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APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY

CALIFORNIA

A PROOF OF THE BIEBERBACH CONJECTURE FOR THE FOURTH COEFFICIENT

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FOR

OFFICE OF NAVAL RESEARCH

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P. R. GARABEDIAN AND M. SCHIFFER

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CHAPTER I

INTRODUCTION

1. Formulation of the problem.

The family of schlicht analytic functions

$$(1.1)\qquad f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

in the unit circle $|z| < 1$ has been widely studied, and considerable interest has focused on the Bieberbach conjecture that $|a_n| \leq n$, with equality holding only for the Koebe function

$$(1.2)\qquad \frac{z}{(1-e^{i\theta}z)^2} = z + 2e^{i\theta}z^2 + 3e^{2i\theta}z^3 + 4e^{3i\theta}z^4 + \dots .$$

Bieberbach [2] showed that $|a_2| \leq 2$ and Loewner [7] established the estimate $|a_3| \leq 3$. The object of the present paper is to present a proof of the

<u>THEOREM</u>. <u>For the general class of schlicht functions</u> (1.1) <u>we have</u>

$$|a_4| \leq 4 , \tag{1.3}$$

<u>with equality holding only for the function</u> (1.2).

While our method does not seem appropriate for all values of the index n, it is our belief that it could be extended to include the next few coefficients, although the significance of the results might not be commensurate with the labor and effort required.

It is a consequence of the theory of normal families of analytic functions that there exists an extremal function $f(z)$ for the problem $|a_4|$ = maximum. Without loss of generality, we can assume that the fourth coefficient of this function is positive, since otherwise we could replace $f(z)$ by $e^{-i\theta}f(ze^{i\theta})$ to obtain, for a suitable real value of θ, a function with $a_4 > 0$. In view of the known value of the fourth coefficient of the function (1.2), we can therefore assume that

$$a_4 \geq 4 . \tag{1.4}$$

We refer to the literature [3, 12, 13, 15] for the proof of

<u>LEMMA 1.1</u>. <u>An extremal function</u> $f(z)$ <u>for the problem</u> $|a_4|$ = maximum <u>with</u> $a_4 > 0$ <u>satisfies the ordinary differential equation</u>

$$(1.5) \qquad \frac{z^2 f'(z)^2}{f(z)^2}\left[\frac{1}{f(z)^3}+\frac{3a_2}{f(z)^2}+\frac{2a_3+a_2^2}{f(z)}\right]$$

$$=\left(\frac{1}{z^3}+\frac{2a_2}{z^2}+\frac{3a_3}{z}+3a_4+3\bar{a}_3 z+2\bar{a}_2 z^2+z^3\right) .$$

<u>The interior of the unit circle</u> $|z| < 1$ <u>is mapped by the extremal function onto the exterior of a set of analytic arcs in the w-plane which satisfy the ordinary differential equation</u>

$$(1.6) \qquad \frac{dw^2}{w^2}\left[\frac{1}{w^3}+\frac{3a_2}{w^2}+\frac{2a_3+a_2^2}{w}\right] \leq 0 \quad .$$

The derivation of the differential equations (1.5) and (1.6) is based on the method of interior variations and it forms a decisive step in our proof of (1.3). Analogous differential equations for the functions maximizing the higher coefficients a_n can be obtained with equal ease, and they are all satisfied by the Koebe function (1.2), with $\theta = 0$.

It is easy to show [12, 14] that the extremal arcs satisfying (1.6) do not fork in the finite part of the w-plane, and we shall also use here the known result [12] that only one extremal arc extends to the point at infinity, although this follows independently from calculations in the present paper. We are thus able to formulate in simple terms

<u>LEMMA 1.2. The extremal function</u> $f(z)$ <u>maps the interior of the unit circle onto the exterior of a single analytic slit. There is a real-valued analytic function</u> $\varphi(t)$ <u>on the interval</u> $0 < t < 1$ <u>such that the coefficients</u> a_2 , a_3 <u>and</u> a_4 <u>of the extremal function</u> $f(z)$ <u>have the Loewner representation</u>

$$a_2 = 2 \int_0^1 e^{i\varphi(t)}dt \quad , \tag{1.7}$$

$$a_3 = -2 \int_0^1 te^{2i\varphi(t)}dt + a_2^2 \quad , \tag{1.8}$$

$$a_4 = 2 \int_0^1 t^2 e^{3i\varphi(t)}dt + 4 \int_0^1 \int_{t_1=t_2}^1 t_1 e^{2i\varphi(t_1)} e^{i\varphi(t_2)} dt_1 dt_2 \\ + 3a_2a_3 - 2a_2^3 \quad . \tag{1.9}$$

For the proof of Lemma 1.2 we refer to the literature [7, 10, 12, 16]. The formulas (1.7), (1.8), and (1.9) are due to Loewner [7], and the analyticity of his function $k(t) = e^{i\varphi(t)}$ for the representation of the extremal mapping $f(z)$ is simply a consequence of the analyticity of the boundary slit in the w-plane described in Lemma 1.1.

Our first innovation is

<u>LEMMA 1.3</u>. <u>The coefficients a_2 and a_3 of the extremal function</u> $f(z)$ <u>satisfy two equations defined by the boundary value problem</u>

$$\dot{a} = -\frac{4}{3}k \quad , \quad \dot{b} = -\frac{2}{3}tk^2 \quad , \tag{1.10}$$

$$\operatorname{Im}\left\{t^2k^3 - atk^2 + bk\right\} = 0 \quad , \tag{1.11}$$

$$a(0) = 2a_2 \quad , \quad b(0) = \frac{2a_3 + a_2^2}{3} \quad , \tag{1.12}$$

$$a(1) = \frac{4}{3}a_2 \quad , \quad b(1) = a_3 \quad , \tag{1.13}$$

<u>for two unknown functions</u> $a(t)$ <u>and</u> $b(t)$ <u>on the interval</u> $0 \leq t \leq 1$, <u>where</u> $k = k(t) = e^{i\varphi(t)}$ <u>with</u> $\varphi(t)$ <u>real, and where the dot indicates differentiation with respect to</u> t .

The proof of Lemma 1.3 is based on the extremal property of the coefficient a_4 represented by formulas (1.7), (1.8), and (1.9) in terms of the function $\varphi(t)$. Loewner's parametric representation [7] of schlicht functions (1.1) in the unit circle $|z| < 1$ shows that for any continuous real-valued function $\varphi(t)$ the expressions (1.7), (1.8), and (1.9) define the first three non-trivial coefficients of a schlicht function. Thus we can substitute for $\varphi(t)$ in these formulas an arbitrary expression of the form $\varphi(t) + rh(t)$, where r is any real number and $h(t)$ is an arbitrary continuous real-valued function on the interval $0 \leq t \leq 1$, and we will obtain a new fourth coefficient $a_4(r)$ whose real part does not exceed a_4 . It follows that for $r = 0$ we must have

$$\frac{d}{dr} \operatorname{Re}\left\{a_4(r)\right\} = 0 \tag{1.14}$$

for every continuous real function $h(t)$. Explicit evaluation of the condition (1.14) yields

$$(1.15)\quad \operatorname{Im}\left\{6\int_0^1 t^2e^{3i\varphi(t)}h(t)dt + 4\int_0^1\int_{t_1=t_2}^1 t_1e^{2i\varphi(t_1)}e^{i\varphi(t_2)}h(t_2)dt_1dt_2\right.$$

$$+\,8\int_0^1\int_0^{t_2=t_1} e^{i\varphi(t_2)}t_1e^{2i\varphi(t_1)}h(t_1)dt_2dt_1 - 12a_2\int_0^1 te^{2i\varphi(t)}h(t)dt$$

$$\left.+\,6a_3\int_0^1 e^{i\varphi(t)}h(t)dt\right\} = 0 \;.$$

Since $h(t)$ is arbitrary, we deduce from (1.15) the identity

$$(1.16)\quad \operatorname{Im}\left\{t^2k(t)^3 + \frac{2}{3}k(t)\int_{t_1=t}^1 t_1k(t_1)^2dt_1\right.$$

$$\left.+\,\frac{4}{3}tk(t)^2\int_0^{t_2=t} k(t_2)dt_2 - 2a_2tk(t)^2 + a_3k(t)\right\} = 0 \;.$$

We set

$$(1.17)\quad a(t) = 2a_2 - \frac{4}{3}\int_0^t k(t_2)dt_2 \;,$$

$$(1.18)\quad b(t) = a_3 + \frac{2}{3}\int_t^1 t_1k(t_1)^2dt_1 \;,$$

and we find that (1.10) follows by direct differentiation of (1.17) and (1.18), while (1.11) is equivalent to (1.16), and (1.12) and (1.13) are consequences of the Loewner formulas (1.7) and (1.8). This proves the lemma. It is an interesting unsolved problem to establish the main result (1.3) by showing directly from Lemma 1.3 that $a_2 = 2$ and $a_3 = 3$.

For the extremal function $f(z)$, we can express a_4 in terms of a_2 and a_3 by using

<u>LEMMA 1.4. There is an angle ψ such that</u>

$$a_3 = \frac{a_4}{2} e^{i\psi} + a_2 e^{-i\psi} + \frac{1}{3} e^{4i\psi} - \frac{2}{3} e^{-2i\psi} - \frac{\bar{a}_2}{3} e^{3i\psi} \quad . \tag{1.19}$$

The derivative $f'(z)$ of the extremal function must have a zero on the unit circle at the point which corresponds to the finite end of the analytic slit in the w-plane bounding the extremal region. We denote this zero by $-e^{i\psi}$ and we deduce from Lemma 1.1 that it must be a double zero of the right-hand side of the differential equation (1.5). Thus

$$-e^{-3i\psi} + 2a_2e^{-2i\psi} - 3a_3e^{-i\psi} + 3a_4 - 3\bar{a}_3e^{i\psi} + 2\bar{a}_2e^{2i\psi} - e^{3i\psi} = 0 \quad , \tag{1.20}$$

and

$$3e^{-3i\psi} - 4a_2e^{-2i\psi} + 3a_3e^{-i\psi} - 3\bar{a}_3e^{i\psi} + 4\bar{a}_2e^{2i\psi} - 3e^{3i\psi} = 0 \quad . \tag{1.21}$$

Subtracting (1.20) from (1.21) and dividing through by $6e^{-i\psi}$, we obtain (1.19). A comparison of (1.21) with (1.11) and (1.13) shows that $\psi = \varphi(1)$.

In addition to the results mentioned so far, we shall need the more elementary

LEMMA 1.5. *The function*

$$f(z^{-2})^{-1/2} = z + \frac{b_1}{z} + \frac{b_3}{z^3} + \frac{b_5}{z^5} + \cdots \tag{1.22}$$

is schlicht and

$$\sum_{\nu=1}^{\infty} (2\nu - 1)\left|b_{2\nu-1}\right|^2 \leq 1 \ . \tag{1.23}$$

This is the classical area theorem and hardly requires proof here. Note that for the extremal mapping, (1.23) is actually an equality.

In order to exploit Lemma 1.5, we derive

LEMMA 1.6. *The coefficients* a_5, a_6 *and* a_7 *of the extremal function* $f(z)$ *can be expressed in terms of the earlier coefficients* a_2, a_3 *and* a_4 *by the formulas*

$$a_5 = \frac{2}{5} a_2 a_4 + \frac{3}{5} \bar{a}_3 \ , \tag{1.24}$$

$$a_6 = \frac{6}{7} a_3 a_4 - \frac{16}{35} a_2^2 a_4 - \frac{3}{7} a_2 a_3^2 + \frac{4}{7} a_2^3 a_3 - \frac{1}{7} a_2^5 + \frac{6}{35} a_2 \bar{a}_3 + \frac{2}{7} \bar{a}_2 \ , \tag{1.25}$$

$$a_7 = a_4^2 - \frac{54}{35} a_2 a_3 a_4 + \frac{64}{105} a_2^3 a_4 - \frac{4}{9} a_3^3 + \frac{40}{21} a_2^2 a_3^2 - \frac{10}{7} a_2^4 a_3 \tag{1.26}$$

$$+ \frac{19}{63} a_2^6 - \frac{8}{35} a_2^2 \bar{a}_3 + \frac{2}{5} a_3 \bar{a}_3 + \frac{4}{63} a_2 \bar{a}_2 + \frac{1}{9} \ .$$

This lemma is obtained by substituting the power series (1.1) into the differential equation (1.5) and equating the coefficients of corresponding powers of z on both sides. Higher coefficients could be obtained by laborious calculations with the same procedure. Note that (1.24) is the elementary result due to Marty [8].

In order to reduce the calculations required to derive Lemma 1.6, we use the expansion

$$(1.27)\qquad -\frac{1}{4}\frac{d}{dz}\left[\frac{1}{f(z)^4}+\frac{4a_2}{f(z)^3}+\frac{4a_3+2a_2^2}{f(z)^2}\right]=\frac{1}{z^5}-\frac{a_4}{z^2}$$

$$+(a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+3a_2a_3^2-4a_2^3a_3+a_2^5)$$

$$+(2a_7-4a_2a_6-6a_3a_5+8a_2^2a_5-5a_4^2+24a_2a_3a_4-16a_2^3a_4+4a_3^3$$

$$-24a_2^2a_3^2+22a_2^4a_3-5a_2^6)z+\cdots,$$

which is based on the theory of Faber polynomials. We substitute (1.27) into the left-hand side of (1.5) and multiply out the resulting power series in z to obtain

$$(1.28)\qquad \frac{1}{z^3}+\frac{2a_2}{z^2}+\frac{3a_3}{z}+3a_4+(5a_5-2a_2a_4)z+(7a_6-2a_2a_5-6a_3a_4+4a_2^2a_4$$

$$+3a_2a_3^2-4a_2^3a_3+a_2^5)z^2+(9a_7-2a_2a_6-6a_3a_5+4a_2^2a_5-9a_4^2+18a_2a_3a_4$$

$$-8a_2^3a_4+4a_3^3-18a_2^2a_3^2+14a_2^4a_3-3a_2^6)z^3+\cdots$$

$$=\frac{1}{z^3}+\frac{2a_2}{z^2}+\frac{3a_3}{z}+3a_4+3\bar{a}_3z+2\bar{a}_2z^2+z^3.$$

Equating the coefficients of z, of z^2, and of z^3 on both sides of (1.28), we establish successively the relations (1.24), (1.25), and (1.26).

