5}(2\beta p) - 2e^{-2\beta p} L_{-7}(2\beta p) \\
&\quad + \text{Real part of } \frac{p^2 e^{ip}}{2(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \Big[e^{-\beta p} Q_5(2\beta) - e^{-2\beta p} Q_5(4\beta)\Big]\Big\},
\end{aligned}\right\} \tag{12.3}$$

where

$$
\left.
\begin{aligned}
L_{-5}(\xi) &= \frac{\xi}{3} e^{\frac{\xi}{2}} \left[\xi K_0\left(\frac{\xi}{2}\right) + (1-\xi) K_1\left(\frac{\xi}{2}\right) \right] , \\
L_{-7}(\xi) &= \frac{2\xi}{15} e^{\frac{\xi}{2}} \left[\xi\left(-\xi + \frac{1}{2}\right) K_0\left(\frac{\xi}{2}\right) \right. \\
&\qquad \left. + \left(\xi^2 - \frac{3}{2}\xi + 2\right) K_1\left(\frac{\xi}{2}\right) \right] , \\
L_{-9}(\xi) &= \frac{2\xi}{105} e^{\frac{\xi}{2}} \left[\xi\left(2\xi^2 - 2\xi + 3\right) K_0\left(\frac{\xi}{2}\right) \right. \\
&\qquad \left. + 2\left(-\xi^3 + 2\xi^2 - 4\xi + 6\right) K_1\left(\frac{\xi}{2}\right) \right] ,
\end{aligned}
\right\} \quad (12.4)
$$

and

$$
\left.
\begin{aligned}
Q_8(\gamma) &= \delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} + \left[-\frac{3}{8}\gamma\delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} + 4i\delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \right] \frac{1}{p} \\
&\quad + \left[\frac{105}{128}\gamma^2\delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} + \frac{374}{64}\gamma\delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta} + \left(\frac{2169}{128} - \frac{3}{2}i\gamma \right) \delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} \right. \\
&\quad \left. - \frac{29}{2} i\delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} - 4\delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \right] \frac{1}{p^2} + \left[-\frac{10395}{3072}\gamma^3\delta^{\frac{13}{2}} e^{\frac{13}{2}i\theta} \right. \\
&\quad - \frac{23625}{1024}\gamma^2\delta^{\frac{11}{2}} e^{\frac{11}{2}i\theta} + \left(\frac{105}{32} i\gamma^2 - \frac{75915}{1024}\gamma \right) \delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} \\
&\quad + \left(\frac{436}{16} i\gamma - \frac{356265}{3072} \right) \delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta} + \left(\frac{3}{2}\gamma + \frac{2865}{32} i \right) \delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} \\
&\quad \left. + \frac{32}{2}\delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} \right] \frac{1}{p^3} + \cdots .
\end{aligned}
\right\} \quad (12.5)
$$

In $Q_8(\gamma)$, $\xi = (\gamma^2 + 1)^{-\frac{1}{2}}$ and $\theta = \arctan 1/\gamma$. $Q_3(\gamma)$ is obtained from Eq. (10.47) as explained after Eq. (10.49). However, when $Q_3(\gamma)$ is used in M_{52}' and M_{53}', $\xi = (\gamma^2 + 1)^{-\frac{1}{2}}$ and $\theta = \arctan 1/\gamma$. The ξ's in the $L(\xi)$ functions take on the values βp and $2\beta p$ as indicated in Eq. (12.3). The functions $K_0(\xi/2)$ and $K_1(\xi/2)$ are modified Bessel functions of the second kind.

The component M_6'' may be written as

$$\left.\begin{aligned} M_6'' &= -\frac{32 \rho K_0^2 b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} (f'-e) h' \, dh' \, df' \int_0^d \int_{L-l}^{L+l} (h-L) \, dh \, df \\ &\times \int_0^{\frac{\pi}{2}} \sec^3\theta \, e^{-K_0 (f+f') \sec^2\theta} \cos\left[K_0 (h'-h) \sec\theta\right] d\theta \end{aligned}\right\} \qquad (12.6)$$

or

$$M_6'' = M_{61}'' + M_{62}'' + M_{63}'' \qquad (12.7)$$

where

$$
\left.
\begin{aligned}
M_{61}'' &= -\frac{32\rho b^2 c^2}{\pi K_o^5 l^4}\Big\{\text{Real part of } \frac{p^2 e^{i\alpha p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_9(0) - 2e^{-\beta p}Q_9(2\beta) + e^{-2\beta p}Q_9(4\beta)\right] \\
&\quad + \text{Real part of } \frac{p^2 e^{-i\alpha' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_{10}(0) - 2e^{-\beta p}Q_{10}(2\beta) + e^{-2\beta p}Q_{10}(4\beta)\right] \\
&\quad + \text{Real part of } \frac{p^2 e^{i\alpha'' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_8(0) - 2e^{-\beta p}Q_8(2\beta) + e^{-2\beta p}Q_8(4\beta)\right]\Big\}, \\
M_{62}'' &= \frac{32\rho b^2 c^2 \epsilon}{\pi K_o^4 l^4}\left[\,2J_1 + J_2 + J_3\,\right], \\
M_{63}'' &= \frac{32\rho b^2 c^2 d}{\pi K_o^4 l^4}\Big\{\text{Real part of } \frac{p^2 e^{i\alpha p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[e^{-\beta p}Q_1(2\beta) - e^{-2\beta p}Q_1(4\beta)\right] \\
&\quad + \text{Real part of } \frac{p^2 e^{-i\alpha' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[e^{-\beta p}Q_2(2\beta) - e^{-2\beta p}Q_{3L}(4\beta)\right] \\
&\quad + \text{Real part of } \frac{p^2 e^{i\alpha'' p}}{4(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[e^{-\beta p}Q_3(2\beta) - e^{-2\beta p}Q_5(4\beta)\right]\Big\}.
\end{aligned}
\right\} \quad (12.8)
$$

