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Technical Report No. 5

RESPONSE OF AN ELASTIC DISK TO IMPACT AND MOVING LOADS

By

A. Cemal Eringen

to

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# RESPONSE OF AN ELASTIC DISK TO IMPACT AND MOVING LOADS

by

A. Cemal Eringen \*

Purdue University

## ABSTRACT

With the use of Fourier transforms a class of elasto-dynamic problems concerning disks have been solved. The disk is subjected to various types of dynamic loadings at the rim. The case of impact and the moving loads are studied in detail.

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(\*) Associate Professor of Division of Engineering Sciences.

## 1. Introduction

Cylindrical roller bearings in high speed mechanism are subject to dynamic loadings. Yet the usual design procedure is based on Hertz formulae which are the result of elasto-static considerations. In many other instances, gears, rollers, or disks are subject to impact or moving loads. If we neglect the coriolis terms we can also bring the rolling disks on contact into the category of disks subject to moving loads. Thus the aim of the present paper is to obtain the solution to this class of elasto-dynamic problems concerning the disk. The dynamic load is applied to the rim of the disk. Two normal concentrated dynamic loads at the two ends of a diameter moving or otherwise are special cases.

Some solutions of free oscillation of cylinders are known since the time of Pochhammer [1], and later Pickett [2], J. Mindlin [3], T. Ghosh [4]. Similarly, the problem of rotating disks has attracted attention of many authors (see, for instance, Lamb and Southwell [5], Timoshenko and Goodier [6], Love [7]). It seems, however, that the forced oscillation problems concerning disks and cylinders have escaped the attention of authors, excepting a paper by J. Mindlin [8], which consists of generalities in a related problem to ours.

The present method is applicable to ring problems and to the plane elasto-dynamic problems concerning circular holes. These problems will be treated in later papers.

## 2. Formulation of the Problem

Equations of motion of plane homogeneous isotropic media in terms of plane polar coordinates  $r$  and  $\theta$  and the time  $t$  are [7]: (See Fig. 1)

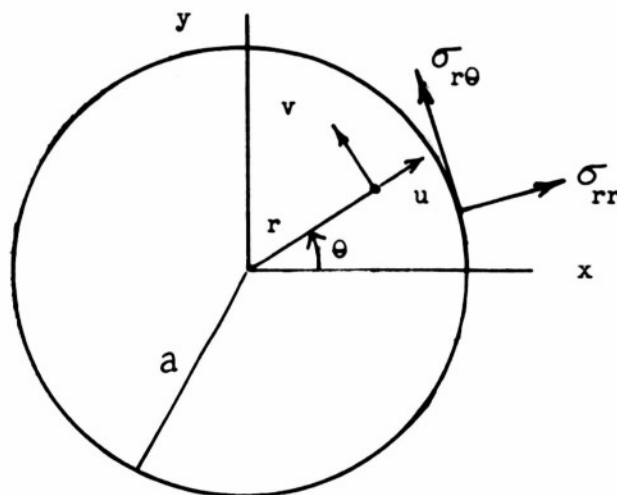


Fig. 1. Circular Disk

$$\begin{aligned} \gamma u_{,tt} &= (\lambda + 2\mu) \Delta_{,r} - 2\mu r^{-1} \omega_{,\theta} \\ \gamma v_{,tt} &= (\lambda + 2\mu) r^{-1} \Delta_{,\theta} + 2\mu \omega_{,r} \end{aligned} \quad (1)$$

where  $u(r, \theta, t)$  and  $v(r, \theta, t)$  are components of the displacement vector,  $\lambda$  and  $\mu$  are the Lamé constants, and  $\gamma$  is the mass density per unit volume. Dilatation  $\Delta$  and rotation  $\omega$  are related to  $u, v$  by:

$$r \Delta = (r u)_{,r} + v_{,\theta}, \quad 2 r \omega = (r v)_{,r} - u_{,\theta} \quad (2)$$

Subscripts after a comma represent differentiation, i.e.,  $u_{,\theta} = \partial u / \partial \theta$  etc. ....

