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O. N. R. RESEARCH MEMORANDUM NO. 10

COMPUTATIONAL THEORY OF LINEAR PROGRAMMING  
I: THE "BOUNDED VARIABLES" PROBLEM

by

A. Charnes and C. E. Lemke

Graduate School of Industrial Administration  
Carnegie Institute of Technology

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COMPUTATIONAL THEORY OF LINEAR PROGRAMMING I:

THE "BOUNDED VARIABLES" PROBLEM

Introduction

This paper is the first of a series<sup>1/</sup> devoted to extension and sharpening of the present methods of linear programming so as to make practicable the calculation of programs involving many more restrictions than can now be conveniently handled. These extensions are developments on the technique first employed by Charnes (ref. [1]) in reducing every linear programming problem to one with a bounded set of solutions.

We here develop an "extended" simplex method for the "bounded variables" problem, e. g., any linear programming problem in which every variable entering is constrained to lie between an upper and a lower bound. This includes as a special case those problems in which not all are so constrained since we can prescribe arbitrarily large (or small) upper (or lower) bounds for those unconstrained. The advantage achieved with the extended method is reduction in size of the computational tableau by suppression of the inequalities expressing the bounds.<sup>2/</sup>

Examples of such constraints occur repeatedly in policy limitations on items of inventory, in market limitations of saleable amounts of products, in production and delivery requirements (cf. ref. [2]), etc. Rather than expound on such examples here (which require individual development for adequate coverage) we shall attempt to present, in each paper, an example seemingly radically divorced from production or industrial connotations in order to gain one of the most important by-products of mathematical formalism, namely,

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<sup>1/</sup> We wish to thank W. W. Cooper for suggestions, stimulation and encouragement of this series.

<sup>2/</sup> We understand that G. B. Dantzig and associates at Rand Corporation had developed a similar method (suggested by Charnes, Ref. [1]) in connection with unspecified, security-restricted problems. Their results are as yet unavailable.

the recognition of an analogous problem in a completely different field from which new insights may be gained into the original problem. The example presented in this paper is that of plastic collapse of structures (frames).

We first review the simplex method in terms suitable for either the "normal" procedure or the "modified" procedure or the "dual" method (ref. [3], [4] and [5]).

The Simplex Method.

The general linear programming problem may be put into the following form, which we shall refer to throughout as the "simplex" problem:

$$\begin{aligned} \text{Maximize: } Z_0 &= \sum_{j=1}^n c_j \rho_j \\ \text{where: } b_i &= \sum_{j=1}^n a_{ij} \rho_j ; \quad i = 1, \dots, m \\ \text{and: } \rho_j &\geq 0 ; \quad j = 1, \dots, n. \end{aligned}$$

It will be convenient, both for the review of the simplex method and for some simple proofs, to restate the problem in vector notation as:

$$\begin{aligned} \text{Maximize: } Z_0 &= \sum_{j=1}^n \rho_j c_j , \\ [1] \quad \text{where: } P_0 &= \sum_{j=1}^n \rho_j P_j , \quad \text{and } \rho_j \geq 0 ; \quad j = 1, \dots, n. \end{aligned}$$

Thus,  $P_0, P_1, \dots, P_n$  are vectors in the Euclidean vector space  $V_m$  of  $m \times 1$  matrices. Throughout we shall refer to  $c_j$  as "the scalar corresponding to the vector  $P_j$ ".

Suppose that a set  $B_m$  of linearly independent vectors, call them  $a_1, a_2, \dots, a_m$  are selected from the vectors  $P_1, \dots, P_n$ . Thus,  $B_m$  is a basis of  $V_m$  and we may write uniquely:

$$[2] \quad P_0 = \sum_{i=1}^m \rho_i a_i .$$

If the  $m$  values  $p_i$  satisfy  $p_i \geq 0$ ;  $i = 1, \dots, m$  then they then form a solution to the system [1]. When the  $p_i$  also satisfy  $p_i > 0$ ;  $i = 1, \dots, m$ , the solution is referred to as a basic solution. It has been shown (ref. [1]) that the problem can be so modified that when one has a solution to [1] of the form [2] the solution is always a basic solution. Hence we may make the initial assumption that the original set of vectors  $P_0, P_1, \dots, P_n$  has this property, which may be characterized by the statement that the vector  $P_0$  is linearly independent of any  $m-1$  vectors chosen from among  $P_1, \dots, P_n$ .