For the $Q(\gamma)$ functions in Eq. (12.8) -- $Q_1(\gamma)$ is given by Eq. (10.42); $Q_2(\gamma)$ is given by Eq. (10.47); and $Q_3(\gamma)$ is given by Eq. (10.47) except for a positive instead of a negative sign on the exponent of the exponential factors, and $\delta = [\gamma^2 + (\alpha'')^2]^{-\frac{1}{2}}$, $\theta = \arctan \alpha''/\gamma$. The definition for $Q_9(\gamma)$ is

$$
\left.
\begin{aligned}
Q_9(\gamma) = {} & \delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} - \left[\frac{3}{8} \gamma \delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} + \frac{25}{8} \delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} \right] \frac{1}{P} \\
& + \left[\frac{105}{128} \delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} + \frac{375}{64} \gamma \delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta} + \frac{2169}{128} \delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} \right. \\
& \left. + 4 \delta^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \right] \frac{1}{P^2} - \left[\frac{10395}{3072} \gamma^3 \delta^{\frac{13}{2}} e^{\frac{13}{2}i\theta} \right. \\
& + \frac{23625}{1024} \gamma^2 \delta^{\frac{11}{2}} e^{\frac{11}{2}i\theta} + \frac{75915}{1024} \gamma \delta^{\frac{9}{2}} e^{\frac{9}{2}i\theta} \\
& \left. + \frac{356265}{3072} \delta^{\frac{7}{2}} e^{\frac{7}{2}i\theta} + \frac{3}{2} \gamma \delta^{\frac{5}{2}} e^{\frac{5}{2}i\theta} + \frac{33}{2} \delta^{\frac{3}{2}} e^{\frac{3}{2}i\theta} \right] \frac{1}{P^3} + \cdots
\end{aligned}
\right\} \quad (12.9)
$$

where $\delta = (\gamma^2 + \alpha^2)^{-\frac{1}{2}}$ and $\theta = \arctan \alpha/\gamma$. $Q_{10}(\gamma)$ is the same as $Q_8(\gamma)$, given by Eq. (12.5), except for a negative instead of a positive exponent on the exponential factors. Also, the definitions for δ and θ in $Q_8(\gamma)$ are $\delta = [\gamma^2 + (\alpha'')^2]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha''/\gamma$. For $Q_{10}(\gamma)$ the definitions are $\delta = [\gamma^2 + (\alpha')^2]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha'/\gamma$.

For the three J functions; J_1 is given by Eq. (10.42), J_2 is given by Eq. (10.46), and J_3 is given by Eq. (10.49).

The component M_{10}''' may be written as

$$\left.\begin{aligned} M_{10}''' &= \frac{32\rho K_0^2 b^2 c^2}{\pi l^4}\int_0^d\int_{-l}^{l}(h')^2\,dh'\,df'\int_0^d\int_{h'}^{l} h\,dh\,df \\ &\times\int_0^{\frac{\pi}{2}}\sec^4\theta\, e^{-K_0(f+f')\sec^2\theta}\sin\left[K_0(h'-h)\sec\theta\right]d\theta \end{aligned}\right\} \quad (12.10)$$

and reduced to

$$\left.\begin{aligned} M_{10}''' = \frac{32\rho b^2 c^2}{\pi K_0^5 l^4}\Big\{ &\text{Imag. part of } \frac{2pe^{ip}}{(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_{11}(0)-2e^{-\beta p}Q_{11}(2\beta)+e^{-2\beta p}Q_{11}(4\beta)\right] \\ &+\text{Real part of } \frac{2e^{ip}}{(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_{12}(0)-2e^{-\beta p}Q_{12}(2\beta)+e^{-2\beta p}Q_{12}(4\beta)\right] \\ &+\frac{1}{3}\left(\frac{p^2}{4}+\frac{6}{5}\right)-\frac{p^2}{2}e^{-\beta p}L_{-6}(\beta p)-4e^{-\beta p}L_{-7}(\beta p) \\ &+\frac{p^2}{4}e^{-2\beta p}L_{-6}(2\beta p)+2e^{-2\beta p}L_{-7}(2\beta p)\Big\} \end{aligned}\right\}$$

(12.11)

where

$$
\left.
\begin{aligned}
Q_{11}(\gamma) = {} & \tfrac{1}{16}\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \left[\tfrac{3}{128}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \tfrac{171}{128}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p} \\
& + \left[\tfrac{105}{2048}\gamma^{2}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \tfrac{195}{1024}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \tfrac{657}{2048}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}\right. \\
& \left. - \delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta}\right]\frac{1}{p^{2}} - \left[\tfrac{10395}{49152}\gamma^{3}\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta} + \tfrac{12285}{16384}\gamma^{2}\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta}\right. \\
& + \tfrac{22995}{16384}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \tfrac{68085}{49152}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} - \tfrac{3}{8}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} \\
& \left. - \tfrac{31}{2}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p^{3}} + \cdots\cdots \\
Q_{12}(\gamma) = {} & \tfrac{3}{8}\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \left[\tfrac{9}{64}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \tfrac{51}{64}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p} \\
& + \left[\tfrac{315}{1024}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \tfrac{765}{512}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \tfrac{3195}{1024}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}\right. \\
& \left. - \delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta}\right]\frac{1}{p^{2}} - \left[\tfrac{10395}{8192}\gamma^{2}\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta}\right. \\
& + \tfrac{48195}{8192}\gamma^{2}\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta} + \tfrac{111825}{8192}\gamma\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} \\
& \left. + \tfrac{131985}{8192}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} - \tfrac{3}{8}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} - \tfrac{25}{8}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p^{3}} + \cdots\cdots
\end{aligned}
\right\} \quad (12.12)
$$

with $\delta = (\gamma^2 + 1)^{-\frac{1}{2}}$ and $\theta = \arctan 1/\gamma$.

The L functions in Eq. (12.11) are given by Eq. (12.4).

