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SIMULTANEOUS NONLINEAR
EQUATIONS OF GROWTH

Dunham Laboratory
Yale University
New Haven, Connecticut

SIMULTANEOUS NONLINEAR EQUATIONS OF GROWTH

W. J. Cunningham

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Report No. 7

Dunham Laboratory
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New Haven, Connecticut

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Preface

This report is the seventh concerned with research accomplished in connection with Navy Contract Nonr-433(00), between Dunham Laboratory, Yale University, and the Office of Naval Research, Department of the Navy. In this report is given a discussion of the solutions for a pair of simultaneous nonlinear differential equations that may apply to phenomena of interest. These equations are studied analytically, and particular examples are solved with an analog computer.

The research was carried on and the report written by the undersigned.

W. J. Cunningham

New Haven, August 1954

Table of Contents

List of Figures and Tables

Abstract

A pair of simultaneous nonlinear equations

$$dx/dt = (\alpha_x/k_x)\left[k_x - x - f_x(y)\right]x$$

$$dy/dt = (\alpha_y/k_y)\left[k_y - y - f_y(x)\right]y$$

may represent problems of interest involving certain biological or physical phenomena. These equations, together with several special cases, are investigated analytically and information about their solutions is obtained. A variety of different solutions can occur, dependent upon the coefficients in the equations and upon the coupling functions f_x and f_y. Criteria are developed from which the properties of the solutions can be predicted. Several numerical examples are solved, making use of an electronic analog computer.

I. Simultaneous Growth Equations

A problem which has attracted the attention of both mathematicians and biologists is that of describing mathematically the effects of environment upon the population of a species of animal. One phase of this problem involves the effect of competition between two different species. Where the two species subsist on a common food supply, or where one species is eaten by the other, or where additional effects exist so that the population of one species influences that of the other, a mathematical description of the situation must require simultaneous differential equations. In general, these equations are nonlinear and are more or less complicated depending upon the number of effects that are considered.

Volterra[1] has studied in some detail the case where the competition equations are

$$dx/dt = \dot{x} = (\alpha_x/k_x)(k_x - \beta_x y)x$$
$$dy/dt = \dot{y} = (\alpha_y/k_y)(k_y - \beta_y x)y$$

Here, t is the independent variable time, x and y are dependent variables representing the two populations, and k_x, α_x, β_x, k_y, α_y, β_y are real constants. Coupling between the variables comes about through the nonlinear product term, xy, in each equation. The variables can be separated in these equations and a solution found by a process partly analytical and partly graphical. The nature of the solution depends upon the coefficients in the equations. Among the possibilities are that one species disappears leaving the other, or that oscillations occur in the populations.

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1. V. Volterra, La Lutte pour la Vie, (Gauthier-Villars, Paris, 1931), ch. I

Somewhat more complicated equations have been investigated by Gause and Witt[2], as

$$\dot{x} = (\alpha_x/k_x)(k_x - x - \beta_x y)x$$
$$\dot{y} = (\alpha_y/k_y)(k_y - y - \beta_y x)y.$$

The extra terms in these equations provide a sort of damping effect and insure that neither population goes to infinity. The method of analysis used by Volterra is no longer applicable. Again, a variety of solutions is possible, depending upon the coefficients.

A still more complicated pair of equations has been suggested by Hutchinson[3] as

$$\dot{x} = (\alpha_x/k_x)\left[k_x - x - f_x(y)\right]x$$
$$\dot{y} = (\alpha_y/k_y)\left[k_y - y - f_y(x)\right]y$$

where $f_x(y)$ and $f_y(x)$ are the functions coupling one variable to the other. These functions are generally well-behaved mathematically, and it is required that $f_x(0) = 0$ and $f_y(0) = 0$. The functions might be polynomials of the sort

$$f_x(y) = \beta_x y + \gamma_x y^2 + \delta_x y^3$$
$$f_y(x) = \beta_y x + \gamma_y x^2 + \delta_y x^3.$$

The equations of Gause and Witt contain only the first terms of these polynomials.

Equations of this same general sort might arise in describing other types of physical phenomena. For example, certain kinds of chemical reactions progress at a rate that depends upon the amount of each component present in the reaction. Simultaneous equations describing the amount of each component are quite similar to the competition equations. Similarly, an electrical system can be conceived which also

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2. G. F. Gause and A. A. Witt, American Naturalist, 69, 596, (1935)
3. G. E. Hutchinson, Ecology, 28, 319, (1947)

would be described by the equations. The two voltages of a pair of d-c generators might build up in this way if the field windings were suitably connected. Two field windings on each machine would be required, one excited by the machine itself and the other excited from a cross connection with the second machine.

In general, the equations apply to any situation where two effects tend to grow with time, but the rate of growth of each is influenced in some way by the other.

II. Most General Form of Equations

II.1 Analysis of equations

The pair of simultaneous equations that are considered here are of the most general form

$$dx/dt = \dot{x} = (\alpha_x/k_x)\left[k_x - x - f_x(y)\right]x \qquad (1)$$

$$dy/dt = \dot{y} = (\alpha_y/k_y)\left[k_y - y - f_y(x)\right]y. \qquad (2)$$

In these equations, t is the independent variable and usually represents time, x and y are the two dependent variables, and α_x, α_y, k_x, k_y are real constants. Functions $f_x(y)$ and $f_y(x)$ are continuous, single-valued, well-behaved functions that can be differentiated with respect to their arguments. Both functions vanish for zero argument, $f_x(0) = 0$ and $f_y(0) = 0$. In many cases the functions take the form of polynomials such as

$$f_x(y) = \beta_x y + \gamma_x y^2 \qquad (3)$$

$$f_y(x) = \beta_y x + \gamma_y x^2 \qquad (4)$$

where β_x, β_y, γ_x, γ_y are real constants. It is through these functions that coupling exists between the two dependent variables.

If there is no coupling between variables x and y, $f_x(y) \equiv 0$, and Eq. (1) becomes the Verhulst-Pearl equation[4]

$$\dot{x} = (\alpha_x/k_x)(k_x - x)x. \tag{5}$$

This is an example of a Bernoulli equation and has the exact solution

$$x = \left[k_x^{-1} + (x_o^{-1} - k_x^{-1}) \exp(-\alpha_x t)\right]^{-1} \tag{6}$$

where $x = x_o$ at $t = 0$. Curves for $\dot{x}$ as a function of x and for x as a function of t are shown in Fig. 1, with several initial conditions. If $|x_o| << |k_x|$, approximately $x = x_o \exp(\alpha_x t)$ and the solution starts off as it would for a simple growth equation having a constant growth factor α_x. With α_x positive, x always approaches the value k_x, regardless of whether x_o is positive or negative. If x_o is negative, x passes through infinity and becomes positive so as to approach a positive k_x. The value $x = 0$ is thus a point of unstable equilibrium, while the value $x = k_x$ is a point of stable equilibrium.

With no coupling, $f_y(x) \equiv 0$, also, and exactly the same sort of solution applies for the equation in y as has just been discussed for the equation in x.

In the more general case where coupling between the variables is present, it is necessary to consider the complete form of Eqs. (1) and (2). The nature of the possible solutions for these equations is most easily studied by considering the single equation obtained as their ratio,

$$\frac{dy}{dx} = \frac{(\alpha_y/k_y)\left[k_y - y - f_y(x)\right]y}{(\alpha_x/k_x)\left[k_x - x - f_x(y)\right]x}. \tag{7}$$

The independent variable t has disappeared in writing this equation.

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4. A. J. Lotka, Elements of Physical Biology, (Williams and Wilkins, Baltimore, 1925), p. 64

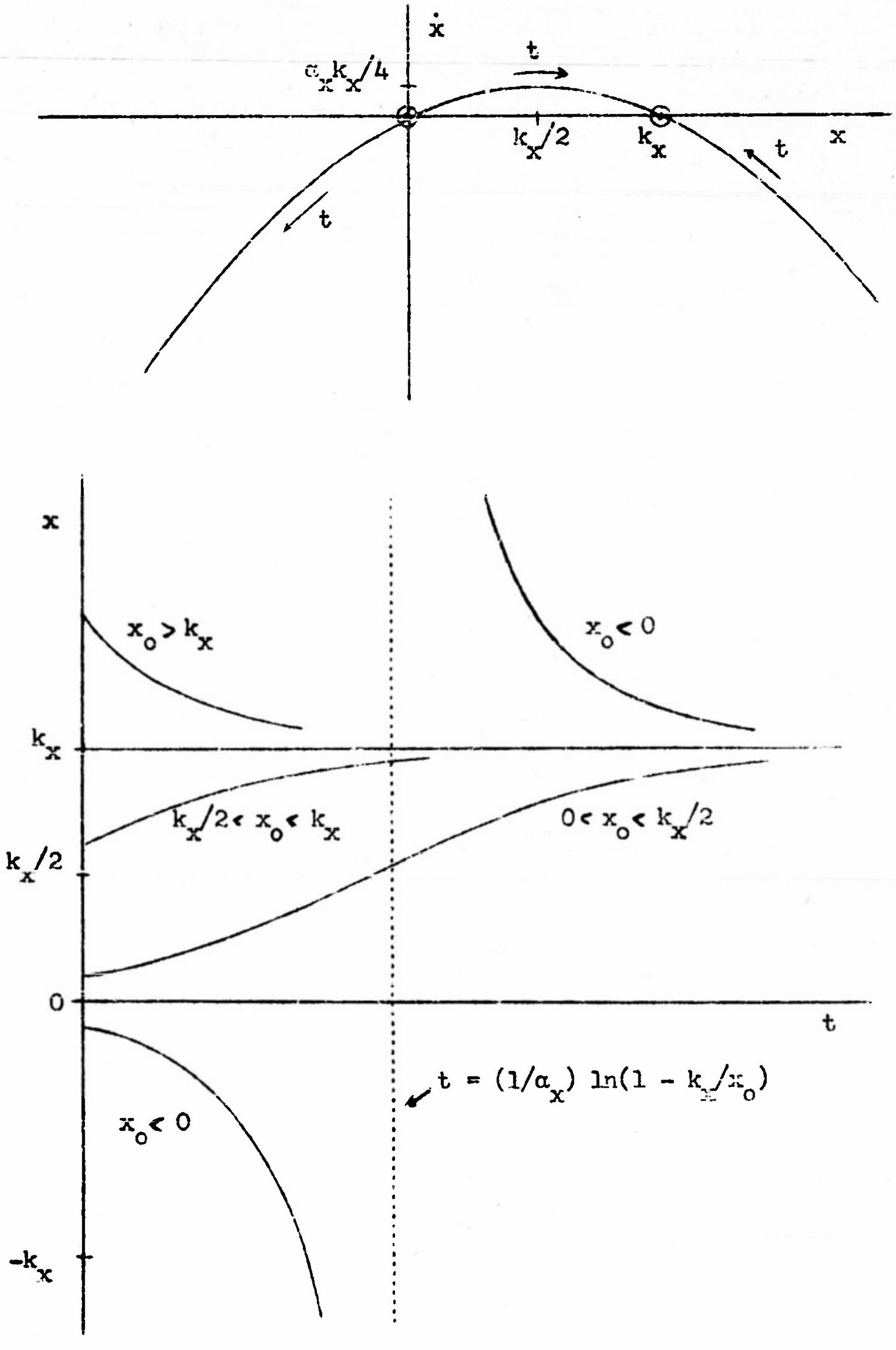

Fig. 1 Solutions for Eq. (5), $\alpha_x > 0$, $k_x > 0$.

