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Interaction of Gravitation with a Particle Vacuum

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We shall examine the interaction of the gravitational field with a vacuum of scalar or pseudoscalar particles (mesons). The Lagrangian of the particles together with the interaction terms can be taken in the form

$$L = \frac{1}{4}\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\,g^{\alpha\beta}}{\partial x^\beta}\left(\psi\frac{\partial\psi}{\partial x^\alpha}+\frac{\partial\psi}{\partial x^\alpha}\psi\right)+ + \frac{1}{4}g^{\alpha\beta}\left(\psi\frac{\partial^2\psi}{\partial x^\alpha\partial x^\beta}+\frac{\partial^2\psi}{\partial x^\alpha\partial x^\beta}\psi\right)-\frac{1}{2}m^2\psi^2 \quad (\hbar = c = 1). \tag{1}$$

The equation of motion in the Heisenberg representation

$$S\psi \equiv g^{\alpha\beta}\frac{\partial^2\psi}{\partial x^\alpha\partial x^\beta}+\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\,g^{\alpha\beta}}{\partial x^\alpha}\frac{\partial\psi}{\partial x^\beta}-m^2\psi = 0 \tag{2}$$

and three-dimensional commutation rules

$$\left[\frac{\partial\sqrt{-g}\,L(r,t)}{\partial(\partial\psi/\partial x^0)},\psi(r',t)\right] = \sqrt{-g}\;g^{\alpha 0}\left[\frac{\partial\psi(r,t)}{\partial x^\alpha},\psi(r',t)\right] = \delta(r-r'), \\ [\psi(r,t),\psi(r',t)] = 0 \qquad \text{etc.} \tag{3}$$

follow from (1).

Using (1) and (2), we can write L in the form

$$L = {}^1/_4\{\psi(x),\ S\psi(x)\}\ ; \tag{4}$$

the wavy brackets denote the anticommutator.

We introduce a one-particle Green's function[1] $G(x, x') = \langle P(\psi(x)\ \psi(x'))\rangle$, where the parentheses denote the average of the vacuum of the particles and P is the chronological ordering operator; using (3) and the formula for the derivative of a discontinuous function,[2] we find for $G(x, x')$ the equation

$$SG(x, x') = g^{\alpha 0}\left\langle\left[\frac{\partial\psi(x)}{\partial x^\alpha},\ \psi(x')\right]\right\rangle\delta(x_0 - x_0') = \frac{1}{\sqrt{-g}}\delta(x - x'). \tag{5}$$

The existence of the additional factor $1/\sqrt{-g}$ on the right-hand side of (5) corresponds to the definition of the Green's function in curvilinear coordinates.

Now it is easy to find that the vacuum value of the Lagrangian L_0 (absence of real particles!) is

$$L_0 = {}^1/_2\,SG(x, x')\,|_{x'\to x}\,. \tag{6}$$

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where, on the right, we take the half-sum of the limits $x' \to x$ for $x'^0 \gtrless x^0$. Since the energy tensor can be obtained even in the absence of the gravitational field by varying the action function W with respect to $g_{\alpha\beta}$, we shall consider the vacuum value of the corresponding quantity

$$\begin{aligned} \langle \delta_{g_{\alpha\beta}} W \rangle_0 &= \langle \int \delta_{g_{\alpha\beta}} (\sqrt{-g} L)(dx) \rangle_0 = \frac{1}{2} \int (\delta_{g_{\alpha\beta}} \sqrt{-g} S)\, G(x, x')|_{x' \to x} (dx) = \\ &= \frac{1}{2} \int (\delta_{g_{\alpha\beta}} (\sqrt{-g} S)) \cdot (\sqrt{-g} S)^{-1} \delta(x - x')|_{x' \to x} (dx) = \\ &= \frac{\delta_{g_{\alpha\beta}}}{2(2\pi)^4} \int e^{-ikx} \ln \sqrt{-g}\, S e^{ikx} (dx)(dk). \end{aligned} \tag{7}$$

Considering $\sqrt{-g}S$ as an operator in Hilbert space, we can write W_{vac} in the form

$$\begin{aligned} W_{vac} \equiv W_0 &= {}^1\!/_2 \operatorname{Sp} \ln \sqrt{-g}\, S + \text{const} = \\ &= -\frac{1}{2} \operatorname{Sp} \ln G + \text{const}, \end{aligned} \tag{8}$$

where Sp (Spur) denotes the trace in Hilbert space of functions which become zero at infinity and can be represented by a Fourier-Plancherel integral. From the computations that led to (8), it is obvious that the established theorem is applicable for the corresponding replacement of S by any fields and particles, including particles with spin one-half; for fermions, the symbol Sp also includes the diagonal summation over the spinor indices. We shall assume temporarily the satisfaction of the condition that it is impossible to produce real pairs *in vacuo* by the gravitational field. Then from (7) we find that

$$W_0 = \frac{1}{2} \frac{1}{(2\pi)^4} \int_0^\infty \tau^{-1} d\tau \int e^{-ikx} e^{i\sqrt{-g} S \tau} e^{ikx} (dx)(dk). \tag{9}$$

This formulation, based on the introduction of the parameter of the proper time (or of the fifth coordinate[1]), has the advantage that all the infinities contained in the theory enter only in the integration over τ at the lower limit.