Finally

$$\left.\begin{aligned} M_{11}'' &= \frac{32\rho K_0^2 b^2 c^2}{\pi l^4} \int_0^d \int_{-l}^{l} (h')^2 \, dh' \, df' \int_0^d \int_{k-l}^{k+l} (h-k) \, dh \, df \\ &\quad \times \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-K_0(f+f')\sec^2\theta} \sin\left[K_0(h'-h)\sec\theta\right] d\theta \end{aligned}\right\} \qquad (12.13)$$

becomes

$$M_{11}'' = \frac{32\rho b^2 c^2}{\pi K_0^5 l^4} \left[J_{13} + J_{14} + J_{15} + J_{16} + J_{17}\right] \qquad (12.14)$$

where

$$\left.\begin{aligned} J_{13} &= \text{Real part of } \frac{p^2 e^{i\alpha p}}{(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \left[Q_{13}(0) - 2e^{-\beta p} Q_{13}(2\beta) + e^{-2\beta p} Q_{13}(4\beta)\right], \\ J_{14} &= -\text{Imag. part of } \frac{p^3 e^{-i\alpha' p}}{(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \left[Q_{14}(0) - 2e^{-\beta p} Q_{14}(2\beta) + e^{-2\beta p} Q_{14}(4\beta)\right], \\ J_{15} &= \text{Imag. part of } \frac{p^3 e^{i\alpha'' p}}{(2p)^{\frac{1}{2}}} \pi^{\frac{1}{2}} \left[Q_{15}(0) - 2e^{-\beta p} Q_{15}(2\beta) + e^{-2\beta p} Q_{15}(4\beta)\right], \end{aligned}\right\} \qquad (12.15)$$

$$\left.\begin{aligned} J_{16} &= \text{Real part of } \frac{p^2 e^{i\alpha' p}}{(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_{16}(0) - 2e^{-\beta p}Q_{16}(2\beta) + e^{-2\beta p}Q_{16}(4\beta)\right], \\ J_{17} &= \text{Real part of } \frac{p^2 e^{i\alpha'' p}}{(2p)^{\frac{1}{2}}}\pi^{\frac{1}{2}}\left[Q_{17}(0) - 2e^{-\beta p}Q_{17}(2\beta) + e^{-2\beta p}Q_{17}(4\beta)\right]. \end{aligned}\right\} \quad (12.15)$$

The $Q(\gamma)$ functions in Eq. (12.15) are defined as follows:

$$\left.\begin{aligned} Q_{13}(\gamma) &= \tfrac{1}{2}\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \left[\tfrac{3}{16}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \tfrac{17}{16}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p} \\ &+ \left[\tfrac{105}{256}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \tfrac{225}{128}\gamma\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \tfrac{1065}{256}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}\right. \\ &\left. + 4\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta}\right]\frac{1}{p^2} - \left[\tfrac{10395}{6144}\gamma^3\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta}\right. \\ &+ \tfrac{16065}{2048}\gamma^2\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta} + \tfrac{37275}{2048}\gamma\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} \\ &\left. + \tfrac{151485}{6144}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \tfrac{3}{2}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \tfrac{25}{2}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p^3} + \cdots \end{aligned}\right\} \quad (12.16)$$

where $\delta = (\gamma^2 + \alpha^2)^{-\frac{1}{2}}$ and $\theta = \arctan \alpha/\gamma$.

$$
\left.
\begin{aligned}
Q_{14}(\gamma) = {} & \tfrac{1}{8}\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta} - \left[\tfrac{3}{64}\gamma\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta} + \tfrac{13}{64}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta}\right]\frac{1}{p} \\
& + \left[\tfrac{105}{1024}\gamma^{2}\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta} + \tfrac{195}{512}\gamma\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} + \tfrac{657}{1024}\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta}\right. \\
& \left. - 2\delta^{\frac{1}{2}}e^{-\frac{1}{2}i\theta}\right]\frac{1}{p^{2}} - \left[\tfrac{10395}{24576}\gamma^{3}\delta^{\frac{13}{2}}e^{-\frac{13}{2}i\theta} + \tfrac{17285}{8192}\gamma^{2}\delta^{\frac{11}{2}}e^{-\frac{11}{2}i\theta}\right. \\
& + \tfrac{22995}{8192}\gamma\delta^{\frac{9}{2}}e^{-\frac{9}{2}i\theta} + \tfrac{68035}{24576}\delta^{\frac{7}{2}}e^{-\frac{7}{2}i\theta} \\
& \left. - \tfrac{3}{4}\gamma\delta^{\frac{5}{2}}e^{-\frac{5}{2}i\theta} - \tfrac{21}{4}\delta^{\frac{3}{2}}e^{-\frac{3}{2}i\theta}\right]\frac{1}{p^{3}} + \cdots\cdot
\end{aligned}
\right\} \quad (12.17)
$$

where $\delta = \left[\gamma^{2} + (\alpha')^{2}\right]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha'/\gamma$.

$Q_{15}(\gamma)$ is obtained from $Q_{14}(\gamma)$ by changing the sign of the exponent of the exponential factors and by using $\delta = \left[\gamma^{2} + (\alpha'')^{2}\right]^{-\frac{1}{2}}$ and $\theta = \arctan \alpha''/\gamma$ for δ and θ.

$$
\left.
\begin{aligned}
Q_{16}(\gamma) = {} & \tfrac{3}{4}\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \left[\tfrac{9}{32}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \tfrac{51}{32}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p} \\
& + \left[\tfrac{315}{512}\gamma^{2}\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta} + \tfrac{765}{256}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} + \tfrac{3195}{512}\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}\right. \\
& \left. - 2\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta}\right]\frac{1}{p^{2}} - \left[\tfrac{31185}{12288}\gamma^{3}\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta}\right.
\end{aligned}
\right\} \quad (12.18)
$$

$$+\frac{48195}{4096}\gamma^2\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta}+\frac{111825}{4096}\gamma\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta}$$

$$\left.+\frac{305955}{12288}\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta}-\frac{3}{4}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}-\frac{35}{4}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right]\frac{1}{p^3}+\cdots\cdots \Bigg)$$

where $\delta=\left[\gamma^2+(\alpha')^2\right]^{-\frac{1}{2}}$ and $\theta=\arctan \alpha'/\gamma$.

$Q_{17}(\gamma)$ is the same as $Q_{16}(\gamma)$ except for the definitions of δ and θ in which α'' is substituted for α', i.e., $\delta=\left[\gamma^2+(\alpha'')^2\right]^{-\frac{1}{2}}$ and $\theta=\arctan \alpha''/\gamma$.

XIII. Numerical Calculations

As mentioned in the introduction, as far as the effect of waves is concerned, the force and moment experienced by the trailing ship may be classed into three parts; the force and moment of the ship as if alone, the force and moment due to mutual interaction between the two ships, and the force and moment due to wave interference.