There are certain singular points[5] for Eq. (7) corresponding to points of equilibrium, where both numerator and denominator vanish simultaneously. The values of x and y at one of these singularities are designated as x_s and y_s. The singularities can be separated into four groups as follows.

$$\text{Gr. 1.} \quad x_s = 0, \quad y_s = 0$$

$$\text{Gr. 2.} \quad x_s = k_x, \quad y_s = 0$$

$$\text{Gr. 3.} \quad x_s = 0, \quad y_s = k_y$$

$$\text{Gr. 4.} \quad x_s = k_x - f_x(y), \quad y_s = k_y - f_y(x)$$

The first three of these singularities are reminiscent of those which would occur for a pair of Verhulst-Pearl equations with no coupling. The fourth group may contain none, or a number of singularities, depending upon the properties of functions f_x and f_y.

The nature of solutions for Eq. (7) near each singular point can be explored by replacing x with $(x_s + u)$ and y with $(y_s + v)$, where u and v are small changes. This substitution gives the equation

$$\frac{dv}{du} = \frac{(\alpha_y/k_y)\left[k_y y_s + k_y v - y_s^2 - 2y_s v - y_s f_y(x_s) - y_s f_y'(x_s)u - f_y(x_s)v\right]}{(\alpha_x/k_x)\left[k_x x_s + k_x u - x_s^2 - 2x_s u - x_s f_x(y_s) - x_s f_x'(y_s)v - f_x(y_s)u\right]} \qquad (8)$$

In writing this equation, series expansions for the coupling functions have been used,

$$f_x(y_s + v) = f_x(y_s) + f_x'(y_s)v + \cdots$$

$$f_y(x_s + u) = f_y(x_s) + f_y'(x_s)u + \cdots$$

with

$$f_x'(y_s) = d/dy\left[f_x(y_s)\right]$$

$$f_y'(x_s) = d/dx\left[f_y(x_s)\right].$$

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5. N. Minorsky, Nonlinear Mechanics, (J. W. Edwards, Ann Arbor, 1947), Part I.

Only linear terms in u and v have been retained. At any singular point, certain terms vanish,

$$\left[k_x - x_s - f_x(y_s)\right]x_s = 0$$
$$\left[k_y - y_s - f_y(x_s)\right]y_s = 0$$

so that Eq. (8) becomes

$$\frac{dv}{du} = \frac{(\alpha_y/k_y)\left\{\left[-y_s f_y'(x_s)\right]u + \left[k_y - 2y_s - f_y(x_s)\right]v\right\}}{(\alpha_x/k_x)\left\{\left[k_x - 2x_s - f_x(y_s)\right]u + \left[-x_s f_x'(y_s)\right]v\right\}} \; . \tag{9}$$

This equation is of the form

$$\frac{dv}{du} = \frac{Au + Bv}{Cu + Dv} \tag{10}$$

where

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) \tag{11}$$
$$B = (\alpha_y/k_y)\left[k_y - 2y_s - f_y(x_s)\right] \tag{12}$$
$$C = (\alpha_x/k_x)\left[k_x - 2x_s - f_x(y_s)\right] \tag{13}$$
$$D = -(\alpha_x/k_x)x_s f_x'(y_s) \; . \tag{14}$$

The nature of solutions for Eq. (9), and also of solutions near the singularities of Eq. (7), depends upon the characteristic exponents

$$(\lambda_1,\lambda_2) = (1/2)\left[(B + C) \pm U^{1/2}\right] \tag{15}$$

where $U = (B + C)^2 + 4(AD - BC)$. It is necessary to examine these exponents near each of the singularities.

<u>Gr. 1. $x_s = 0$, $y_s = 0$</u>

For this singularity

$$A = 0 \qquad C = \alpha_x$$
$$B = \alpha_y \qquad D = 0$$
$$(B + C) = (\alpha_x + \alpha_y)$$
$$(AD - BC) = -\alpha_x\alpha_y$$
$$U = (B - C)^2 \geq 0$$

If $\alpha_x\alpha_y < 0$, a saddle point exists; if $\alpha_x\alpha_y > 0$, a nodal point exists, stable if both α_x and α_y are negative and unstable if both α_x and α_y are positive. The situation is summarized in the stability diagram of Fig. 2.

Gr. 2. $x_s = k_x$, $y_s = 0$

For this singularity

$$A = 0$$

$$B = (\alpha_y/k_y)Y \quad \text{where} \quad Y = k_y - f_y(k_x)$$

$$C = -\alpha_x$$

$$D = -\alpha_x f_x'(0)$$

$$(B + C) = (\alpha_y Y/k_y - \alpha_x)$$

$$(AD - BC) = \alpha_x\alpha_y Y/k_y$$

$$U = (B - C)^2 \geq 0$$

In this case, the quantity Y is important. It can be found easily from a graphical plot relating to Eq. (7). In Fig. 3 is plotted the curve $y = k_y - f_y(x)$ upon axes of x and y. The curve is the locus of those values of x and y which make the bracket in the numerator of Eq. (7) vanish. Any curve representing a solution for Eq. (7) must cross this locus with horizontal slope. Thus, this curve is the isocline for zero slope, $dy/dx = 0$, or for $\dot{y} = 0$ in Eq. (2). The ordinate for this curve, evaluated at $x = k_x$, is the quantity Y, and is positive in Fig. 3.

If $\alpha_x\alpha_y Y/k_y > 0$, a saddle point exists; if $\alpha_x\alpha_y Y/k_y < 0$, a nodal point exists, stable if $(B + C) < 0$ and unstable if $(B + C) > 0$. The situation is summarized in the stability diagram of Fig. 4.

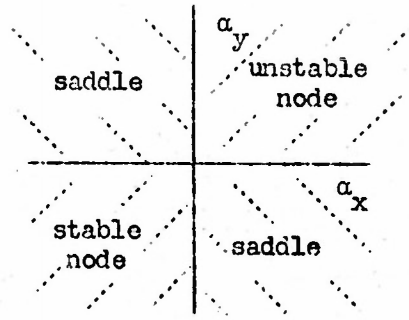


Fig. 2 Stability of Gr. 1, Eqs (1-2)

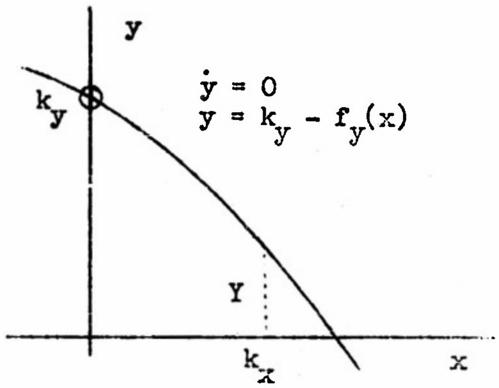


Fig. 3 Determination of Y

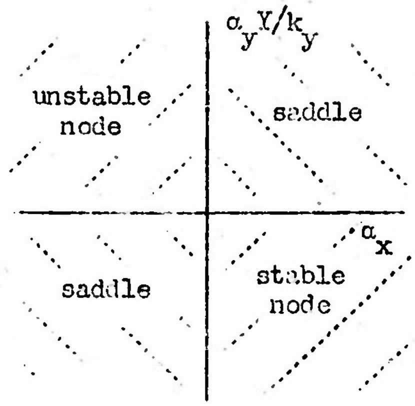


Fig. 4 Stability of Eqs.(1-2),Gr.2

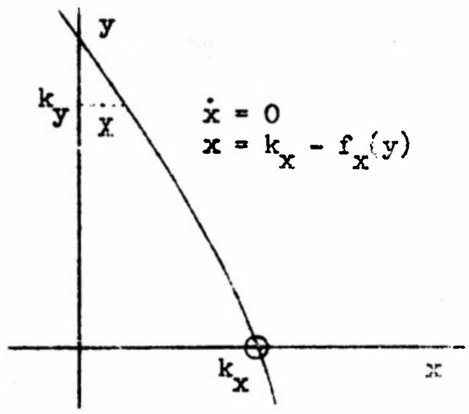


Fig. 5 Determination of X

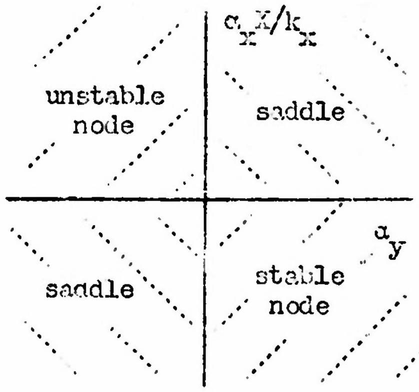


Fig. 6 Stability of Eqs.(1-2), Gr.3

Gr. 3. $x_s = 0$, $y_s = k_y$

For this singularity

$$A = -\alpha_y f_y'(0)$$

$$B = -\alpha_y$$

$$C = (\alpha_x/k_x)X \quad \text{where} \quad X = k_x - f_x(k_y)$$

$$D = 0$$

$$(B + C) = (\alpha_x X/k_x - \alpha_y)$$

$$(AD - BC) = \alpha_x \alpha_y X/k_x$$

$$U = (B - C)^2 \geq 0$$

This case is analogous to the one just preceding. Here, a plot of the curve $x = k_x - f_x(y)$ as shown in Fig. 5 is useful. This curve is the isocline for infinite slope of a solution curve for Eq. (7), $dy/dx = \infty$, or for $\dot{x} = 0$ in Eq. (1). The abscissa for this curve, evaluated at $y = k_y$, is the quantity X, and is positive in Fig. 5. The situation is summarized in the stability diagram of Fig. 6.

Gr. 4. $x_s = k_x - f_x(y_s)$, $y_s = k_y - f_y(x_s)$

The number of singularities determined by these relations depends upon the functions f_x and f_y. If these functions are nonlinear, as for example Eqs. (3) and (4), numerical determination of x_s and y_s may be a fairly tedious process, requiring the solution for the roots of an equation of high degree. Probably a simpler and more informative approach is to resort to another gra ical construction.

Singular points in general are located at the intersection of isocline curves corresponding to different slopes, i.e. to different values of dy/dx. Thus, if the isoclines for zero slope, $\dot{y} = 0$ or $y = k_y - f_y(x)$, and the isoclines for infinite slope, $\dot{x} = 0$ or

$x = k_x - f_x(y)$, are plotted, singularities are found at their intersections. The number and location of the intersections will depend upon f_x and f_y.

Quadratic functions, such as Eqs. (3) and (4), lead to parabolic curves for the isoclines, as in the example of Fig. 7. This figure is like a combination of Figs. 3 and 5, but here the functions are such that both X and Y are negative. A maximum of four intersections, and thus four singularities, can occur with these parabolas. If the coefficients of Eqs. (3) and (4) are allowed to vary, the parabolic curves change in shape and location, and the number of singularities may be any integer from zero to four inclusive. More complicated forms of functions f_x and f_y might lead to even more singularities.