With this assumption, the simplex method may be described as one which proceeds from basic solution to basic solution until a maximal basic solution is found, i. e., one which yields the maximum value for  $Z_0$ . We shall review the procedure for going from one basic solution to another by the simplex method. Thus, suppose one has located a basis  $B_m$  such that, expressing  $P_0$  as in [2] one has  $p_i > 0$ ;  $i = 1, \dots, m$ . Let  $c_{r_1}$  be the scalar corresponding to the vector  $a_1$ .

We introduce the unique vectors  $a^i$ ;  $i = 1, \dots, m$  satisfying:

$$[3] \quad a_i \cdot a^j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}; \quad \text{for } i, j = 1, \dots, m.$$

We shall call the vectors  $a^i$  the "dual vectors" for the set  $a_i$ . If  $A$  is the matrix whose  $i$ th column is  $a_i$  then the  $j$ th row of  $A^{-1}$  is  $a^j$ . Having the dual vectors and denoting the inner product of vectors  $x$  and  $y$  by  $x \cdot y$  any vector  $x$  in  $V_m$  can be expressed by:

$$[4] \quad x = \sum_{i=1}^m (x \cdot a^i) a_i.$$

In particular, we have:

$$[5] \quad P_0 = \sum_{i=1}^m (P_0 \cdot a^i) a_i,$$

$$[6] \quad P_j = \sum_{i=1}^m (P_j \cdot a^i) a_i; \quad j = 1, \dots, n.$$

Having the computed data  $P_0'a^i$  and  $P_j'a^i$ , the first step in the procedure is to compute the quantities:

$$[7] \quad Z(P_j) = \sum_{i=1}^m (P_j'a^i)c_{r_i} - c_j \quad ; \quad j = 1, \dots, n.$$

$Z(P_j)$  is formed by replacing each  $a_i$  by its corresponding scalar  $c_{r_i}$  in the expression [6] for  $P_j$  and then subtracting the scalar corresponding to  $P_j$ .

When the  $n$  quantities [7] are all non-negative, one may demonstrate that the problem is finished: the basic solution is a maximal one. When at least one of the quantities [7] is negative, a new basic solution may be secured which increases the functional  $Z_0$ . This is accomplished by forming a new basis  $B_m'$  from  $B_m$  by replacing one of the  $a_i$  by another vector. The replacing vector may be any  $P_k$  for which  $Z(P_k) < 0$ . To decide upon the vector to be replaced one finds the value of  $i$ , say  $r$ , such that:

$$[8] \quad \theta = (P_0'a^r)/(P_k'a^r) = \min_i (P_0'a^i)/(P_k'a^i) \quad \text{for } P_k'a^i > 0.$$

The following facts may then be demonstrated (ref. [6]):

- i) The value  $r$  which yields  $\theta$  is unique.
- ii) The set  $B_m'$  consisting of  $a_1, \dots, a_{r-1}, P_k, a_{r+1}, \dots, a_m$  is a basis and again yields a basic solution.
- iii) The new basic solution yields a larger value of the functional  $Z_0$ .

Having decided upon the values of  $k$  and  $r$ , when the vectors  $P_0, P_1, \dots, P_n$  are expressed in terms of the new basis  $B_m'$ , the transition to the new basic solution may be considered complete. (There is a simple algorithm for this transition which will be presented later).

One may then show that in a finite number of steps one will obtain a maximal basic solution, namely one for which all of the quantities  $Z(P_j)$  are non-negative.

It is usual, both for computational and explanatory purposes, to construct a computational tableau which exhibits the computations pertaining to a given stage and at the same time clarifies, for the computer, the mechanics of proceeding to the next stage.

Computational Tableau

Corresponding scalars →		$c_1 \dots c_k \dots c_n$			
		$B_m$	$P_0$	$P_1 \dots P_k \dots P_n$	
$c_{r_1}$	$a_1$	$P_0' a^1$	$P_1' a^1 \dots P_k' a^1 \dots P_n' a^1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$c_{r_s}$	$a_s$	$P_0' a^s$	$P_1' a^s \dots P_k' a^s \dots P_n' a^s$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$c_{r_m}$	$a_m$	$P_0' a^m$	$P_1' a^m \dots P_k' a^m \dots P_n' a^m$		
$Z(F_j) \rightarrow$	$Z_0$	$Z(P_1) \dots Z(P_k) \dots Z(P_n)$			

We shall refer to such a tableau as an "mxn tableau" although there are a few additional rows and columns, and shall likewise refer to the problem as an "mxn problem".