For the ship when alone, the horizontal force, or wave resistance, has been calculated and plotted by Havelock (4). It is an increasing oscillating function of the velocity. As the draft-length ratio is increased this force increases. To the authors' knowledge, the vertical force and moment, due to waves, on the ship when alone has neither been calculated nor plotted. The form of the functions involved in the vertical force and moment are very similar to those in the expression for the horizontal force. Thus it is to be expected that the vertical force and moment are also increasing oscillating functions of the velocity.

The force and moment components which are due to mutual interaction vary inversely as the square of the distance between the two ships, and hence, they may be expected to diminish very rapidly with the distance between the two ships. For any practical distance between the two ships, these components can be neglected.

Due to the motivation of this problem, viz., the problem of the effect of a leading ship on a trailing ship; of major interest here are the force and moment components due to wave interference. The significant ones are designated by X_6'', Z_5'' and M_{61}'', M_{62}'', M_{63}'', M_{11}''. The interpretation of these various components in terms of the distribution of sources and sinks which replace the ship is given in Section VII and VIII. The numerical calculations and comments are confined to these force and moment components.

The parameters involved in the numerical calculations are: (1) the ratio of the distance between ship centers and ship-length, $\alpha = L/2l$, and the corresponding quantities $\alpha' = L_1/2l = \alpha - 1$ and $\alpha'' = L_2/2l = \alpha + 2$; (2) draft ship-length ratio $\beta = d/2l$; (3) ratio of the distance from the origin of the rectangular axes to the center of mass of the ship and the ship-length, $\epsilon/2l$; and (4) the inverse of the square of Froude's number, $p = 1/F^2$, where $F = c/\sqrt{2gl}$. The values chosen for these various parameters are as follows:

$$\alpha = 2 \text{ through } 7$$
$$\beta = 0.1$$
$$\epsilon/2l = 1/40$$
$$p = 20$$

Omitting the details, Figures (13.1) through (13.7) show the variation of the various components listed above with respect to the spacing between ships. The curves are all of an oscillatory nature and are camped very slowly with respect to spacing between ships. The frequency per ship-length of the oscillations in the curves remains practically constant with increasing distance between ships. It is surprising that even for a relatively large distance of six ship-lengths, where one might expect negligible wave interference effect, there is an appreciable value for the separate components.

To obtain the relative magnitude of the wave interference effect on the trailing ship, consider X_6''. From Fig. (13.1), the value for the wave interference effect at a ship spacing of one ship-length is about $0.00375\,\rho b^2 g\,l$. At a spacing of six ship-lengths, the value decreases to about $0.002\,\rho b^2 g\,l$. The wave-resistance, for the same parameters, for the trailing ship as if alone is approximately equal to $0.005\,\rho b^2 g\, l$ (4). Thus this component of the wave interference effect at one and six ship-lengths is about 75% and 40%, respectively, of the wave-resistance of the ship as if alone. Then, since the wave interference effect is oscillatory, by a judicious spacing of the two ships, there is an appreciable reduction in the wave-resistance of the trailing ship. For the other force and moment components, one may expect similar relative magnitudes.

At the speed chosen in the calculations, all the curves oscillate at the rate of about three cycles per ship-length. The equations for the various forces and moments indicate that as the speed increases, the rate of oscillation of the curves decreases. Thus, at a low speed the wave interference effect on the trailing ship may change three times, per ship length spacing, from a positive to a negative value. At a higher speed the change from a positive to negative value may occur only once per ship-length spacing between ships. This would indicate that if there is to be any sustained saving in power which would result from reduced resistance, it would more likely be at the higher speeds.

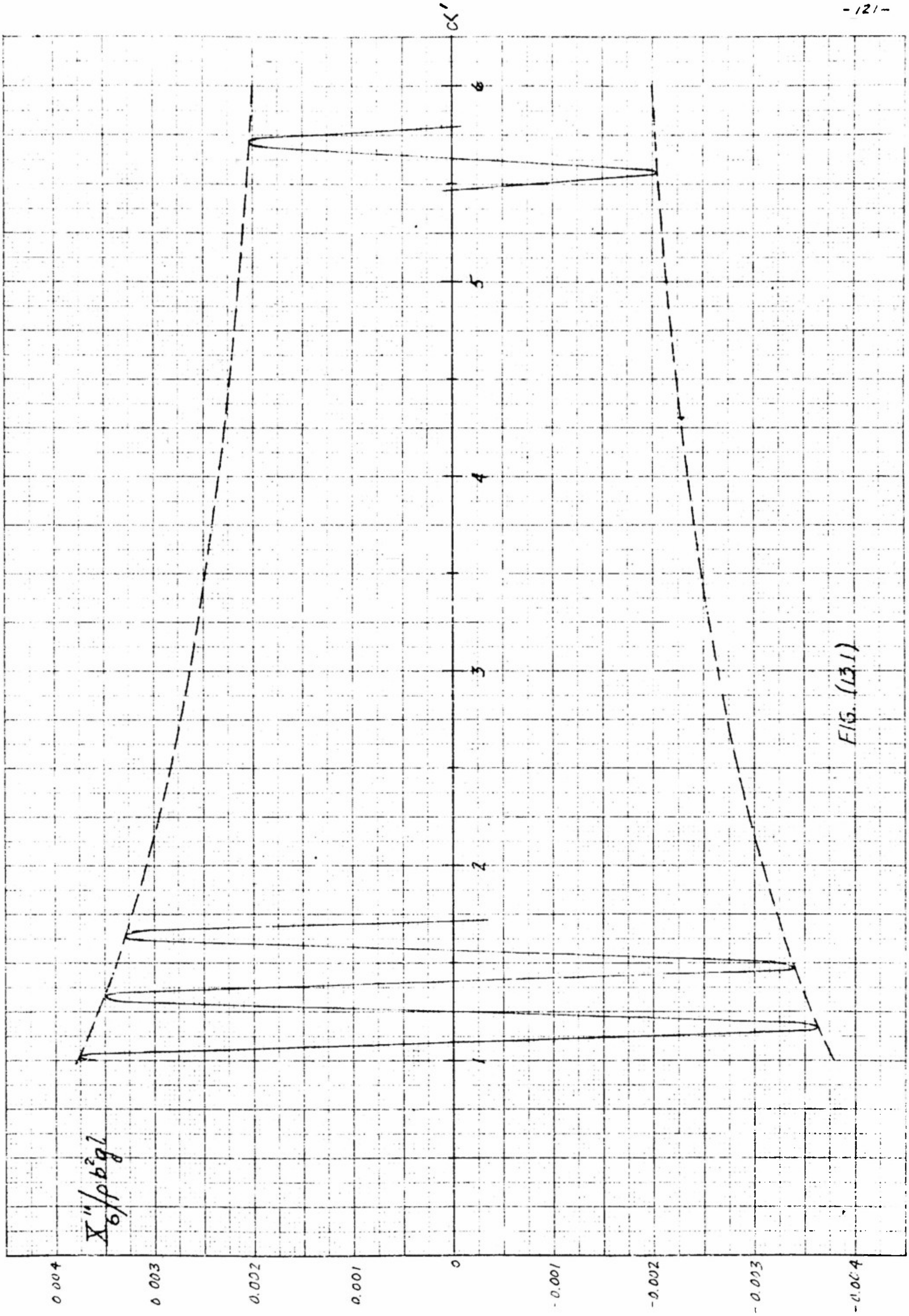
$X''_6/\rho b^2 g l$
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
-0.004
1
2
3
4
5
6
α'

