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Hydromagnetic Effects of Upwelling Near a Boundary

BY

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EARTH'S MAGNETISM AND MAGNETOHYDRODYNAMICS

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HYDROMAGNETIC EFFECTS OF UPWELLING NEAR A BOUNDARY

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Editor's note:

Dr. Bade, who joined this project early in June, 1954, had to leave us in September pursuant to action of his draft board. The present report, while not in as complete a shape as Dr. Bade would wish, nevertheless contains a number of contributions and theorems of value for hydromagnetic theory. (W. M. E.)

1. Introduction

The geomagnetic secular variations may be attributed to the distortion of pre-existing magnetic fields in the earth's core by upwelling motions of the fluid toward the core-mantle boundary (Elsasser, 1950). The purpose of the work reported here is to compute the hydromagnetic effects of such an upwelling. The main problem is to determine the effects at the surface of the core, since the field in the mantle is a solution of $\nabla^2 \underline{\underline{B}} = 0$ determined by the boundary conditions at the surface of the core and at infinity.

Hydromagnetic effects in a homogeneous conducting fluid are described by the equation*

$$\frac{\partial \underline{\underline{B}}}{\partial t} = \nabla \times (\underline{\underline{v}} \times \underline{\underline{B}}) + \nu_m \nabla^2 \underline{\underline{B}}, \quad (1)$$

where as usual $\nu_m = (\mu\sigma)^{-1}$. It will be assumed that $\underline{\underline{v}} = \underline{\underline{v}}(\underline{\underline{r}})$ is a given stationary velocity field; we are not concerned with the problem of the ponderomotive reaction of the field on the fluid motion. Equation (1) is in general difficult to solve exactly. Here a rather crude approximation is adopted. Instead of letting the induction term $\nabla \times (\underline{\underline{v}} \times \underline{\underline{B}})$ and the diffusion term $\nu_m \nabla^2 \underline{\underline{B}}$ act

*We use the rationalized mks system, see Elsasser (1954)

concurrently on the field $\underline{B}$, we let them act consecutively. That is, the problem is to be solved approximately in two steps:

1. The magnetic viscosity ν_m is set equal to zero and the fluid motion $\underline{v}$ is permitted to operate on the field for a time t .

2. The fluid motion is stopped and diffusion is permitted to act for an equal time t on the field which was produced by the fluid motion in step 1.

The final field resulting from these two steps is taken as an approximation to the field which would be produced by the simultaneous action of induction and diffusion for a time t .

In view of the unprecise nature of the method used for solving equation (1), and since the upwelling should be fairly well localized, it would be pointless to complicate the problem by working in spherical coordinates. Instead, the surface of the core will be represented by the infinite plane $z = 0$; the core fluid will be assumed to occupy the region $z < 0$, the mantle the region $z > 0$.

The local rectangular coordinate system is taken with the z -axis upward and the x -axis pointing south. Two types of pre-existing fields, upon which the induction process acts, will be considered:

1. A poloidal dipole field will appear in the local system as a uniform field

$$b_x = -B \sin \Psi, \quad b_y = 0, \quad b_z = -B \cos \Psi. \quad (2a)$$

Here Ψ is the angle between the positive z -axis and the direction of the vector $-\underline{b}$; Ψ is positive in the northern hemisphere.

Assuming that the field is due to a dipole of moment a located at

the earth's center and directed toward the south pole,

$$B = ar^{-3}(1 + 3 \cos^2 \theta)^{1/2}$$

$$\sin \psi = (1 + 3 \cos^2 \theta)^{-1/2} \sin \theta \quad (3)$$

$$\cos \psi = 2(1 + 3 \cos^2 \theta)^{-1/2} \cos \theta.$$

Here r is the radius of the core and θ is the colatitude from the north pole of the point under consideration. In the local cylindrical coordinate system, the dipole field becomes

$$b_\rho = -B \sin \psi \cos \varphi_0$$

$$b_\varphi = B \sin \psi \sin \varphi_0 \quad (2b)$$

$$b_z = -B \cos \psi.$$

II. A toroidal field in the y -direction would be represented in general by

$$b_x = b_z = 0, \quad b_y = B \sin kz_0 e^{-k^2 \nu_{mt}},$$

as can be shown by separation of variables in (1), with $\underline{v} = 0$ and the boundary condition $\underline{b} = 0$ on the surface $z_0 = 0$. If $k = 0$, the solution reduces to

$$b_x = b_z = 0, \quad b_y = \beta z_0. \quad (4a)$$

This last form is the one which will be used here. It should be a good enough approximation to the general solution near the surface of the core. It is stable against diffusion. In local cylindrical coordinates, (4a) becomes

$$b_\rho = \beta z_0 \sin \varphi_0, \quad b_\varphi = \beta z_0 \cos \varphi_0, \quad b_z = 0. \quad (4b)$$

The poloidal ~~dipole~~ field (2b) can be derived from the vector potential

$$\begin{aligned} a_p &= B(z_0 \sin \psi - \rho_0 \cos \psi \cos \varphi_0) \sin \varphi_0 \\ a_\varphi &= B(z_0 \sin \psi - \rho_0 \cos \psi \cos \varphi_0) \cos \varphi_0 \\ a_z &= 0, \end{aligned} \quad (2c)$$

and the toroidal field (4b) can be derived from

$$a_p = a_\varphi = 0, \quad a_z = -\beta z_0 \rho_0 \cos \varphi_0 \quad (4c)$$

2. The Induction Problem

Parker (1954) has shown that the hydromagnetic induction equation (1) (with $v_m = 0$) has a formal integral which is exactly analogous to Cauchy's integral of Helmholtz's equation for the vorticity in hydrodynamics. A similar integral can easily be derived for the vector potential. It is simplest to start from the integral theorem

$$\frac{d}{dt} \int \underline{B} \cdot d\underline{S} = 0, \quad (5a)$$

which implies that

$$\frac{d}{dt} \oint \underline{A} \cdot d\underline{r} = 0, \quad (5b)$$

where the integration is along an arbitrary closed path. Here d/dt denotes the substantial derivative, i.e. the surface or path of integration is assumed to move with the fluid. It is convenient to work in general coordinates, as this ensures that the results will be valid in the curvilinear coordinates which will be used in the induction calculations. Equation (5b) becomes

$$\oint (A_m dx^m - a_l dx^l_0) = 0,$$

where a_l are the covariant components of the vector potential at

the location x_0^L of a given fluid particle at time 0, and A_m are those of the vector potential at the location x^m of the same fluid particle at time t . Now

$$dx_0^L = dx^m \frac{\partial x_0^L}{\partial x^m}.$$

Hence

$$A_m = a_L \frac{\partial x_0^L}{\partial x^m} + \frac{\partial \psi}{\partial x^m},$$