The formulas (1.24), (1.25), and (1.26) are used in conjunction with Lemma 1.5 to prove

<u>LEMMA 1.7.</u> *The coefficients a_2, a_3 and a_4 of the extremal function $f(z)$ satisfy the inequality*

$$(1.29) \qquad 1 \geq \left|\frac{1}{2} a_2\right|^2 + 3\left|\frac{1}{2} a_3 - \frac{3}{8} a_2^2\right|^2 + 5\left|\frac{1}{2} a_4 - \frac{3}{4} a_2 a_3 + \frac{5}{16} a_2^3\right|^2$$

$$+ 7\left|\frac{11}{20} a_2 a_4 - \frac{3}{10} \bar{a}_3 + \frac{3}{8} a_3^2 - \frac{15}{16} a_2^2 a_3 + \frac{35}{128} a_2^4\right|^2$$

$$+ 9\left|\frac{9}{28} a_3 a_4 - \frac{229}{560} a_2^2 a_4 - \frac{1}{7} \bar{a}_2 + \frac{51}{140} a_2 \bar{a}_3 - \frac{81}{112} a_2 a_3^2 + \frac{181}{224} a_2^3 a_3 - \frac{313}{1792} a_2^5\right|^2$$

$$+ 11\left|\frac{1}{8} a_4^2 + \frac{9}{56} a_2 a_3 a_4 - \frac{239}{3360} a_2^3 a_4 + \frac{1}{18} - \frac{23}{126} a_2 \bar{a}_2 - \frac{1}{4} a_3 \bar{a}_3 + \frac{179}{560} a_2^2 \bar{a}_3\right.$$

$$\left. + \frac{13}{144} a_3^3 - \frac{493}{1344} a_2^2 a_3^2 + \frac{157}{1792} a_2^4 a_3 + \frac{2087}{64512} a_2^6\right|^2 .$$

To derive (1.29), we notice that (1.22) yields

$$(1.30) \qquad b_1 = -\frac{1}{2} a_2 , \quad b_3 = -\frac{1}{2} a_3 + \frac{3}{8} a_2^2 , \quad b_5 = -\frac{1}{2} a_4 + \frac{3}{4} a_2 a_3 - \frac{5}{16} a_2^3 ,$$

$$(1.31) \qquad b_7 = -\frac{1}{2} a_5 + \frac{3}{4} a_2 a_4 + \frac{3}{8} a_3^2 - \frac{15}{16} a_2^2 a_3 + \frac{35}{128} a_2^4 ,$$

$$(1.32) \qquad b_9 = -\frac{1}{2} a_6 + \frac{3}{4} a_2 a_5 + \frac{3}{4} a_3 a_4 - \frac{15}{16} a_2^2 a_4 - \frac{15}{16} a_2 a_3^2 + \frac{35}{32} a_2^3 a_3 - \frac{63}{256} a_2^5 ,$$

$$(1.33) \qquad b_{11} = -\frac{1}{2}a_7 + \frac{3}{4}a_2a_6 + \frac{3}{4}a_3a_5 + \frac{3}{8}a_4^2 - \frac{15}{16}a_2^2a_5 - \frac{15}{8}a_2a_3a_4$$

$$-\frac{5}{16}a_3^3 + \frac{35}{32}a_2^3a_4 + \frac{105}{64}a_2^2a_3^2 - \frac{315}{256}a_2^4a_3 + \frac{231}{1024}a_2^6 \ .$$

By recourse to tedious algebra one can substitute the expressions (1.24), (1.25) and (1.26) into (1.30), (1.31), (1.32) and (1.33) in order to obtain formulas for the coefficients $b_1, b_3, \ldots, b_{11}$ in terms of a_2, a_3 and a_4 alone. These formulas, together with the first six terms of the inequality (1.23), establish the lemma.

The remaining inequalities required for our analysis will be formulated as

<u>LEMMA 1.8</u>. <u>The coefficients</u> a_2, a_3 <u>and</u> a_4 <u>satisfy the inequalities</u>

$$(1.34) \qquad |a_2| \leq 2 \ , \quad |a_3| \leq 3 \ , \quad |a_2^2 - a_3| \leq 1 \ ,$$

$$(1.35) \qquad \left| a_4 - 2a_2a_3 + \frac{13}{12}a_2^3 \right| \leq \frac{2}{3} \ .$$

The estimates (1.34) follow in a familiar way [7] from Loewner's formulas (1.7) and (1.8). A proof of inequality (1.35) can be found in reference [9], but for the sake of completeness we sketch here an alternate demonstration.

It suffices to discuss the extremal problem

$$(1.36) \qquad \mathrm{Re}\left\{ a_4 - 2a_2a_3 + \frac{13}{12}a_2^3 \right\} = \text{maximum} \ .$$

The extremal region for this problem is bounded by analytic arcs which satisfy the differential equation

$$\frac{dw^2}{w^2}\left[\frac{1}{w^3}+\frac{a_2}{w^2}+\frac{a_2^2}{4w}\right] \leq 0 \quad , \tag{1.37}$$

analogous to (1.6). Since the left-hand side of (1.37) is a perfect square, we can integrate in an elementary fashion, and we find that along the extremal arcs

$$\operatorname{Re}\left\{2w^{-3/2} + 3a_2w^{-1/2}\right\} = 0 \quad . \tag{1.38}$$

From this relation and the Schwarz principle of reflection it can be established that the only extremal functions for (1.36) are the Koebe function (1.2) and the mapping

$$w = (z^{-3} - 2 + z^3)^{-1/3} \quad . \tag{1.39}$$

Both functions give equality in (1.35) and hence the bound is correct.

We have now developed sufficient background to describe our proof of the conjecture (1.3). In the next chapter we use the inequalities (1.4) and (1.29) to establish bounds on $|a_2 - 2|$ and $|a_3 - 3|$ which are of the order of a hundredth. In the final chapter of the paper we show that if a_2 and a_3 are thus limited, then the equations (1.10), (1.11), (1.12) and (1.13) imply that $a_2 = 2$ and $a_3 = 3$. This is enough to prove (1.3). The first part of the method amounts to solving the differential equation (1.5) by recursion in a power series. The second part consists in solving the non-linear boundary value problem of Lemma 1.3 by perturbations in the neighborhood of the known solution $k \equiv 1$.

CHAPTER II

PRELIMINARY ESTIMATION OF a_2 AND a_3

1. Application of the triangle inequality.

In this chapter we use the inequality (1.29) in order to restrict the region of possible values of the coefficients a_2 and a_3 of the extremal function. As a first step we introduce the new variable $\lambda = a_3 - \frac{3}{4} a_2^2$ and derive

LEMMA 2.1. *If* $\lambda = a_3 - \frac{3}{4} a_2^2$, *then*

$$
\begin{aligned}
(2.1) \qquad 1 \geq\ & \frac{1}{4}\left|a_2\right|^2 + \frac{3}{4}\left|\lambda\right|^2 + \frac{5}{4}\left|a_4 - \frac{3}{2} a_2\lambda - \frac{1}{2} a_2^3\right|^2 \\
& + 7\left|\frac{11}{20} a_2 a_4 - \frac{7}{32} a_2^4 - \frac{9}{40} \bar{a}_2^2 - \frac{3}{10} \bar{\lambda} + \frac{3}{8}\lambda^2 - \frac{3}{8} a_2^2 \lambda\right|^2 \\
& + 9\left|\frac{47}{280} a_2^2 a_4 + \frac{1}{7} \bar{a}_2 - \frac{153}{560} a_2 \bar{a}_2^2 - \frac{11}{448} a_2^5 + \left(\frac{31}{112} a_2^3 - \frac{9}{28} a_4\right)\lambda \right. \\
& \left. - \frac{51}{140} a_2 \bar{\lambda} + \frac{81}{112} a_2 \lambda^2\right|^2 + 11\left|\frac{1}{8} a_4^2 + \frac{83}{1680} a_2^3 a_4 + \frac{1}{18} + \frac{111}{1120} a_2^2 \bar{a}_2^2 \right. \\
& - \frac{23}{126} a_2 \bar{a}_2 - \frac{283}{4032} a_2^6 + \frac{13}{144}\lambda^3 - \frac{1}{4}\lambda\bar{\lambda} \\
& \left. + \left(\frac{9}{56} a_2 a_4 - \frac{3}{16} \bar{a}_2^2 - \frac{139}{448} a_2^4\right)\lambda + \frac{37}{280} a_2^2 \bar{\lambda} - \frac{55}{336} a_2^2 \lambda^2\right|^2 .
\end{aligned}
$$

This lemma is a direct consequence of the inequality (1.29) when we substitute for a_3 the value

$$a_3 = \lambda + \frac{3}{4} a_2^2 \quad . \tag{2.2}$$

Our aim in the present section will be to show from (2.1) that $|a_2|$ is very close to 2 . We have

<u>LEMMA 2.2.</u> <u>The terms</u>

$$|b_1|^2 + 3|b_3|^2 + 5|b_5|^2 \leq 1 \tag{2.3}$$

<u>alone in</u> (2.1) <u>imply the estimates</u>

$$|a_2| \geq 1.67 \quad , \quad |\lambda| \leq .64 \quad . \tag{2.4}$$

For the proof, we set

$$A = |a_2| \quad , \quad L = |\lambda| \quad , \tag{2.5}$$

and we find by (1.4) and (2.1) that

$$(4 - A^2 - 3L^2)^{1/2} - 5^{1/2}(4 - 1.5AL - .5A^3) \geq 0 \quad . \tag{2.6}$$

We maximize the left-hand side of (2.6) with respect to L and find for L

$$2L(4 - A^2 - 3L^2)^{-1/2} = 5^{1/2}A \quad ,$$

whence

$$L = 5^{1/2}A\left(\frac{4 - A^2}{4 + 15A^2}\right)^{1/2} , \quad 4 - A^2 - 3L^2 = 4\,\frac{4 - A^2}{4 + 15A^2} .$$

Therefore A will satisfy the inequality

$$(2.7) \qquad (\tfrac{4}{5} + 3A^2)^{1/2}\,(4 - A^2)^{1/2} + A^3 \geq 8 .$$

This relation is not fulfilled in the interval $0 \leq A \leq 1.67$, and thus the bound (2.4) on $|a_2|$ is established. The bound on $|\lambda|$ is now obtained from the estimate

$$(2.8) \qquad 3|\lambda|^2 \leq 4 - A^2 \leq 4 - (1.67)^2 = 1.2111 ,$$

and the lemma is proved.

LEMMA 2.3. *Let* $a_2 = Ae^{i\alpha}$ *and normalize the extremal function* $f(z)$ *so that* $0 \leq \alpha \leq \pi/3$. *Then*

$$(2.9) \qquad 0 \leq \alpha \leq .22 .$$

The normalization is obtained by means of the substitutions $\overline{f(\overline{z})}$ or $e^{-i\theta}f(ze^{i\theta})$, $\theta = \pm 2\pi/3$: By Lemma 2.2

$$\left|a_4 - 1.5a_2\lambda - .5a_2^3\right| \leq (.24222)^{1/2} < .493 ,$$

whence

$$\left|a_4 - .5a_2^3\right| \leq 2.413$$

and

(2.10) $$\left|\sin 3\alpha\right| \leq .604 \quad ,$$

and (2.9) follows.