It is worth noting that in Fig. 7, the horizontal axis corresponds to an isocline where $\dot{y} = 0$, and the vertical axis corresponds to an isocline where $\dot{x} = 0$. The intersections of the axes with each other and with the parabolic curves lead to the first three singularities.

For the singularities of the fourth group,

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) = -SM$$
$$B = -(\alpha_y/k_y)y_s = -S$$
$$C = -(\alpha_x/k_x)x_s = -R$$
$$D = -(\alpha_x/k_x)x_s f_x'(y_s) = -RN$$

where

$$R \equiv \alpha_x x_s/k_x \qquad G \equiv R/S = \alpha_x k_y x_s/\alpha_y k_x y_s$$
$$S \equiv \alpha_y y_s/k_y$$
$$M \equiv f_y'(x_s) = d/dx\left[f_y(x_s)\right]$$
$$N \equiv f_x'(y_s) = d/dy\left[f_x(y_s)\right]$$

Quantities R and S can be found from known values of α_x, α_y, k_x, and k_y together with values of x_s and y_s taken from the

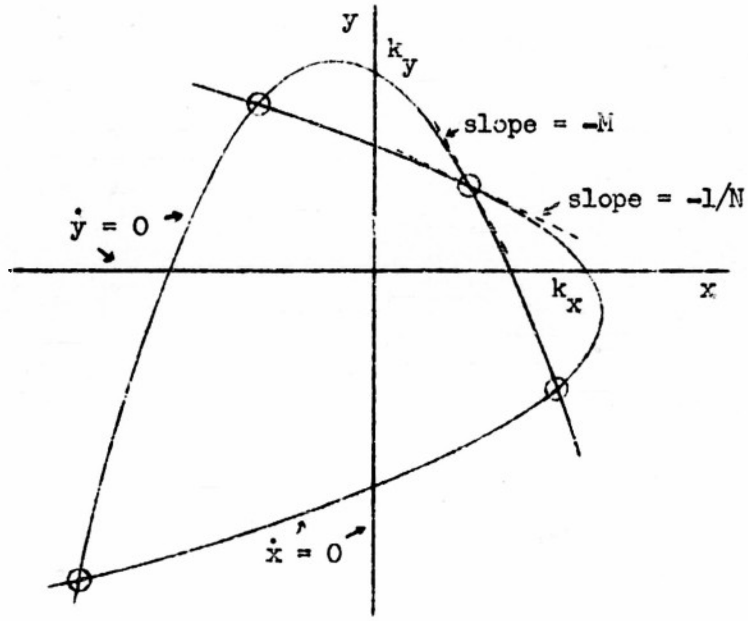


Fig. 7 Isoclines for Eqs. (1-2); determination of M and N

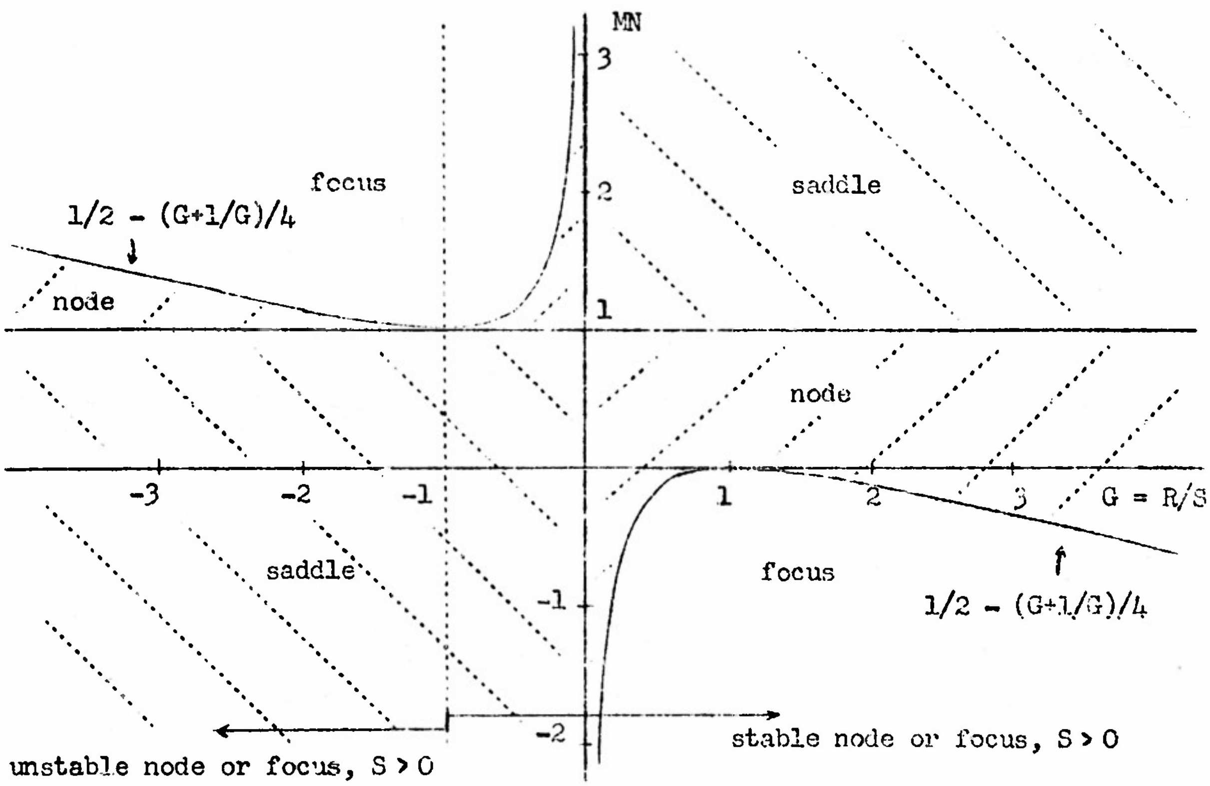


Fig. 8 Stability of Eqs. (1-2), Gr. 4

graphical construction. Quantities M and N can be found from the slopes of the isocline curves for $\dot{y} = 0$ and $\dot{x} = 0$, as measured at the singularity. The isocline for $\dot{y} = 0$ is $y = k_y - f_y(x)$, which has the slope $dy/dx = -d/dx\left[f_y(x)\right] = -f_y'(x)$. This slope evaluated at the singularity is $dy/dx\Big]_{(x = x_s)} = -f_y'(x_s) = -M$. Similarly, the isocline for $\dot{x} = 0$ is $x = k_x - f_x(y)$ with the slope $dy/dx = (dx/dy)^{-1} = \left\{-d/dy\left[f_x(y)\right]\right\}^{-1} = \left[-f_x'(y)\right]^{-1}$. This slope, evaluated at the singularity is $dy/dx\Big]_{(y = y_s)} = \left[-f_x'(y_s)\right]^{-1} = -1/N$. Thus, M and N can be determined from direct measurement of the slopes of the curves, or by numerical substitution in the derivatives of the isocline curves, $M = f_y'(x_s)$ and $N = f_x'(y_s)$.

Important combinations of the coefficients are the following.

$$(B + C) = -(R + S) = -S(G + 1)$$

$$(AD - BC) = RS(MN - 1) = S^2G(MN - 1)$$

$$U = (R - S)^2 + 4RSMN = 4S^2G\left[(G + 1/G)/4 - 1/2 + MN\right]$$

The following solutions may exist.

a. <u>Saddle</u>

$$G(MN - 1) > 0$$

b. <u>Node</u>

Both $G(MN - 1) < 0$

and $G\left[(G + 1/G)/4 - 1/2 + MN\right] > 0$

Stable if $S(G + 1) > 0$

Unstable if $S(G + 1) < 0$

c. <u>Focus</u>

$G\left[(G + 1/G)/4 - 1/2 + MN\right] < 0$

Stable if $S(G + 1) > 0$

Unstable if $S(G + 1) < 0$

The situation is summarized in the stability diagram of Fig. 8.

II.2 Numerical examples from computer

In order to check these conclusions concerning the solutions for Eqs. (1) and (2) a particular pair of equations was studied with an electronic analog computer. The equations used were Eqs. (1) and (2) with the quadratic forms of Eqs. (3) and (4). These quadratics are the most complicated functions that can be handled with the available computer. The connections for the computer are shown in Fig. 9. The numerical values for the coefficients of the equations were

α_x varied	$\alpha_y = 1$
$k_x = 3$	$k_y = 4$
$\beta_x = 1/2$	$\beta_y = 1$
$\gamma_x = 1/6$	$\gamma_y = 1/4$

These coefficients yield parabolic isoclines that intersect at four points, one in each quadrant, and thus give a total of seven singularities.

The solution curves for the equations, as plotted directly with the analog computer, are shown in Fig. 10, 11, and 12, for which $\alpha_x = +1, -1,$ and $-1/2$, respectively. The isoclines for $\dot{y} = 0$ and $\dot{x} = 0$ are shown in Fig. 10. These same isoclines apply also to Figs. 11 and 12, since a change in α_x does not change the isocline curves. Various initial values of x and y were used in each case, and the resulting solution curves are shown. The direction of increasing time is indicated by the arrowheads on these curves.

Important numerical data applying to the seven singularities for each of the three values of α_x are given in Table I. These data used with the stability diagrams of Figs. 2, 4, 6, and 8 allow the prediction of the kinds of solutions that apply near each singularity. The predictions are listed in Table I. The data for the Gr. 4 singularities are plotted in Fig. 13, which is similar to Fig. 8.