The Bounded Variables Problem

First an illustration. The following problem arises in the theory of plastic collapse for structures (refs. [7], [8]).

Consider a structure composed of elastic, perfectly plastic beams joined together rigidly. We are interested in plastic collapse of this structure under the action of concentrated applied forces. We assume that each beam is homogeneous so that the same yield conditions apply to any cross-section of the same beam.

Consider a set of numbers  $p_1, \dots, p_m$  which represent the magnitudes of external forces which are to be applied to the structure. The structure

will be in equilibrium when these forces are balanced by the reactions at the supports. These reactions, together with the applied forces, give rise to a system of internal forces which determine the bending moment at every point of the system (the effects of shear and axial forces can be neglected here).

For a given cross-section there will be a maximum and a minimum bending moment which the cross-section can carry due to the assumption of perfect plasticity. Thus, at each cross-section the bending moment  $M$  will be subject to an inequality  $-M^0 \leq M \leq M^0$ .  $M^0$  is the "fully plastic moment" or the moment for which plastic flow would first begin. Now, the bending moment distribution will be linear along each member. Hence, we need only be concerned with those points of the structure where plastic yielding will first occur. These points are finite in number and depend only on the structure and on the points of application of the forces. They are points including those for which the bending moment has a turning point, as well as the ends of the beams, or points where a support is present, or points at which one of the forces is applied. These points are called "test stations". Thus, at each test station the bending moment  $M_j$  is subject to the restrictions:

$$[9] \quad -M_j^0 \leq M_j \leq M_j^0 \quad ; \quad j = 1, \dots, n.$$

Now, considering the forces  $p_1, \dots, p_m$  as a base load, we apply instead the loads  $\rho p_1, \dots, \rho p_m$ . Then, besides the restrictions [9], the bending moments  $M_j$  will be subject to the equilibrium conditions:

$$[10] \quad \sum_{j=1}^n b_{ij} M_j = \rho p_i \quad ; \quad i = 1, \dots, m,$$

where the  $b_{ij}$  depend on the structure and on the points of application of the forces.

We seek the largest value  $\bar{\rho}$  of  $\rho$  permissible in order that the structure remain in equilibrium under the action of the loads  $\rho p_i$ .

We make the following transformations and notational changes:

Let  $x_j = (M_j/M_j^0) + 1$ ;  $x_{n+1} = \rho$ ;  $a_{ij} = b_{ij}M_j^0$ ; and  $b_i = \sum_{j=1}^n a_{ij}$ . The problem then becomes:

$$\begin{aligned} \text{Maximize: } & Z_0 = x_{n+1} \\ \text{where: } & b_i = \sum_{j=1}^n a_{ij}x_j + (-p_i)x_{n+1} \quad ; \quad i = 1, \dots, m, \\ & \text{and: } \quad 0 \leq x_j \leq 2 \quad ; \quad j = 1, \dots, n. \end{aligned}$$

This is an example of the following problem which we shall state in vector form and which we shall call the "bounded variables" problem:

$$\begin{aligned} \text{Maximize: } & Z_0 = \sum_{j=1}^n \rho_j c_j, \\ \text{where: } & P_0 = \sum_{j=1}^n \rho_j P_j, \quad \text{and } \quad 0 \leq \rho_j \leq b_j \quad ; \quad j = 1, \dots, n. \end{aligned}$$

We shall set up and treat this problem as a simplex problem. In this manner we shall have an  $(m+n) \times (2n)$  problem. Such a problem would ordinarily employ  $(m+n) \times (2n)$  tableaux. We shall show, however, that it will be sufficient to employ  $m \times n$  tableaux; that is, that the problem can be handled in the space  $V_m$  of the vectors  $P_j$ .