where ψ is an arbitrary function of x^n, t . The gradient term will contribute nothing to B , and can be omitted:

$$A_m = a_L \frac{\partial x_0^L}{\partial x^m}. \quad (6)$$

The Cauchy-Parker integral for B can be obtained either by taking the curl of (6) or by direct calculation from (5a). The latter method was used by Parker in his original derivation. Here the former method will be used, as it is somewhat shorter:

$$\begin{aligned} B^i &= \frac{1}{2} g^{-1/2} \epsilon^{ijk} \left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) \\ &= \frac{1}{2} g^{-1/2} \epsilon^{ijk} \frac{\partial a_L}{\partial x_0^m} \left[\frac{\partial x_0^m}{\partial x^j} \frac{\partial x_0^L}{\partial x^k} - \frac{\partial x_0^m}{\partial x^k} \frac{\partial x_0^L}{\partial x^j} \right]. \end{aligned}$$

Multiplying both sides by $\partial x_0^p / \partial x^i$ and utilizing the definition of a determinant in terms of the ϵ^{ijk} , one finds

$$B^i \frac{\partial x_0^p}{\partial x^i} = g^{-1/2} \left| \frac{\partial x_0^a}{\partial x^\beta} \right| \epsilon^{pml} \frac{\partial a_L}{\partial x_0^m} = \frac{1}{2} g^{-1/2} \left| \frac{\partial x_0^a}{\partial x^\beta} \right| \epsilon^{pml} \left(\frac{\partial a_L}{\partial x_0^m} - \frac{\partial a_m}{\partial x_0^L} \right)$$

Hence

$$B^r = \frac{1}{J} \frac{\partial x^r}{\partial x_0^p} b^p, \quad (7a)$$

where

$$J = \left(\frac{g}{g_0} \right)^{1/2} \left| \frac{\partial x^\beta}{\partial x_0^a} \right|. \quad (7b)$$

The quantities b^p are the contravariant components of the field at the location x_0^p of a given fluid particle at time 0, and the B^r are those of the field at the location x^r of the same particle at time t .

Let ξ^i be a system of Cartesian coordinates. Then

$$\delta_{ij} = g_{mn} \frac{\partial x^m}{\partial \xi^i} \frac{\partial x^n}{\partial \xi^j},$$

and hence

$$\left| \frac{\partial \xi^i}{\partial x^m} \right| = g^{1/2}.$$

Now the quantity $|\partial \xi_i / \partial \xi_j|$ is known in the Lagrangian form of hydrodynamics to have the value unity in case of incompressible flow.

This quantity is just J (7b):

$$\left| \frac{\partial \xi^i}{\partial \xi_j} \right| = \left| \frac{\partial \xi^i}{\partial x^m} \right| \left| \frac{\partial x^m}{\partial x^n} \right| \left| \frac{\partial x^n}{\partial \xi_j} \right| = \left(\frac{g}{g_0} \right)^{1/2} \left| \frac{\partial x^m}{\partial x^n} \right| = J.$$

Hence $J = 1$ for incompressible flow.

It will prove convenient to work in cylindrical coordinates ρ, φ, z , where (6) and (7) become

$$\begin{cases} A_\rho = \frac{\partial \rho_0}{\partial \rho} a_\rho + \rho_0 \frac{\partial \varphi_0}{\partial \rho} a_\varphi + \frac{\partial z_0}{\partial \rho} a_z \\ A_\varphi = \frac{1}{\rho} \frac{\partial \rho_0}{\partial \varphi} a_\rho + \frac{\rho_0}{\rho} \frac{\partial \varphi_0}{\partial \varphi} a_\varphi + \frac{1}{\rho} \frac{\partial z_0}{\partial \varphi} a_z \\ A_z = \frac{\partial \rho_0}{\partial z} a_\rho + \rho_0 \frac{\partial \varphi_0}{\partial z} a_\varphi + \frac{\partial z_0}{\partial z} a_z \end{cases} \quad (8)$$

$$\begin{cases} B_\rho = \frac{1}{J} \left[\frac{\partial \rho}{\partial \rho_0} b_\rho + \frac{1}{\rho_0} \frac{\partial \rho}{\partial \varphi_0} b_\varphi + \frac{\partial \rho}{\partial z_0} b_z \right] \\ B_\varphi = \frac{\rho}{J} \left[\frac{\partial \varphi}{\partial \rho_0} b_\rho + \frac{1}{\rho_0} \frac{\partial \varphi}{\partial \varphi_0} b_\varphi + \frac{\partial \varphi}{\partial z_0} b_z \right] \\ B_z = \frac{1}{J} \left[\frac{\partial z}{\partial \rho_0} b_\rho + \frac{1}{\rho_0} \frac{\partial z}{\partial \varphi_0} b_\varphi + \frac{\partial z}{\partial z_0} b_z \right]. \end{cases} \quad (9)$$

The equations (6) to (9) are based on the Lagrangian form

of hydrodynamics, in which the motion is described in terms of the trajectories $x^i = x^i(x_0^j, t)$ of the individual fluid particles. However, it is easier to investigate the character of the fluid motion as a whole when using the Eulerian method of velocity fields. For this reason, we begin by using the latter method and shift over to the Lagrangian viewpoint only when we have formulated the description of a suitable fluid motion. It will be assumed that the flow is incompressible, $\nabla \cdot \underline{v} = 0$. The formula for the divergence in cylindrical coordinates is

$$\nabla \cdot \underline{v} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{1}{\rho} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Hence a velocity field defined by

$$v_\rho = -\frac{\alpha}{\rho} R(\rho) Z'(z), \quad v_\phi = 0, \quad v_z = \frac{\alpha}{\rho} R'(\rho) Z(z) \quad (10)$$

is a case of incompressible flow exhibiting rotational symmetry about the z-axis. Here $\alpha = \text{constant}$ and R and Z are arbitrary differentiable functions. Since $v_\phi = 0$, the streamlines all lie in planes $\phi = \text{constant}$. The equation of the streamlines is $R(\rho)Z(z) = \text{constant}$. Since the fluid motion is steady, these streamlines are also the trajectories of the fluid particles.

In order to get a model which is physically reasonable and which matches the problem, it is necessary to impose some conditions on the functions R, Z . It will be assumed that

1. the fluid does not cross the surface $z = 0$; hence $v_z = 0$ at $z = 0$, or $Z(0) = 0$.
2. on the z-axis, there is no radial motion; hence $v_\rho = 0$ at $\rho = 0$, or $\frac{1}{\rho} R(\rho) \rightarrow 0$ as $\rho \rightarrow 0$.