FIG. (13.1)

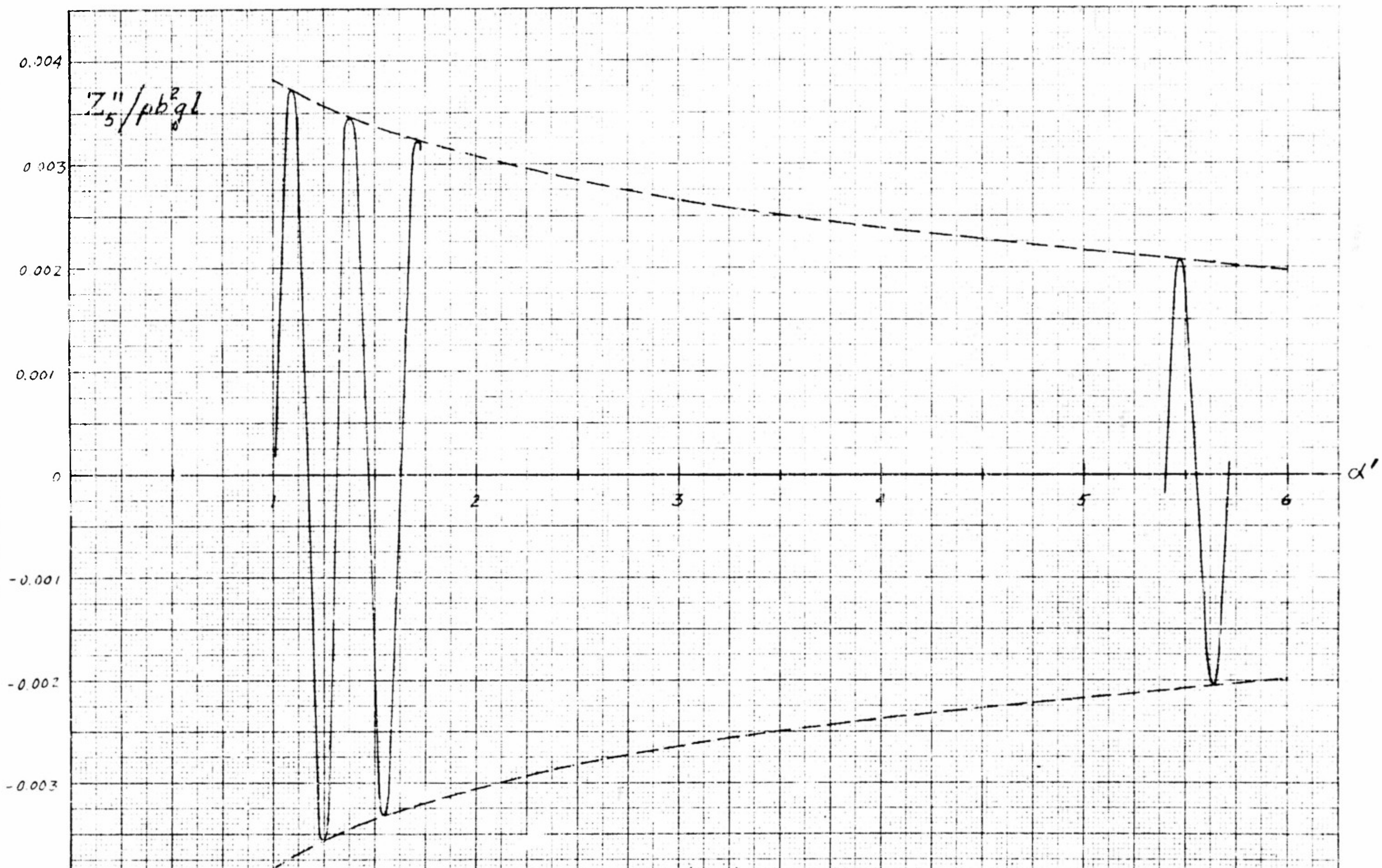
$Z''_5/\rho b_0^2 g l$
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
-0.004
1
2
3
4
5
6
α'

FIG. (13.2)