If we introduce the new variables  $x_j$  defined by:

$$[11] \quad \rho_j + x_j = b_j \quad ; \quad j = 1, \dots, n,$$

we may phrase the problem as follows in simplex form:

$$\begin{aligned} \text{Maximize: } & Z_0 = \sum_{j=1}^n \rho_j c_j, \\ \text{where: } & P_0 = \sum_{j=1}^n \rho_j P_j, \\ & b_j = \rho_j + x_j \quad \text{and} \quad \rho_j, x_j \geq 0 \quad ; \quad j = 1, \dots, n. \end{aligned}$$

We wish to define vectors in  $V_{m+n}$ . These will be designated by putting bars above the letters, as  $\bar{z}$ . At times we shall use the notation  $\bar{z} = \begin{bmatrix} x \\ y \end{bmatrix}$ , where  $x$  is a vector in  $V_m$  and  $y$  is a vector in  $V_n$ .

We introduce the following vectors in  $V_n$ :

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad Q_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{-(jth place)} ; \quad j = 1, \dots, n.$$

And the following vectors in  $V_{m+n}$ :

$$\bar{P}_0 = \begin{bmatrix} P_0 \\ b \end{bmatrix}, \quad \bar{P}_j = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad \bar{Q}_j = \begin{bmatrix} 0 \\ Q_j \end{bmatrix}; \quad j = 1, \dots, n.$$

The  $(m+n) \times (2n)$  problem may then be phrased as:

$$\begin{aligned} \text{Maximize:} \quad Z_0 &= \sum_{j=1}^n \rho_j c_j, \\ [12] \quad \text{where:} \quad \bar{P}_0 &= \sum_{j=1}^n \rho_j \bar{P}_j + \sum_{j=1}^n x_j \bar{Q}_j ; \quad \rho_j, x_j \geq 0 ; \quad j = 1, \dots, n. \end{aligned}$$

Suppose that a basic solution for this problem has been found. This implies that a basis  $B_{m+n}$  of  $V_{m+n}$  has been chosen from among the vectors  $\bar{P}_1, \dots, \bar{P}_n, \bar{Q}_1, \dots, \bar{Q}_n$ . Recall that the simplex method requires that each of these  $2n$  vectors, together with  $\bar{P}_0$ , be expressed in terms of the basis  $B_{m+n}$ . We show that the coefficients involved can be obtained by merely expressing the vectors  $P_1, \dots, P_n$  in terms of a certain basis  $B_m$  of  $V_m$ .

To do this let us first note some properties of an arbitrary basis  $B_{m+n}$  chosen as above.

- i) The number  $s$  of vectors  $\bar{P}_j$  in  $B_{m+n}$  satisfies  $m \leq s \leq n$ .
- ii) For each  $j$ , either  $\bar{P}_j$  or  $\bar{Q}_j$  or both are in  $B_{m+n}$ .
- iii) By i and ii there are precisely  $m$  values of  $j$  such that both  $\bar{P}_j$  and  $\bar{Q}_j$  are in  $B_{m+n}$ . Furthermore, the set  $B_m$  of the  $m$  vectors  $P_j$  such that both  $\bar{P}_j$  and  $\bar{Q}_j$  are in  $B_{m+n}$  is a basis of  $V_m$ .

In particular, suppose that  $B_{m+n}$  yields a basic solution for the problem [12]. For notational ease we suppose that  $\bar{P}_1, \dots, \bar{P}_s$  are in  $B_{m+n}$  and

that  $\bar{P}_i$  and  $\bar{Q}_i$  are both in  $E_{m+n}$  for  $i = 1, \dots, m$ . As a result we have that  $\bar{Q}_{s+1}, \dots, \bar{Q}_n$  complete the basis  $E_{m+n}$  and that  $P_1, \dots, P_m$  form a basis  $E_m$  of  $V_m$ .

Let  $a^1, \dots, a^m$  be the vectors dual to the basis  $E_m$ . Thus, we have:

$$[13] \quad P_i \cdot a^j = \delta_{ij} \quad ; \quad i, j = 1, \dots, m.$$

Suppose that we have computed the quantities  $P_0 \cdot a^i$  and  $P_j \cdot a^i$  satisfying:

$$[14] \quad P_0 = \sum_{i=1}^m (P_0 \cdot a^i) P_i$$

$$[15] \quad P_j = \sum_{i=1}^m (P_j \cdot a^i) P_i \quad ; \quad j = 1, \dots, n.$$