Elimination of  $u$  and  $v$  among (1) and (2) leads to:

$$\begin{aligned}\alpha_1^2 r^2 \Delta_{,tt} &= r (r \Delta_{,r})_{,r} + \Delta_{,\theta\theta}, \quad \alpha_1^2 = \gamma / (\lambda + 2\mu) \\ \alpha_2^2 r^2 \omega_{,tt} &= r (r \omega_{,r})_{,r} + \omega_{,\theta\theta}, \quad \alpha_2^2 = \gamma / \mu\end{aligned}\quad (3)$$

These are the equations of dilatational and rotational waves.

Components  $\sigma_{rr}$ ,  $\sigma_{r\theta}$ ,  $\sigma_{\theta\theta}$  of the stress tensor are given by:

$$\begin{aligned}\sigma_{rr} &= \lambda \Delta + 2\mu u_{,r}, \quad \sigma_{r\theta} = \mu r^{-1} u_{,\theta} + \mu r(v/r)_{,r} \\ \sigma_{\theta\theta} &= \lambda \Delta + 2\mu r^{-1} (v_{,\theta} + u)\end{aligned}\quad (4)$$

The problem is to solve (1), (2) under a given  $\sigma_{rr}$  and  $\sigma_{r\theta}$  at the rim  $r = a$  of the disk:

$$\sigma_{rr}(a, \theta, t) = \sigma_0(\theta, t), \quad \sigma_{r\theta}(a, \theta, t) = \tau_0(\theta, t) \quad (5)$$

subject to the condition that these surface tractions are in equilibrium at each instant.

### 3. The Solution

The periodic solution of (3) with respect to  $\theta$  is obtained to be

$$\begin{aligned}\bar{\Delta} &= \sum_{n=0}^{\infty} (A_{1n} \sin n\theta + A_{2n} \cos n\theta) Z_n(\rho_1), \\ \bar{\omega} &= \sum_{n=0}^{\infty} (B_{1n} \cos n\theta - B_{2n} \sin n\theta) Z_n(\rho_2),\end{aligned}\quad (6)$$

$$Z_n(\rho_j) = C_{jn} J_n(\rho_j) + D_{jn} Y_n(\rho_j), \quad \rho_j = \alpha_j \tau r, \quad (j = 1, 2)$$

where  $Z_n(\rho)$  is the cylinder function  $[g]$ ;  $A_{jn}$ ,  $B_{jn}$ ,  $C_{jn}$  and  $D_{jn}$  are

constants of integration; and the barred quantities represent the Fourier transforms, i.e.

$$\bar{F}(\tau) = \int_{-\infty}^{\infty} e^{i\tau t} F(t) dt, \quad i = \sqrt{-1} \quad (7)$$

The inversion formula for (7) is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\tau} \bar{F}(\tau) d\tau \quad (8)$$

Substituting (6) into (2) after taking the Fourier transforms of (2) and solving the resulting partial differential equations we obtain:

$$\begin{aligned} \bar{u} &= \sum_{n=0}^{\infty} U_{1n}(r) \sin n\theta + U_{2n}(r) \cos n\theta \\ \bar{v} &= \sum_{n=0}^{\infty} V_{1n}(r) \cos n\theta - V_{2n}(r) \sin n\theta \end{aligned} \quad (9)$$

where

$$\begin{aligned} -r^{-1} U_{jn}(r) &= A_{jn} \rho_1^{-1} Z'_n(\rho_1) + B_{jn} 2n \rho_2^{-2} Z_n(\rho_2) \\ -r^{-1} V_{jn}(r) &= A_{jn} n \rho_1^{-2} Z_n(\rho_1) + B_{jn} 2 \rho_2^{-1} Z'_n(\rho_2) \end{aligned} \quad (10)$$

where prime represents differentiation.