These two assumptions are essential. One can also impose the less important but still plausible assumptions that

3. the only radial motion is the spreading-out as the field approaches the plane $z = 0$, hence $v_p = 0$ at $z = -\infty$, or $Z'(-\infty) = 0$.
4. in any plane $z = \text{constant}$, the motion at large distances from the point $\rho = 0$ resembles that from a point source, hence $v_p \sim \rho^{-1}$ at $\rho = \infty$, or $R(\infty) = \text{constant}$.

It is easy to find functions R, Z satisfying these conditions. However, it is simplest not to specialize these functions at this stage, but rather to carry the calculation through (approximately) in terms of the symbols R, Z themselves. We will use power series expansions in t , discarding terms of higher than the second degree. From (10), in this approximation,

$$\begin{aligned} \rho_0 &= \rho + \frac{a}{c} RZ't + \frac{a^2}{2\rho^2} \left[RR'Z'^2 - RR'ZZ'' - \frac{1}{\rho} R^2Z'^2 \right] t^2 \\ z_0 &= z - \frac{a}{\rho} R'Zt + \frac{a^2}{2\rho^2} \left[R'^2 - RR'' + \frac{1}{\rho} RR' \right] ZZ't^2 \end{aligned} \quad (11)$$

and correspondingly

$$\begin{aligned} \rho &= \rho_0 - \frac{a}{\rho_0} RZ't + \frac{a^2}{2\rho_0^2} \left[RR'Z'^2 - RR'ZZ'' - \frac{1}{\rho_0} R^2Z'^2 \right] t^2 \\ z &= z_0 + \frac{a}{\rho_0} R'Zt + \frac{a^2}{2\rho_0^2} \left[R'^2 - RR'' + \frac{1}{\rho} RR' \right] ZZ't^2 \end{aligned} \quad (12)$$

In (11), R, Z and their derivatives are functions of the final coordinates ρ, z of the particle; in (12), they are functions of the initial coordinates ρ_0, z_0 . If one plans to use (8) in computing the final field $\underline{B}$, one calculates the derivatives $\partial x_0^i / \partial x^j$ as functions of ρ, z directly from (11):

$$\begin{aligned} \frac{\partial \rho_0}{\partial \rho} &= 1 + \frac{a}{\rho} \left(R' - \frac{1}{\rho} R \right) Z't + \frac{a^2}{2\rho^2} \left[R'^2 Z'^2 + RR'' Z'^2 - R'^2 ZZ'' \right. \\ &\quad \left. - RR'' ZZ'' - \frac{4}{\rho} RR' Z'^2 + \frac{2}{\rho} RR' ZZ'' + \frac{3}{\rho^2} R^2 Z'^2 \right] t^2 \end{aligned} \quad (13)$$

$$\begin{aligned}
\frac{\partial \rho_0}{\partial z} &= \frac{a}{\rho} RZ''t + \frac{a^2}{2\rho^2} \left[RR'Z'Z'' - RR'ZZ''' - \frac{2}{\rho} R^2Z'Z''' \right] t^2 \\
\frac{\partial z_0}{\partial \rho} &= -\frac{a}{\rho} \left(R'' - \frac{1}{\rho} R' \right) Zt + \frac{a^2}{2\rho^2} \left[R'R'' - RR''' - \frac{1}{\rho} R'^2 + \frac{3}{\rho} RR'' - \frac{3}{\rho^2} RR' \right] ZZ't^2 \\
\frac{\partial z_0}{\partial z} &= 1 - \frac{a}{\rho} R'Z't + \frac{a^2}{2\rho^2} \left[R'^2 - RR''' + \frac{1}{\rho} RR' \right] (Z'^2 + ZZ'')t^2
\end{aligned} \tag{13}$$

$$\frac{\partial \rho_0}{\partial \phi} = \frac{\partial z_0}{\partial \phi} = \frac{\partial \phi_0}{\partial \rho} = \frac{\partial \phi_0}{\partial z} = 0, \quad \frac{\partial \phi_0}{\partial \phi} = 1.$$

On the other hand, if one plans to use (9) in computing B_m , one must calculate the derivatives $\partial x^i / \partial x_0^j$ from (12), and then eliminate ρ_0 and z_0 from the resulting expressions by means of substitutions of the sort

$$\begin{aligned}
\frac{1}{\rho_0} &= \frac{1}{\rho} - \frac{a}{\rho^3} RZ't \\
R(\rho_0) &= R(\rho) \left[1 + \frac{a}{\rho} R'(\rho) Z'(z)t \right],
\end{aligned}$$

which give the final result correct to terms in t^2 . One finds

$$\begin{aligned}
\frac{\partial \rho}{\partial \rho_0} &= 1 - \frac{a}{\rho} \left(R' - \frac{1}{\rho} R \right) Z't + \frac{a^2}{2\rho^2} \left[(R'^2 - RR'')(Z'^2 + ZZ'') - \frac{1}{\rho^2} R^2Z'^2 \right] t^2 \\
\frac{\partial \rho}{\partial z_0} &= -\frac{a}{\rho} RZ''t - \frac{a^2}{2\rho^2} RR'(Z'Z''' - ZZ''')t^2 \\
\frac{\partial z}{\partial \rho_0} &= \frac{a}{\rho} \left(R'' - \frac{1}{\rho} R' \right) Zt + \frac{a^2}{2\rho^2} \left[RR''' - R'R'' + \frac{1}{\rho} R'^2 \right. \\
&\quad \left. - \frac{1}{\rho} RR''' + \frac{1}{\rho^2} RR' \right] ZZ't^2 \\
\frac{\partial z}{\partial z_0} &= 1 + \frac{a}{\rho} R'Z't + \frac{a^2}{2\rho^2} (R'^2 + RR'' - \frac{1}{\rho} RR')(Z'^2 - ZZ'')t^2 \\
\frac{\partial \rho}{\partial \phi_0} &= \frac{\partial z}{\partial \phi_0} = \frac{\partial \phi}{\partial \rho_0} = \frac{\partial \phi}{\partial z_0} = 0, \quad \frac{\partial \phi}{\partial \phi_0} = 1.
\end{aligned} \tag{14}$$