By means of these quantities we may write down the expressions for  $\bar{P}_0, \bar{P}_j$  for  $s < j \leq n$  and  $\bar{Q}_j$  for  $m < j \leq s$  in terms of  $E_{m+n}$ : namely, for those vectors not already in the basis  $E_{m+n}$ . Thus, we have:

$$[15] \quad \bar{P}_j = \bar{Q}_j + \begin{bmatrix} P_j \\ 0 \end{bmatrix} = \bar{Q}_j + \sum_{i=1}^m (P_j \cdot a^i) [\bar{P}_i - \bar{Q}_i] \quad ; \quad s < j \leq n,$$

$$[16] \quad \bar{Q}_j = \bar{P}_j - \begin{bmatrix} P_j \\ 0 \end{bmatrix} = \bar{P}_j - \sum_{i=1}^m (P_j \cdot a^i) [\bar{P}_i - \bar{Q}_i] \quad ; \quad m < j \leq s,$$

$$[17] \quad \bar{P}_0 = \begin{bmatrix} P_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \sum_{i=1}^m (P_0 \cdot a^i) [\bar{P}_i - \bar{Q}_i] + \sum_{j=1}^n b_j \bar{Q}_j =$$

$$= \sum_{i=1}^m (P_0 \cdot a^i) [\bar{P}_i - \bar{Q}_i] + \sum_{i=1}^m b_i \bar{Q}_i + \sum_{i=s+1}^n b_i \bar{Q}_i + \sum_{j=m+1}^s b_j [\bar{P}_j - \sum_{i=1}^m P_j \cdot a^i (\bar{P}_i - \bar{Q}_i)] =$$

$$= \sum_{i=1}^m \phi_i \bar{P}_i + \sum_{i=1}^m (b_i - \phi_i) \bar{Q}_i + \sum_{j=m+1}^s b_j \bar{P}_j + \sum_{j=s+1}^n b_j \bar{Q}_j,$$

where we have set  $\phi_i = P_0 \cdot a^i - \sum_{j=m+1}^s b_j (P_j \cdot a^i)$  for  $i = 1, \dots, m$ .

We recall now that the first step in applying the simplex method is to calculate the quantities  $Z(\bar{P}_j)$  and  $Z(\bar{Q}_j)$ , obtained (see eqn. [7]) from the vector expressions [15] and [16] by replacing each of the basis vectors by its corresponding scalar.

Thus, in view of [15] and [16], these quantities become:

$$[18] \quad Z(\bar{P}_j) = 0 + \sum_{i=1}^m (P_j \cdot a^i) c_i - c_j = Z(P_j) \quad ; \quad s < j \leq n,$$

$$[19] \quad Z(\bar{Q}_j) = c_j - \sum_{i=1}^m (P_j \cdot a^i) c_i - 0 = -Z(P_j) \quad ; \quad m < j \leq s.$$

The values in [18] and [19] will determine whether or not the basic solution corresponding to the basis  $B_{m+n}$  is maximal. If it is not, one of these quantities will be negative and we may proceed to a new basic solution. In this case, having decided which vector will replace a vector in  $B_{m+n}$ , we need to find the vector which is to be replaced. For this we need the quantity  $\theta$  (see eqn. [8]). The ratios competing for  $\theta$  are found by means of equations [15], [16], and [17]. We need to distinguish two cases.

i) If  $\bar{Q}_k$  for  $m < k \leq s$  is to replace some vector in  $B_{m+n}$ :

$$\theta = \text{Min.} \left\{ \underset{i}{\text{Min.}} \phi_i / (-P_k \cdot a^i) \text{ for } P_k \cdot a^i < 0, \underset{i}{\text{Min.}} (b_i - \phi_i) / P_k \cdot a^i \text{ for } P_k \cdot a^i > 0, b_k \right\}.$$

ii) If  $\bar{P}_k$  for  $s < k \leq n$  is to replace some vector in  $B_{m+n}$ :

$$\theta = \text{Min.} \left\{ \underset{i}{\text{Min.}} \phi_i / P_k \cdot a^i \text{ for } P_k \cdot a^i > 0, \underset{i}{\text{Min.}} (b_i - \phi_i) / (-P_k \cdot a^i) \text{ for } P_k \cdot a^i < 0, b_k \right\}.$$

Having determined the vector to be replaced, one can complete the replacement and proceed to the new basic solution.

We have thus shown that essentially all of the computations in one stage of the  $(m+n) \times (2n)$  problem may be handled by means of an  $m \times n$  tableau. To complete the discussion of the bounded variables problem, we shall exhibit the computational tableau associated with one stage of the problem and list the rules of procedure to be followed in passing to the next stage.