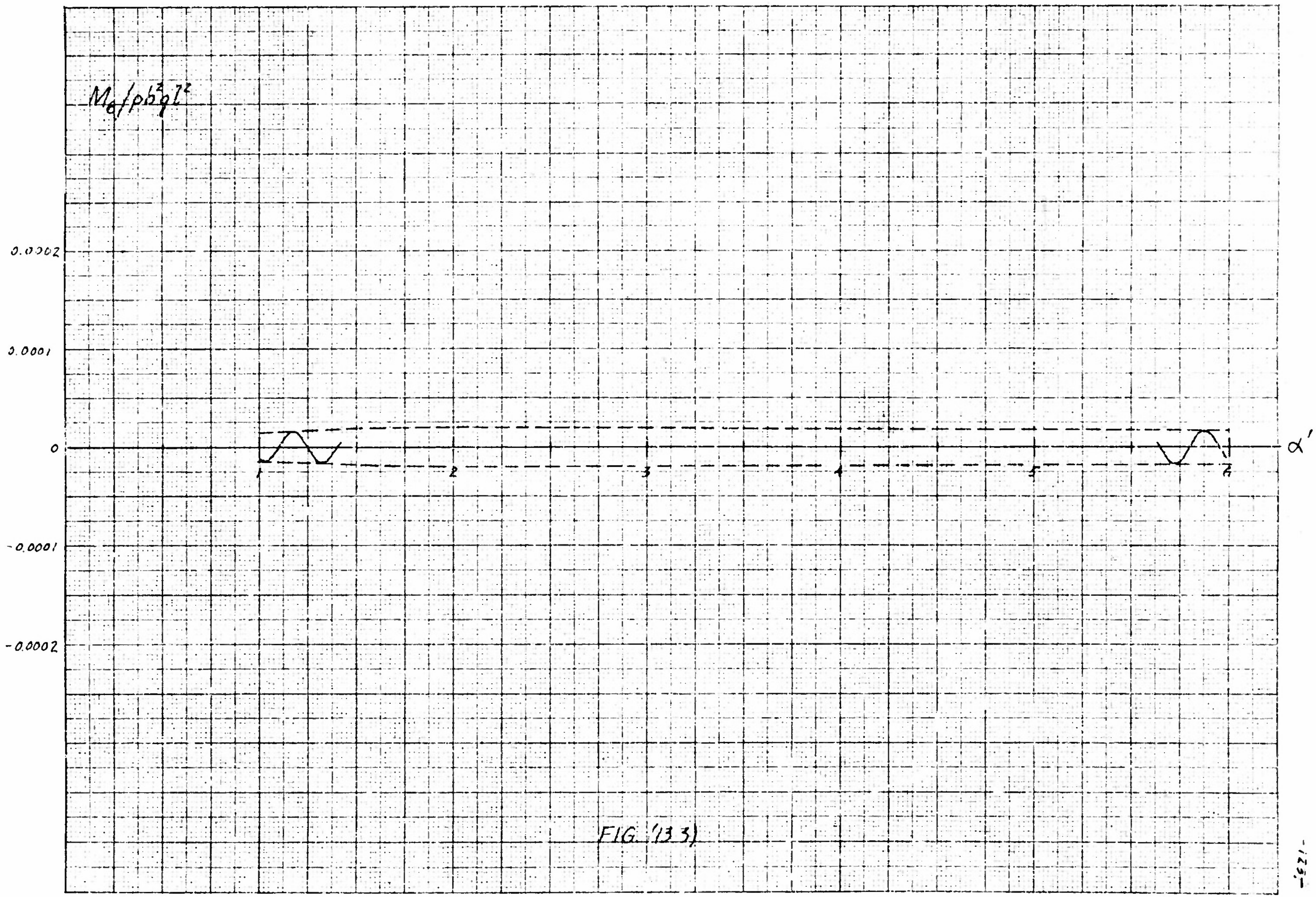

FIG. (13.3)

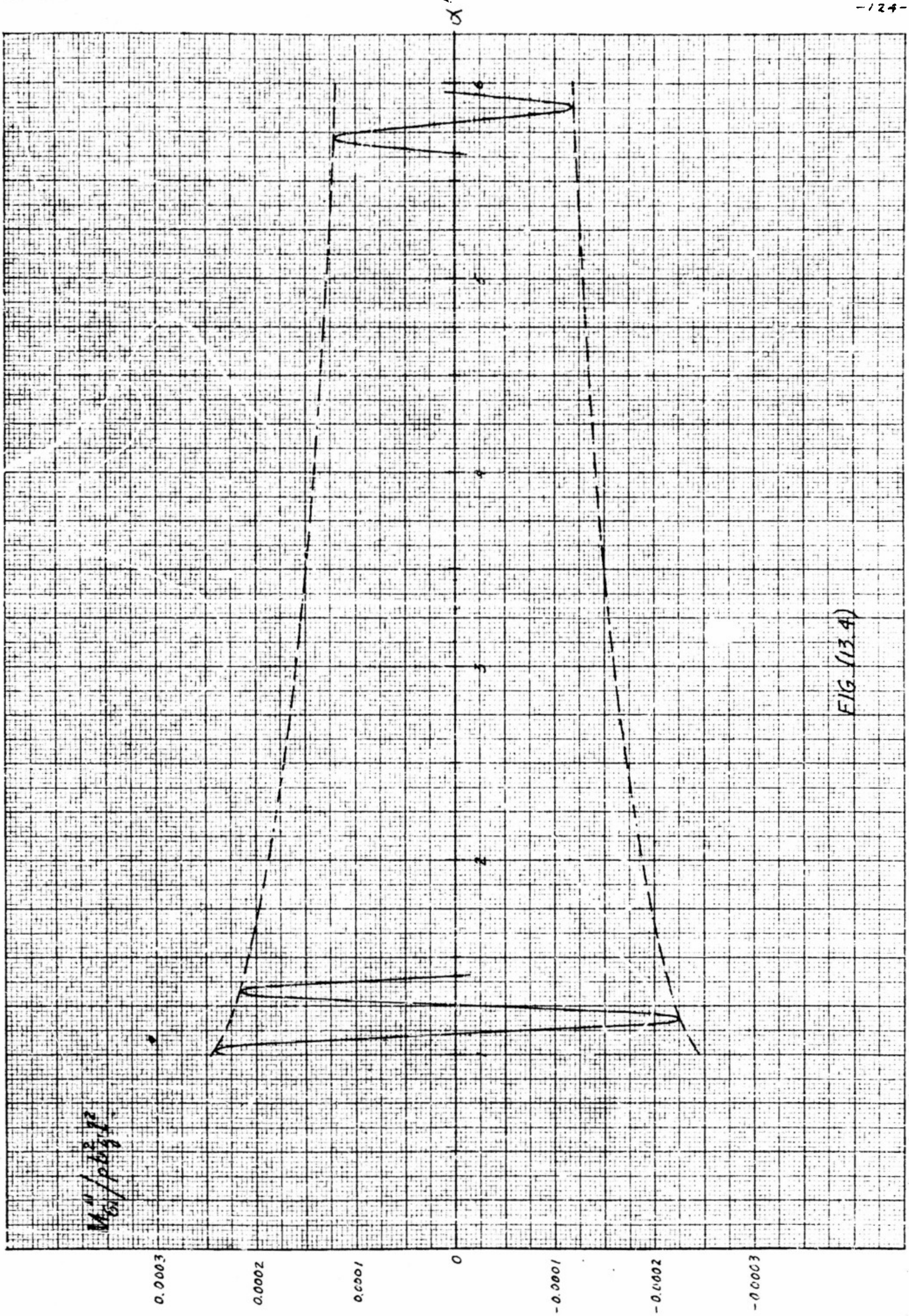

FIG (13.4)