Substituting from (13) into (8) and taking the curl, or substituting from (14) into (9) (with $J = 1$ since the flow is

incompressible), one finds in the case of an initial poloidal field (2) that the field as modified by the fluid motion is given by

$$B_{\rho} = -B \sin \psi \cos \varphi \left\{ 1 - \frac{a}{\rho} (R' - \frac{1}{\rho} R) Z' t \right. \\ \left. + \frac{a^2}{2\rho^2} \left[(R'^2 - RR'') (Z'^2 + ZZ'') - \frac{1}{\rho^2} R^2 Z'^2 \right] t^2 \right\} \quad (15a) \\ + B \cos \psi \left\{ \frac{a}{\rho} RZ'' t + \frac{a^2}{2\rho^2} RR' (Z'Z'' - ZZ''') t^2 \right\}.$$

$$B_{\varphi} = B \sin \psi \sin \varphi \left\{ 1 - \frac{a}{\rho^2} RZ' t \right. \\ \left. + \frac{a^2}{2\rho^3} \left[RR'ZZ'' - RR'Z'^2 + \frac{3}{\rho} R^2 Z'^2 \right] t^2 \right\} \quad (15b)$$

$$B_z = -B \sin \psi \cos \varphi \left\{ \frac{a}{\rho} (R'' - \frac{1}{\rho} R') Z t \right. \\ \left. + \frac{a^2}{2\rho^2} \left[RR''' - R'R'' + \frac{1}{\rho} R'^2 - \frac{1}{\rho} RR'' + \frac{1}{\rho^2} RR' \right] ZZ' t^2 \right\} \quad (15c) \\ - B \cos \psi \left\{ 1 + \frac{a}{\rho} R'Z' t + \frac{a^2}{2\rho^2} (R'^2 + RR'' - \frac{1}{\rho} RR') (Z'^2 - ZZ'') t^2 \right\}$$

In the case of an initial toroidal field (4) one finds for the final field

$$B_{\rho} = \beta \sin \varphi \left\{ z - \frac{az}{\rho} (R' - \frac{1}{\rho} R) Z' t \right. \\ \left. + \frac{a^2 z}{2\rho^2} \left[(R'^2 - RR'') (Z'^2 + ZZ'') - \frac{1}{\rho^2} R^2 Z'^2 \right] t^2 \right. \\ \left. - \frac{a}{\rho} R'Z t + \frac{a^2}{2\rho^2} (3R'^2 - RR'' - \frac{1}{\rho} RR') ZZ' t^2 \right\} \quad (16a)$$

$$B_{\varphi} = \beta \cos \varphi \left\{ z - \frac{az}{\rho^2} RZ' t + \frac{a^2 z}{2\rho^3} \left[RR' (ZZ'' - Z'^2) + \frac{3}{\rho} R^2 Z'^2 \right] t^2 \right. \\ \left. - \frac{a}{\rho} R'Z t + \frac{a^2}{2\rho^2} (R'^2 - RR'' + \frac{3}{\rho} RR') ZZ' t^2 \right\} \quad (16b)$$

$$B_z = \beta \sin \phi \left\{ \frac{\alpha z}{\rho} (R'' - \frac{1}{\rho} R') Z t - \frac{\alpha^2}{2} (R' R'' - \frac{1}{\rho} R'^2) Z^2 t^2 \right. \\ \left. + \frac{\alpha^2 z}{2\rho^2} [R R'' - R' R'' + \frac{1}{\rho} R'^2 - \frac{1}{\rho} R R'' + \frac{1}{\rho^2} R R'] Z Z' t^2 \right\} \quad (16c)$$

If the diffusion process is slow in comparison with the induction process (i.e. if v_m is sufficiently small in the region $z < 0$), then (15) and (16) already give a crude approximation to the solution of equation (1). Since $Z(0) = 0$, it is apparent that the field (16), produced by action of the fluid motion on the initial toroidal field (4), is zero on the boundary $z = 0$. Thus the induction term $\nabla \times (\underline{v} \times \underline{B})$ in equation (1) is, by itself, incapable of causing the toroidal field to manifest itself outside of the core.

The condition $Z(0) = 0$ does not bring about such a marked simplification in the field (15), which is produced by action of the fluid motion on the initial poloidal field (2). A few terms vanish in (15a) and (15b), and the first two lines of equation (15c) disappear. In order to gain a certain insight into the meaning of these formulas, let us compute the average values

$$\bar{B}_i(r) = (\pi r^2)^{-1} \int_0^r \int_0^{2\pi} B_i(\rho, \phi, 0) \rho d\rho d\phi \quad (17)$$

of the Cartesian components B_i of $\underline{B}$ over a circle of radius r in the plane $z = 0$. These are

$$\bar{B}_x = -B \sin \psi + B \sin \psi \cdot \frac{1}{r^2} \int_0^r \left\{ \alpha R' Z'(0) t - \frac{\alpha^2}{2} (R'^2 - R R'' - \frac{1}{\rho} R R' + \frac{2}{\rho^2} R^2) Z'^2(0) t^2 \right\} d\rho \quad (18a)$$

$$\bar{B}_y = 0 \quad (18b)$$

$$\begin{aligned} \bar{B}_z = & -B \cos \psi - 2B \cos \psi \cdot \frac{1}{r^2} \int_0^r \left\{ aR'Z'(0)t \right. \\ & \left. + \frac{a^2}{2\rho} (R'^2 + RR'' - \frac{1}{\rho} RR')Z'^2(0)t^2 \right\} d\rho. \end{aligned} \quad (18c)$$

Thus $\bar{B}_1 = b_1 + r^{-2}I_1(r)$, where b_1 is the initial field (2a) and the quantities $I_1(r)$ are the changes in the components of b_1 due to the action of the fluid motion, integrated over a circle of radius r about the origin in the plane $z = 0$.

In the mantle ($z > 0$), the conductivity σ is small and hence we can set $\nu_m = \infty$, approximately. Then in this region $\bar{B}$ is a solution of $\nabla^2 \bar{B} = 0$, matching the field of the core at the boundary $z = 0$ and reverting to the uniform poloidal field (2) at infinity. A field whose average (17) is zero over every circle about the origin in the plane $z = 0$ will vanish more rapidly as $z \rightarrow \infty$ than a field whose average is in general non-zero. In fact, it can be shown* that $I_1(\infty)$ can be taken as a rough measure of the amount of field which will penetrate to the surface of the mantle and produce an observable variation in the field there:

$$I_x = B \sin \psi \left\{ aR(\infty)Z'(0)t - a^2Z'^2(0)t^2 \int_0^\infty \rho^{-1}R'^2 d\rho \right\} \quad (19a)$$

$$I_y = 0 \quad (19b)$$

$$I_z = -2B \cos \psi \left\{ aR(\infty)Z'(0)t \right\}. \quad (19c)$$

Here the conditions 2 and 4 (pp. 7-8) on $R(\rho)$ have been used to simplify and evaluate some of the integrals.