<u>LEMMA 2.4</u>. <u>The terms</u>

(2.11) $$\left|b_1\right|^2 + 3\left|b_3\right|^2 + 5\left|b_5\right|^2 + 7\left|b_7\right|^2 \leq 1$$

<u>alone in</u> (2.1) <u>imply the estimates</u>

(2.12) $$\left|a_2\right| \geq 1.92 \quad , \quad \left|\lambda\right| \leq .324 \quad .$$

We adopt again the notation (2.5) and in addition we set

(2.13) $$\mu = a_4 - \frac{3}{2}\, a_2 \lambda - \frac{1}{2}\, a_2^3 \quad , \quad M = \left|\mu\right| .$$

Applying the triangle inequality to (2.11), we find

$$(2.14)\quad (4 - A^2 - 3L^2 - 5M^2)^{1/2} \geq (28)^{1/2} \left| \frac{3}{10} a_2 a_4 - \frac{3}{32} a_2^4 - \frac{9}{40} \bar{a}_2^2 + \frac{1}{4} a_2 \mu - \frac{3}{10} \bar{\lambda} + \frac{3}{8} \lambda^2 \right|$$

$$\geq (28)^{1/2} \left[\frac{6}{5} A \cos\alpha - \frac{3}{32} A^4 \cos 4\alpha - \frac{9}{40} A^2 \cos 2\alpha - \frac{1}{4} AM - \frac{3}{10} L \cos\theta + \frac{3}{8} L^2 \cos 2\theta \right] ,$$

where $\lambda = L e^{i\theta}$. We can set $\alpha = 0$ in (2.14), since the right-hand side is an increasing function of α in the relevant interval $0 \leq \alpha \leq .22$. Indeed, we have there, since $A \geq 1.67$,

$$- \frac{6}{5} A \sin\alpha + \frac{3}{8} A^4 \sin 4\alpha + \frac{9}{20} A^2 \sin 2\alpha \geq 0 .$$

Also we find

$$\frac{3}{8} L^2 \cos 2\theta - \frac{3}{10} L \cos\theta = \frac{3}{4}(L \cos\theta - \frac{1}{5})^2 - \frac{3}{8} L^2 - \frac{3}{100} \geq - \frac{3}{8} L^2 - \frac{3}{100} ,$$

and hence

$$(2.15)\quad (4 - A^2 - 3L^2 - 5M^2)^{1/2} - (28)^{1/2}[1.2A - .09375A^4 - .225A^2 - .03 - .25AM - .375L^2] \geq 0 .$$

We maximize the left-hand side of (2.15) and find for M the worst value

$$M = \left(\frac{28A^2 - 7A^4 - 21A^2L^2}{100 + 35A^2}\right)^{1/2} ,$$

whence (2.15) yields

$$(2.16) \qquad (4 - A^2 - 3L^2)^{1/2}(100 + 35A^2)^{1/2}$$

$$- 7^{1/2}[24A - 1.875A^4 - .6 - 4.5A^2 - 7.5L^2] \geq 0 .$$

Maximizing with respect to L^2 gives for L

$$\frac{100 + 35A^2}{4 - A^2 - 3L^2} = 175 ,$$

or

$$(2.17) \qquad L^2 = \frac{8}{7} - \frac{2}{5}A^2 .$$

There are two cases, according as

$$(2.18) \qquad A^2 < \frac{20}{7}$$

or

$$(2.19) \qquad A^2 \geq \frac{20}{7} .$$

In the first case, (2.16) reduces to the inequality

$$\frac{80}{7} - 24A + 1.875A^4 + 2.5A^2 + .6 \geq 0 \quad .$$

This inequality is not fulfilled in the interval $1.67 \leq A \leq (20/7)^{1/2}$, and thus by Lemma 2.2 the alternative (2.19) must prevail. But then $L = 0$ is the least favorable possibility and (2.16) yields

$$[(4 - A^2)(100 + 35A^2)/7]^{1/2} - 24A + 1.875A^4 + 4.5A^2 + .6 \geq 0 \quad . \tag{2.20}$$

In the interval $(20/7)^{1/2} \leq A \leq 1.92$ the relation (2.20) is not satisfied, and this establishes the lower bound (2.12) on A. The upper bound (2.12) on L follows again from our lower estimate on A and the inequality $3L^2 \leq 4 - A^2$. This proves the lemma.

<u>LEMMA 2.5</u>. <u>We have</u>

$$0 \leq \alpha \leq .104 \quad , \quad 0 \leq \operatorname{Im} a_2 \leq .208 \quad . \tag{2.21}$$

Using (2.3), we derive the inequality

$$\left|a_4 - .5\, a_2^3\right| \leq 3L + (.8 - .2A^2)^{1/2} \leq 1.2225 \quad ,$$

whence $\sin 3\alpha \leq .30563$ and the lemma follows.

LEMMA 2.6. *The terms*

$$(2.22) \qquad |b_1|^2 + 3|b_3|^2 + 5|b_5|^2 + 7|b_7|^2 + 9|b_9|^2 \leq 1$$

alone in (2.1) *imply the estimate*

$$(2.23) \qquad |a_2| \geq 1.95 \quad .$$

We treat first the case $|\theta - \pi| \leq \pi/4$, where again $\lambda = L e^{i\theta}$. With this hypothesis we have $Re\{a_2 \lambda\} \leq 0$, by Lemma 2.5, and hence by (2.3)

$$(.8 - .2A^2)^{1/2} \geq 4 - .5A^3 \quad .$$

This inequality is not satisfied in the interval $1.92 \leq A \leq 1.95$, and therefore we can restrict ourselves in the remainder of the proof of Lemma 2.6 to the case $|\theta - \pi| \geq \pi/4$.

Since $|\lambda| \leq .324$ and $A \geq 1.92$, we have

$$Re\left\{\frac{47}{280} a_2^2 e^{-i\alpha}(a_4 - 4) - \frac{9}{28} e^{-i\alpha}(a_4 - 4)\lambda\right\} \geq 0 \quad ,$$

and therefore by (2.1)

$$(2.24) \qquad \frac{1}{3}\left(1 - \frac{1}{4} A^2 - \frac{3}{4} L^2 - \frac{5}{4} M^2\right)^{1/2} \geq \frac{47}{70} A^2 \cos\alpha + \frac{1}{7} A \cos 2\alpha$$

$$- \frac{153}{560} A^3 \cos 2\alpha - \frac{11}{448} A^5 \cos 4\alpha - \mathrm{Re}\left\{\frac{9}{7} \lambda e^{-i\alpha} - \frac{31}{112} A^3 e^{2i\alpha}\lambda + \frac{51}{140} A\bar{\lambda} - \frac{81}{112} A \lambda^2\right\}$$

$$= \frac{47}{70} A^2 \cos\alpha + \frac{1}{7} A \cos 2\alpha - \frac{153}{560} A^3 \cos 2\alpha - \frac{11}{448} A^5 \cos 4\alpha$$

$$- L\left\{\frac{9}{7} \cos(\theta - \alpha) - \frac{31}{112} A^3 \cos(2\alpha + \theta) + \frac{51}{140} A \cos\theta - \frac{81}{56} AL \cos^2\theta + \frac{81}{112} AL\right\}$$

$$= \frac{47}{70} A^2 \cos\alpha + \frac{1}{7} A \cos 2\alpha - \frac{153}{560} A^3 \cos 2\alpha - \frac{11}{448} A^5 \cos 4\alpha - \frac{81}{112} AL^2$$

$$- L\left\{\left[\frac{9}{7} \cos\alpha - \frac{31}{112} A^3 \cos 2\alpha + \frac{51}{140} A\right]\cos\theta - \frac{81}{56} AL \cos^2\theta\right.$$

$$\left. + \left[\frac{9}{7} \sin\alpha + \frac{31}{112} A^3 \sin 2\alpha\right]\sin\theta\right\} .$$

If $\sin\theta \geq 0$, then $\sin(\theta - \alpha) \geq 0$, since $|\theta - \pi| \geq \pi/4$, and we can use the formula

$$|a_4 - 1.5a_2\lambda - .5a_2^3| = M$$

to derive the estimate

$$\sin 3\alpha = (4 \cos^2\alpha - 1)\sin\alpha \leq \frac{M}{.5A^3} .$$

Using this inequality, the estimate $M < .251$ based on Lemma 2.4, and Lemma 2.5, we find the improved bound $\sin\alpha \leq .024$ and this yields in turn

$$\sin\alpha \leq .6673 \frac{M}{A^3}$$

when $\sin\theta \geq 0$. Also we find by completing the square that

$$\frac{81}{56} AL^2 \cos^2\theta - [\frac{9}{7}\cos\alpha - \frac{31}{112}A^3 \cos 2\alpha + \frac{51}{140}A]L\cos\theta$$

$$\geq -\frac{14}{81A}[\frac{9}{7}\cos\alpha + \frac{51}{140}A - \frac{31}{112}A^3\cos 2\alpha]^2 ,$$

and hence (2.24) yields

$$(2.25)\quad \frac{1}{3}(1 - \frac{A^2}{4} - \frac{3L^2}{4} - \frac{5M^2}{4})^{1/2} \geq \frac{47}{70}A^2\cos\alpha + \frac{1}{7}A\cos 2\alpha - \frac{153}{560}A^3\cos 2\alpha$$

$$- \frac{11}{448}A^5\cos 4\alpha - \frac{81}{112}AL^2 - .6673[\frac{9}{7A^3} + \frac{31}{56}]ML$$

$$- \frac{14}{81A}[\frac{9}{7}\cos\alpha + \frac{51}{140}A - \frac{31}{112}A^3\cos 2\alpha]^2$$

$$\geq .6714A^2\cos\alpha + .1428A\cos 2\alpha - .2733A^3\cos 2\alpha - .02456A^5\cos 4\alpha$$

$$- .723215AL^2 - .490616ML$$

$$- .1729A[1.2857\frac{\cos\alpha}{A} + .364286 - .276786A^2\cos 2\alpha]^2 ,$$

since $A \geq 1.92$. The right-hand side of (2.25) is an increasing function of α in the relevant interval $0 \leq \alpha \leq .104$, since

$$- .6714A^2 \sin \alpha - .2856A \sin 2\alpha + .5466A^3 \sin 2\alpha + .09824A^5 \sin 4\alpha$$

$$- .1729A \frac{d}{d\alpha} [1.2857 \frac{\cos \alpha}{A} + .364286 - .276786A^2 \cos 2\alpha]^2 \geq 0$$

there. Hence we need only consider the case $\alpha = 0$. Furthermore, if we assume $A \leq 1.95$ and use the inequality between arithmetic and geometric means, we obtain

$$.723215AL^2 + .490616ML \leq 1.41027L^2 + (.245308)(.102538)L^2$$

$$+ \frac{.245308}{.102538} M^2 \leq 1.9139(.75L^2 + 1.25M^2) .$$

Thus we set $2N^2 = .75L^2 + 1.25M^2$ and (2.25) yields

$$\frac{1}{3}(1 - \frac{1}{4} A^2 - 2N^2)^{1/2} + 3.8278N^2 \geq .6714A^2 + .1428A - .2733A^3$$

$$- .02456A^5 - .1729A[1.2857A^{-1} + .364286 - .276786A^2]^2 .$$

We maximize with respect to N^2 and find that the least favorable value is

$$N^2 = .49621 - \frac{A^2}{8} .$$

Hence

(2.26) $$1.92843 \geq 1.149875A^2 + .1428A - .2733A^3 - .02456A^5$$

$$- .1729A[1.2857A^{-1} + .364286 - .276786A^2]^2 \quad .$$

The final inequality (2.26) is not satisfied in the interval $1.92 \leq A \leq 1.95$, and therefore the lemma is proved.

2. Estimation of imaginary parts.

Our later estimates will be simpler to derive if we expand the main inequality (1.29) in terms of the differences between the coefficients a_2, a_3 and a_4 and the conjectured values 2, 3 and 4. We have

LEMMA 2.7. *If*

(2.27) $$a_2 = 2 - \delta \ , \quad a_3 = 3 - \eta \ , \quad a_4 = 4 + \epsilon \ ,$$

then

(2.28)

$$1 \geq \left|1 - \tfrac{1}{2}\delta\right|^2 + 3\left|\tfrac{3}{2}\delta - \tfrac{1}{2}\eta - \tfrac{3}{8}\delta^2\right|^2 + 5\left|\tfrac{3}{2}\delta - \tfrac{3}{2}\eta - \tfrac{1}{2}\epsilon - \tfrac{15}{8}\delta^2 + \tfrac{3}{4}\eta\delta + \tfrac{5}{16}\delta^3\right|^2$$

$$+ 7\left|\tfrac{3}{10}\delta + \tfrac{3}{2}\eta + \tfrac{3}{10}\bar{\eta} + \tfrac{11}{10}\epsilon + \tfrac{15}{4}\delta^2 - \tfrac{15}{4}\delta\eta + \tfrac{3}{8}\eta^2 - \tfrac{11}{20}\delta\epsilon + \tfrac{15}{16}\delta^2\eta - \tfrac{35}{16}\delta^3 + \tfrac{35}{128}\delta^4\right|^2$$

$$+ 9\left|\tfrac{221}{70}\delta - \tfrac{1}{7}\bar{\delta} - \tfrac{13}{14}\eta + \tfrac{51}{70}\bar{\eta} + \tfrac{47}{70}\epsilon + \tfrac{149}{140}\delta^2 - \tfrac{75}{14}\delta\eta - \tfrac{51}{140}\delta\bar{\eta} + \tfrac{81}{56}\eta^2 + \tfrac{9}{28}\eta\epsilon\right.$$

$$\left. - \tfrac{229}{140}\delta\epsilon - \tfrac{1022}{224}\delta^3 + \tfrac{543}{112}\delta^2\eta - \tfrac{81}{112}\delta\eta^2 + \tfrac{229}{560}\delta^2\epsilon - \tfrac{181}{224}\delta^3\eta + \tfrac{1565}{896}\delta^4 - \tfrac{313}{1792}\delta^5\right|^2$$

$$+ 11\left|\tfrac{2143}{630}\delta - \tfrac{23}{63}\bar{\delta} - \tfrac{31}{7}\eta + \tfrac{37}{70}\bar{\eta} - \tfrac{293}{210}\epsilon + \tfrac{23}{126}\delta\bar{\delta} - \tfrac{8419}{840}\delta^2 - \tfrac{179}{140}\delta\bar{\eta} + \tfrac{75}{14}\delta\eta + \tfrac{1}{4}\eta\bar{\eta}\right.$$

$$+ \tfrac{55}{84}\eta^2 - \tfrac{1}{8}\epsilon^2 - \tfrac{13}{35}\delta\epsilon + \tfrac{9}{28}\eta\epsilon + \tfrac{35251}{5040}\delta^3 - \tfrac{11}{112}\delta^2\eta + \tfrac{179}{560}\delta^2\bar{\eta} - \tfrac{493}{336}\delta\eta^2$$

$$+ \tfrac{13}{144}\eta^3 + \tfrac{239}{560}\delta^2\epsilon - \tfrac{9}{56}\delta\eta\epsilon - \tfrac{1481}{672}\delta^4 - \tfrac{157}{224}\delta^3\eta + \tfrac{493}{1344}\delta^2\eta^2 - \tfrac{239}{3360}\delta^3\epsilon$$

$$\left. + \tfrac{157}{1792}\delta^4\eta + \tfrac{2087}{5376}\delta^5 - \tfrac{2087}{64512}\delta^6\right|^2 .$$

This lemma is proved by direct substitution of the new variables δ, η and ϵ into the inequality (1.29).