Combining (4), (6), and (9) we obtain

$$\begin{aligned} \bar{\sigma}_{rr}/2\mu &= \sum_{n=0}^{\infty} [A_{1n} N_{1n}(r\tau) + B_{1n} N_{2n}(r\tau)] \sin n\theta + [A_{2n} N_{1n}(r\tau) \\ &\quad + B_{2n} N_{2n}(r\tau)] \cos n\theta \\ \bar{\sigma}_{r\theta}/2\mu &= \sum_{n=0}^{\infty} [A_{1n} S_{1n}(r\tau) + B_{1n} S_{2n}(r\tau)] \cos n\theta - [A_{2n} S_{1n}(r\tau) \\ &\quad + B_{2n} S_{2n}(r\tau)] \sin n\theta \end{aligned} \quad (11)$$

(cont'd.)

(11, cont'd.)

$$\begin{aligned} \bar{\sigma}_{\theta\theta}/2\mu = \sum_{n=0}^{\infty} [A_{1n} T_{1n}(r\tau) + B_{1n} T_{2n}(r\tau)] \sin n\theta + [A_{2n} T_{1n}(r\tau) \\ + B_{2n} T_{2n}(r\tau)] \cos n\theta \end{aligned}$$

where:

$$\begin{aligned} N_{1n}(r\tau) &= (\lambda/2\mu) z_n(\rho_1) + (1-n^2) \rho_1^{-2} z_n(\rho_1) + \rho_1^{-1} z'_n(\rho_1) \\ N_{2n}(r\tau) &= 2n \rho_2^{-2} z_n(\rho_2) - 2n \rho_2^{-1} z'_n(\rho_2) \\ S_{1n}(r\tau) &= n \rho_1^{-2} z_n(\rho_1) - n \rho_1^{-1} z'_n(\rho_1) \\ S_{2n}(r\tau) &= (1-2n^2) \rho_2^{-2} z_n(\rho_2) + 2 \rho_2^{-1} z'_n(\rho_2) \\ T_{1n}(r\tau) &= (\lambda/2\mu) z_n(\rho_1) + n^2 \rho_1^{-2} z_n(\rho_1) - \rho_1^{-1} z'_n(\rho_1) \\ T_{2n}(r\tau) &= -N_{2n}(r\tau) \end{aligned} \quad (12)$$

#### 4. Dynamic tractions applied to the rim of a disk

In the case of a disk, the stress and deformation components must be finite at  $r = 0$ . Hence  $D_{jn} = 0$ . Without loss of generality we also take  $C_{jn} = 1$ . This means in all of our formulas we must replace  $Z_n$  by  $J_n$ .

We use boundary conditions (5) to determine the constants  $A_{jn}$  and  $B_{jn}$ . The Fourier's theorem thus leads to:

$$\begin{aligned} A_{1n} &= [2\mu D_n(a\tau)]^{-1} [S_{2n}(a\tau) \tilde{\sigma}_{os} - N_{2n}(a\tau) \tilde{\tau}_{oc}] \\ B_{1n} &= [2\mu D_n(a\tau)]^{-1} [S_{1n}(a\tau) \tilde{\sigma}_{os} + N_{1n}(a\tau) \tilde{\tau}_{oc}] \\ A_{2n} &= [2\mu D_n(a\tau)]^{-1} [S_{2n}(a\tau) \tilde{\sigma}_{oc} + N_{2n}(a\tau) \tilde{\tau}_{os}] \end{aligned}$$

(cont'd.)

$$B_{2n} = [2 \mu D_n(a\tau)]^{-1} [S_{1n}(a\tau) \tilde{\sigma}_{oc} - N_{1n}(a\tau) \tilde{\tau}_{os}],$$

$$(n=1, 2, \dots),$$

$$A_{jo} = \frac{1}{2} [A_{jn}]_{n=0}, \quad B_{jo} = \frac{1}{2} [B_{jn}]_{n=0}, \quad (13)$$

$$D_n(a\tau) = N_{1n}(a\tau) S_{2n}(a\tau) - N_{2n}(a\tau) S_{1n}(a\tau)$$

where:

$$\tilde{\sigma}_{oc} = \pi^{-1} \int_0^{2\pi} \bar{\sigma}_o(\theta, \tau) \cos n\theta d\theta$$

$$\tilde{\sigma}_{os} = \pi^{-1} \int_0^{2\pi} \bar{\sigma}_o(\theta, \tau) \sin n\theta d\theta$$

similarly  $\tilde{\tau}_{oc}$  and  $\tilde{\tau}_{os}$  are defined.