According to this model, the east horizontal component of $\bar{B}$ at the surface should not change appreciably, and to linear terms

*See Appendix A.

in t the percentage change in the vertical component should be twice that in the horizontal component and should have opposite sign. It should perhaps be emphasized that the above conclusions have only been demonstrated in the approximation in which cubic and higher terms in t are neglected, and in which the effects of diffusion are ignored.

3. The Diffusion Problem.

The problem of computing the effects of diffusion of a magnetic field across a surface of discontinuity in σ is in general a formidable exercise in analysis. In this section we will give this problem a precise formulation and attempt to draw some conclusions about the diffusion of the fields (15) and (16).

When $\underline{v} = 0$ in (1), that equation reduces to the diffusion equation

$$\frac{\partial \underline{B}}{\partial t} = \nu_m \nabla^2 \underline{B}. \quad (20)$$

This equation is valid throughout any region where $\sigma = \text{constant}$. If σ is a function of position, an additional term $\sigma^{-1} \nabla \sigma \times \underline{E}$ appears on the right; and if σ is discontinuous at any point, equation (1) is not valid at that point.

If σ is constant throughout space, the magnetic diffusion problem can be solved by a method exactly analogous to that used in the corresponding scalar diffusion problem (e.g. that of conduction of heat in an infinite homogeneous medium). In this case, the problem can be split into three independent diffusion problems for the Cartesian components B_n of $\underline{B}$. Using the method

of Fourier transforms*, one finds

$$B_n(\underline{r}, t) = (4\pi\nu t)^{-3/2} \int_{-\infty}^{\infty} B_n(\underline{r}', 0) \exp \left[-\frac{(\underline{r} - \underline{r}')^2}{4\nu t} \right] d^3 \underline{r}'. \quad (21)$$

From

$$\frac{\partial}{\partial x_n} \exp \left[-\frac{(\underline{r} - \underline{r}')^2}{4\nu t} \right] = -\frac{\partial}{\partial x'_n} \exp \left[-\frac{(\underline{r} - \underline{r}')^2}{4\nu t} \right]$$

and from $\nabla \cdot \underline{B}(\underline{r}', 0) = 0$, integrating by parts, one can show that the vector field defined by (21) is solenoidal, $\nabla \cdot \underline{B}(\underline{r}, t) = 0$.

The same result follows directly from $\partial \underline{B} / \partial t = -\nu_m \nabla \times (\nabla \times \underline{B})$.

The problem under consideration here is more intricate, as σ is discontinuous at the surface $z = 0$:

$$\begin{aligned} \sigma &= \sigma_1 & \text{for } z < 0 \\ \sigma &= 0 & \text{for } z > 0 \end{aligned}$$

Consequently,

$$\begin{aligned} \nu_m &= \nu = (\mu\sigma)^{-1} & \text{for } z < 0 \\ \nu_m &= \infty & \text{for } z > 0 \end{aligned}$$

Hence in the region $z > 0$, $\underline{B}$ satisfies the equation $\nabla^2 \underline{B} = 0$. At $z = 0$, equation (20) is not valid. According to the analogy with scalar diffusion theory, one would seek to determine six functions $B_{x1}, B_{y1}, B_{z1}, B_{x2}, B_{y2}, B_{z2}$ of position and time, such that $\underline{B}_1(\underline{r}, 0)$ is the initial field in $z < 0$, $\underline{B}_2(\underline{r}, 0)$ is the initial field in $z > 0$, the vector $\underline{B}_1(\underline{r}, t)$ satisfies equation (20) with $\nu_m = \nu$ in the region $z < 0$, and the vector $\underline{B}_2(\underline{r}, t)$ satisfies $\nabla^2 \underline{B}_2 = 0$ in the region $z > 0$. Since the diffusion equation does not hold at $z = 0$, the behavior of the functions $\underline{B}_1, \underline{B}_2$ on the boundary

*See Sneddon (1951), chapter 5.

must be determined from boundary conditions. There should be six of these, as there are six functions to be determined.

In order to find these boundary conditions, one may first note that the behavior of the field at the boundary is certainly determined by the fundamental equations of electromagnetic theory in a medium at rest:

$$\nabla \times \underline{\underline{E}} = - \partial \underline{\underline{B}} / \partial t, \quad \epsilon \nabla \cdot \underline{\underline{E}} = \rho \quad (22a)$$

$$\nabla \times \underline{\underline{B}} = \epsilon \mu \partial \underline{\underline{E}} / \partial t + \mu \underline{\underline{J}}, \quad \nabla \cdot \underline{\underline{B}} = 0 \quad (22b)$$

$$\underline{\underline{J}} = \sigma \underline{\underline{E}} \quad (22c)$$

$$\nabla \cdot \underline{\underline{J}} + \partial \rho / \partial t = 0. \quad (22d)$$

In diffusion theory, the boundary conditions are relations which must be satisfied for $t > 0$, but not necessarily for $t = 0$. When $t > 0$, $\underline{\underline{E}}$ and $\underline{\underline{B}}$ and their time derivatives are certainly finite. Hence it follows from (22b) that at $z = 0$, the vector $\underline{\underline{B}}$ is continuous: $\underline{\underline{B}}_1 = \underline{\underline{B}}_2$. From the first equation of (22a), it follows that E_x and E_y are continuous at $z = 0$. From the second equation (22a), $E_{z2} - E_{z1} = \tau / \epsilon$, where τ is the surface density of charge on $z = 0$. According to (22d), $J_{z2} - J_{z1} = - \partial \tau / \partial t$. Combining these last two relations with the first equation of (22b), one finds that $(\nabla \times \underline{\underline{B}})_z$ is continuous at $z = 0$.

In the approximation usually used in hydromagnetic theory, the displacement current $\epsilon \partial \underline{\underline{E}} / \partial t$ is neglected in comparison with the conduction current $\underline{\underline{J}}$. In this same approximation, the quantity $\partial \tau / \partial t = \epsilon (\partial E_{z2} / \partial t - \partial E_{z1} / \partial t)$ must also be neglected. But then τ may be regarded as constant in time, and the corresponding longitudinal part of $\underline{\underline{E}}$ may be split off and discarded. The justification for this procedure is that the longitudinal

part of $\underline{E}$ will be a slowly varying electric field, and hence will have little effect on the magnetic field $\underline{B}$. In this approximation, J_z is continuous across the boundary. But in the region $z > 0$, $\underline{J} = 0$ because $\sigma = 0$; hence $J_z = E_z = 0$ at $z = 0$.