We shall turn to the case of a weak gravitational field and examine it with an accuracy of the second order. In our case it is more convenient to use the following invariant condition instead of the usually applied four conditions of de Donder [7], which in a weak field are transformed into the Hilbert-Lorentz conditions:

$$\partial \sqrt{-g} / \partial x^\alpha = 0, \quad \text{or} \quad \sqrt{-g} = \text{const}. \tag{10}$$

Under this condition, substituting $\tau \to \sqrt{-g}\tau$, we can write an expression accurate to within a constant term,

$$\begin{aligned} W_0 = \frac{1}{2(2\pi)^4} \int_0^\infty \exp(-im^2\tau)\, \tau^{-1} d\tau \int \exp(-ikx) \Big[\exp\Big(i \Big(\frac{\partial}{\partial x^\alpha} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \Big) \tau \Big) - \\ - \exp\Big(i e_{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \Big) \Big] \exp(ikx)(dx)(dk). \end{aligned} \tag{11}$$

Using the relationships for a weak field $g_{\alpha\beta} = e_{\alpha\beta} + h_{\alpha\beta}$ ($e_{\alpha\beta} = 1, 1, 1, -1$) etc., and using the property of the trace

$$\mathrm{Sp}\, AB = \mathrm{Sp}\, BA, \tag{12}$$

we can, in the integrand in (11), replace the expression in square brackets by

$$i\tau \frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta} - e_{\alpha\beta}\right)\frac{\partial}{\partial x^{\beta}}\int_0^1 \exp\left[i\left(e_{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} + \lambda\frac{\partial}{\partial x^{\beta}}\left(g^{\alpha\beta} - e_{\alpha\beta}\right)\frac{\partial}{\partial x^{\alpha}}\right)\tau\right] d\lambda. \tag{13}$$

It is easily verified that the exponent under the integral sign in (13) satisfies the operator integral equation

$$\exp\left[i\left(e_{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} + \lambda\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta} - e_{\alpha\beta}\right)\frac{\partial}{\partial x^{\beta}}\right)\tau\right] = \exp\left[i\tau e_{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\right] +$$
$$+ i\lambda\int_0^{\tau}\left\{\exp\left[-ie_{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}(\tau' - \tau)\right]\times\right. \tag{14}$$
$$\left.\times\frac{\partial}{\partial x^{\beta}}\left(g^{\alpha\beta} - e_{\alpha\beta}\right)\frac{\partial}{\partial x^{\alpha}}\exp\left[i\left(e_{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} + \lambda\frac{\partial}{\partial x^{\beta}}\left(g^{\alpha\beta} - e_{\alpha\beta}\right)\frac{\partial}{\partial x^{\alpha}}\right)\tau'\right]\right\} d\tau'.$$

Solving (14) by means of iteration to accuracy of order $h_{\alpha\beta}$ and substituting the solution into (11), we finally find for W_{vac} the following expression (in rationalized units):

$$W_0 = -\frac{m^2c^2}{3\cdot 2^4\cdot \pi\hbar}\int_0^{\infty} s^{-2}e^{-s}ds\cdot\int(dx)\left[2\frac{\partial h_{\alpha\beta}(x)}{\partial x^{\nu}}\frac{\partial h_{\gamma\delta}(x)}{\partial x^{\pi}}e_{\alpha\gamma}e_{\beta\pi}e_{\delta\nu} -\right.$$
$$\left.-\frac{\partial h_{\alpha\beta}(x)}{\partial x^{\nu}}\frac{\partial h_{\gamma\delta}(x)}{\partial x^{\mu}}e_{\alpha\gamma}e_{\beta\delta}e_{\mu\nu}\right] +$$
$$+\frac{\hbar}{2^7\cdot 15\pi^2}\int_0^{\infty}s^{-1}e^{-s}ds\cdot\int(dx)\left[2\frac{\partial^2 h_{\alpha\beta}(x)}{\partial x^{\nu}\partial x^{\mu}}\frac{\partial^2 h_{\gamma\delta}(x)}{\partial x^{\pi}\partial x^{\theta}}e_{\alpha\gamma}e_{\beta\pi}e_{\delta\nu}e_{\mu\theta} -\right.$$
$$\left.-\frac{\partial h_{\alpha\beta}(x)}{\partial x^{\nu}\partial x^{\mu}}\frac{\partial h_{\gamma\delta}(x)}{\partial x^{\pi}\partial x^{\theta}}e_{\alpha\gamma}e_{\beta\delta}e_{\nu\pi}e_{\mu\theta} - \frac{\partial^2 h_{\alpha\beta}(x)}{\partial x^{\nu}\partial x^{\mu}}\frac{\partial^2 h_{\gamma\delta}(x)}{\partial x^{\pi}\partial x^{\theta}}e_{\alpha\nu}e_{\beta\mu}e_{\gamma\pi}e_{\delta\theta}\}\right] +$$
$$+\frac{\hbar^3}{15\cdot 2^6\cdot\pi^2 m^2c^2}\int(dk)\int_{-1}^{1}dv\, v^2 h_{\alpha\beta}(-k)\,h_{\gamma\delta}(k)\frac{1}{1 + \frac{\hbar^2k^2}{4m^2c^2}(1 - v^2) - i\epsilon}\times$$
$$\times\left[v^4 e_{\alpha\gamma}e_{\beta\pi}e_{\delta\nu}k_{\pi}k_{\nu}(k^2)^2 - \frac{1}{2}v^4 e_{\alpha\gamma}e_{\beta\delta}(k^2)^3 -\right.$$
$$\left.-\frac{1}{16}\left(3v^4 - 10v^2 + 15\right)e_{\alpha\theta}e_{\beta\pi}e_{\gamma\mu}e_{\delta\nu}k^2k_{\theta}k_{\pi}k_{\mu}k_{\nu}\right](\epsilon\to +0). \tag{15}$$