Various special cases are of interest:

$$(a) \text{ Zero surface shear: } \tilde{\tau}_{oc} = \tilde{\tau}_{os} = 0 \quad (14)$$

$$(b) \text{ Normal traction with central symmetry and (a): } \tilde{\sigma}_{os} = 0 \quad (15)$$

$$(c) \text{ (b) with constant amplitude over } 0 \leq \theta \leq \alpha \text{ :}$$

$$(15) \text{ and } \tilde{\sigma}_o(\theta, \tau) = \bar{\sigma}(\tau) \begin{cases} 0 \leq \theta \leq \alpha \\ \pi - \alpha \leq \theta \leq \pi \\ 0 \quad \alpha < \theta \leq \pi - \alpha \end{cases} \quad (16)$$

Hence

$$\tilde{\sigma}_{oc} = 2 \bar{\sigma}_o(\tau) \cdot (\pi n)^{-1} \sin n\alpha \quad (17)$$

$$(d) \text{ Impact load and (c):$$

$$(16) \text{ and } \lim_{\alpha \rightarrow 0} 2 \bar{\sigma}_o(\tau) \alpha = \bar{P}_o(\tau)$$

$$\begin{aligned} \alpha &\rightarrow 0 \\ \bar{\sigma}_o &\rightarrow \infty \end{aligned}$$

Hence  $\tilde{\sigma}_{oc} = \bar{P}_o(\tau)/\pi a$  (18)

(e) Impulsive concentrated load and (d):

(18) and  $P_o(t) = P_o \delta(t)$ , where  $\delta(t)$  is the Dirac delta function defined by:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

In this case,

$$\bar{P}_o(\tau) = P_o, \quad \tilde{\sigma}_{oc} = P_o/\pi a \quad (19)$$

where  $P_o$  is the amplitude of the concentrated load.

### 5. Moving load

(a) Moving normal and tangential stresses:

Moving loads can be represented by

$$\sigma_o(\theta, t) = \sigma_o(\theta - \Omega t), \quad \tau_o(\theta, t) = \tau_o(\theta - \Omega_1 t) \quad (20)$$

Fourier Transforms of these are:

$$\bar{\sigma}_o(\theta, \tau) = 2 e^{i\tau\theta/\Omega} s_o(\tau/\Omega), \quad \bar{\tau}_o(\theta, \tau) = 2 e^{i\tau\theta/\Omega_1} s_1(\tau/\Omega_1) \quad (21)$$

$$s_o(\tau/\Omega) = \frac{1}{2\Omega} \int_{-\infty}^{\infty} \sigma_o(\phi) e^{-i\tau\phi/\Omega} d\phi \quad (\text{in general}) \quad (22)$$

$$= \frac{1}{\Omega} \int_0^{\infty} \sigma_o(\phi) \cos(\tau\phi/\Omega) d\phi \quad \text{when } \sigma_o(-\phi) = \sigma_o(\phi)$$

where  $s_1(\tau/\Omega_1)$  follows from  $s_o(\tau/\Omega)$  by replacing  $\Omega$  and  $\sigma_o(\phi)$

by  $\Omega_1$  and  $\tau_o(\phi)$  respectively. Hence:

$$\tilde{\sigma}_{oc} = (2\tau/\pi\Omega^2 i) \int (\tau/\Omega)^2 - n^2 \int^{-1} (e^{2\pi\tau i/\Omega} - 1) s_0(\tau/\Omega) \quad (23)$$

$$\tilde{\sigma}_{os} = (2n/\pi\Omega) \int (\tau/\Omega)^2 - n^2 \int^{-1} (e^{2\pi\tau i/\Omega} - 1) s_0(\tau/\Omega)$$

Quantities  $\tilde{\tau}_{oc}$  and  $\tilde{\tau}_{os}$  are obtained from (23) by writing  $\Omega_1$  and  $s_1(\tau/\Omega_1)$  in place of  $\Omega$  and  $s_0(\tau/\Omega)$  respectively.