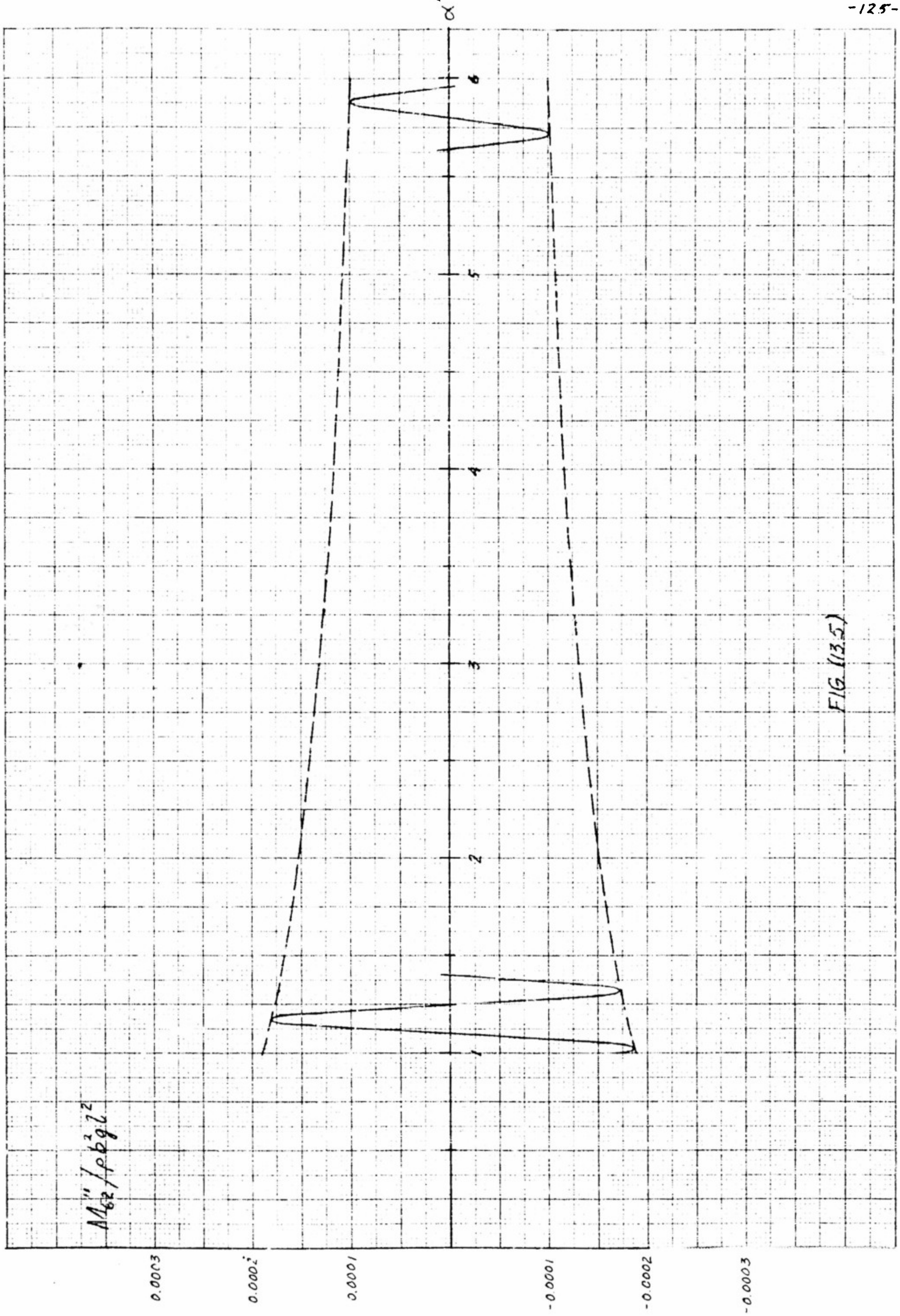

FIG (13.5)