The preceding discussion has established the six boundary conditions

$$\begin{aligned} B_{x1} &= B_{x2}, & B_{y1} &= B_{y2}, & B_{z1} &= B_{z2} \\ E_{x1} &= E_{x2}, & E_{y1} &= E_{y2}, & E_z &= (\nabla \times \underline{B})_z = 0 \end{aligned} \quad (23)$$

In order to employ these conditions in the solution of the magnetic diffusion problem in the form outlined above, it would be necessary to express them in terms of $\underline{B}_1$, $\underline{B}_2$, and the derivatives of these vectors. However, this is not possible for the conditions on E_x and E_y , because (22b) and (22c) give the only direct relation between $\underline{E}$ and the derivatives of $\underline{B}$, and this relation breaks down in the region $z > 0$, where $\sigma = 0$.

Here appears the essential complicating distinction between magnetic and scalar diffusion problems. The boundary conditions of the magnetic problem may each involve several components of $\underline{B}$, as in a condition on $\nabla \times \underline{B}$; as a consequence, the problem cannot generally be split into three independent problems for the Cartesian components. Moreover, the boundary conditions are conditions on the electromagnetic field which cannot be reduced to conditions on the magnetic field alone.

This last difficulty, however, can be circumvented in a simple way; if the problem is formulated in terms of the potentials rather than the field intensities, one can express all

of the boundary conditions in terms of the same dependent variables that appear in the diffusion equation. As mentioned above, if the displacement current is neglected (as is always done in hydro-magnetism), then the quantity $\partial\tau/\partial t$ must also be neglected, and the longitudinal component of $\underline{E}$ due to a constant surface charge τ can be split off and discarded. Then the field intensities can be represented in terms of the vector potential $\underline{A}$,

$$\underline{B} = \nabla \times \underline{A}, \quad \underline{E} = -\partial \underline{A} / \partial t \quad (24)$$

where

$$\nabla \cdot \underline{A} = 0. \quad (25)$$

Then all the equations (22) are satisfied identically (with $\rho = 0$ everywhere) except the first equation of (22b), which becomes

$$\frac{\partial \underline{A}}{\partial t} = \nu_m \nabla^2 \underline{A}. \quad (26)$$

Since $\underline{B}$ is finite, it follows from the first equation of (24) and from (25) that

$$A_{x1} = A_{x2}, \quad A_{y1} = A_{y2}, \quad A_{z1} = A_{z2} \quad (27a)$$

The former boundary conditions (23) on $\underline{B}$, namely $\underline{B}_1 = \underline{B}_2$, become

$$\nabla \times \underline{A}_1 = \nabla \times \underline{A}_2. \quad (27b)$$

In view of the second equation (24), (27a) implies that E_x and E_y are continuous at the boundary. The remaining condition (23), namely $E_z = 0$, becomes

$$\frac{\partial A_z}{\partial t} = 0 \quad \text{at } z = 0. \quad (27c)$$

We now have exactly six independent conditions on $\underline{A}$. Equation (27a) gives three conditions; (27b) gives only two more,

since the continuity of A_x and A_y implies the continuity of $(\nabla \times \underline{A})_z$; and (27c) gives the sixth condition.

The fact that five of these six conditions are statements of continuity means that it will be necessary to solve simultaneously for the two vector fields $\underline{A}_1, \underline{A}_2$, where $\underline{A}_1(r, 0)$ and $\underline{A}_2(r, 0)$ are given and

$$\nu \nabla^2 \underline{A}_1 = \partial \underline{A}_1 / \partial t \text{ in } z < 0 \quad (28a)$$

$$\nabla^2 \underline{A}_2 = 0 \quad \text{in } z > 0 \quad (28b)$$

and the boundary conditions (27) are satisfied at $z = 0$.

The solution of this problem can be obtained by means of the following procedure. According to (27c), $A_z(x, y, 0)$ is not a function of t . In the region $z \leq 0$ we may set

$$A_{z1}(r, t) = u(r) + v(r, t), \quad (29)$$

where $v(x, y, 0, t) = 0$. Then

$$\nabla^2 u = 0, \quad \frac{\partial v}{\partial t} = \nu \nabla^2 v.$$

The solution of this diffusion problem for v is known from the theory of heat conduction (semi-infinite region, temperature zero on the boundary). It is

$$v(r, t) = (4\pi\nu t)^{-3/2} \iiint_{-\infty}^{\infty} v(r', 0) \left[e^{-\xi_1^2/4\nu t} - e^{-\xi_2^2/4\nu t} \right] d^3 r', \quad (30)$$

where

$$\xi_1^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$\xi_2^2 = (x-x')^2 + (y-y')^2 + (z+z')^2.$$

Now let

$$\rho^2 = (x-x')^2 + (y-y')^2 + z^2.$$

Then according to equation (1) of Appendix A, the field $u(\underline{r})$ in the region $z < 0$ is given by

$$u = - \frac{z}{2\pi} \iint_{-\infty}^{\infty} \zeta^{-3} A_z(x', y', 0, 0) dx' dy' \quad (31)$$

and the field A_{z2} in the region $z > 0$ is given by

$$A_{z2} = \frac{z}{2\pi} \iint_{-\infty}^{\infty} \zeta^{-3} A_z(x', y', 0, 0) dx' dy' \quad (32)$$

The field A_z is now determined throughout space by equations (29), (30), (31), and (32).

The problem of determining A_x and A_y is more difficult, because of the nature of the boundary conditions on these quantities. According to equation (1) of Appendix A and the boundary conditions (27a), A_x and A_y can be represented in the region $z > 0$ by

$$\begin{aligned} A_{x2} &= \frac{z}{2\pi} \iint_{-\infty}^{\infty} \zeta^{-3} A_{x1}(x', y', 0, t) dx' dy' \\ A_{y2} &= \frac{z}{2\pi} \iint_{-\infty}^{\infty} \zeta^{-3} A_{y1}(x', y', 0, t) dx' dy'. \end{aligned} \quad (33)$$

The remaining boundary conditions, which state that $\partial A_x / \partial z$ and $\partial A_y / \partial z$ are continuous at the boundary, must now be used to obtain the solution in the region $z < 0$. This problem can be formulated most simply in terms of Green's function, which is here defined as the field $G(x, y, z; x', y', z'; t)$ at the point (x', y', z') at time t due to an instantaneous point source of unit strength generated at the point (x, y, z) at time zero. It is assumed that $z, z' < 0$, that the field in the region $z < 0$ at $t = 0$ is zero everywhere except at (x', y', z') and that the boundary conditions are

satisfied at $z = 0$. Then one can show* that the field at the point (x, y, z) at time t due to an initial field $A_x(x', y', z', 0)$ at time zero is given by

$$A_{x1}(x, y, z, t) = \iiint_{-\infty}^{\infty} A_{x1}(x', y', z', 0) G(x, y, z; x', y', z'; t) dx' dy' dz'. \quad (34)$$

Thus if one can determine Green's function G for the problem, (34) gives the solution in the region $z < 0$ for an arbitrary initial field $A_{x1}(x', y', z', 0)$. Once this field is known, the field in the region $z > 0$ is determined by (33).