(b) Moving periodic loads:

$$\sigma_0(\theta - \Omega t) = (Q_0/\pi a) \sum_{n=0}^{\infty} p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t) \quad (24)$$

$$\tau_0(\theta - \Omega_1 t) = (Q_0/\pi a) \sum_{n=0}^{\infty} r_n \cos n(\theta - \Omega_1 t) + s_n \sin n(\theta - \Omega_1 t)$$

After taking Fourier transforms of  $\sigma_0$  and  $\tau_0$ , we substitute into (13) to obtain  $\sigma_{oc}$ , ...,  $\tau_{os}$ . This gives:

$$\begin{aligned} a \tilde{\sigma}_{oc}/Q_0 &= (p_n - iq_n) \delta(\Omega n - \tau) + (p_n + iq_n) \delta(-\Omega n - \tau) \\ a \tilde{\sigma}_{os}/Q_0 &= (ip_n + q_n) \delta(\Omega n - \tau) + (-ip_n + q_n) \delta(-\Omega n - \tau) \end{aligned} \quad (25)$$

Quantities  $\tau_{oc}$  and  $\tau_{os}$  follow from (25) by writing  $r_n$  and  $s_n$  in place of  $p_n$  and  $q_n$ . In obtaining (25) we used the formal relation\* [10]

$$2\pi \delta(u) = \int_{-\infty}^{\infty} e^{-itu} dt$$

Components of displacements and stress tensor can now be obtained by combining (9), (11), (13) and (25) and taking inverse Fourier transforms. Thus

\* This is justified in the sense of a distribution function [11].

$$-(2\pi a \mu / Q_0 r) u = \sum_{n=0}^{\infty} u_n^{(1)}(r, \Omega n) \left[ p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t) \right] \\ + u_n^{(2)}(r, \Omega_1 n) \left[ -r_n \sin n(\theta - \Omega_1 t) + s_n \cos n(\theta - \Omega_1 t) \right]$$

$$-(2\pi a \mu / Q_0 r) v = \sum_{n=0}^{\infty} v_n^{(1)}(r, \Omega n) \left[ -p_n \sin n(\theta - \Omega t) + q_n \cos n(\theta - \Omega t) \right] \\ + v_n^{(2)}(r, \Omega_1 n) \left[ -r_n \cos n(\theta - \Omega_1 t) - s_n \sin n(\theta - \Omega_1 t) \right]$$

$$(\pi a / Q_0) \sigma_{rr} = \sum_{n=0}^{\infty} \sigma_{1n}^{(1)}(r, \Omega n) \left[ p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t) \right] \\ + \sigma_{1n}^{(2)}(r, \Omega_1 n) \left[ -r_n \sin n(\theta - \Omega_1 t) + s_n \cos n(\theta - \Omega_1 t) \right] \quad (26)$$

$$(\pi a / Q_0) \sigma_{r\theta} = \sum_{n=0}^{\infty} \tau_n^{(1)}(r, \Omega n) \left[ -p_n \sin n(\theta - \Omega t) + q_n \cos n(\theta - \Omega t) \right] \\ + \tau_n^{(2)}(r, \Omega_1 n) \left[ -r_n \cos n(\theta - \Omega_1 t) - s_n \sin n(\theta - \Omega_1 t) \right]$$

$$(\pi a / Q_0) \sigma_{\theta\theta} = \sum_{n=0}^{\infty} \sigma_{2n}^{(1)}(r, \Omega n) \left[ p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t) \right] \\ + \sigma_{2n}^{(2)}(r, \Omega_1 n) \left[ -r_n \sin n(\theta - \Omega_1 t) + s_n \cos n(\theta - \Omega_1 t) \right]$$