Extended Tableau No. 1

$b_j$ corresponding scalars $\rightarrow$				$b_1$ ..	$b_{m+1}$ ..	$b_s$	$b_{s+1}$ ..	$b_n$
				$c_1$ .. $c_m$	$c_{m+1}$ ..	$c_s$	$c_{s+1}$ ..	$c_n$
$\downarrow$	$B_m$	a	$P_0$ b	$P_1$ .. $P_m$	$P_{m+1}$ ..	$P_s$	$P_{s+1}$ ..	$P_n$
$c_1$	$P_1$	$\phi_1$	$b_1 - \phi_1$	$P_1'a^1$ .. $P_m'a^1$	$P_{m+1}'a^1$ ..	$P_s'a^1$	$P_{s+1}'a^1$ ..	$P_n'a^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$c_r$	$P_r$	$\phi_r$	$b_r - \phi_r$	$P_1'a^r$ .. $P_m'a^r$	$P_{m+1}'a^r$ ..	$P_s'a^r$	$P_{s+1}'a^r$ ..	$P_n'a^r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$c_m$	$P_m$	$\phi_m$	$b_m - \phi_m$	$P_1'a^m$ .. $P_m'a^m$	$P_{m+1}'a^m$ ..	$P_s'a^m$	$P_{s+1}'a^m$ ..	$P_n'a^m$
$Z(P_j) \rightarrow$	$Z_0$			$Z(P_1)$ .. $Z(P_m)$	$Z(P_{m+1})$ ..	$Z(P_s)$	$Z(P_{s+1})$ ..	$Z(P_n)$
$B_{m+n} \rightarrow$				+, - .. +, -	+ ..	+ ..	- ..	-

Explanations and Rules of Procedure:

1. The basis  $B_{m+n}$  is visualized in the tableau as follows:

$\bar{P}_j$  is in  $B_{m+n}$  if the last entry in the column headed by  $P_j$  is a +.

$\bar{Q}_j$  is in  $B_{m+n}$  if the last entry in the column headed by  $P_j$  is a -.

Thus, both + and - at the bottom of the column headed by  $P_j$  indicates that both  $\bar{P}_j$  and  $\bar{Q}_j$  are in  $B_{m+n}$ .

2. For the procedure, one first checks the n values  $Z(P_j)$  in the next to the last row. A maximum has been reached if both:

$$Z(P_j) \geq 0 \text{ in all of those columns having + in the last entry;}$$

$$Z(P_j) \leq 0 \text{ in all of those columns having - in the last entry.}$$

[This would mean that all of the quantities  $Z(\bar{P}_j)$  and  $Z(\bar{Q}_j)$  were non-negative.]

3. If some  $Z(P_j)$  violates the above, we proceed to a new basic solution.

Having decided upon the replacing vector, we need the value  $\theta$  given in either i or ii on the preceding page. It is for this reason that the columns a and b under  $P_0$  and the first row were added to the tableau.

4. Having decided upon the replacing vector and the vector to be replaced we may pass on to the new basic solution and the new tableau. The following cases may arise:

i. The vector  $\bar{Q}_k$  replaces the vector  $\bar{P}_k$  in  $B_{m+n}$  for some  $k$  with  $m < k \leq s$ . This induces no change in the basis  $B_m$ . The only changes in the tableau in this case are the following:

a. The + under the column headed by  $P_k$  is changed to -.

b. The quantity  $\phi_i' = \phi_i + b_k(P_k'a^i)$ ;  $i = 1, \dots, m$  replaces  $\phi_i$  in the a and b columns under  $P_0$ . (see the expression for  $\phi_i$  in eqn. 17).

c. The quantity  $b_k Z(P_k)$  is added to  $Z_0$ . ( $Z_0$ , of course, need not be carried along. The maximum value of  $Z_0$  may be obtained at the end of the problem from eqn. 17.)

ii. The vector  $\bar{P}_k$  replaces the vector  $\bar{Q}_k$  in  $B_{m+n}$  for some  $k$  with  $s < k \leq n$ . This induces no change in  $B_m$ . The only changes in the tableau in this case are the following:

a. The - under the column headed by  $P_k$  is changed to +.

b. The quantity  $\phi_i' = \phi_i - b_k(P_k'a^i)$ ;  $i = 1, \dots, m$  replaces  $\phi_i$  in the a and b columns under  $P_0$ .