Here, since we limit ourselves to the linear approximation, the condition (10) can be used in the form

$$\partial h(x)/\partial x^{\alpha} = 0, \quad \text{or} \quad k_{\alpha}h(\pm k) = 0 \tag{16}$$

everywhere except when analyzing the term linear in h in W_{vac}, in which case we must take (10) to second order in $h_{\alpha\beta}(x)$, which enables us to reduce the obtained linear term in W_{vac} to a quadratic term. Here the usual substitution $s \to -is$ was made. From the very beginning we could have taken an expansion in $e^{+s\tau}$ instead of the expansion in $e^{+is\tau}$, without applying the condition of the impossibility of pair production; this, however, would have made the intermediate computations more complicated. The first term of (15), up to the linearly divergent factor, is proportional to the action function of the free gravitational field W_G (when the condition (16) is satisfied[3]), and can be eliminated by a simultaneous change in scale (renormalization) of the coordinates and of the "gravitational charge," which role is played by the rest mass:

$$x_\nu \to \left(1 + \frac{km^2}{2\pi\hbar c}\frac{1}{6}\int_0^\infty s^{-2}e^{-s}ds\right)^{1/2} x_\nu,$$

$$m \to \left(\frac{1}{1 + \frac{km^2}{2\pi\hbar c}\frac{1}{6}\int_0^\infty s^{-2}e^{-s}ds}\right)^{1/2} m \tag{17}$$

(here k is the Newtonian gravitation constant); the remaining terms of the action function preserve their previous form. We note that the renormalization of the gravitational charge causes a difference between the gravitational field and the inert masses of elementary particles. The second term in (15) is proportional to the product of the generalized D'Alembertian and W_G up to the logarithmically divergent coefficient; for its isolation, by renormalization, there would have to be a term of similar type with higher derivatives in Einstein's initial Lagrangian of the gravitational field. We note that the appearance of linear divergence in W_{vac} (unlike the case of electrodynamics) is due to the quadrupole character of the gravitational field. When condition (16) is taken into account, the vacuum addition to the action of the gravitational field (15) can be rewritten by replacing the potentials $h_{\alpha\beta}(x)$ by the field "intensities" $\Gamma_{\alpha\beta,\gamma}(x)$; the calibration invariance (in the sense of reference 2) of the expression obtained and the fact that the field mass of the graviton is zero follow directly.

Eq. (15) enables us to examine in the first nonvanishing order the various effects connected with the polarization of the vacuum of the particles by the gravitational field. In particular, from (15), by means of the usual interpretation of the imaginary part of the action function, we obtain the probability of pair production of particles by the gravitational field (which becomes zero at $\pm\infty$)

$$2\,\mathrm{Im}\,(W_0 + W_G) = -\frac{1}{2^8\pi\cdot 15}\int\limits_{-k^2 < 4m^2c^2/\hbar^2}(dk)\left(1 - \frac{4m^2c^2}{\hbar^2(-k^2)}\right)^{1/2} h_{\alpha\beta}(-k)\,h_{\gamma\delta}(k)\times$$
$$\times\Big[\left(1 - \frac{4m^2c^2}{\hbar^2(-k^2)}\right)^2 e_{\alpha\gamma}e_{\beta\pi}e_{\delta\nu}k_\pi k_\nu k^2 - \frac{1}{2}\left(1 - \frac{4m^2c^2}{\hbar^2(-k^2)}\right)^2 e_{\alpha\gamma}e_{\beta\delta}(k^2)^2 -$$
$$-\frac{1}{2}\left(1 + 2\frac{m^2c^2}{\hbar^2(-k^2)} + 6\left(\frac{m^2c^2}{\hbar^2(-k^2)}\right)^2\right)e_{\alpha\theta}e_{\beta\pi}e_{\gamma\mu}e_{\delta\nu}k_\theta k_\pi k_\mu k_\nu\Big]. \tag{18}$$

Hence it is easy to obtain the effective cross section of pair production.[4] We note that the study of W_{vac} in the next approximation must lead us to nonlinear vacuum additions to the gravitational field Lagrangian. Thus, along with the arguments of the general theory of relativity, the nonlinearity in the gravitational field equations is, curiously, also of necessity deduced from the quantum vacuum corrections.

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