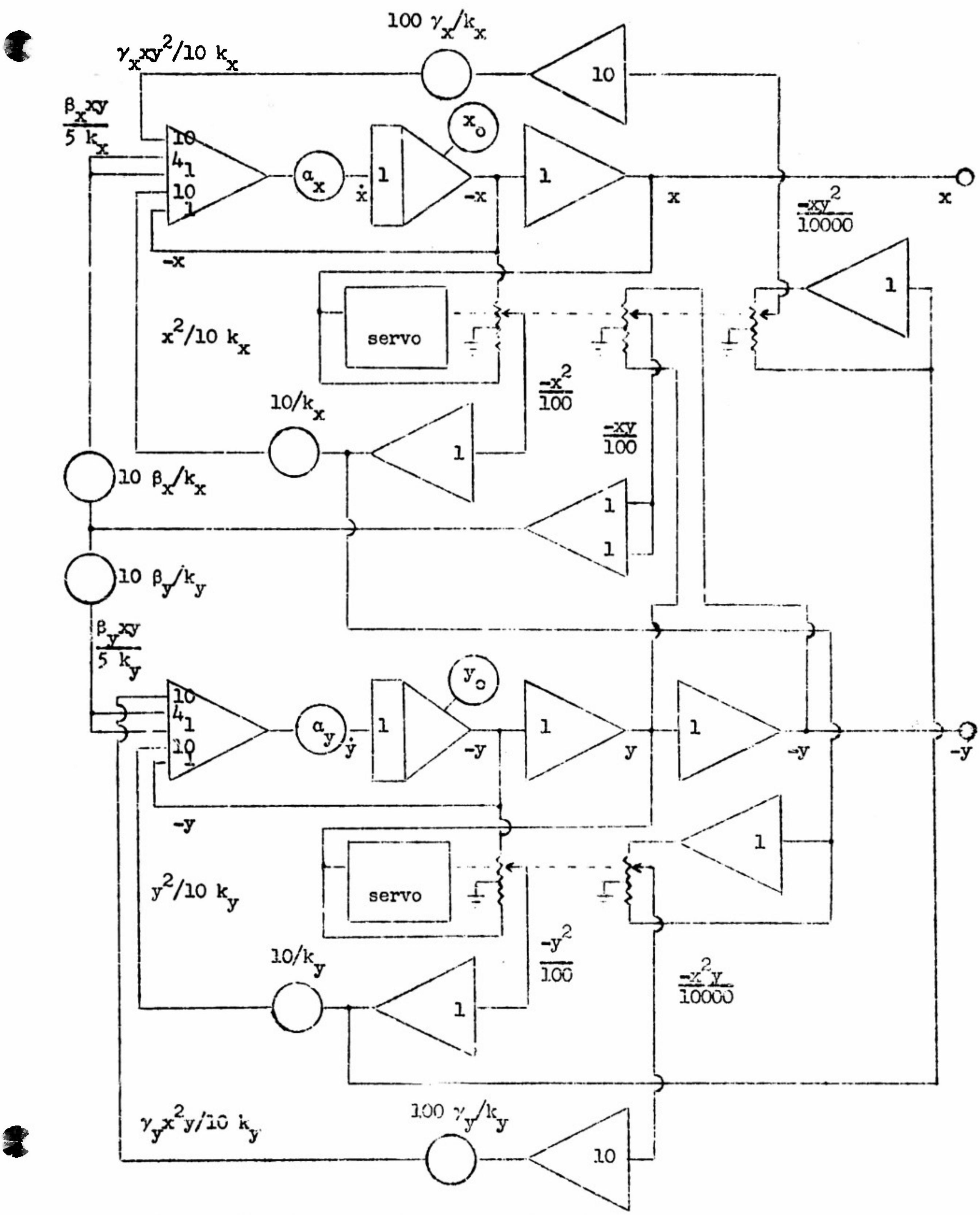
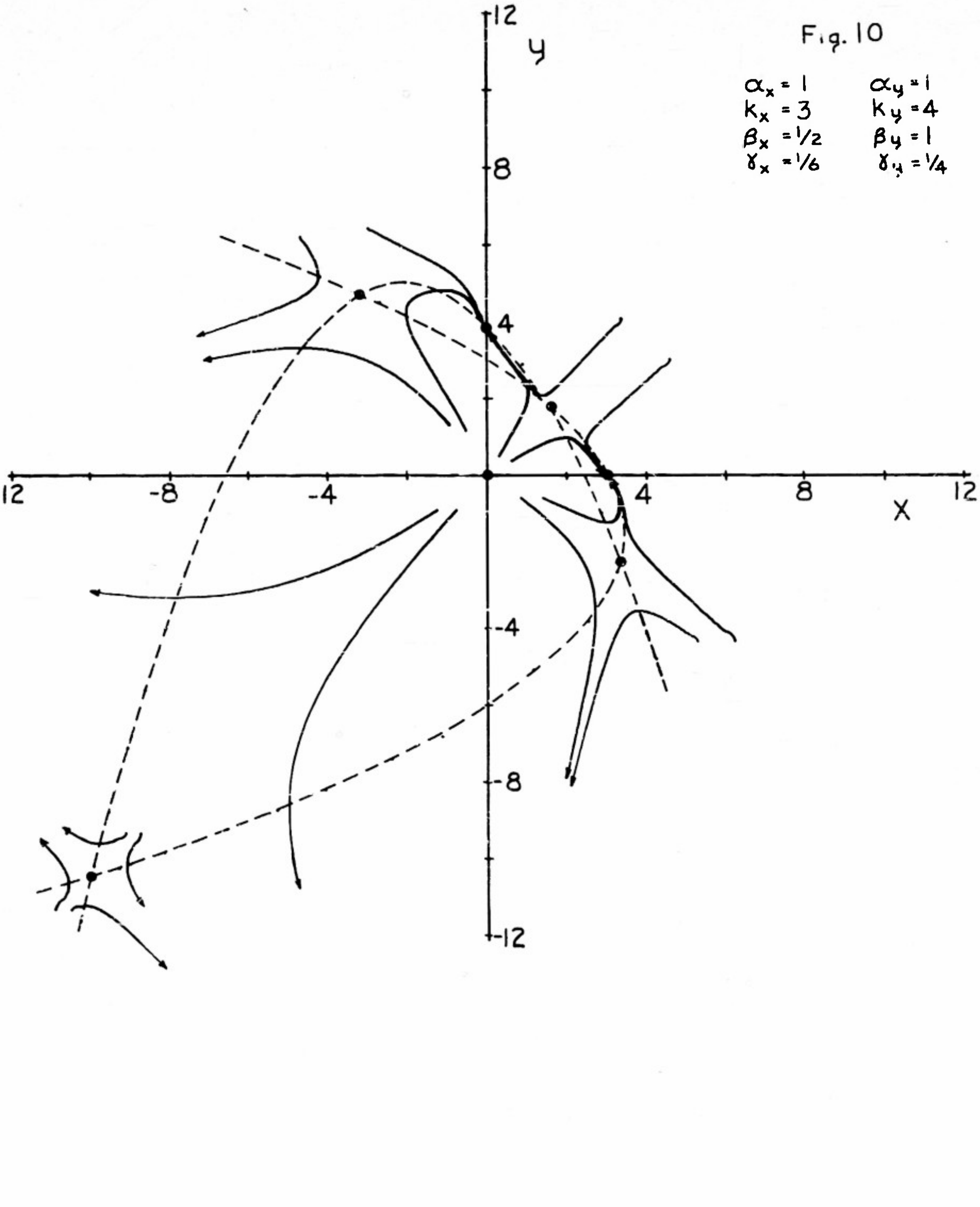


Fig. 9 Computer setup for solving Eqs (1-4)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$\beta_x = 1/2$ $\beta_y = 1$
$\gamma_x = 1/6$ $\gamma_y = 1/4$

Fig. 10

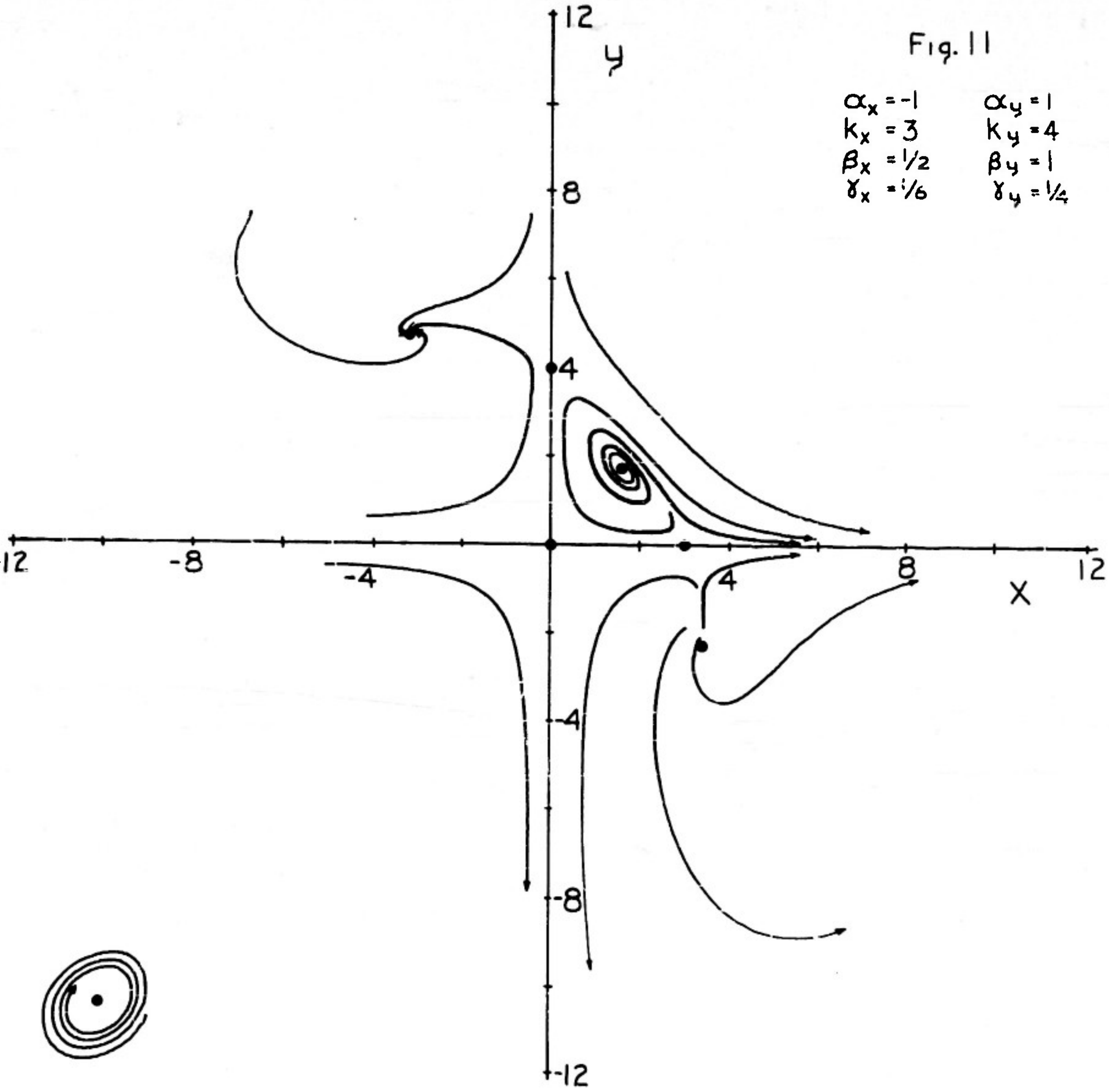

Fig. 11

$\alpha_x = -1$ $\alpha_y = 1$
$k_x = 3$ $k_y = 4$
$\beta_x = 1/2$ $\beta_y = 1$
$\gamma_x = 1/6$ $\gamma_y = 1/4$