where

$$\begin{aligned}
u_n^{(1)}(r, \Omega n) &= \left[ D_n(a, \Omega n) \right]^{-1} \left[ (\alpha_1 r, \Omega n)^{-1} J'_n(\alpha_1 r, \Omega n) S_{2n}(a, \Omega n) \right. \\
&\quad \left. - 2n(\alpha_2 r, \Omega n)^{-2} J_n(\alpha_2 r, \Omega n) S_{1n}(a, \Omega n) \right] \\
v_n^{(1)}(r, \Omega n) &= \left[ D_n(a, \Omega n) \right]^{-1} \left[ n(\alpha_1 r, \Omega n)^{-2} J_n(\alpha_1 r, \Omega n) S_{2n}(a, \Omega n) \right. \\
&\quad \left. - 2(\alpha_2 r, \Omega n)^{-1} J'_n(\alpha_2 r, \Omega n) S_{1n}(a, \Omega n) \right] \quad (27)
\end{aligned}$$

$$\begin{aligned}
\sigma_{1n}^{(1)}(r, \Omega n) &= \left[ \bar{D}_n(a, \Omega n) \right]^{-1} \left[ \bar{N}_{1n}(r, \Omega n) S_{2n}(a, \Omega n) - N_{2n}(r, \Omega n) S_{1n}(a, \Omega n) \right] \\
\tau_n^{(1)}(r, \Omega n) &= \left[ \bar{D}_n(a, \Omega n) \right]^{-1} \left[ \bar{S}_{1n}(r, \Omega n) S_{2n}(a, \Omega n) - S_{2n}(r, \Omega n) S_{1n}(a, \Omega n) \right] \\
\sigma_{2n}^{(1)}(r, \Omega n) &= \left[ \bar{D}_n(a, \Omega n) \right]^{-1} \left[ \bar{T}_{1n}(r, \Omega n) S_{2n}(a, \Omega n) - T_{2n}(r, \Omega n) S_{1n}(a, \Omega n) \right]
\end{aligned}$$

where

$$D_n(a, \Omega n) = N_{1n}(a, \Omega n) S_{2n}(a, \Omega n) - N_{2n}(a, \Omega n) S_{1n}(a, \Omega n) \quad (28)$$

Functions  $u_n^{(2)}$ ,  $v_n^{(2)}$ ,  $\sigma_{1n}^{(2)}$ ,  $\tau_n^{(2)}$  and  $\sigma_{2n}^{(2)}$  are obtained from the corresponding ones with superscripts (1) above by replacing  $\Omega$  by  $\Omega_1$  and  $S_{2n}(a, \Omega n)$  and  $S_{1n}(a, \Omega n)$  by  $N_{2n}(a, \Omega_1 n)$  and  $N_{1n}(a, \Omega_1 n)$  respectively, except in  $D_n(a, \Omega n)$ , where we replace  $\Omega$  by  $\Omega_1$ .

(c) Two diametrically opposite, moving, concentrated loads

(Fig. 2).

This is a case of technical importance.

In this case we have

$$\tau_0 = 0 \text{ hence}$$

$$r_n = s_n = 0. \text{ The con-}$$

centrated loads can

formally be represent-

ed by

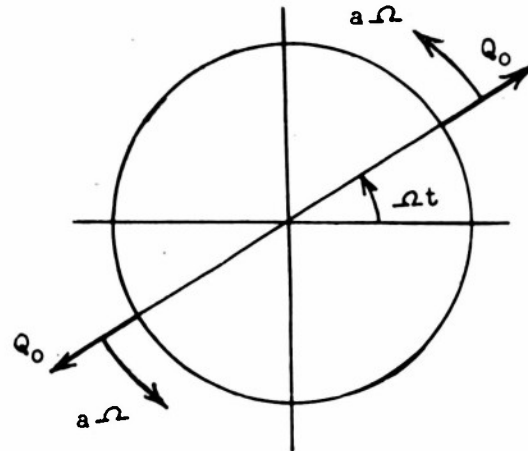


Fig. 2. Moving radial load.

$$\begin{aligned} \sigma_0(\theta - \Omega t) &= (Q_0/\pi a) [\delta(\theta - \Omega t) + \delta(\pi - \theta + \Omega t)] \\ &= (Q_0/\pi a) \sum_{n=0}^{\infty} p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t) \end{aligned} \quad (29)$$

where  $Q_0$  is the amplitude of each of the concentrated radial load.