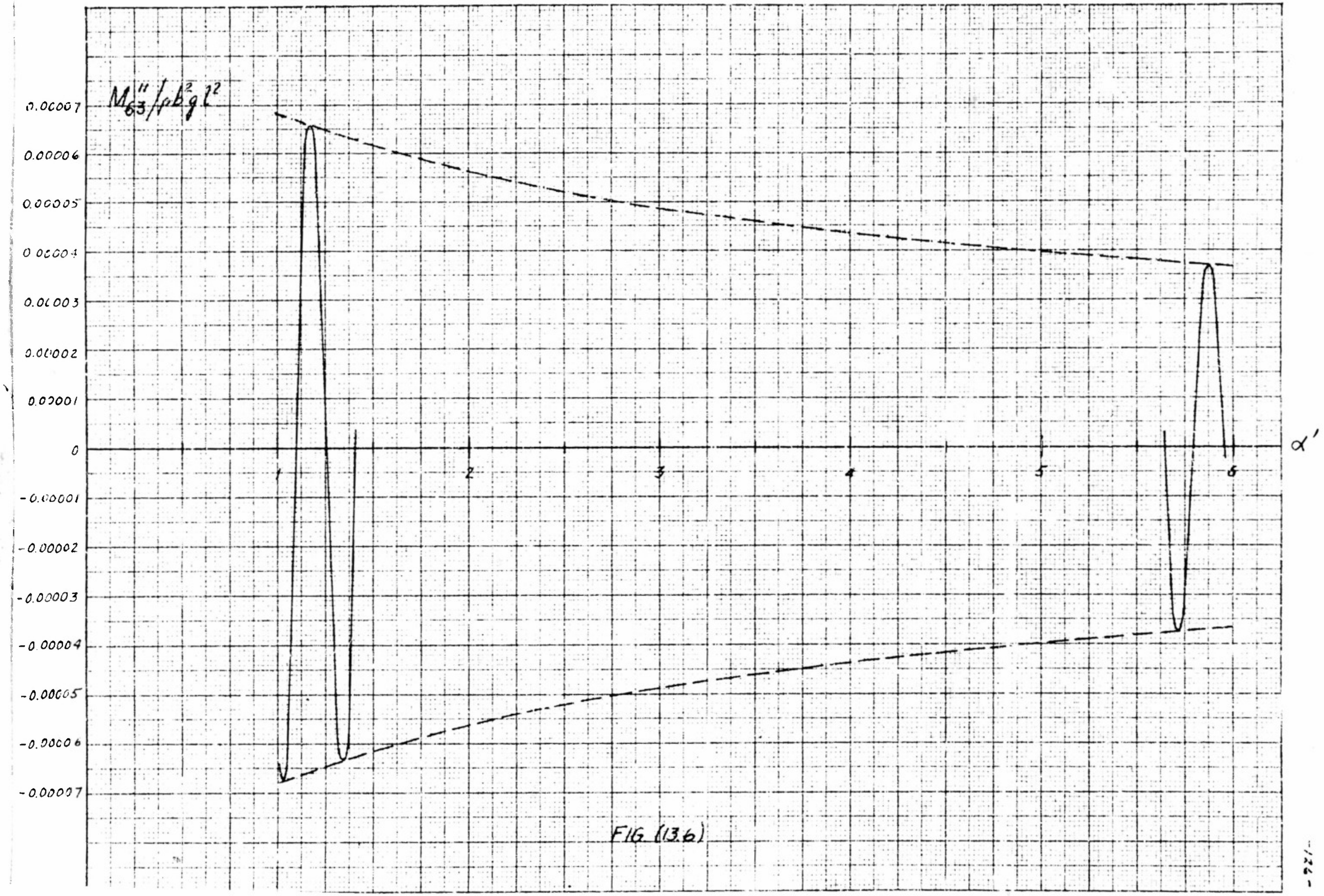
$M''_{63}/\rho b^2 g l^2$
0.00007
0.00006
0.00005
0.00004
0.00003
0.00002
0.00001
0
-0.00001
-0.00002
-0.00003
-0.00004
-0.00005
-0.00006
-0.00007
1
2
3
4
5
6
α'

FIG (13.6)

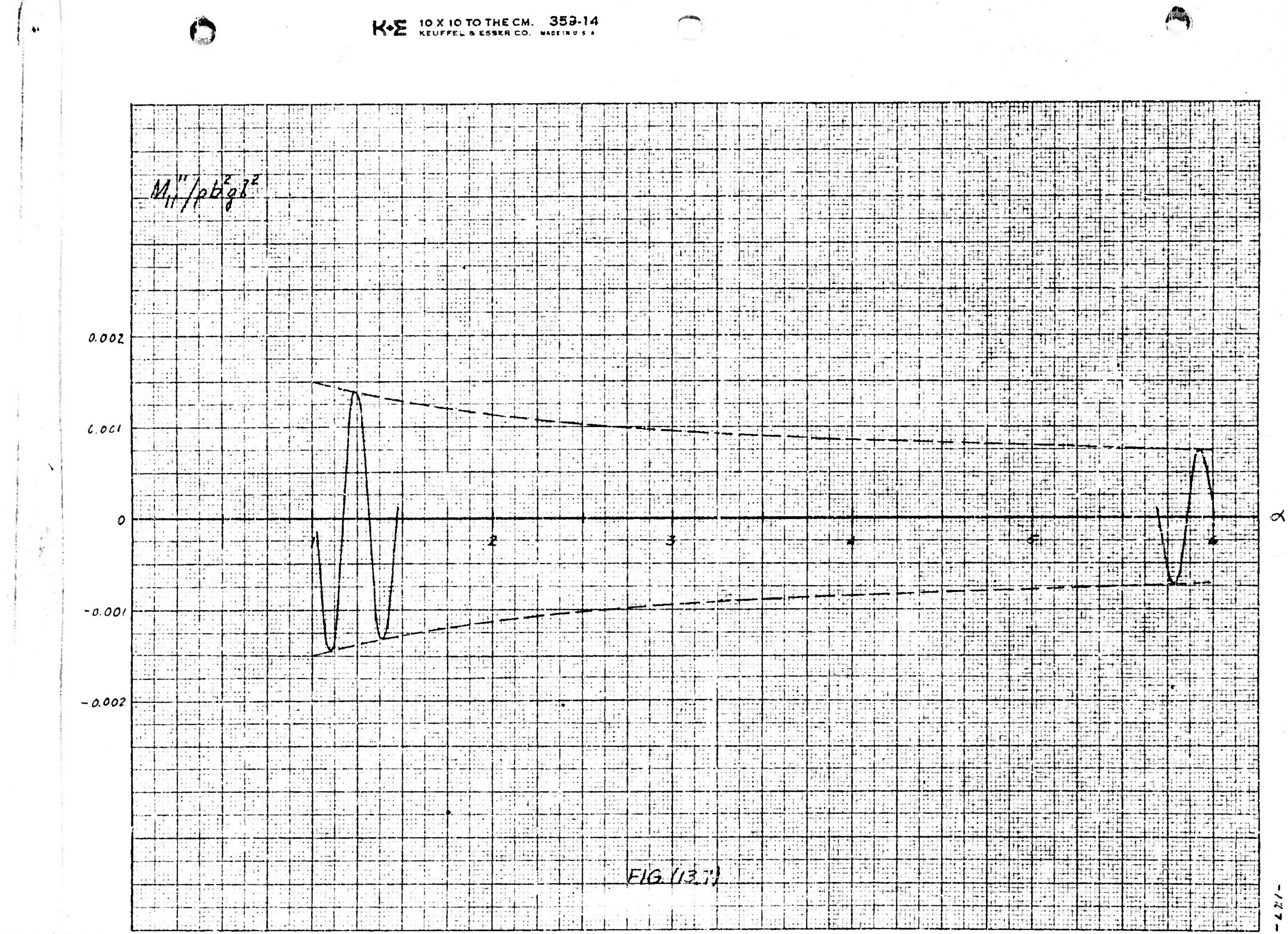
K+E 10 X 10 TO THE CM. 359-14
KEUFFEL & ESSER CO. MADE IN U.S.A.
$M_{11}''/pb^2q^2\gamma^2$
0.002
0.001
0
-0.001
-0.002
1
2
3
4
5
6
α'

FIG (13.7)

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