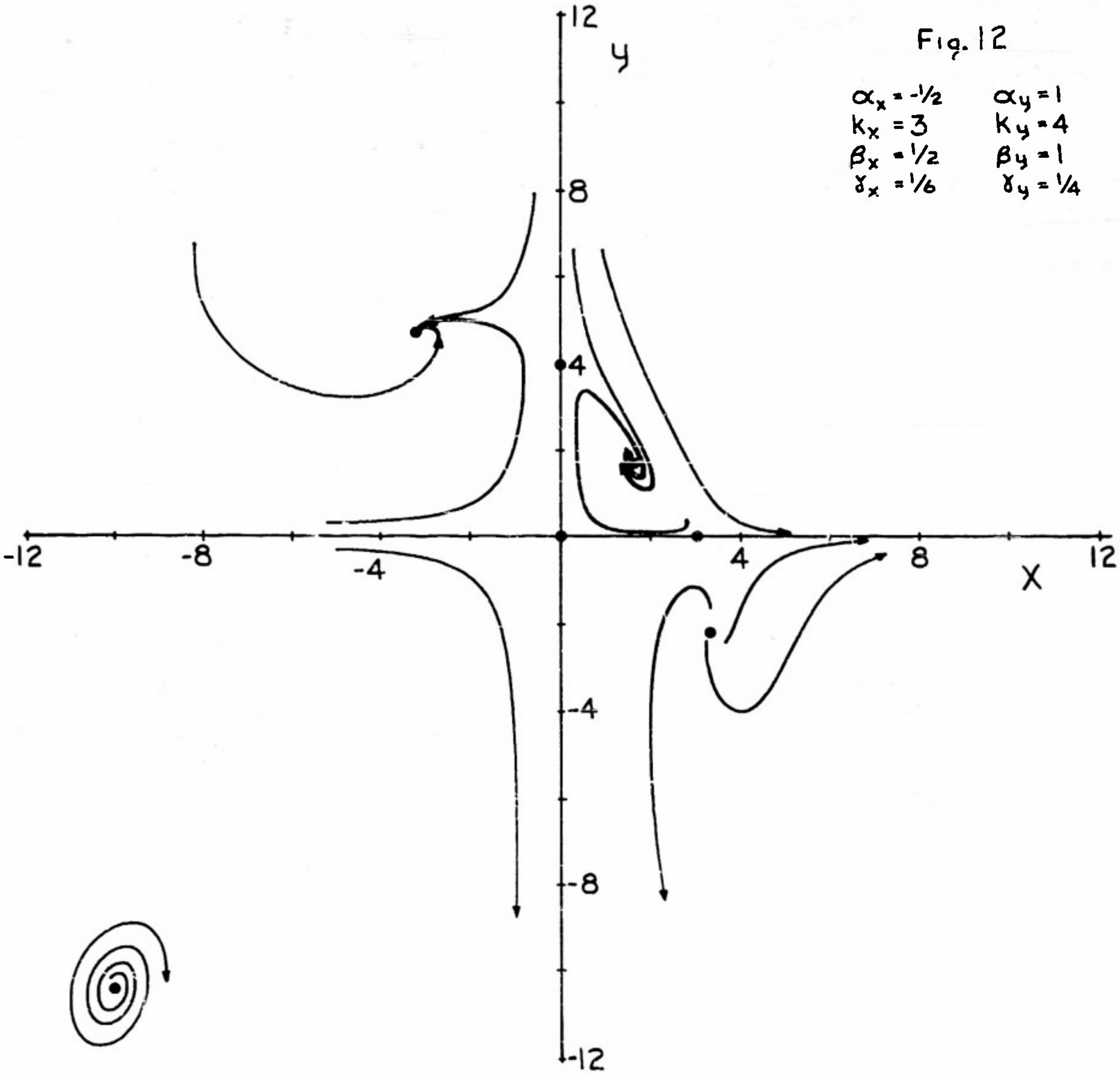

Fig. 12

$\alpha_x = -1/2$	$\alpha_y = 1$
$k_x = 3$	$k_y = 4$
$\beta_x = 1/2$	$\beta_y = 1$
$\gamma_x = 1/6$	$\gamma_y = 1/4$

Table I

Location and Types of Singularities for Examples

$$\dot{x} = (\alpha_x/k_x)(k_x - x - \beta_x y - \gamma_x y^2)x$$

$$\dot{y} = (\alpha_y/k_y)(k_y - y - \beta_y x - \gamma_y x^2)y$$

α_x listed below	$\alpha_y = 1$
$k_x = 3$	$k_y = 4$
$\beta_x = 1/2$	$\beta_y = 1$
$\gamma_x = 1/6$	$\gamma_y = 1/4$

singular point	α_x	x_s	y_s	X	Y	M	N	MN	R	S	G	type
1.a	1	0	0									unst node
2.a	1	3	0		-1.25							st node
3.a	1	0	4	-1.67								st node
4.1.a	1	1.60	1.76			1.80	1.09	1.96	.53	.44	1.21	saddle
4.2.a	1	3.32	-2.08			2.16	-.19	-.41	1.11	-.52	-2.14	saddle
4.3.a	1	-3.06	4.72			-.53	2.07	-1.10	-1.02	1.18	-.86	saddle
4.4.a	1	-9.86	-10.4			-3.93	-2.97	11.7	-3.29	-2.61	1.26	saddle
1.b	-1	0	0									saddle
2.b	-1	3	0		-1.25							saddle
3.b	-1	0	4	-1.67								saddle
4.1.b	-1	1.60	1.76			1.80	1.09	1.96	-.53	.44	-1.21	unst focus
4.2.b	-1	3.32	-2.08			2.16	-.19	-.41	-1.11	-.52	2.14	unst focus
4.3.b	-1	-3.06	4.72			-.53	2.07	-1.10	1.02	1.18	.86	st focus
4.4.b	-1	-9.86	-10.4			-3.93	-2.97	11.7	3.29	-2.61	-1.26	st focus
1.c	-1/2	0	0									saddle
2.c	-1/2	3	0		-1.25							saddle
3.c	-1/2	0	4	-1.67								saddle
4.1.c	-1/2	1.60	1.76			1.80	1.09	1.96	-.26	.44	-.60	st focus
4.2.c	-1/2	3.32	-2.08			2.16	-.19	-.41	-.55	-.52	1.07	unst focus
4.3.c	-1/2	-3.06	4.72			-.53	2.07	-1.10	.51	1.18	.43	st focus
4.4.c	-1/2	-9.86	-10.4			-3.93	-2.97	11.7	1.64	-2.61	-.63	unst focus

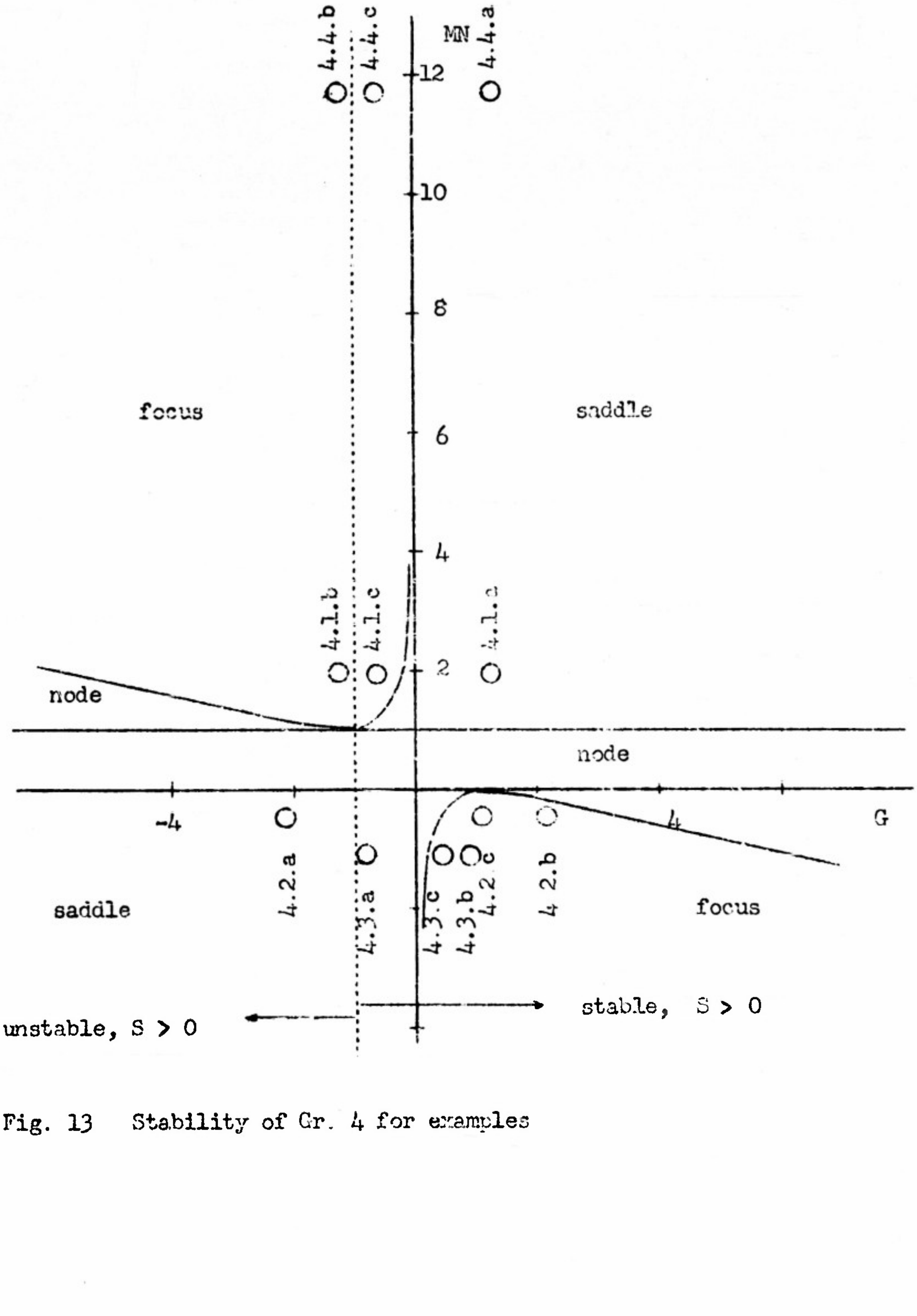

Fig. 13 Stability of Gr. 4 for examples

The types of solutions predicted in this way are the same as those found with the computer.

II.3 Sketch of solution curves

It is possible to sketch the general shapes of the solution curves without having recourse to a computer. This can be done from a knowledge of the location and nature of the singularities, of the isocline curves for $\dot{y} = 0$ and $\dot{x} = 0$, and of the asymptotes for solution curves near a node or a saddle. Information relating to the singularities and the isoclines has been given above. The asymptotes are considered here.

It can be shown that the solution curves near the singularity corresponding to a node or a saddle approach a definite slope as time approaches either plus or minus infinity.[6,7] As time approaches plus infinity, the slope is

$$dy/dx\Big]_{t \to +\infty} = m_1 = A/(\lambda_1 - B) = (\lambda_1 - C)/D. \qquad (16)$$

As time approaches minus infinity, the slope is

$$dy/dx\Big]_{t \to -\infty} = m_2 = A/(\lambda_2 - B) = (\lambda_2 - C)D. \qquad (17)$$

In these equations, A, B, C, D are the coefficients from Eq. (10), λ_1 is the more positive characteristic exponent and λ_2 is the less positive characteristic exponent, as found from Eq. (15).

As an example, a sketch of the solution curves corresponding to the case of Fig. 10 is shown in Fig. 14. Isocline curves for $\dot{y} = 0$ and $\dot{x} = 0$ are plotted from the equations

$$\dot{y} = 0: \qquad y = k_y - \beta_y x - \gamma_y x^2; \qquad y = 0$$

$$\dot{x} = 0: \qquad x = k_x - \beta_x y - \gamma_x y^2; \qquad x = 0$$

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6. A. A. Andronow and C. E. Chaikin, Theory of Oscillations, (Princeton University Press, Princeton, 1949), ch. V.

7. B. G. Farley, Proc. IRE, 40, 1497, (1952)

Their intersections locate the seven singularities, the natures of which have been predicted in Table I.