<u>LEMMA 2.8.</u> <u>In the notation</u> (2.27), set

(2.29) $$\delta = p - iP \;, \quad \eta = q - iQ \;.$$

<u>Then</u>

$$(2.30) \qquad 0 \leq P \leq .078 \quad , \quad -.067 \leq Q \leq .078 \; .$$

For the proof we have by Lemma 2.7, using only the imaginary parts of b_3, b_5 and b_7,

$$(2.31)\ 1 - \frac{1}{4}A^2 \geq 3\left(-\frac{3}{2}P + \frac{1}{2}Q + \frac{3}{4}pP\right)^2$$

$$+ 5\left(-\frac{3}{2}P + \frac{3}{2}Q + \frac{15}{4}pP - \frac{3}{4}qP - \frac{3}{4}pQ - \frac{15}{16}p^2P + \frac{5}{16}P^3\right)^2$$

$$+ 7\left(-\frac{3}{10}P - \frac{3}{2}Q + \frac{3}{10}Q - \frac{15}{2}pP + \frac{15}{4}pQ + \frac{15}{4}qP - \frac{3}{4}qQ + \frac{11}{20}\epsilon P - \frac{15}{8}pqP - \frac{15}{16}p^2Q\right.$$

$$\left.+ \frac{15}{16}P^2Q + \frac{105}{16}p^2P - \frac{35}{16}P^3 - \frac{35}{32}p^3P + \frac{35}{32}pP^3\right)^2 = 3[.5Q - 1.5P(1 - .5p)]^2$$

$$+ 5[1.5Q(1 - .5p) - 1.5P(1 - 2.5p + .5q + .625p^2 - .208333P^2)]^2$$

$$+ 7[.3P(1 + 25p - 12.5q - 1.8333\epsilon + 6.25pq - 6.25PQ - 21.875p^2$$

$$+ 7.291666P^2 + 3.6458333p^3 - 3.6458333pP^2) + 1.2Q(1 - 3.125p + .625q$$

$$+ .78125P^2 + .78125p^2)]^2 \; .$$

In order to estimate the coefficients which appear here, we use the inequality

$$(2.32) \qquad \epsilon + 4q \leq 7p + 2pq - 2PQ + 6.5P^2 - 6.5p^2 + 1.08333p^3 - 3.25pP^2 \; ,$$

which results from (1.35) when we make the substitutions (2.27) and (2.29). We shall also need the inequalities

(2.33) $$\epsilon \geq 0 \quad , \quad p \geq 0 \quad , \quad q \geq 0 \quad , \quad P \geq 0 \quad ,$$

which follow from (1.4), from Lemma 1.8, and from our normalization of the extremal function. By Lemmas 2.5 and 2.6 we have

(2.34) $$p \leq .0606 \quad , \quad P \leq .208 \quad ,$$

and thus

$$1 - .5p \geq .9697 \quad ,$$

$$1 - 2.5p + .5q + .625p^2 - .208333P^2 \geq .841 \quad ,$$

$$1 + 25p - 12.5q - 1.8333\epsilon + 6.25pq - 6.25PQ - 21.875p^2$$

$$+ 7.291666P^2 + 3.6458333p^3 - 3.6458333pP^2$$

$$\geq 1 + 3.125p - 1.5625p^2 - 13.021P^2 \geq .436 \quad ,$$

$$1 - 3.125p + .625q + .78125P^2 + .78125p^2 \geq .813 \quad .$$

After these preparations, we can proceed to apply (2.31) systematically in order to prove the lemma. For $Q \leq 0$ we get

$$.0494 \geq 3(.5Q)^2 + 5(1.454Q)^2 \geq 11.32Q^2 ,$$

which establishes the lower bound (2.30) on Q. On the other hand, if $Q \geq 0$ and $P \geq .6Q$, then

$$.0494 \geq 3(.3727Q)^2 + 7(1.054Q)^2 \geq 8.193Q^2 ,$$

while if $Q \geq 0$ and $P \leq .6Q$, then

$$.0494 \geq 5(1.5Q[.4 + p - .3q - .375p^2 + .125P^2])^2 + 7(.9756Q)^2$$

$$\geq 5(.57Q)^2 + 7(.9756Q)^2 \geq 8.287Q^2 ,$$

by virtue of the inequality

$$q + p^2 \leq 4p + P^2$$

resulting from (1.34). The upper bound (2.30) on Q follows from these estimates.

To estimate P, we must again consider several different cases. If $Q \leq 0$, then

$$.0494 \geq 3(1.454P)^2 + 5(1.261P)^2 \geq 14P^2$$

and there is no difficulty. If $P \leq Q$ there is also no difficulty in view of the bound on Q already established. If $Q \leq P \leq 1.3Q$ we find

$$.0494 \geq 3(.95455P)^2 + 7(.88126P)^2 \geq 8.1698P^2$$

and $P \leq .078$. Finally, in the case $P \geq 1.3Q \geq 0$ we have

$$.0494 \geq 3(.5Q - 1.4545P)^2 + 5(1.5Q - 1.2615P)^2 + 7(.1308P + .9756Q)^2$$

$$= 14.4233P^2 - 21.4995PQ + 18.6625Q^2$$

$$= 8.23143P^2 + 18.6625(Q - .576007P)^2$$

$$\geq 8.23P^2 \quad ,$$

and this serves to establish the bound (2.30) on P and the lemma.

LEMMA 2.9. *The increments q and ϵ of the coefficients a_3 and a_4 satisfy*

(2.35) $$0 \leq q \leq .116 \quad , \quad 0 \leq \epsilon \leq .45 \quad .$$

These estimates follow from inequality (2.32), which yields because of Lemmas 2.6 and 2.8 the bound

$$\epsilon + 3.8788q \leq .4494 \quad .$$

<u>LEMMA</u> <u>2.10</u>. <u>We</u> <u>have</u>

$$-\ .059 \leq .75Q - P \leq .0402 \quad . \tag{2.36}$$

We set $S = .75Q - P$ and we find from (2.31) that for $S \geq 0$

$$.0494 \geq 5(1.9394S)^2 + 7(1.3S)^2 \geq 30.635S^2 \quad ,$$

and the upper bound on S is established. If $S \leq 0$ and $-S \geq .343P$, then by (2.31)

$$.0494 \geq 3(.66666S - .78788P)^2 + 5(1.9394S + .66512P)^2$$

$$= 20.13968S^2 + 9.7478173PS + 4.07417P^2$$

$$\geq 14.309S^2$$

and $S \geq -\ .059$. On the other hand, when $0 \geq S \geq -\ .343P$, then $S \geq -\ .027$ by Lemma 2.8, and this proves the lemma.

We close this section with a remark which will turn out to be useful later. In applying formula (2.31), we have estimated the term $1 - .25A^2$ by the number $.0494$. If, instead, we make use of the fact that this term cannot exceed p, our present calculations yield the final

LEMMA 2.11. *Setting again* $S = .75Q - P$, *we have the inequalities*

$$P^2 \leq .123p \quad , \quad Q^2 \leq .123p \quad , \quad S^2 \leq .07p \quad . \tag{2.37}$$

3. The discriminant condition.

In this section we develop a procedure based on Lemma 1.4 which yields good bounds for q and ϵ when a bound on p is known. We derive first

LEMMA 2.12. *The angle* Ψ *satisfies*

$$1.5\epsilon + 3q\cos\Psi - 2p\cos 2\Psi = -(1-\cos\Psi)[(2\cos\Psi - 1)^2 + 1] + (3Q - 4P)\sin\Psi + 4P(1-\cos\Psi)\sin\Psi \tag{2.38}$$

and

$$h(\Psi) = -(\tfrac{1}{2}\epsilon + \tfrac{2}{3}p + \tfrac{4}{3}p\cos^2\Psi)\sin\Psi + Q - \tfrac{4}{3}P + \tfrac{4}{3}P(1-\cos^3\Psi) \quad , \tag{2.39}$$

where

$$h(\Psi) = (\tfrac{2}{3} - \tfrac{8}{3}\cos^2\Psi + \tfrac{8}{3}\cos^3\Psi)\sin\Psi \quad . \tag{2.40}$$

Formula (2.38) is obtained from (1.20) and formula (2.39) is obtained from the imaginary part of (1.19) by replacing a_2 , a_3 and a_4 by the new variables p , q , P , Q and ϵ .

LEMMA 2.13. The angle ψ lies in the interval

$$- .122 \leq \psi \leq .087 \quad . \tag{2.41}$$

From the real part of (1.19) we find

$$3 - q - \frac{1}{3} p \cos 3\psi - \frac{4}{3} P \sin^3 \psi - \frac{1}{3} \cos 4\psi + \frac{2}{3} \cos 2\psi + \frac{2}{3} \cos 3\psi$$

$$= (4 + \frac{1}{2} \epsilon - p) \cos \psi \quad ,$$

whence by Lemmas 2.6, 2.8 and 2.9

$$3 - 1.9069 \leq 4.225 \cos \psi$$

and $|\psi| \leq 1.31$. Hence for $\psi \geq 0$ by (2.39)

$$h(\psi) \leq Q - \frac{4}{3} P + \frac{4}{3} P \leq .078 \quad ,$$

in view of Lemma 2.8. But in the interval $.121 \leq \psi \leq 1.31$ we find $h(\psi) > .078$ because of the definition (2.40). Thus $\psi \leq .121$ and (2.39) yields

$$h(\Psi) \leq Q - \frac{4}{3}P + .03P \leq .056 \quad .$$

By (2.40) we have $h(\Psi) > .056$ in the interval $.087 \leq \Psi \leq .121$ and hence the upper bound (2.41) is established.

If $\Psi \leq 0$, (2.39) and Lemma 2.10 give

$$h(\Psi) \geq Q - \frac{4}{3}P \geq -.0787 \quad ,$$

and since $h(\Psi) < -.0787$ in the interval $-1.31 \leq \Psi \leq -.122$, the lemma is proved.

<u>LEMMA</u> <u>2.14</u>. <u>The quantities</u> p, q, P, Q <u>and</u> ϵ <u>satisfy the inequality</u>

$$\frac{3}{2}\epsilon + 3q - 2p \leq \frac{1}{2}(3Q - 4P)^2 \quad . \tag{2.42}$$

For the proof of this lemma, we combine (2.38) and (2.39) to obtain

$$\begin{aligned} 1.5\epsilon + 3q\cos\Psi - 2p\cos 2\Psi &= -(1-\cos\Psi)[(2\cos\Psi - 1)^2 + 1] \\ &+ \frac{[3Q - 4P + 4P(1-\cos\Psi)][3Q - 4P + 4P(1-\cos^3\Psi)]}{(2 - 8\cos^2\Psi + 8\cos^3\Psi) + (1.5\epsilon + 2p + 4p\cos^3\Psi)} \\ &\leq -1.97(1-\cos\Psi) + \frac{(3Q-4P)^2 + .203(1-\cos\Psi)}{2 - 8\cos^2\Psi + 8\cos^3\Psi} \quad , \end{aligned}$$

by virtue of Lemmas 2.8, 2.10 and 2.13. But

$$\frac{1}{1 - 4\cos^2\Psi + 4\cos^3\Psi} - 1 = \frac{4(1 - \cos\Psi)\cos^2\Psi}{1 - 4\cos^2\Psi + 4\cos^3\Psi}$$

$$\leq 4.124(1 - \cos\Psi) \quad ,$$

and therefore

$$1.5\epsilon + 3q - 2p \leq (1 - \cos\Psi)[- 1.97 + 3q + 2.062(.236)^2 + .4186(.0075)]$$

$$+ \tfrac{1}{2}(3Q - 4P)^2 \quad .$$

The lemma now follows, since the coefficient of $(1 - \cos\Psi)$ on the right is negative.

<u>LEMMA 2.15</u>. <u>We have</u>

$$q \leq .05 \quad , \quad \epsilon \leq .1 \quad . \tag{2.43}$$

These bounds result immediately from Lemma 2.14 and the inequalities (2.34) and (2.36).

4. <u>Linearization</u>.