If $G(x, y, z, t)$ is any solution in the region $z < 0$ of the diffusion equation and the boundary conditions on A_x and A_y (in particular, it may be the Green's function), then $G(x, y, 0, t)$ is the field on the boundary. Substituting into (33) and applying the boundary condition on $\partial G / \partial z$, we obtain

$$\left[\frac{\partial G}{\partial z} \right]_{z=0} = \frac{1}{2\pi} \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \iiint_{-\infty}^{\infty} \frac{z G(x'', y'', 0, t)}{[(x-x'')^2 + (y-y'')^2 + z^2]^{3/2}} dx'' dy''. \quad (35)$$

The author has not yet succeeded in determining a Green's function which satisfies (35) in the general three-dimensional case.

There is one special case in which the entire problem can be solved with ease: the case in which A_x and A_y are functions of z only. Then in equations (33), the quantities A_{x1} and A_{y1} can be brought out from under the integral signs and the integrations can be performed explicitly. The results are that A_{x2} and A_{y2} are independent of position in the region $z > 0$. Thus $\partial A_x / \partial z$ and $\partial A_y / \partial z$ are zero at $z = 0$, and in the region $z < 0$ the diffusion

* Carslaw and Jaeger (1947), chapter XIII

problem for A_x and A_y is formally exactly analogous to the problem of heat conduction in a semi-infinite medium with the adiabatic boundary condition. The solution for A_{x1} is thus

$$A_{x1} = (4\pi\nu t)^{-1/2} \int_{-\infty}^0 A_x(z', 0) \left[e^{-(z-z')^2/4\nu t} + e^{-(z+z')^2/4\nu t} \right] dz',$$

and A_{y1} is given by a similar equation.

A slightly different approach to the general (three-dimensional) problem is obtained if it is assumed that ν_1 , and ν_2 are both finite, but that $\nu_2 \gg \nu_1$. The boundary conditions (27) remain exactly the same. All components of $\underline{A}$ and $\nabla \times \underline{A}$ are continuous across the boundary. From the electromagnetic equations (22), one finds that $(\nabla \times \underline{B})_z$ must be continuous, and hence that $\nu^{-1} \partial A_z / \partial t$ must be continuous across the boundary. But A_z itself is continuous, so that (unless $\nu_1 = \nu_2$) one can conclude that $\partial A_z / \partial t = 0$, as before. Now the diffusion equation is satisfied both below and above the boundary (but with different values of ν), so that both fields $\underline{A}_1$ and $\underline{A}_2$ can be represented in terms of Green's functions. It may prove to be an advantage to have functions of the same type on both sides of the boundary.

In the composite medium with finite ν_2 , the boundary conditions can also be stated in terms of $\underline{B}$ rather than of $\underline{A}$. They are

$$\underline{B}_1 = \underline{B}_2$$

$$\nu_1 (\nabla \times \underline{B}_1)_x = \nu_2 (\nabla \times \underline{B}_2)_x, \quad \nu_1 (\nabla \times \underline{B}_1)_y = \nu_2 (\nabla \times \underline{B}_2)_y$$

$$(\nabla \times \underline{B}_1)_z = (\nabla \times \underline{B}_2)_z = 0.$$

(The last of these holds in the approximation used here, in which E_z is assumed to be continuous at $z = 0$.) In the limit $\nu_2 \rightarrow \infty$, the conditions on $(\nabla \times \underline{B})_x$ and $(\nabla \times \underline{B})_y$ in their present form give no information about $\underline{B}_1$. However, these conditions can be transformed in this case into conditions somewhat resembling (35). In the region $z > 0$, these conditions state that when $\nu_2 = \infty$, $\nabla \times \underline{B}_2 = 0$, and hence $\underline{B}_2 = \nabla V$. Now since $B_{z2} = \partial V / \partial z$ is given on the boundary, V is determined throughout the region $z > 0$ by*

$$V = - \frac{1}{2\pi} \iint_{-\infty}^{\infty} \zeta^{-1} B_{z1}(x', y', 0) dx' dy'.$$

Hence

$$B_{x1}(x, y, 0, t) = \frac{\partial V(x, y, 0)}{\partial x}, \quad B_{y1}(x, y, 0, t) = \frac{\partial V(x, y, 0)}{\partial y}$$

are the required conditions. It should be noted that here the various components of $\underline{B}$ are very much mixed together in the boundary conditions. One great advantage of the formulation based on the vector potential is that it proves possible to separate the problem into three independent diffusion problems for A_x , A_y , A_z .

It is possible to extend the preceding results of this section to the case of a spherical boundary between the two media. The boundary conditions on $\underline{A}$ are that $\underline{A}$ and $\nabla \times \underline{A}$ are continuous across the boundary, and that the normal component $\partial A_r / \partial t = 0$ at the boundary $r = a$. The field $\nabla^2 \underline{B}_2 = 0$ for $r > a$ can be expressed in terms of the field on the surface by means of integrals similar to equation (1) of Appendix A; see the same

*Jeffreys (1950), p. 221.

reference to Jeffreys. The field A_r can be determined by a method similar to that used in the case of A_z with a plane boundary; the solution of the diffusion problem in the interior of the sphere with $v = 0$ on the surface is very much more complicated than the corresponding solution in a semi-infinite region, but it is given in the literature*.

*Carslaw and Jaeger (1947), pp. 210-212.

Appendix A: Solution of Laplace's Equation in a Semi-infinite Region.

In the case of the semi-infinite region $z > 0$, the well-known integral

$$f(x,y,z) = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \nabla f - f \nabla \frac{1}{r} \right] \cdot d\mathbf{S}$$

of Laplace's equation can be transformed* into the expression

$$f(x,y,z) = \frac{z}{2\pi} \iint_{-\infty}^{\infty} \frac{f(x',y',0) dx' dy'}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}, \quad (A1)$$

which gives the values of the harmonic function f throughout $z > 0$ in terms of its values on the plane $z = 0$.

One application of this formula is the justification of the use of equations (19) to approximate the field at the surface of the mantle due to an upwelling at the surface of the core. We can split the total field $\underline{B}$ into the constant (initial) field $\underline{b}$ and a variable field $\underline{\beta}$ produced by the action of the fluid motion on $\underline{b}$. We assume that the upwelling is sufficiently well localized that the improper integrals

$$I_1 = \pi^{-1} \int_0^{\infty} \int_0^{2\pi} \beta_1(\rho, \phi, 0) \rho d\rho d\phi \quad (A2)$$

exist. (This will be the case, for instance, if condition 4, p. 8, on $R(\rho)$ is satisfied.) Now the variable field at the surface of the mantle is given by (A1) with $\beta_1(x,y,z)$ substituted for $f(x,y,z)$; here z is regarded as representing the height of the surface of the mantle above that of the core. A representative

*A proof is given in Jeffreys (1950), p. 221.