The slope of a solution curve must change algebraic sign whenever it crosses one of the isoclines, $\dot{y} = 0$ or $\dot{x} = 0$. Thus, with the knowledge that the origin is an unstable node, so that the solution curves near the origin in the first quadrant must have positive slope, the signs of the slopes of the solution curves in each region can be assigned, as shown. Because the axes are also particular isoclines, a solution curve can never cross either axis, and must remain within the quadrant in which it starts.

The values for the slopes of the asymptotes m_1 and m_2 are given in Table II. The equality of α_x and α_y in this example causes m_1 and m_2 for the node at the origin to be indeterminate. The asymptotes are plotted for each singularity in Fig. 14.

The asymptotes are also separatrix curves near a singularity. Thus, the asymptote m_2 near Singularity 4.1 separates those solution curves that tend toward Singularity 2 from those that tend toward Singularity 3.

A number of solution curves are sketched in Fig. 14, making use of the information collected there. These solution curves represent quite accurately the true solutions, as found with the computer and shown in Fig. 10.

A solution curve near a focal point spirals about that point and approaches it from no definite direction. Thus, there are no asymptotes associated with a focal point, and this aid to sketching solution curves is not available. This situation applies near the focal points of the examples shown in Figs. 11 and 12.

Table II

Asymptotic Slopes for Example

singular point	A	B	C	D	λ_1	λ_2	m_1	m_2
1.a	0	1	1	0	1	1	---	---
2.a	0	-.31	-1	-.5	-.31	-1	-1.38	0
3.a	-1	-1	-.56	0	-.56	-1	-2.25	∞
4.1.a	-.79	-.44	-.53	-.58	.19	-1.16	-1.25	1.08
4.2.a	1.12	.52	-1.11	.21	.65	-1.24	8.5	-.63
4.3.a	.63	-1.18	1.02	2.12	1.51	-1.67	.23	-1.27
4.4.a	-10.3	2.61	3.29	-9.78	13.0	-7.0	-.99	1.05

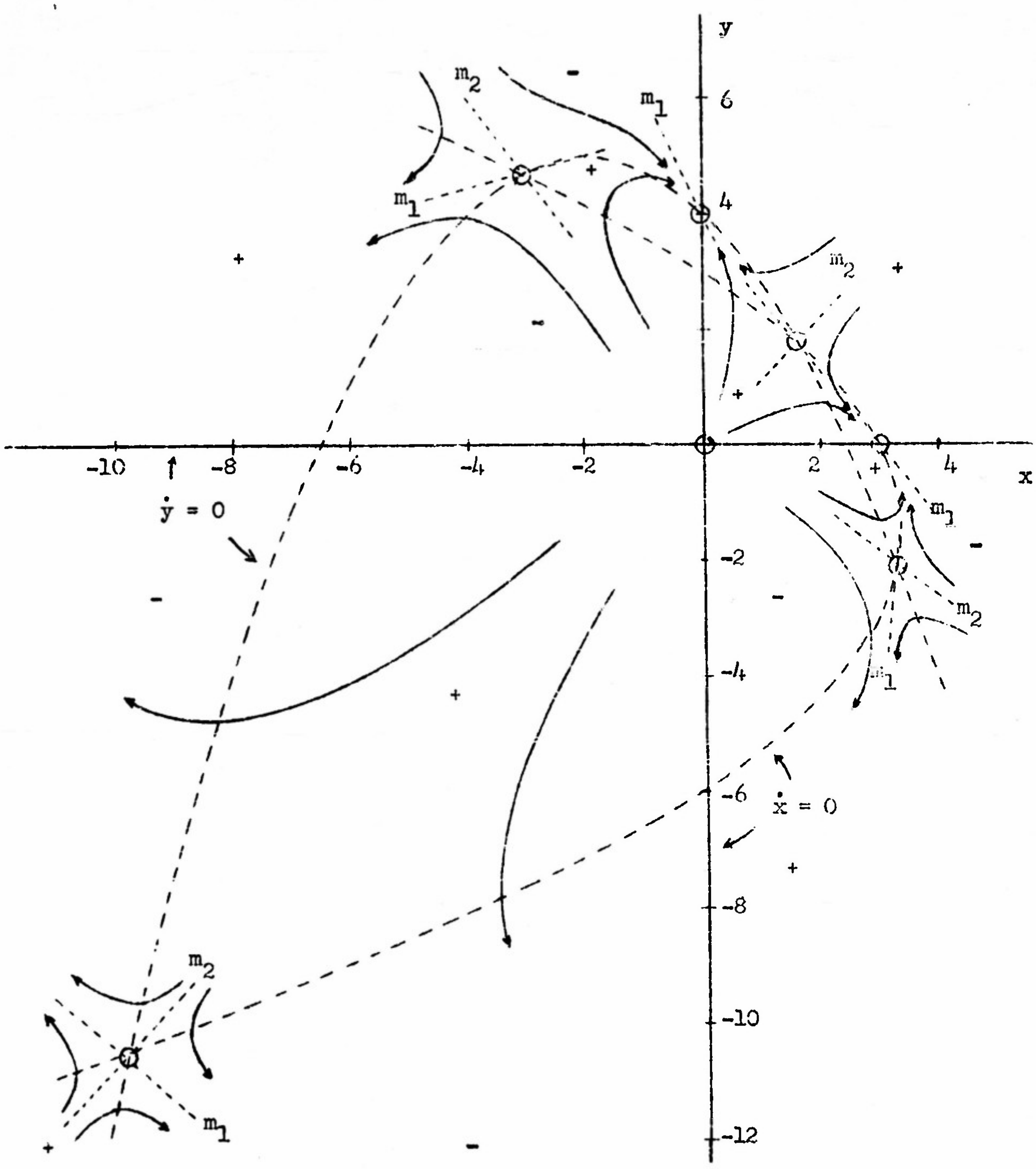


Fig. 14 Sketched solution curves, example of Fig. 10

III. Special Case with no Squared Terms

The form used in writing Eqs. (1) and (2) is convenient in that the parameters k_x and k_y represent the ultimate values of x and y when there is no coupling between the variables. This form is not well adapted to apply to the case in which the terms in x^2 and y^2 are missing on the right sides of the equations. It is simpler to consider this latter case by itself, rather than to attempt a modification of the preceding analysis to include it.

The equations with no squared terms are

$$\dot{x} = (\alpha_x/k_x)\left[k_x - f_x(y)\right]x \tag{18}$$

$$\dot{y} = (\alpha_y/k_y)\left[k_y - f_y(x)\right]y \tag{19}$$

where, again, $f_x(0) = 0$, $f_y(0) = 0$, and these functions can be differentiated. The ratio of these two equations is

$$\frac{dy}{dx} = \frac{(\alpha_y/k_y)\left[k_y - f_y(x)\right]y}{(\alpha_x/k_x)\left[k_x - f_x(y)\right]x} \tag{20}$$

As before, near a singularity, x is replaced by $(x_s + u)$ and y by $(y_s + v)$, giving

$$\frac{dv}{du} = \frac{(\alpha_y/k_y)\left\{\left[-y_s f_y'(x_s)\right]u + \left[k_y - f_y(x_s)\right]v\right\}}{(\alpha_x/k_x)\left\{\left[k_x - f_x(y_s)\right]u + \left[-x_s f_x'(y_s)\right]v\right\}} \tag{21}$$

where $f_y'(x_s) = d/dx\left[f_y(x_s)\right]$ and $f_x'(y_s) = d/dy\left[f_x(y_s)\right]$, and only linear terms are retained. The coefficients are

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) \tag{22}$$

$$B = (\alpha_y/k_y)\left[k_y - f_y(x_s)\right] \tag{23}$$

$$C = (\alpha_x/k_x)\left[k_x - f_x(y_s)\right] \tag{24}$$

$$D = -(\alpha_x/k_x)x_s f_x'(y_s). \tag{25}$$

There are only two groups of singularities.

Gr. 1. $x_s = 0, y_s = 0$

For this singularity

$$A = 0 \qquad C = \alpha_x$$
$$B = \alpha_y \qquad D = 0$$

The stability diagram is shown in Fig. 15.

Gr. 2. $f_x(y_s) = k_x, \quad f_y(x_s) = k_y$

For this singularity

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) \qquad C = 0$$
$$B = 0 \qquad D = -(\alpha_x/k_x)x_s f_x'(y_s)$$

Quantities A and D must be calculated from the equations. The stability diagram is shown in Fig. 16.

An example of this special case is that for which

$$f_x(y) = \beta_x y + \gamma_x y^2 \tag{3}$$
$$f_y(x) = \beta_y x + \gamma_y x^2 \tag{4}$$

the same forms used previously. If the quantities $\gamma_x k_x/\beta_x^2$ and $\gamma_y k_y/\beta_y^2$ are both positive, there are four singularities in Gr. 2, one occurring in each of the four quadrants. Furthermore, the algebraic signs of x_s and $f_y'(x_s)$ are the same, as are the signs of y_s and $f_x'(y_s)$. With these conditions, the sign of product AD is the same as that of product $\alpha_x\alpha_y$. Typical solution curves are as shown in Fig. 17. With other relations of the parameters in the equations, the solutions may be different, of course.

If in Eqs. (3) and (4), both $\gamma_x = 0$ and $\gamma_y = 0$, the original equations reduce to

$$\dot{x} = (\alpha_x/k_x)\left[k_x - \beta_x y\right]x \tag{26}$$
$$\dot{y} = (\alpha_y/k_y)\left[k_y - \beta_y x\right]y. \tag{27}$$