In this section we refine our estimates of p , q , P , Q and ϵ by expanding the fundamental inequality (2.28) and examining the quadratic terms. We have

LEMMA 2.16. The terms (2.22) alone in the inequality (2.28) imply

$$
\begin{aligned}
(2.44)\quad p \geq\ & 100.653p^2 - 30.291pq + 35.044q^2 + 13.777\epsilon^2 + 33.55p\epsilon + 32.803q\epsilon \\
& + 116.889P^2 - 120.394PQ + 46.795Q^2 + p[41.995p^2 - 195.01pq - 5.100q^2] \\
& + 4.243q^3 + \epsilon[- 11.071p^2 - 121.179pq + 22.099q^2 - 28.239\epsilon p + 3.885\epsilon q] \\
& + \epsilon[- 179.459P^2 + 180.485PQ - 32.843Q^2] + p[52.942P^2 + 95.328PQ - 5.375Q^2] \\
& + q[- 464.27P^2 + 359.173PQ - 77.933Q^2] - 247.548p^4 + 279.472p^3q - 17.453p^2q^2 \\
& - 55.141\epsilon p^3 + 58.594\epsilon p^2q + 165.423\epsilon pP^2 - 58.594\epsilon qP^2 - 117.188\epsilon pPQ \\
& + P^2[271.012P^2 - 424.074PQ + 144.615Q^2] + p^2[- 70.392P^2 + 170.162PQ \\
& - 144.615Q^2] + 263.644pqP^2 + 17.453q^2P^2 - 254.324pqPQ \\
& - \max(|\delta|^2, |\eta|^2)[190|\delta|^2 + 64.6|\delta\eta| + 107.3|\delta|\epsilon] - 5|\delta|^2\epsilon^2 .
\end{aligned}
$$

For each $\nu > 1$ we can write the coefficient b_ν in the form

$$b_\nu = \ell_\nu + q_\nu + h_\nu ,$$

where ℓ_ν indicates terms of degree one, q_ν indicates terms of degree two, and h_ν indicates terms of degree three or higher in the expansion of b_ν as a polynomial in δ, η, $\bar{\delta}$, $\bar{\eta}$ and ϵ. In deriving (2.44), we use the inequality

$$|b_\nu|^2 \geq |\ell_\nu|^2 + 2\mathrm{Re}\{\ell_\nu \bar{q}_\nu\} + 2\mathrm{Re}\{\ell_\nu \bar{h}_\nu\}$$

and we expand in all detail the quadratic and cubic terms $|\ell_\nu|^2$ and $2\mathrm{Re}\{\ell_\nu \bar{q}_\nu\}$. On the other hand, we work out explicitly only the largest contributions from the term $2\mathrm{Re}\{\ell_9 \bar{h}_9\}$, and we estimate the remaining terms of higher degree through the inequality

$$\begin{aligned}(2.45)\quad & 10\mathrm{Re}\left\{\left(\tfrac{3}{2}\delta - \tfrac{3}{2}\eta - \tfrac{1}{2}\epsilon\right)\tfrac{5}{16}\bar{\delta}^3\right\} + 14\mathrm{Re}\left\{\left(\tfrac{3}{10}\delta + \tfrac{3}{2}\eta + \tfrac{3}{10}\bar{\eta} + \tfrac{11}{10}\epsilon\right)\left(\tfrac{15}{16}\bar{\delta}^2\bar{\eta}\right.\right. \\ & \left.\left. - \tfrac{35}{16}\bar{\delta}^3 + \tfrac{35}{128}\bar{\delta}^4\right)\right\} + 18\mathrm{Re}\left\{\left(\tfrac{221}{70}\delta - \tfrac{1}{7}\bar{\delta} - \tfrac{13}{14}\eta + \tfrac{51}{70}\bar{\eta}\right.\right. \\ & \left.\left. + \tfrac{47}{70}\epsilon\right)\left(-\tfrac{81}{112}\bar{\delta}\bar{\eta}^2 + \tfrac{229}{560}\bar{\delta}^2\epsilon - \tfrac{181}{224}\bar{\delta}^3\bar{\eta} + \tfrac{1565}{896}\bar{\delta}^4 - \tfrac{313}{1792}\bar{\delta}^5\right)\right\} \\ & \geq -\max(|\delta|^2, |\eta|^2)[190|\delta|^2 + 64.6|\delta\eta| + 107.3|\delta|\epsilon] - 5|\delta|^2\epsilon^2 ,\end{aligned}$$

in which terms of degree larger than four have been estimated by means of the known bound $|\delta| \leq .0988$.

LEMMA 2.17. *The quantities* p , q , P *and* Q *satisfy*

$$(2.46)\quad p \geq 100.653p^2 - 30.291pq + 35.04q^2 - 156.2p^3 - 493.6p^4 + 93.056P^2$$

$$- 101.810PQ + 41.681Q^2 + \min(0\ ,\ 5.777PQ)\ .$$

Our first step in proving Lemma 2.17 is to show that the terms involving ϵ on the right in inequality (2.44) are non-negative and can therefore be neglected. This follows from the set of estimates

$$13.777\epsilon^2 - 28.239p\epsilon^2 + 3.885q\epsilon^2 - 5|\delta|^2\epsilon^2 \geq 0\ ,$$

$$32.803q\epsilon - 121.179pq\epsilon + 22.099q^2\epsilon + 58.594p^2q\epsilon - 58.594P^2q\epsilon \geq 0\ ,$$

$$8.55p\epsilon - 11.071p^2\epsilon - 55.141p^3\epsilon + 165.423P^2p\epsilon - 117.188PQp\epsilon$$

$$- 107.3\max(|\delta|^2\ ,|\eta|^2)|\delta|\epsilon \geq 0\ ,$$

$$\epsilon[25p - 179.459P^2 + 180.485PQ - 32.843Q^2] \geq \epsilon[25p - 120.324S^2 - 59.136P^2] \geq 0\ ,$$

which depend strongly on Lemma 2.11. Next we note that by (2.34) and (2.43)

$$q[464.27P^2 - 359.173PQ + 77.933Q^2] + p^2[70.392P^2 - 170.162PQ + 144.615Q^2]$$

$$\leq 23.473P^2 - 18.584PQ + 4.428Q^2\ ,$$

since the quadratic forms in P and Q which appear are positive-definite. Also

$$95.328pPQ + P^2[271.012P^2 - 424.074PQ + 144.615Q^2] - 254.324pqPQ ,$$

$$\geq \min(0 , 95.328pPQ) ,$$

and

$$52.942pP^2 - 5.375pQ^2 + 263.644pqP^2 + 17.453q^2P^2 \geq - .326Q^2 .$$

Finally, by the inequality between arithmetic and geometric means,

$$\max(|\delta|^2 , |\eta|^2)[190|\delta|^2 + 64.6|\delta\eta|]$$

$$\leq (p^2 + .123p)[246p^2 + 32.3(P^2 + Q^2)]$$

$$\leq 246p^4 + 30.26p^3 + .36(P^2 + Q^2) ,$$

since $q \leq .854p$ by Lemmas 2.11 and 2.14. Therefore the terms involving P and Q on the right in (2.44) exceed the quadratic form

$$93.056P^2 - 101.810PQ + 41.681Q^2 + \min(0 , 5.777PQ) .$$

It remains to estimate the terms in p and q. We find

$$1.533p^2q - 5.100pq^2 + 4.243q^3 \geq 0 \quad ,$$

$$279.472p^3q - 17.453p^2q^2 \geq 0 \quad ,$$

$$11.735p^3 - 196.543p^2q - 493.548p^4 \geq - 156.2p^3 - 493.6p^4 \quad ,$$

and thus Lemma 2.17 is established.

LEMMA 2.18. *From the inequality* (2.22) *alone we have*

(2.47) $p \leq .0109 \quad , \quad q \leq .0073 \quad , \epsilon \leq .0146 \quad , \quad P \leq .0094 \; , \; - .0081 \leq Q \leq .0141 \; .$

By (2.46) we find

$$p \geq 94.106p^2 + [6.547p^2 - 30.291pq + 35.04q^2] - 156.2p^3 - 493.6p^4 \geq 82.827p^2 \quad ,$$

using the known estimate $p \leq .0606$. Hence $p \leq .01208$ and therefore in turn

$$p \geq 92.14p^2 \quad ,$$

which establishes the upper bound (2.47) on p . We have

$$93.056P^2 - 101.81PQ + 41.681Q^2 = 30.865P^2 + 41.681(Q - 1.2213P)^2$$

$$= 13.833Q^2 + 93.056(P - .547014Q)^2 \quad ,$$

and thus (2.46) yields

$$30.885P^2 \leq p - 92.14p^2 \leq .0027133 \quad ,$$

$$13.833Q^2 \leq p - 92.14p^2 \leq .0027133 \quad .$$

The upper bounds (2.47) on P and Q follow. For the lower bound on Q we can use the estimate

$$41.681Q^2 \leq .0027133 \quad ,$$

since only negative values of Q are relevant. To estimate q and ϵ, we note that

$$93.056P^2 - 101.81PQ + 41.681Q^2 + \min(0 \ , \ 5.777PQ) \geq 71.5S^2 \quad ,$$

whence by Lemma 2.17

$$71.5S^2 + p \leq \frac{1}{92.14} < .0109 \quad .$$

Lemma 2.14 now gives

$$1.5\epsilon + 3q \leq 2p + 8S^2 \leq .0218 \quad ,$$

and this estimate suffices to complete the proof of Lemma 2.18.

Thus far we have made no actual use of the formula (1.26) for the coefficient a_7, and, indeed, we could finish the proof of the main theorem without recourse to terms involving a_7. However, it is a simple matter to improve our estimates at this stage by using (1.26), and we proceed to do so in order to throw additional light on the inner workings of our method.

LEMMA 2.19. *The complete inequality* (2.28) *yields*

$$
\begin{aligned}
(2.48)\quad p \geq\ & 97.581p^2 + 202.35(q + .37685\epsilon - .718617p)^2 \\
& + \epsilon[49.94p + 6.452\epsilon + 266.16p^2 - 193.031pq - 33.252q^2 - 25.188p\epsilon + 4.743q\epsilon \\
& \quad + 3.836\epsilon^2 - 269.829p^3 + 58.594p^2q - 523.488P^2 + 451.313PQ - 55.472Q^2 \\
& \quad + 809.488pP^2 - 58.594qP^2 - 117.188pPQ] + P^2[272.955 - 926.402p - 1001.895q \\
& \quad - 308.575P^2 + 338.694PQ + 144.615Q^2 + 266.664p^2 + 2063.951pq + 17.453q^2] \\
& - PQ[531.176 - 2387.998p - 592.236q + 2118.142p^2 + 254.324pq] \\
& + Q^2[317.100 - 756.084p - 186.018q - 144.615p^2] - 615.354p^3 + 921.731p^2q \\
& \quad - 294.6pq^2 - 73.386q^3 + 219.687p^4 - 320.63p^3q - 17.453p^2q^2 \\
& - [.0028\epsilon^2 + .0332p\epsilon + 3.377p^2 - 528.7p^3 + 20695.3p^4] .
\end{aligned}
$$

This lemma is proved in the same way that we proved Lemma 2.16. The quadratic and cubic terms on the right in (2.48) are obtained by straightforward algebra, and higher degree terms which have not been worked out explicitly are estimated by (2.45) and the inequality

$$22\mathrm{Re}\Big\{\Big(\frac{2143}{630}\delta - \frac{23}{63}\bar{\delta} - \frac{31}{7}\eta + \frac{37}{70}\bar{\eta} - \frac{293}{210}\epsilon\Big)\Big(-\frac{11}{112}\bar{\delta}^2\bar{\eta} + \frac{179}{560}\bar{\delta}^2\eta$$

$$- \frac{493}{336}\bar{\delta}\bar{\eta}^2 + \frac{13}{144}\bar{\eta}^3 + \frac{239}{560}\bar{\delta}^2\epsilon - \frac{9}{56}\bar{\delta}\bar{\eta}\epsilon - \frac{1481}{672}\bar{\delta}^4 - \frac{157}{224}\bar{\delta}^3\bar{\eta}$$

$$+ \frac{493}{1344}\bar{\delta}^2\bar{\eta}^2 - \frac{239}{3360}\bar{\delta}^3\epsilon + \frac{157}{1792}\bar{\delta}^4\bar{\eta} + \frac{2087}{5376}\bar{\delta}^5 - \frac{2087}{64512}\bar{\delta}^6\Big)\Big\}$$

$$\geq - \max(|\delta|^2,|\eta|^2)[391.4\max(|\delta|^2,|\eta|^2) + 175.7\epsilon\max(|\delta|,|\eta|) + 18.1\epsilon^2] \quad .$$

Hence the total contribution of the terms not calculated explicitly exceeds

$$- \max(|\delta|^2,|\eta|^2)[646\max(|\delta|^2,|\eta|^2) + 283\epsilon\max(|\delta|,|\eta|) + 23.1\epsilon^2]$$

$$\geq - .0028\epsilon^2 - .0332p\epsilon - 3.377p^2 + 528.7p^3 - 20695.3p^4 \quad ,$$

since we have the estimate

$$\max(|\delta|^2,|\eta|^2) \leq .0723p - 5.66p^2 \leq .000117$$

based on Lemma 2.17.

<u>LEMMA 2.20. The quantities</u> p , q , P , Q <u>and</u> ϵ <u>satisfy the inequality</u>

$$p \geq 94.2p^2 + 255P^2 - 501PQ + 303Q^2 \quad . \tag{2.49}$$

To prove this lemma, we remark first that

$$\begin{aligned} (2.50)\quad & 202.35(q + .37685\epsilon - .718617p)^2 - 312.233p^3 + 868.983p^2q \\ & - 604.622pq^2 - 85.866p\epsilon^2 + 327.475p^2\epsilon - 455.704pq\epsilon \\ & = (202.35 - 604.622p)(q + .37685\epsilon - .718617p)^2 \geq 0 \quad . \end{aligned}$$

Thus when we subtract the quantity on the left in (2.50) from the right-hand side of (2.48), we either improve or do not alter the inequality (2.48). After the subtraction, the terms involving ϵ which remain have a non-negative sum and can be neglected, since by Lemma 2.17

$$\begin{aligned} & [49.9p + 6.44\epsilon - 61.315p^2 + 262.673pq - 33.252q^2 + 60.678p\epsilon + 4.743q\epsilon \\ & + 3.836\epsilon^2 - 269.829p^3 + 58.594p^2q - 523.488P^2 + 451.313PQ - 55.472Q^2 \\ & + 809.488pP^2 - 58.594qP^2 - 117.188pPQ] \geq 0 \quad . \end{aligned}$$

The terms of degree three or more in p and q alone which are left also form a non-negative contribution and can be neglected, since

$$225.579p^3 + 52.748p^2q + 310.022pq^2 - 73.386q^3$$

$$- 20475.7p^4 - 320.63p^3q - 17.453p^2q^2 \geq 0 \quad .$$

Thus the term in p^2 on the right in (2.49) is all that remains, except for terms involving P and Q. These terms in (2.48) are estimated to exceed the quadratic form

$$255P^2 - 501PQ + 303Q^2 \quad ,$$

since

$$- 308.575P^2 + 338.694PQ + 144.615Q^2 \geq - 7p$$

and since we have diminished the term in Q^2 so generously that we are sure to have neglected a positive-definite form in P and Q. This proves the lemma.

<u>LEMMA 2.21.</u> <u>The estimates</u>

(2.51) $\quad p \leq .0107 \ , \ q \leq .0071 \ , \ \epsilon \leq .0142 \ , \ P \leq .0075 \ , \ - .003 \leq Q \leq .0069 \ ,$

$$|S| \leq .0036$$

are satisfied.