From (29) Fourier coefficients  $p_n$  and  $q_n$  are calculated to be:

$$p_n = \begin{cases} 2/\pi & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}, \quad q_n = 0 \quad (30)$$

Hence (26) together with (29) and (30) and  $r_n = s_n = 0$  gives the displacement and stress components. Below we give displacement components. The rest is obtained in an obvious manner.

$$\begin{aligned} -(\pi^2 a \mu/Q_0 r) u &= \sum_{0,2,4,\dots} u_n^{(1)}(r, \Omega n) \cos n(\theta - \Omega t) \\ (\pi^2 a \mu/Q_0 r) v &= \sum_{0,2,4,\dots} v_n^{(1)}(r, \Omega n) \sin n(\theta - \Omega t) \end{aligned} \quad (31)$$

where  $u_n^{(1)}$  and  $v_n^{(1)}$  are given by (26).

Concentrated moving shear loads and other types of load combinations may easily be obtained from (24) and (26).

## 6. Computation and Discussion

Computations have been carried out to determine the roots of  $D_n(a_2 n) = 0$  given by (28). The roots of this equation give the resonance speed for the concentrated load.

The ratio of the resonance speed  $C_r = a_2$  of the concentrated load to the dilatational wave velocity  $C_1 = \alpha_1^{-1}$  has been solved from  $D_n = 0$  for a steel cylinder with  $E = 30 \times 10^6$  psi and  $\nu = 0.3$ . First four roots of  $D_n = 0$  for  $n = 1, 2, 3, 4$  are listed on the table given below\*.

The computation is carried out on I.B.M. Card Program Calculator machine and are correct up to two decimal places.

The values of  $n C_r / C_1 = n a_2 \alpha_1$  (obtained by calculating the roots of  $D_n(a_2 n) = 0$ ). Steel:  $E = 30 \times 10^6$ ,  $\nu = 0.3$ .

| n | $D_1 = 0$ | $D_2 = 0$ | $D_3 = 0$ | $D_4 = 0$ |
|---|-----------|-----------|-----------|-----------|
| 1 | 1.51      | 1.26      | 1.94      | 2.52      |
| 2 | 3.43      | 2.35      | 3.21      | 4.08      |
| 3 | 3.80      | 4.22      | 4.95      | 5.67      |
| 4 | 5.33      | 5.06      | 6.22      | 7.19      |

\* For  $n = 0$  we take the limit of the functions in (27) and find finite values.

From this table it is seen that the smallest  $C_r/C_1$  is obtained for  $n=2$ . This value is  $C_r/C_1 = 1.26/2 = .63$ . We therefore expect a critical speed for the moving load in the neighborhood of  $0.63 C_1$  which will create resonance in the disk.

The Rayleigh surface wave velocity for  $\nu = .29$  is 0.9258 times the velocity of shear waves or .503 times the velocity of dilational waves. Thus  $0.63 C_1$  represents a velocity between the Rayleigh surface wave velocity and the shear wave velocity. Further we notice that this velocity is minimum for  $n=2$  rather than  $n=1$ . Examining (31) we found that we have only even terms. Hence diametrically opposite moving loads give a smaller critical speed than the case of a single load. In the former case the fundamental mode  $n=0$  represents a uniform lateral extension which is not dependent on time. Thus it is static in nature and has no resonance frequency associated with it. The second mode is of  $\cos 2 \theta$  -type and gives the minimum critical speed mentioned above. It is interesting to know that the minimum critical speed in the case of a flat semi-infinite plate (the Rayleigh surface wave velocity) is less than that of the curved surface (the disk in the present case). With this point in mind perhaps one can classify the curved surfaces from the elasto-dynamic point of view. This process will, no doubt, present mathematical difficulties.

The present results, of course, must be accepted tentatively until experimental verification.

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