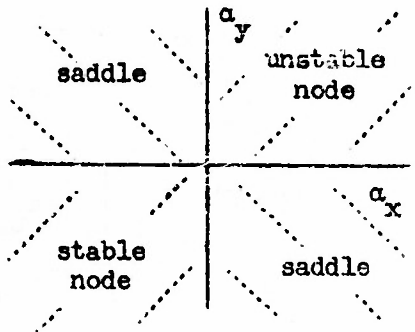


Fig. 15 Stability of Eqs. (18-19), Gr. 1

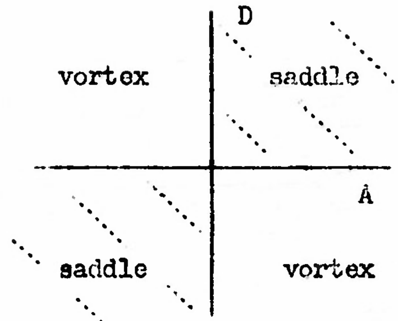


Fig. 16 Stability of Eqs. (18-19), Gr. 2

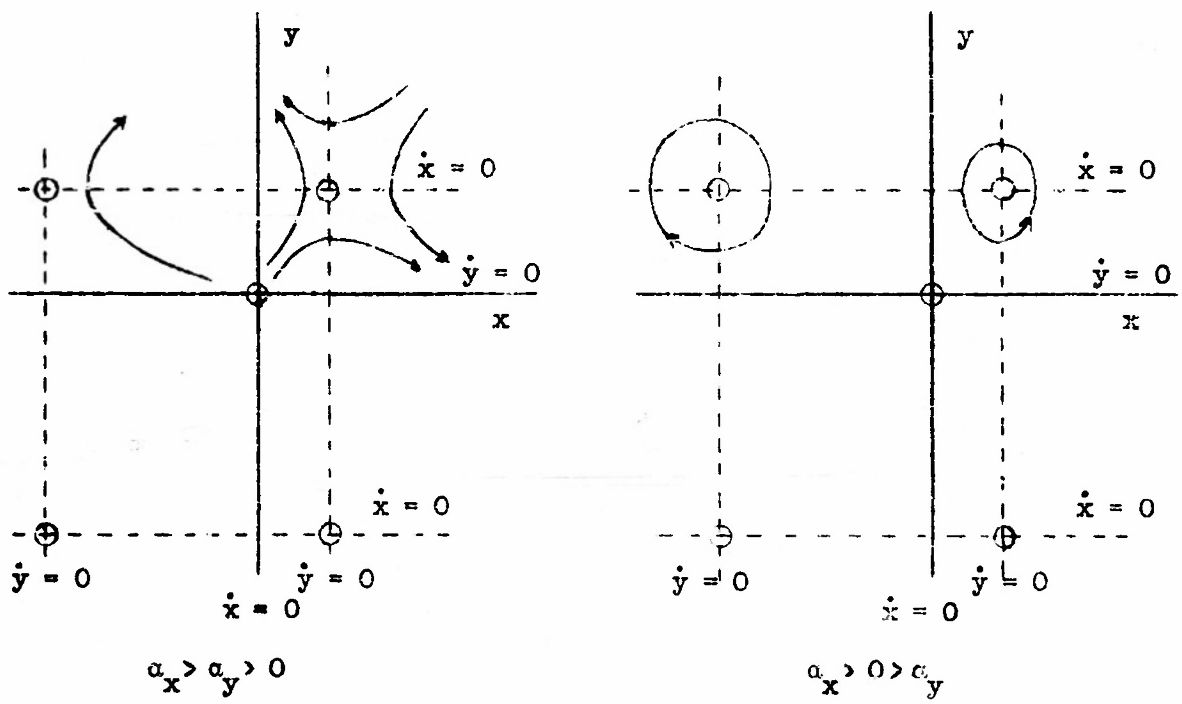


Fig. 17 Solution curves for Eqs. (18-19)

These equations are those studied by Volterra. Only two singularities exist, one at the origin and the other at the point $x_s = k_y/\beta_y$, $y_s = k_x/\beta_x$.

IV. Special Case with one Squared Term

A special case, intermediate in complexity between the most general case considered first, and that just discussed, is the following,

$$\dot{x} = (\alpha_x/k_x)\left[k_x - x - f_x(y)\right]x \tag{28}$$

$$\dot{y} = (\alpha_y/k_y)\left[k_y - f_y(x)\right]y \; . \tag{29}$$

The x^2 term is present in the first equation but there is no y^2 term in the second equation.

The coefficients applying near a singularity are

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) \tag{30}$$

$$B = (\alpha_y/k_y)\left[k_y - f_y(x_s)\right] \tag{31}$$

$$C = (\alpha_x/k_x)\left[k_x - 2x_s - f_x(y_s)\right] \tag{32}$$

$$D = -(\alpha_x/k_x)x_s f_x'(y_s) \tag{33}$$

There are three groups of singularities.

Gr. 1. $x_s = 0, \; y_s = 0$

$$A = 0 \qquad C = \alpha_x$$

$$B = \alpha_y \qquad D = 0$$

The stability diagram is shown in Fig. 18.

Gr. 2. $x_s = k_x, \; y_s = 0$

$$A = 0 \qquad C = -\alpha_x$$

$$B = (\alpha_y/k_y)\left[k_y - f_y(k_x)\right] \qquad D = -\alpha_x f_x'(0)$$

It is necessary to calculate the value of B from the equation. The stability diagram is shown in Fig. 19.

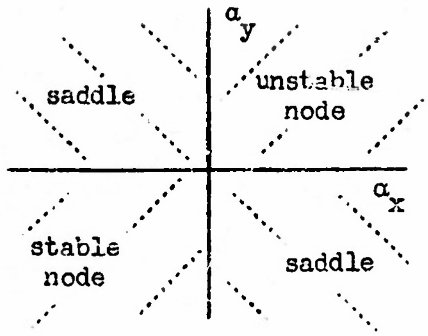


Fig. 18 Stability of Eqs (28-29),Gr.1

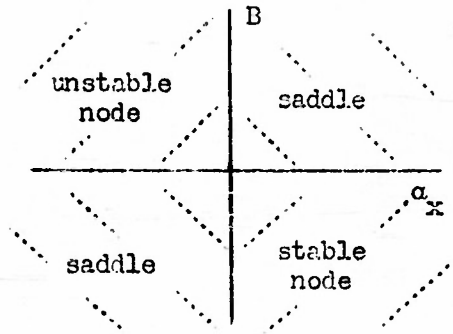


Fig. 19 Stability of Eqs.(28-29),Gr.2

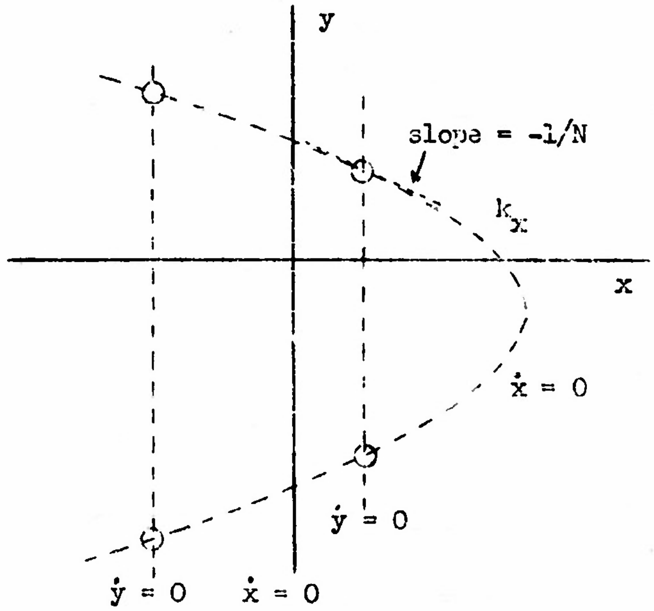
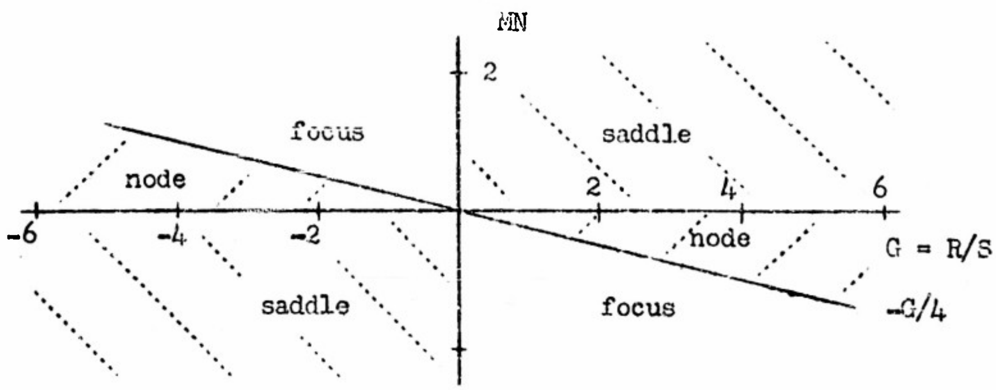


Fig. 20 Isoclines for Eqs. (28-29)



stable focus or node for R > 0

Fig. 21 Stability of Eqs.(28-29), Gr. 3

Gr. 3. $x_s = k_x - f_x(y_s)$, $f_y(x_s) = k_y$

$$A = -(\alpha_y/k_y)y_s f_y'(x_s) = -SM$$

$$B = 0$$

$$C = -(\alpha_x/k_x)x_s = -R$$

$$D = -(\alpha_x/k_x)x_s f_x'(y_s) = -RN$$

where

$$R = \alpha_x x_s/k_x \qquad G = R/S$$

$$S = \alpha_y y_s/k_y$$

$$M = f_y'(x_s) \qquad N = f_x'(y_s)$$

The isoclines for $\dot{y} = 0$ in this case are $y = 0$ and $f_y(x_s) = k_y$, and for $\dot{x} = 0$ are $x = 0$ and $x = k_x - f_x(y)$. These isoclines are plotted in Fig. 20, where $f_x(y)$ has been chosen to give a parabolic curve and $f_y(x)$ has been chosen to give a pair of vertical lines. Four singularities belonging to Gr. 3 exist. Quantity N can be found from the slope of the isocline for $\dot{x} = 0$ at the singular point, as was described in discussing the most general case. Quantity M must be calculated from its defining equation.

The stability diagram for Gr. 3 is shown in Fig. 21.

V. Degenerate Cases

One degenerate case is that described by the equations

$$\dot{x} = (\alpha_x/k_x)(k_x - x - y)x \qquad (34)$$

$$\dot{y} = (\alpha_y/k_y)(k_y - y - x)y. \qquad (35)$$

These equations are of the most general type, but the coefficients are particularly simple. The isoclines for $\dot{y} = 0$ and $\dot{x} = 0$, other than the two axes, are straight lines parallel to one another, so that there are no singularities in Gr. 4. If $\alpha_x > 0$, $\alpha_y > 0$, and $k_x > k_y > 0$, the only stable singularity is that at $x_s = k_x$, $y_s = 0$, and the solution curves are as shown in Fig. 22.

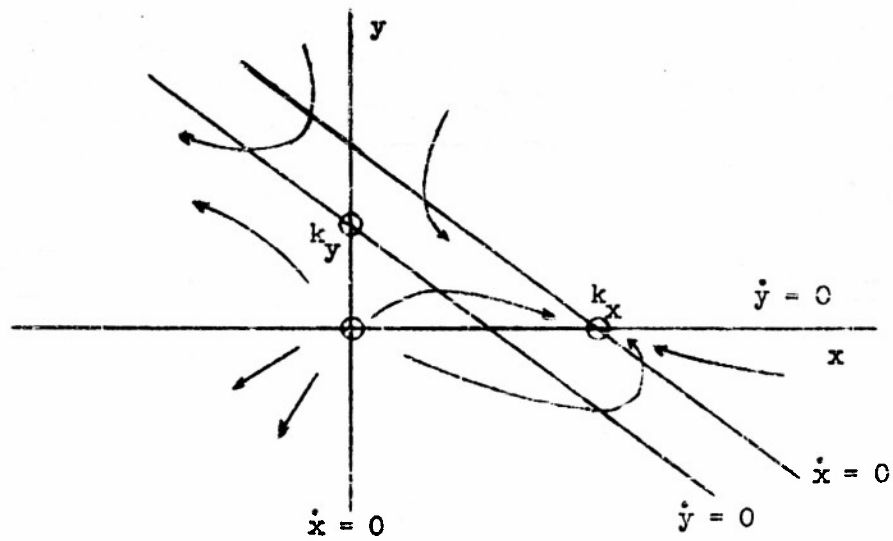


Fig. 22 Solution curves for Eqs.(34-35)