The estimate (2.51) on p follows immediately from (2.49). To obtain the upper bounds on P and Q we complete the square in (2.49) to find

$$47.9P^2 \leq p - 94.2p^2 \leq .002654 ,$$

$$56.9Q^2 \leq p - 94.2p^2 \leq .002654 .$$

For the lower bound on Q we need consider only the case $Q \leq 0$, and hence we have directly

$$303Q^2 \leq .002654 .$$

For S we find

$$205S^2 \leq p - 94.2p^2 \leq .002654 ,$$

whence also

$$8S^2 + 2p \leq \frac{2}{94.2} ,$$

and the bounds on q and ϵ follow by Lemma 2.14. This proves Lemma 2.21.

We can now formulate the principal lemma of this chapter, upon which the remainder of the proof of (1.3) will be based. Using the inequalities (2.51), we obtain

<u>LEMMA 2.22. The combinations of</u> a_2 <u>and</u> a_3 <u>appearing in the boundary conditions</u> (1.12) <u>and</u> (1.13) <u>satisfy</u>

$$(2.52)\qquad 0 \leq \mathrm{Re}\left\{4 - 2a_2\right\} \leq .0214 \;, \quad 0 \leq \mathrm{Im}\left\{2a_2\right\} \leq .015 \;,$$

$$(2.53)\qquad 0 \leq \mathrm{Re}\left\{\frac{10}{3} - \frac{2a_3 + a_2^2}{3}\right\} \leq .019 \;, \quad -.002 \leq \mathrm{Im}\left\{\frac{2a_3 + a_2^2}{3}\right\} \leq .0146 \;,$$

$$(2.54)\qquad 0 \leq \mathrm{Re}\left\{\frac{8}{3} - \frac{4}{3}a_2\right\} \leq .0143 \;, \quad 0 \leq \mathrm{Im}\left\{\frac{4}{3}a_2\right\} \leq .01 \;,$$

$$(2.55)\qquad 0 \leq \mathrm{Re}\left\{3 - a_3\right\} \leq .0071 \;, \quad -.003 \leq \mathrm{Im}\left\{a_3\right\} \leq .0069 \;.$$

Although it plays no part in the proof of the main theorem, the following lemma, making explicit the bound we have obtained at this stage on a_4, is of interest.

<u>LEMMA 2.23. From the discriminant condition</u> (1.19) <u>and the inequality</u> (1.29) <u>based on the area theorem</u> (1.23) <u>and the recursion formulas</u> (1.24), (1.25) <u>and</u> (1.26), <u>we get the bound</u>

$$(2.56)\qquad |a_4| \leq 4.0142$$

<u>on the fourth coefficient of schlicht functions</u> (1.1).

CHAPTER III

LOCAL UNIQUENESS

1. The fundamental equations.

By means of a non-linear boundary value problem in ordinary differential equations we developed in Lemma 1.3 of Chapter I a set of equations for the coefficients a_2 and a_3 of the extremal function $f(z)$. We observe that this set of conditions is indeed fulfilled in the case of the Koebe function

$$f(z) = \sum_{n=1}^{\infty} n\, z^n \quad , \tag{3.1}$$

for which we have $a_2 = 2$, $a_3 = 3$ and

$$k(t) \equiv 1 \;,\; a(t) = 4 - \frac{4}{3}\, t \;,\; b(t) = \frac{10}{3} - \frac{1}{3}\, t^2 \quad . \tag{3.2}$$

Our problem is to investigate whether there exist other solutions of the same set of equations. It will be shown in this chapter that when the values a_2 and a_3 are sufficiently near to the values 2 and 3, respectively, these latter values are the only ones compatible with the above boundary value problem. In order to give this uniqueness proof, some formal preparation is necessary.

We put

$$k = e^{i\varphi} \,,\; a(t) = u(t) + iv(t) \;,\; b(t) = x(t) + iy(t) \quad , \tag{3.3}$$

and we bring the boundary value problem of Lemma 1.3 into the real form

(3.4) $\quad u'(t) = -\frac{4}{3}\cos\varphi \;, \qquad x'(t) = -\frac{2}{3}t\cos 2\varphi \;,$

(3.5) $\quad v'(t) = -\frac{4}{3}\sin\varphi \;, \qquad y'(t) = -\frac{2}{3}t\sin 2\varphi \;,$

(3.6) $\quad t^2\sin 3\varphi - t(u\sin 2\varphi + v\cos 2\varphi) + x\sin\varphi + y\cos\varphi = 0 \;,$

with the boundary conditions

(3.7) $\quad u(1) = \frac{2}{3}u(0) \;, \qquad x(1) = \frac{3}{2}x(0) - \frac{1}{8}[u(0)^2 - v(0)^2] \;,$

(3.8) $\quad v(1) = \frac{2}{3}v(0) \;, \qquad y(1) = \frac{3}{2}y(0) - \frac{1}{4}u(0)v(0) \;.$

It will be useful to introduce the functions

(3.9) $$p(t) = \frac{16}{3}t^2 - 8t + \frac{10}{3}$$

and

(3.10) $\quad U(t) = u(t) - 4 + \frac{4}{3}t \;, \qquad X(t) = x(t) - \frac{10}{3} + \frac{1}{3}t^2 \;.$

The polynomial (3.9) will occur frequently in our estimates; it is positive-definite and attains its minimum $1/3$ at $t = 3/4$. The functions (3.10) represent the deviation of the solutions $u(t)$ and $x(t)$ from the corresponding known solutions (3.2). By means of the above notation, we can bring (3.6) into the form

$$(3.11) \quad (\sin\varphi)[p(t) + X(t) - 2tU(t) - 16(\tfrac{4}{3}t^2 - t)\sin^2\tfrac{\varphi}{2} + 16t^2\sin^4\tfrac{\varphi}{2}$$

$$+ 4tU(t)\sin^2\tfrac{\varphi}{2} + 2tv(t)\sin\varphi - y(t)\tan\tfrac{\varphi}{2}] = (tv(t) - y(t)) \quad .$$

We have further the differential equations

$$(3.12) \qquad U'(t) = \frac{8}{3}\sin^2\frac{\varphi}{2} \quad , \quad X'(t) = \frac{4}{3}t\sin^2\varphi \quad .$$

The structure of equations (3.11) and (3.12) indicates clearly the estimation procedure to be followed in our uniqueness proof. We will be able to estimate most quantities easily if an estimate of the form

$$(3.13) \qquad \left|\sin\frac{\varphi}{2}\right| \leq \frac{\rho}{2}$$

with sufficiently small ρ is available. We shall make this assumption in the following section and derive from it various consequences.

2. Estimates by means of ρ .

By (3.6) and Lemma 2.22

$$|\tan\varphi(0)| = \frac{|y(0)|}{|x(0)|} \leq .0045 \quad ,$$

and hence either $|\varphi(0)| < .0045$ or $|\varphi(0) - \pi| < .0045$. In both cases we deduce from (3.11) and Lemma 2.22 that

$$2.4|\sin \varphi(t)| \leq .04$$

in the interval $0 \leq t \leq .1$. Thus if $|\varphi(0) - \pi| < .0045$, we would have $\cos \varphi(t) < 0$ in the interval $0 \leq t \leq .1$, since $\varphi(t)$ is continuous. By (1.7) this would imply $\mathrm{Re}\{a_2\} < 1.8$, which contradicts (2.52). Thus $|\varphi(0)| < .0045$.

We take ρ in the interval $.01 < \rho < 1$ and we deduce that

$$\left|\sin \frac{\varphi(0)}{2}\right| < \frac{\rho}{2} .$$

Let next $[0 , T_1]$ be the largest interval in $0 \leq t \leq 1/2$ for which the inequality (3.13) is still fulfilled. We deduce from (3.12) at once

<u>LEMMA 3.1</u>. <u>We have in the interval</u> $0 \leq t \leq T_1$

$$(3.14) \qquad 0 \leq U(t) - U(0) \leq \frac{2}{3} t \rho^2 , \quad 0 \leq X(t) - X(0) \leq \frac{2}{3} t^2 \rho^2 .$$

<u>The left-hand inequalities hold in the entire interval</u> $0 \leq t \leq 1$.

From the differential equations (3.5) for $v(t)$ and $y(t)$ we infer

LEMMA 3.2. *We have in the interval* $0 \le t \le T_1$

$$|v(t) - v(0)| \le \frac{4}{3}\rho t \quad , \quad |y(t) - y(0)| \le \frac{2}{3}\rho t^2 \ . \tag{3.15}$$

We utilize next the fact that (3.7) implies the boundary condition $U(1) = \frac{2}{3}U(0)$. By (2.52), $U(0) \le 0$ and, since $U(t)$ increases monotonically, we have the estimate

$$U(0) \le U(t) \le \frac{2}{3}U(0) \quad , \quad 0 \le t \le 1 \ . \tag{3.16}$$

We now rewrite (3.11) in the form

$$F(t) \sin\varphi = tv - y \ , \tag{3.17}$$

with

$$\begin{aligned} F(t) &= p(t) + X(0) + [X(t) - X(0)] - 2tU(t) \\ &+ 16(t - \frac{4}{3}t^2)\sin^2\frac{\varphi}{2} + 16t^2\sin^4\frac{\varphi}{2} + 4tU(t)\sin^2\frac{\varphi}{2} \\ &+ 2tv(0)\sin\varphi - y(0)\tan\frac{\varphi}{2} + 2t[v(t) - v(0)]\sin\varphi - [y(t) - y(0)]\tan\frac{\varphi}{2} \ . \end{aligned} \tag{3.18}$$

Using Lemmas 3.1 and 3.2, and (3.18), we obtain

$$F(t) \ge p(t) - |X(0)| - 2t|v(0)|\rho - \frac{|y(0)|\rho}{(4-\rho^2)^{1/2}} - \frac{8}{3}\rho^2 t^2 - \frac{2}{3}\frac{\rho^2 t^2}{(4-\rho^2)^{1/2}} \tag{3.19}$$

for t in the interval $0 \leq t \leq T_1$. Since $p(t)$ attains its minimum at $t = 3/4$, this yields for $0 \leq t \leq T_1$ the estimate

$$F(t) \geq \frac{2}{3} - |X(0)| - |v(0)|\rho - \frac{|y(0)|\rho}{(4-\rho^2)^{1/2}} - \frac{2}{3}\rho^2 - \frac{1}{6}\frac{\rho^2}{(4-\rho^2)^{1/2}} .$$

Applying Lemma 2.22 and taking $\rho = 1/7$, we obtain

<u>LEMMA 3.3. In the entire interval $0 \leq t \leq T_1$ in which</u>

$$(3.20) \qquad \left|\sin \frac{\varphi}{2}\right| \leq \frac{1}{14}$$

<u>holds, we have</u>

$$(3.21) \qquad F(t) \geq .629 .$$

Thus far we have made estimates which are valid in an interval extending to the right from $t = 0$. However, since the boundary conditions on $u(t)$, $v(t)$, $x(t)$ and $y(t)$ yield as much information at the point $t = 1$ as they do at $t = 0$, we proceed to establish similar estimates valid in an interval extending to the left from $t = 1$.

From Lemma 2.13 and the identity $\varphi(1) = \Psi$, we know that $|\varphi(1)| \leq .122$. By (2.39), (2.40) and Lemma 2.21 we can improve this bound to obtain $|\varphi(1)| < .008$. Since we take ρ in the interval $.01 < \rho < 1$, we find

$$\left|\sin \frac{\varphi(1)}{2}\right| < \frac{\rho}{2} ,$$

and we denote by $[T_2, 1]$ the largest interval in $1/2 \leq t \leq 1$ for which the inequality (3.13) is still fulfilled. From (3.5) we derive

<u>LEMMA 3.4. We have in the interval</u> $T_2 \leq t \leq 1$

$$|v(t) - v(1)| \leq \frac{4}{3}\rho(1 - t) \quad , \quad |y(t) - y(1)| \leq \frac{2}{3}(1 - t^2)\rho \qquad (3.22)$$

We now replace (3.18) by the equivalent formula

$$F(t) = p(t) + X(0) + [X(t) - X(0)] - 2tU(t) + 16(t - \frac{4}{3}t^2)\sin^2\frac{\varphi}{2}$$

$$+ 16t^2 \sin^4\frac{\varphi}{2} + 4tU(t)\sin^2\frac{\varphi}{2} + 2tv(1)\sin\varphi$$

$$- y(1)\tan\frac{\varphi}{2} + 2t[v(t) - v(1)]\sin\varphi - [y(t) - y(1)]\tan\frac{\varphi}{2} \quad ,$$

and we are led to the estimate

$$F(t) \geq \frac{16}{3}t^2 - 8t + \frac{10}{3} - |X(0)| - 4\rho^2 \max(0 , t - \frac{4}{3}t^2)$$

$$- 2|v(1)|\rho t - \frac{|y(1)|\rho}{(4 - \rho^2)^{1/2}} - \frac{8}{3}\rho^2 t(1 - t) - \frac{2}{3}(1 - t^2)\frac{\rho^2}{(4 - \rho^2)^{1/2}}$$

in the interval $T_2 \leq t \leq 1$. We set $\rho = 1/7$ again and find that by Lemma 2.22

$$F(t) \geq 3.307 - 8.0573t + 5.3945t^2 - \max(0 , \frac{12 - 16t}{147} t) \geq .298 \quad ,$$

since the minimum of the quadratic on the right occurs at approximately $t = .746807$. Thus we obtain

LEMMA 3.5. *In the entire interval* $T_2 \leq t \leq 1$ *in which* (3.20) *holds, we have*

$$F(t) \geq .298 \quad . \tag{3.23}$$

3. *Integration of the differential system.*

Let us define a new function of t by

$$\omega(t) = tv(t) - y(t) \quad . \tag{3.24}$$

Then equation (3.17) may be written in the form

$$\sin \varphi = \frac{\omega(t)}{F(t)} , \tag{3.25}$$

and equations (3.5) lead to the system of differential equations

$$\frac{dv}{dt} = - \frac{4}{3F(t)} \omega(t) \quad , \quad \frac{d\omega}{dt} = v(t) - \omega(t) \frac{8t \sin^2(\varphi/2)}{3F(t)} \quad . \tag{3.26}$$

We introduce the functions

$$r_1(t) = \exp[\frac{8}{3} \int_0^t \frac{t \sin^2(\varphi/2)}{F(t)} dt] , \quad r_2(t) = \exp[- \frac{8}{3} \int_t^1 \frac{t \sin^2(\varphi/2)}{F(t)} dt] \tag{3.27}$$

and we define

$$w_j(t) = r_j(t)\omega(t) \quad , \quad j = 1 , 2 \; . \tag{3.28}$$

Then for both $j = 1$ and $j = 2$ the system (3.26) reduces to

$$\frac{dv}{dt} = - R_j(t)w_j \quad , \quad \frac{dw_j}{dt} = r_j(t)v \quad , \tag{3.29}$$

with

$$R_j(t) = \frac{}{3F(t)r_j(t)} \; . \tag{3.30}$$

We use the assumption (3.20) that $\rho = 1/7$, and we find

$$1 \leq r_1(t) \leq 1.003 \quad , \quad 0 < R_1(t) \leq 2.12 \tag{3.31}$$

in the interval $0 \leq t \leq T_1$ and

$$.983 \leq r_2(t) \leq 1 \quad , \quad 0 < R_2(t) \leq 4.552 \tag{3.32}$$

in the interval $T_2 \leq t \leq 1$.