c. The quantity  $b_k Z(P_k)$  is subtracted from  $Z_0$ .

iii. If either  $\bar{P}_r$  or  $\bar{Q}_r$  with  $1 \leq r \leq m$  is replaced either by some  $\bar{P}_k$  with  $s < k \leq n$  or some  $\bar{Q}_k$  with  $m < k \leq s$  then  $P_k$  replaces  $P_r$  in the basis  $B_m$  to form a new basis  $B_m'$  and, as in the usual simplex procedure, the new tableau must be based upon the basis  $B_m'$ . The simplex algorithm for changing the tableau elements, which is included here for the sake of completeness, is as follows. The element  $P_j'a^i$  is replaced by the element:

$$(P_j'a^i)' = P_j'a^i - (P_k'a^i/P_k'a^r)P_j'a^r \quad \text{for } j = 1, \dots, n \text{ but } j \neq k,$$

$$\text{and for } i = 1, \dots, m \text{ but } i \neq r,$$

$$(P_j'a^r)' = P_j'a^r/P_k'a^r \quad ; \quad j = 1, \dots, n,$$

$$(P_k'a^i)' = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k. \end{cases}$$

The  $n$  quantities  $Z(P_j)$  are replaced by:

$$Z(P_j)' = Z(P_j) - [P_j'a^r/P_k'a^r]Z(P_k) \quad ; \quad j = 1, \dots, n.$$

Under  $P_0$  the columns headed a and b are changed by replacing  $\phi_i$  by:

$$\phi_i' = \phi_i - \theta P_k' a^i; \quad \text{for } i = 1, \dots, m, \text{ but } i \neq r, \text{ and } \theta \text{ replaces}$$

$\phi_r$  in the rth place of the a column;  $b_k - \theta$  replaces  $b_r - \phi_r$  in the rth place of the b column.

Finally,  $P_k$  replaces  $P_r$  in the  $B_m$  column;  $c_k$  replaces  $c_r$  in the corresponding scalars column; and both + and - appear in the last entry of the  $P_k$  column, whereas the  $P_r$  column has a + if  $\bar{Q}_r$  has been replaced in  $B_{m+n}$  or a - if  $\bar{P}_r$  has been replaced.

This completes the discussion of the bounded variables problem.

As a final remark, it should be noted that when a change in the tableau (case iii above) is necessary, the modified simplex method (ref. [4]) may be employed, whereby only the vectors  $(a^i)'$  dual to the new basis  $B_m'$  are computed from the old vectors  $a^i$ , and then the quantities  $(P_j' a^i)'$  are obtained, if needed, by taking the scalar product of  $P_j$  and  $(a^i)'$ . The algorithm for obtaining the new dual vectors is:

$$(a^r)' = (1/P_k' a^r) a^r$$

$$(a^i)' = a^i - (P_k' a^i) (a^r)' \quad \text{for } i = 1, \dots, m \text{ but } i \neq r.$$

#### References

- [1] Charnes, A., "Optimality and Degeneracy in Linear Programming", *Econometrica*, Vol. 20, 2, April 1952, pp. 169.
- [2] Charnes, A., Cooper, W. W., Mellon, B., "A Model for Programming and Sensitivity Analysis in an Integrated Oil Company", *Econometrica*, Forthcoming.
- [3] Charnes, A., Cooper, W. W., Henderson, A., "Introduction to Linear Programming", John Wiley and Sons, 1953.
- [4] Charnes, A., Lemke, C. E., "A Modified Simplex Method for Control of Round-off Error in Linear Programming", *Proc. Assn. for Computing Machinery*, 1952, p. 97.
- [5] Lemke, C. E., "The Dual Method of Solving the Linear Programming Problem", *Naval Research Logistics Quarterly*, Jan. 1954.
- [6] Dantzig, G. B., "Activity Analysis of Production and Allocation", Chapter XXI, *Cowles Commission Monograph 13*, New York, John Wiley and Sons, 1951.

References (contd.)

- [7] Greenberg, H. J., Prager, W., "Limit Design of Beams and Frames", Proc. Amer. Soc. of Civil Engineers, Vol. 77, February, 1951.
- [8] Charnes, A., Greenberg, H. J., "Plastic Collapse and Linear Programming", abstract presented at Summer Meeting of Amer. Math. Society, Sept. 1951.

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