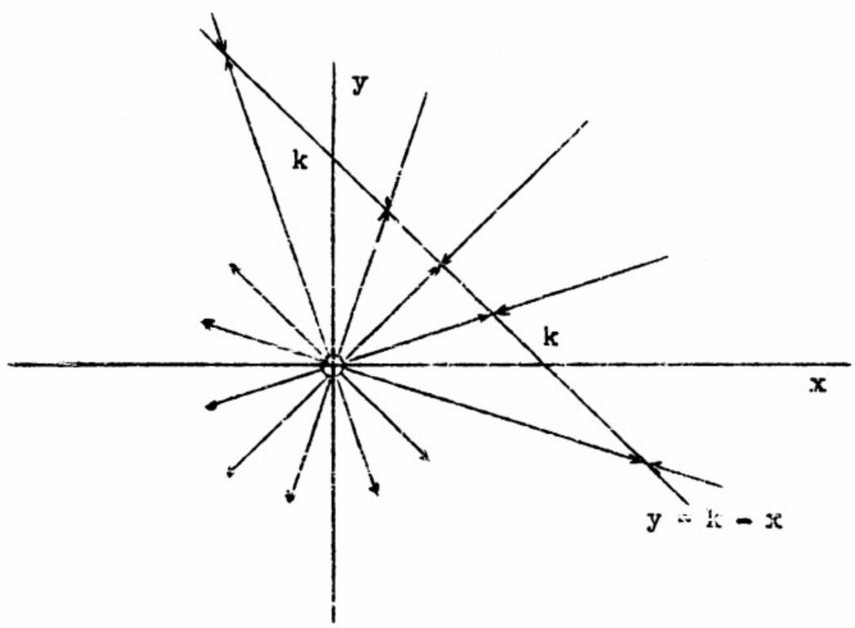


Fig. 23 Solution curves for Eqs.(34-35), $a_x = a_y = a$, $k_x = k_y = k$

A second degenerate case occurs if $\alpha_x = \alpha_y = \alpha$ and $k_x = k_y = k$ in Eqs. (34) and (35). This is the only combination of parameters in the general equations for which $[k_x - x - f_x(y)] = [k_y - y - f_y(x)]$, so that $dy/dx = y/x$. If $\alpha > 0$, the origin is an unstable node. An infinity of stable points are located along the line $y = k - x$, and the solution curves are shown in Fig. 23.

VI. General Effects of Parameters

Some comments can be made about the effects of the parameters that appear in the general equations, Eqs. (1) and (2). The coefficients α_x and α_y are the basic quantities in determining the growth rates. Large positive values of these quantities tend to produce rapid growth. Quantities k_x and k_y determine the final values of x and y when α_x and α_y are positive and there is no coupling between x and y.

The functions $f_x(y)$ and $f_y(x)$ determine the shape of isocline curves for $\dot{x} = 0$ and $\dot{y} = 0$, respectively. If these functions are those of Eqs. (3) and (4), the isoclines are parabolic in shape. The parabola for $\dot{x} = 0$ always passes through the point $x = k_x$, $y = 0$. At this point the slope of the parabola is $dy/dx = -1/\beta_x$. If β_x varies, the parabola moves as shown in Fig. 24, but its shape does not change. The shape is determined by γ_x, and the effect of changes in γ_x is shown in Fig. 25. The dotted line in each of these figures is the locus of the vertex of the parabola as it moves.

One type of solution that is of particular interest is that which is oscillatory in nature. The simplest pair of equations that gives oscillations is the following

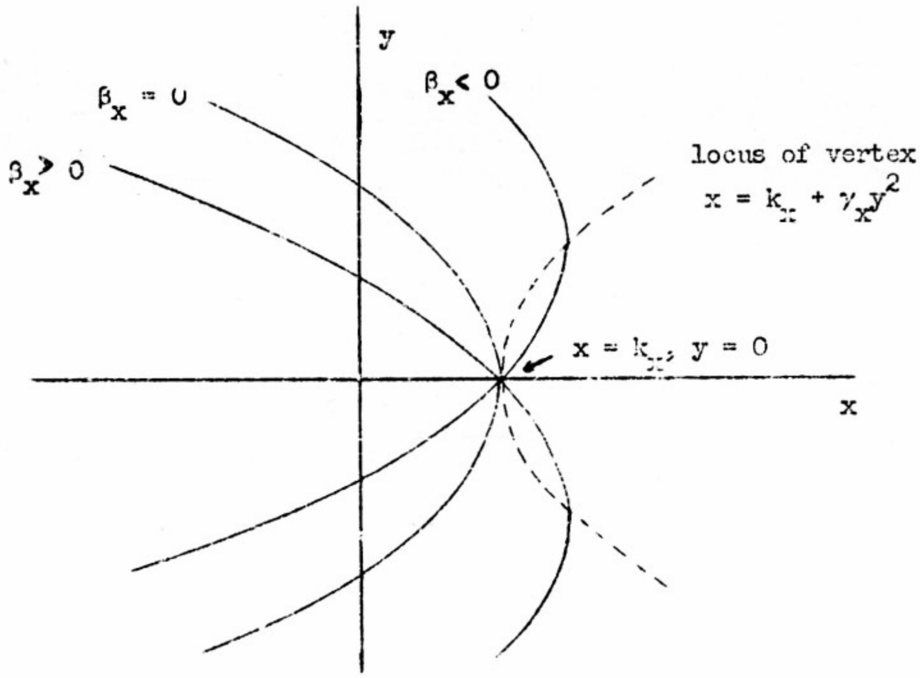


Fig. 24 Parabola for $\dot{x} = 0$, β_x varies

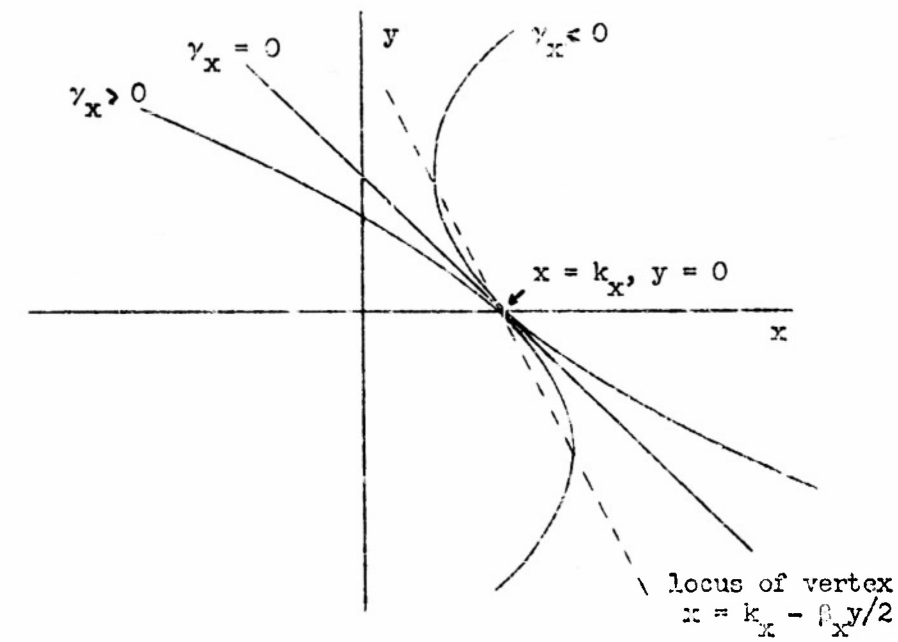


Fig. 25 Parabola for $\dot{x} = 0$, γ_x varies

$$\dot{x} = (a_x - b_x y)x \tag{36}$$

$$\dot{y} = -(a_y - b_y x)y. \tag{37}$$

These are examples of Eqs. (18) and (19), where $\alpha_x = a_x$, $\alpha_y = -a_y$, $\alpha_x/k_x = b_x$, $\alpha_y/k_y = -b_y$, $f_x(y) = y$, $f_y(x) = x$. A singularity exists at $x_s = a_y/b_y = k_y$, $y_s = a_x/b_x = k_x$. For small changes near this singularity, the following equations apply

$$\dot{u} = -(a_y b_x/b_y)v \tag{38}$$

$$\dot{v} = (a_x b_y/b_x)u \tag{39}$$

so that

$$\ddot{u} + a_x a_y\, u = 0. \tag{40}$$

The solution for this equation is a periodic oscillation having the angular frequency $\omega = (a_x a_y)^{1/2}$. In terms of the coefficients used originally, α_x and α_y determine the frequency of oscillation while k_x and k_y locate the mean values of x and y about which oscillation takes place.

Only linear terms appear in the functions f_x and f_y in Eqs. (36) and (37). If additional terms are introduced, a more complicated solution results. A simple example illustrating the effects has extra terms in only one of the two equations,

$$\dot{x} = (a_x - b_x y - c_x y^2 - d_x x)x \tag{41}$$

$$\dot{y} = -(a_y - b_y x)y. \tag{42}$$

The singularity about which oscillation may occur is located at x_s, y_s, given by

$$a_x - b_x y_s - c_x y_s^2 - d_x x_s = 0 \tag{43}$$

$$a_y - b_y x_s = 0; \qquad x_s = a_y/b_y. \tag{44}$$

For small changes near this singularity

$$\dot{u} = -d_x x_s u - (b_x x_s + 2c_x x_s y_s)v \tag{45}$$

$$\dot{v} = b_y y_s u \tag{46}$$

so that

$$\ddot{u} + d_x x_s \dot{u} + a_y(a_x + c_x y_s^2 - d_x x_s)u = 0. \qquad (47)$$

The term $d_x x^2$ in Eq. (41) introduces damping, while both this term and the term $c_x y^2 x$ change the location of the singularity and modify the frequency of oscillation.

In general, quadratic terms in functions f_x and f_y, as in Eqs. (3) and (4), lead to a modification of the frequency of oscillation, to a change in the location of the singularity, and also, perhaps, to additional singularities. So long as the terms x^2 and y^2 are missing, as in Eqs. (18) and (19), coefficients B and C are identically zero near the singularity where oscillation occurs. Thus, there can be no damping and the oscillation remains periodic.

In the more general case of Eqs. (1) and (2), oscillation may occur also, as was true in Figs. 11 and 12. However, the presence of the terms x^2 and y^2 introduces damping, the algebraic sign of which depends upon the signs and magnitudes of terms in the equations. In general, quantity (B + C) is not zero, and the oscillation either builds up or decays. With a particular adjustment of coefficients, it is possible to make (B + C) = 0, in which case a periodic solution would exist. Any small changes in coefficients would destroy this periodicity, however. The amplitude of the periodic solution would depend upon initial conditions.

It does not appear that a limit cycle, representing a periodic solution with amplitude determined solely by the equations, can exist. A limit cycle might occur about an unstable focal point if a suitable positive damping effect came into play as the amplitude of the oscillation increased. In all the examples that have been studied, an unstable focal point always leads to a solution that ultimately runs away, as in Fig. 11.

All of the preceding discussion has been concerned with the solution of Eqs. (1) and (2) in which only the relation between variables x and y at given instants in time is considered. Curves of y as a function of x have been obtained with time as a parameter, but a scale of time along these curves is not available. In general, it appears to be quite difficult to find a solution for x, say, as a function of time. Such a solution would require that y be eliminated from the two equations, leaving a single equation in x and t. This single equation would be of second order and would contain a number of nonlinear terms. In all but very simple cases such as Eqs. (36-37) and (41-42), the solution of this equation by conventional analytical methods appears to be a hopeless task.