In order to simplify our calculations, we make a slight change of scale by introducing the new independent variables

$$s_1 = \int_0^t r_1(t)dt \quad , \quad s_2 = 1 - \int_t^1 r_2(t)dt \; . \tag{3.33}$$

In terms of the new variable s_j the system (3.29) simplifies to

$$\frac{dv}{ds_j} = -\frac{R_j}{r_j} w_j \quad , \quad \frac{dw_j}{ds_j} = v \quad , \tag{3.34}$$

or

$$\frac{d^2 w_j}{ds_j^2} + \frac{R_j}{r_j} w_j = 0 \quad . \tag{3.35}$$

Thus we bring our differential system into a form convenient for application of the Sturm-Liouville theory.

We shall prove

LEMMA 3.6. The solution $w_1(s_1)$ has at most one zero in the interval $[0 \, , \, s_1(T_1)]$ and the solution $w_2(s_2)$ has at most one zero in the interval $[s_2(T_2) \, , \, 1]$.

Suppose there were two points s_j' and s_j'' at which w_j vanished in one of these intervals. Then we would find by integration by parts the identity

$$\int_{s_j'}^{s_j''} w_j'(s_j)^2 ds_j = \int_{s_j'}^{s_j''} \frac{R_j}{r_j} w_j(s_j)^2 ds_j \quad ,$$

and therefore by (3.31) and (3.32)

$$\frac{\int_{s_j'}^{s_j''} (w_j')^2 ds_j}{\int_{s_j'}^{s_j''} w_j^2 \, ds_j} \leq 4.7 \quad , \quad w_j(s_j') = w_j(s_j'') = 0 \quad .$$

On the other hand, it is well known from the calculus of variations that the left-hand ratio is at least $\pi^2(s_j'' - s_j')^{-2}$. Hence

$$\frac{\pi^2}{4.7} \leq (s_j'' - s_j')^2 < 1.1 \quad ,$$

which is absurd. Thus $w_j(s_j)$ vanishes at most once in the interval considered. The lemma implies, of course, that $\omega(t)$ vanishes at most once in the corresponding intervals.

We state now

LEMMA 3.7. In the interval $0 \leq s_1 \leq s_1(T_1)$ we have the estimate

$$|w_1(s_1)| \leq s_1|v(0)| + |y(0)| \quad . \tag{3.36}$$

Observe, at first, that by definitions (3.24) and (3.28) and by the differential equations (3.34), we have

$$w_1(0) = - y(0) \quad , \quad w_1'(0) = v(0) \quad . \tag{3.37}$$

We have always assumed $v(0) = \mathrm{Im}\{2a_2\} \geq 0$ and we shall now deal with the two possibilities $w_1(0) \geq 0$ and $w_1(0) < 0$. We shall use as function of comparison a solution of the differential equation

$$\frac{d^2W_1}{ds_1^2} + (1.46)^2 W_1 = 0 \quad . \tag{3.38}$$

We require that the solution $W_1(s_1)$ of (3.38) have the same initial values for $s_1 = 0$ as $w_1(s_1)$, and since by (3.31)

$$\frac{R_1}{r_1} \leq (1.46)^2 \quad ,$$

we will have $|W_1(s_1)| \leq |w_1(s_1)|$ as long as $W_1(s_1)$ does not change its sign.

We have clearly

$$W_1(s_1) = w_1(0)\cos(1.46s_1) + \frac{1}{1.46} w_1'(0)\sin(1.46s_1) \quad .$$

Let us start with the case $w_1(0) \geq 0$. We see by direct calculation that $W_1(s_1)$ cannot vanish before the point $s_1 = s_1(T_1)$. Thus we are also sure that $w_1(s_1)$ will remain non-negative in the entire interval. On the other hand, $w_1(s_1)$ is always convex towards the s_1-axis and hence it can be estimated by the inequality

$$w_1(s_1) \leq w_1(0) + s_1 w_1'(0) \quad ,$$

which proves the lemma in this case by virtue of (3.37).

In the case $w_1(0) < 0$, the solution $w_1(s_1)$ will increase at first because $w_1'(0) \geq 0$. Let s_1^* denote the first point at which it vanishes. We have

$$(3.39) \qquad |w_1(s_1)| \leq |w(0)| \quad , \quad 0 \leq s_1 \leq s_1^* \ .$$

We can write because of (3.35)

$$w_1'(s_1^*) = w_1'(0) + \int_0^{s_1^*} \frac{R_1}{r_1}|w_1(s_1)|\,ds_1 \leq w_1'(0) + |w_1(0)| \int_0^{s_1^*} \frac{R_1}{r_1}\,ds_1 \ .$$

For $s_1 \geq s_1^*$ we use the fact that $w_1(s_1)$ must stay positive because of Lemma 3.6 and is, therefore, concave. Thus

$$(3.40) \quad w_1(s_1) \leq (s_1-s_1^*)w_1'(s_1^*) \leq (s_1-s_1^*)w_1'(0) + (s_1-s_1^*)|w_1(0)| \int_0^{s_1^*} \frac{R_1}{r_1}\,ds_1 \ .$$

Observe that in view of the estimates (3.31) we have

$$(s_1-s_1^*) \int_0^{s_1^*} \frac{R_1}{r_1}\,ds_1 \leq 1 \quad , \quad 0 \leq s_1^* \leq s_1 \leq s_1(T_1) \ .$$

Hence the inequalities (3.39) and (3.40) imply the desired result (3.36). Thus the lemma has been proved in all cases.

Since by the definition (3.33) we have the inequality

$$s_1(t) \leq tr_1(t) \quad ,$$

we may bring (3.36) into the form

$$|\omega(t)| \leq t\,|v(0)| + |y(0)| \quad . \tag{3.41}$$

We can now use formula (3.25) in order to estimate $\sin\varphi$ by means of (3.41) in the entire interval $[0\ ,\ T_1]$. We find by use of Lemma 2.22 the inequality

$$|\sin\varphi| \leq .03514 \quad , \quad 0 \leq t \leq T_1 \quad .$$

In the same interval, consequently,

$$\left|\sin\frac{\varphi}{2}\right| \leq .0176 \quad .$$

Thus $|\sin(\varphi/2)|$ can never attain the upper limit $1/(2\rho) = 1/(14)$ in the interval $[0\ ,\ T_1]$ and hence there is no point in the interval $[0\ ,\ 1/2]$ where the estimate (3.20) could break down. Thus $T_1 = 1/2$ and we have established

LEMMA 3.8. *The estimate*

$$|\sin \frac{\varphi}{2}| \leq \frac{.0352}{2} \tag{3.42}$$

holds in the entire interval $0 \leq t \leq 1/2$.

We proceed to prove

LEMMA 3.9. *In the interval* $s_2(T_2) \leq s_2 \leq 1$ *we have*

$$|w_2(s_2)| \leq |w_2(1)| + (1 - s_2)|w_2'(1)| \quad . \tag{3.43}$$

Using the definitions (3.24) and (3.28) we find

$$w_2(1) = v(1) - y(1) \quad , \tag{3.44}$$

and using the differential equations (3.34) we find

$$w_2'(1) = v(1) \quad . \tag{3.45}$$

We know that $v(1) = \operatorname{Im}\{(4/3)a_2\} \geq 0$, whence $w_2'(1) \geq 0$, and we have to distinguish the two cases $w(1) > 0$ and $w(1) \leq 0$.

For $w(1) \leq 0$ we use as a function of comparison the solution $W_2(s_2)$ of the differential equation

$$\frac{d^2W_2}{ds_2^2} + (2.152)^2 W_2 = 0$$

which has at $s_2 = 1$ the same initial values as $w_2(s_2)$. Since by (3.32)

$$\frac{R_2}{r_2} \leq (2.152)^2 ,$$

we will have $|W_2(s_2)| \leq |w_2(s_2)|$ as long as $W_2(s_2)$ does not change its sign. Clearly

$$W_2(s_2) = w_2(1)\cos(2.152[1 - s_2]) - \frac{w_2'(1)}{2.152}\sin(2.152[1 - s_2]) .$$

But neither the sine nor the cosine here change their sign if $s_2 \geq s_2(T_2) \geq 1/2$. Hence in the interval considered $w_2(s_2)$ will remain negative and will be convex from below. Thus we obtain the estimate

$$|w_2(s_2)| \leq |w_2(1)| + (1 - s_2)w_2'(1) ,$$

which proves the lemma in this particular case.

We discuss next the possibility $w_2(1) > 0$. As s_2 decreases from the value $s_2 = 1$, the function $w_2(s_2)$ will at first decrease, because $w_2'(1) \geq 0$. Let s_2^* be the first point at which w_2 vanishes, if such a point exists. In the interval $s_2^* \leq s_2 \leq 1$ we have

$$0 \leq w_2(s_2) \leq w_2(1) . \tag{3.46}$$

At the point s_2^* we find by (3.35)

$$w_2'(s_2^*) = w_2'(1) + \int_{s_2^*}^{1} \frac{R_2}{r_2} w_2 \, ds_2 \quad ,$$

and hence

$$w_2'(s_2^*) \leq w_2'(1) + w_2(1) \int_{s_2^*}^{1} \frac{R_2}{r_2} \, ds_2 \quad .$$

In case $s_2(T_2) \leq s_2 \leq s_2^*$ we have $w_2(s_2) \leq 0$ by Lemma 3.6, whence $w_2(s_2)$ is convex and

$$|w_2(s_2)| \leq w_2'(s_2^*)(s_2^*-s_2) \leq w_2'(1)(s_2^*-s_2) + w_2(1)(s_2^*-s_2) \int_{s_2^*}^{1} \frac{R_2}{r_2} \, ds_2 \quad .$$

By (3.32)

$$(s_2^* - s_2) \int_{s_2^*}^{1} \frac{R_2}{r_2} \, ds_2 \leq 1 \quad ,$$

and therefore we obtain the estimate

$$(3.47) \qquad |w_2(s_2)| \leq w_2(1) + w_2'(1)(s_2^* - s_2) \quad .$$

Inequalities (3.46) and (3.47) complete the proof of Lemma 3.9.

In view of the inequality

$$1 - s_2 = \int_{t}^{1} r_2 dt \leq (1 - t)$$

and the definition (3.28), we deduce from (3.43) that

$$.983\,|\omega(t)| \leq |w_2(1)| + (1 - t)|w_2'(1)|$$

for $T_2 \leq t \leq 1$. Thus by (3.25), (3.44) and (3.45)

$$|\sin \varphi(t)| \leq 3.414[|v(1) - y(1)| + .5|v(1)|]$$

$$= 3.414[|\tfrac{4}{3} P - Q| + \tfrac{2}{3}|P|]$$

in this interval. But Lemma 2.21 yields the estimate

$$|\tfrac{4}{3} P - Q| + \tfrac{2}{3}|P| = \tfrac{4}{3}|S| + \tfrac{2}{3}|P| \leq .0098 \quad ,$$

and therefore

$$|\sin \frac{\varphi}{2}| \leq .0168$$

in the interval $T_2 \leq t \leq 1$. It follows that $|\sin(\varphi/2)|$ never attains the upper limit $1/(2\rho) = 1/(14)$ in the interval $[T_2 , 1]$, and hence $T_2 = 1/2$. Thus we have established

LEMMA 3.10. *The estimate*

(3.48) $$|\sin \frac{\varphi}{2}| \leq \frac{.0336}{2}$$

holds in the entire interval $1/2 \leq t \leq 1$.

Lemmas 3.8 and 3.10 establish that the inequality (3.13) holds throughout the interval $0 \leq t \leq 1$ with $\rho = .0352$. From this stage forward we shall use the functions $r_1(t)$, $R_1(t)$ and $w_1(t)$ in the entire interval $0 \leq t \leq 1$ and we therefore introduce the simplified notation

$$r = r_1 \quad , \quad R = R_1 \quad , \quad w = w_1 \quad , \quad s = s_1 \quad . \tag{3.49}$$

We have

<u>LEMMA 3.11. In the interval</u> $0 \leq t \leq 1/2$

$$0 < R(t) \leq 2.07 \quad , \tag{3.50}$$

<u>and in the complete interval</u> $0 \leq t \leq 1$

$$1 \leq r(t) \leq 1.002 \quad , \quad 0 < R(t) \leq 4.48 \quad . \tag{3.51}$$

By means of our revised estimate (3.42), we can sharpen (3.21) to obtain in the interval $0 \leq t \leq 1/2$

$$F(t) \geq .645 \quad ,$$

and the bound (3.50) follows by (3.30). To establish the bounds (3.51), we use (3.23) and (3.27) directly.