Multiplying equation (1) by $\underline{B}$, one obtains

$$\frac{1}{2} \frac{\partial (\underline{B}^2)}{\partial t} = \underline{B} \cdot \left[\nabla \times (\underline{v} \times \underline{B}) \right] - \nu_m \underline{B} \cdot \left[\nabla \times (\nabla \times \underline{B}) \right]. \quad (B1)$$

The first term on the right can be transformed using vector identities in the following way:

$$\begin{aligned} \underline{B} \cdot \left[\nabla \times (\underline{v} \times \underline{B}) \right] &= \underline{B} \cdot \left[(\underline{B} \cdot \nabla) \underline{v} - (\underline{v} \cdot \nabla) \underline{B} - \underline{B} (\nabla \cdot \underline{v}) \right] \\ &= B_i B_j \frac{\partial v_i}{\partial x_j} - \frac{1}{2} \underline{v} \cdot \nabla (\underline{B}^2) - \underline{B}^2 (\nabla \cdot \underline{v}). \end{aligned}$$

Let

$$V_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (B2)$$

Then

$$\underline{B} \cdot \left[\nabla \times (\underline{v} \times \underline{B}) \right] = B_i B_j V_{ij} - \frac{1}{2} \underline{v} \cdot \nabla (\underline{B}^2) - V_{ii} \underline{B}^2.$$

The second term on the right in (B1) can be transformed as follows:

$$\begin{aligned} -\underline{B} \cdot \left[\nabla \times (\nabla \times \underline{B}) \right] &= \nabla \cdot \left[\underline{B} \times (\nabla \times \underline{B}) \right] - (\nabla \times \underline{B})^2 \\ &= \frac{1}{2} \nabla^2 (\underline{B}^2) - (\nabla \times \underline{B})^2 - \nabla \cdot \left[(\underline{B} \cdot \nabla) \underline{B} \right]. \end{aligned}$$

The last term can be written in various forms:

$$\nabla \cdot \left[(\underline{B} \cdot \nabla) \underline{B} \right] = \frac{\partial B_i}{\partial x_j} \frac{\partial B_j}{\partial x_i} = \frac{\partial^2 (B_i B_j)}{\partial x_i \partial x_j} = \frac{\partial B_i}{\partial x_j} \frac{\partial B_j}{\partial x_i} - (\nabla \times \underline{B})^2.$$

Selecting the last of these, and substituting into (B1), we find

$$\frac{d}{dt} \left(\frac{\underline{B}^2}{2} \right) = B_i B_j V_{ij} - V_{ii} \underline{B}^2 + \nu_m \left[\nabla^2 \left(\frac{\underline{B}^2}{2} \right) - \frac{\partial B_i}{\partial x_j} \frac{\partial B_j}{\partial x_i} \right].$$

Here $d/dt = \partial/\partial t + \underline{v} \cdot \nabla$. Introducing the magnetic stress tensor

$$T_{ij} = \frac{1}{\mu} (B_i B_j - \frac{1}{2} B^2 \delta_{ij}) \quad (B3)$$

value of the field at the surface can be obtained by setting $x = y = 0$ in (A1); the corresponding value $\beta_1(0,0,z)$ is the field at that point on the surface of the mantle which is directly above the center of the upwelling in the core. This value can be expressed in the form

$$\beta_1(0,0,z) = \frac{z}{2\pi} \int_0^\infty \int_0^{2\pi} \beta_1(\rho, \phi, 0) (\rho^2 + z^2)^{-3/2} \rho d\rho d\phi. \quad (A3)$$

Now $\beta_1(0,0,z)$ is certainly finite. Hence there is a distance c such that "most" of the field $\beta_1(0,0,z)$ comes from sources lying within a circle of radius c about the origin in the plane $z = 0$. This quantity c could be regarded as the "radius of the upwelling," so far as its magnetic effects are concerned. If we now make the additional assumption that z is larger than c , we can (in a very crude approximation) neglect the term ρ^2 in the denominator of (A3). We then obtain

$$\beta_1(0,0,z) \sim \frac{1}{2\pi z^2} \int_0^c \int_0^{2\pi} \beta_1(\rho, \phi, 0) \rho d\rho d\phi \sim \frac{1}{2z^2} I_1. \quad (A4)$$

Supplement : Diffusion-Induction Equation for the Energy Density.

In scalar diffusion theory, there is often available a conservation principle which may be of some use in predicting or visualizing the course of the diffusion process. In magnetic diffusion theory there is no field quantity which is actually conserved, but one can write the differential equation for the energy density $B^2/2\mu$ in the form of a diffusion equation with a loss term.

we can reduce this expression to the slightly more symmetrical form

$$\frac{d}{dt} \left(\frac{\underline{B}^2}{2\mu} \right) = T_{ij} V_{ij} - V_{ii} \left(\frac{\underline{B}^2}{2\mu} \right) + \nu_m \left[\nabla^2 \left(\frac{\underline{B}^2}{2\mu} \right) - \frac{1}{\mu} \frac{\partial B_i}{\partial x_j} \frac{\partial B_j}{\partial x_i} \right]. \quad (B4)$$

The first two terms represent energy transfer from the fluid motion to the field. The quantity V_{ii} is simply $\nabla \cdot \underline{v}$, so that if the fluid is incompressible the entire contribution of motional induction is contained in the scalar product $T_{ij} V_{ij}$ of the magnetic stress tensor (B3) and the strain tensor (B2) of the fluid motion.

If the fluid is at rest, the first two terms on the right in (B4) vanish, the symbol d/dt on the left reverts to $\partial/\partial t$, and the equation reduces to a diffusion equation for $\underline{B}^2/2\mu$ with a loss term $-(\nu_m/\mu)(\partial B_i/\partial x_j)(\partial B_j/\partial x_i)$.

Under some circumstances, the flux of $\underline{B}$ through a surface may be conserved during diffusion of the field. If one can somehow establish, without first solving the entire diffusion problem, that for some specified curve C the right side of the equation

$$\frac{\partial}{\partial t} \int_S \underline{B} \cdot d\underline{S} = -\nu_m \oint_C (\nabla \times \underline{B}) \cdot d\underline{r}$$

will remain zero during a given time interval, then one can conclude that the flux through the corresponding surface S will remain constant during that interval.

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