4. The boundary value problem.

After the preceding preliminary estimates we come now to the uniqueness proof for the non-linear boundary value problem of Lemma 1.3.

We return to the differential system

$$\frac{dv}{dt} = - R(t)w \quad , \quad \frac{dw}{dt} = r(t)v \tag{3.52}$$

with the boundary conditions

$$v(1) = \frac{2}{3} v(0) \quad , \quad w(1) = r(1)[\frac{2}{3} v(0) + \frac{3}{2} w(0) + \frac{1}{4} u(0)v(0)] \quad , \tag{3.53}$$

which we obtain from (3.8), (3.24), (3.28), (3.29) and (3.49). By integration by parts we derive the identity

$$\int_{t_1}^{t_2} \frac{1}{R(t)} \left(\frac{dv}{dt}\right)^2 dt = - [v(t)w(t)] \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} r(t)v(t)^2 \, dt \quad , \tag{3.54}$$

or, with $t_1 = 0$ and $t_2 = 1$,

$$\int_0^1 \frac{1}{R(t)} \left(\frac{dv}{dt}\right)^2 dt + \frac{4}{9} r(1)v(0)^2 + [r(1) - 1]v(0)w(0) + \frac{1}{6} r(1)u(0)v(0)^2 = \int_0^1 r(t)v(t)^2 \, dt \quad . \tag{3.55}$$

We have

LEMMA 3.12. *A non-trivial solution* $v(t)$ *of the system* (3.52) *can vanish at most once in the interval* $0 \leq t \leq 1$.

In fact, suppose that $v(t_1) = v(t_2) = 0$, with $0 \leq t_1 < t_2 \leq 1$. Using identity (3.54) we find

$$\int_{t_1}^{t_2} \frac{1}{R}\left(\frac{dv}{dt}\right)^2 dt = \int_{t_1}^{t_2} rv^2\, dt \quad ,$$

and in view of the estimates (3.51) we obtain the inequality

$$\frac{1}{4.48} \int_{t_1}^{t_2} \left(\frac{dv}{dt}\right)^2 dt \leq 1.002 \int_{t_1}^{t_2} v^2\, dt \quad . \tag{3.56}$$

Using again the inequality

$$\int_{t_1}^{t_2} \left(\frac{dv}{dt}\right)^2 dt \geq \frac{\pi^2}{(t_2 - t_1)^2} \int_{t_1}^{t_2} v^2\, dt \quad ,$$

which holds for all continuously differentiable functions $v(t)$ vanishing at t_1 and t_2, we derive from (3.56) the absurd relation

$$\frac{\pi^2}{4.48} \leq 1.002(t_2 - t_1)^2 \leq 1.002 \quad .$$

Thus the lemma is proved.

From the beginning we have made the normalization $v(0) \geq 0$. If $v(0) = 0$, then by (3.53) we would also have $v(1) = 0$, and therefore $v(t) \equiv 0$ by Lemma 3.12. Thus our extremal function would be given by (3.1) and there would be nothing more to prove. It follows that we may restrict ourselves to the case $v(0) > 0$. By (3.53) this implies $v(1) > 0$ and hence $v(t)$ cannot vanish in the interval $0 \leq t \leq 1$, by Lemma 3.12.

Since $v(t)$ decreases from $v(0)$ to $(2/3)v(0)$, we conclude from the first equation (3.52) that $w(t)$ must be positive somewhere in the interval. From the second equation (3.52) we deduce that $w(t)$ increases monotonically, and hence

$$w(1) \geq 0 \quad . \tag{3.57}$$

This implies by (3.53) the inequality

$$w(0) \geq -\left(\frac{4}{9} + \frac{u(0)}{6}\right)v(0) \quad . \tag{3.58}$$

Therefore we can replace the identity (3.55) by the inequality

$$\int_0^1 \frac{1}{R(t)}\left(\frac{dv}{dt}\right)^2 dt + v(0)^2\left(\frac{4}{9} + \frac{u(0)}{6}\right) \leq \int_0^1 r(t)v(t)^2\, dt \quad .$$

Using the estimate (3.51) and putting

$$\ell = \frac{4}{9} + \frac{u(0)}{6} = \frac{10}{9} - \frac{1}{3}p \quad , \tag{3.59}$$

we find

$$\int_0^1 \frac{1}{R(t)}\left(\frac{dv}{dt}\right)^2 dt + \ell v(0)^2 \leq 1.002 \int_0^1 v(t)^2\, dt \quad . \tag{3.60}$$

By Lemma 2.22, we have

$$\ell \geq 1.107 \quad ,$$

and this establishes

LEMMA 3.13. *If* $v(t) \not\equiv 0$, *then*

$$(3.61) \qquad \frac{\int_0^1 \frac{1}{R(t)}\left(\frac{dv}{dt}\right)^2 dt + 1.107 v(0)^2}{\int_0^1 v(t)^2 \, dt} \leq 1.002 \quad .$$

On the other hand, we shall now prove in an elementary way

LEMMA 3.14. *For all functions* $V(t)$ *which are piece-wise continuously differentiable and not identically zero in the interval* $0 \leq t \leq 1$ *and which satisfy the boundary condition*

$$(3.62) \qquad V(1) = \frac{2}{3} V(0) \quad ,$$

we have

$$(3.63) \qquad \frac{\int_0^1 \frac{1}{R(t)}\left(\frac{dV}{dt}\right)^2 dt + 1.107 V(0)^2}{\int_0^1 V(t)^2 \, dt} > 1.08 \quad .$$

By Lemma 3.11 we have

$$\frac{1}{R(t)} \geq (.695)^2 \quad , \quad 0 \leq t \leq 1/2 \quad ,$$

$$\frac{1}{R(t)} \geq (.472)^2 \quad , \quad 1/2 \leq t \leq 1 \quad ,$$

and therefore it suffices to calculate

$$(3.64) \quad k^2 = \min \frac{\int_0^{1/2} (.695)^2 V'(t)^2 \, dt + \int_{1/2}^{1} (.472)^2 V'(t)^2 \, dt + 1.107 V(0)^2}{\int_0^1 V(t)^2 \, dt}$$

and to show that $k^2 > 1.08$. From the elements of the calculus of variations, we obtain the following characterization for the extremal function $V(t)$ which yields the minimum in (3.64). We have for $V(t)$ the differential equations

$$(3.65) \qquad V'' + \frac{k^2}{(.695)^2} V = 0 \quad , \quad 0 \leq t < \frac{1}{2} \quad ,$$

$$(3.66) \qquad V'' + \frac{k^2}{(.472)^2} V = 0 \quad , \quad \frac{1}{2} < t \leq 1 \quad ,$$

the saltus conditions

$$(3.67) \quad (.695)^2 V'(\tfrac{1}{2} - 0) = (.472)^2 V'(\tfrac{1}{2} + 0) \quad , \quad V(\tfrac{1}{2} - 0) = V(\tfrac{1}{2} + 0) \quad ,$$

and the natural boundary condition

$$\frac{2}{3}(.472)^2 V'(1) = (.695)^2 V'(0) - 1.107V(0) \quad , \tag{3.68}$$

together, of course, with the side condition (3.62), which is presupposed all the time.

We deduce from the differential equations (3.65) and (3.66) that

$$V(t) = C_1 \cos \frac{k}{.695}(t - \tfrac{1}{2}) + \frac{1}{.695} D_1 \sin \frac{k}{.695}(t - \tfrac{1}{2}) \ , \ 0 \leq t < \tfrac{1}{2} \ ,$$

$$V(t) = C_2 \cos \frac{k}{.472}(t - \tfrac{1}{2}) + \frac{1}{.472} D_2 \sin \frac{k}{.472}(t - \tfrac{1}{2}) \ , \ \tfrac{1}{2} < t \leq 1 \ .$$

The saltus conditions (3.67) lead to the equations

$$C_1 = C_2 \quad , \quad D_1 = D_2 \quad .$$

On the other hand, the boundary conditions (3.62) and (3.68) yield the system of linear equations

$$C_1 \cos \frac{k}{.944} + \frac{1}{.472} D_1 \sin \frac{k}{.944} = \frac{2}{3}[C_1 \cos \frac{k}{1.39} - \frac{1}{.695} D_1 \sin \frac{k}{1.39}] \ ,$$

$$-\frac{2}{3}(.472)C_1 k \sin \frac{k}{.944} + \frac{2}{3} D_1 k \cos \frac{k}{.944} = C_1[.695k \sin \frac{k}{1.39} - 1.107 \cos \frac{k}{1.39}]$$

$$+ D_1[k \cos \frac{k}{1.39} + \frac{1.107}{.695} \sin \frac{k}{1.39}] \ .$$

This is a homogeneous system of linear equations which has non-trivial solutions only if its determinant

$$(3.69)\quad \Delta = -\frac{4}{3}k + \left[\frac{13}{18}k + \frac{.695}{.944}k + \frac{2}{9}\,\frac{.472}{.695}k\right]\cos\frac{k}{2}\left(\frac{1}{.695}+\frac{1}{.472}\right)$$

$$+\left[\frac{13}{18}k - \frac{.695}{.944}k - \frac{2}{9}\,\frac{.472}{.695}k\right]\cos\frac{k}{2}\left(\frac{1}{.695}-\frac{1}{.472}\right)$$

$$+ 1.107\left[\frac{1}{2}\left(\frac{1}{.695}+\frac{1}{.472}\right)\right]\sin\frac{k}{2}\left(\frac{1}{.695}+\frac{1}{.472}\right)$$

$$+ 1.107\left[\frac{1}{2}\left(\frac{1}{.695}-\frac{1}{.472}\right)\right]\sin\frac{k}{2}\left(\frac{1}{.695}-\frac{1}{.472}\right)$$

vanishes. Thus we obtain for k the transcendental relation

$$(3.70)\qquad 1 = 1.20703\cos(1.77875k) - .123694\cos(.339897k)$$

$$+ 1.4768\,\frac{\sin(1.77875k)}{k} + .2822\,\frac{\sin(.339897k)}{k}\ .$$

The equation (3.70) has no roots in the interval $0 \leq k \leq 1.04$, whence $k > 1.04$ and the lemma is proved.

The inequalities (3.61) and (3.63) are contradictory, and therefore we obtain

<u>LEMMA 3.15</u>. <u>There does not exist a solution of the boundary value problem (3.4)-(3.8) whose initial values satisfy the estimates of Lemmas 2.21 and 2.22, except for the solution with the exact initial values</u>

$$(3.71)\qquad u(0) = 4\ ,\quad v(0) = 0\ ,\quad x(0) = \frac{10}{3}\ ,\quad y(0) = 0\ .$$

5. Conclusion.

Lemma 3.15 completes our proof of the inequality $|a_4| \leq 4$. Since our derivation of the principal Lemmas 2.22 and 3.15 turned out to involve a multitude of laborious calculations, it would seem appropriate to discuss from a broader point of view the main features of our method and to suggest alternative procedures which could have been used.

As we mentioned in Chapter II, it is not actually necessary to introduce terms involving a_7 in the preliminary estimation of a_2 and a_3 , for the terms based on a_5 and a_6 yield sufficiently refined bounds for the uniqueness proof of this chapter. To be precise, if we carried through all the arguments of this chapter with the same basic formulas, but used everywhere numerical bounds derived from Lemma 2.17 rather than from Lemma 2.20, we would be led to a contradiction which would again establish Lemma 3.15. The chief difference in the proof would be that the new bounds appearing in Lemmas 3.13 and 3.14 would be much closer. Also, the value $\rho = 1/7$ would turn out to be barely large enough for the success of our estimates in Section 3. The presence of more refined, and therefore more convincing, bounds is not, however, so much the motive for exploiting terms in a_7 as is a deeper understanding of our method and of its applicability to further coefficient problems.

In the opposite direction, we could have worked out preliminary bounds based on a_8 and on higher coefficients, and with the improved estimates thus obtained our study of the non-linear boundary value problem of Lemma 1.3 would not have required such care. On the other hand, we could emphasize less the method of Chapter II and refine instead the uniqueness proof in this chapter. Such a refinement could be obtained by subdividing the interval $0 \leq t \leq 1$ into more than two subintervals. This improvement amounts to a more exact numerical integration of the differential equations (1.10) and provides a routine tool suitable for more difficult problems.

We have found no way to avoid altogether the estimation procedure of Chapter II. However, Loewner's formulas could be side-stepped by expressing the equations for a_2 and a_3 near the point $a_2 = 2$, $a_3 = 3$ in terms of the periods of hyperelliptic integrals. This is done by separating variables in the differential equation (1.5), integrating, and writing down the conditions for the singularities of the integrals on either side of the equation to correspond under the conformal mapping by the extremal function. We have already mentioned this alternative when we studied another coefficient problem [5]. We find our present approach more appealing because of its connection with non-linear differential equations.

We should remark that the inequality (1.23) is also equivalent to the above equations for a_2 and a_3 if all the terms are retained and all the coefficients $b_{2\nu-1}$ are expressed in terms of a_2 , a_3 and a_4 . This equivalence follows because boundedness of the coefficients $b_{2\nu-1}$ implies convergence of the solution of the differential equation (1.5) in the entire interior of the unit circle. It is this insight which leads us to persevere in the tedious calculations of Chapter II.

Finally, we are hopeful that the general method of our proof will find application to a wide variety of the more difficult extremal problems for schlicht functions which have thus far defied analysis. However, we must resist the temptation to spend a lifetime working out a wealth of examples, each with its own special twist!

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