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A METHOD FOR SYNTHESIS OF FACTOR ANALYSIS STUDIES

by

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FOREWORD

This report is of primary interest to research workers who have occasion to use factor analysis techniques. Since factor analysis is a basic research tool, the developments described in this report will be of interest to research workers in personnel management as well as in other fields.

The technique of factor analysis is a widely used and important research tool aimed at giving a better understanding of the underlying abilities that personnel classification tests measure. In the area of test construction and administration, the method of factor analysis can be used to answer the question: How many traits, or abilities-- or for convenience, factors-- are measured by a given test or set of tests? Such information is used in improving the effectiveness of tests and test batteries. To date, the technique of factor analysis has been developed to produce this information for the tests used in only one study at a time. The problem arises, then, of comparing the factors isolated in different experimental studies to answer the further question: Are the factors isolated in each of several studies identical or different? To answer such questions, it has been necessary to incorporate the tests from several studies into a larger, over-all study on a representative sample of people and then apply factorial analysis techniques. The difficulties and cost of doing this have discouraged its being done.

It is, therefore, desirable to have a technique for making more immediate comparisons of factors. The present investigation, primarily methodological in nature, is concerned with a technique that will allow comparison of factors when certain conditions are met and will also enable adjustment of the factors obtained in separate studies so as to synthesize the findings. Computational details of the technique are described, and several numerical examples are presented to show the practicability of the technique.

CONTENTS

LIST OF TABLES

A METHOD FOR SYNTHESIS OF FACTOR ANALYSIS STUDIES*

Ledyard R Tucker

I. Introduction

After several factor analysis studies in a particular domain have been completed, one of the major questions that occurs is how to synthesize results from these studies. Often there are a number of identical tests in the several batteries employed in the studies. It is hoped that these common tests will aid in a more firm synthesis of the studies. The problem has been how to make use of this identity of tests. Claims have been reported in the literature that rotation to simple structure will yield invariance of factors, and it has been held that the common tests could assist in identifying corresponding factors in the two studies. However, some difficulty has been encountered in this approach (as has also been reported in the literature); consequently, the need for devising a more definite method has become clear. In particular, it has been hoped that by some new method two studies which have overlapping tests and have been factored in accordance with L. L. Thurstone's general theories to loadings on reference axes could be separately rotated into congruence and then jointly rotated to simple structure.

An illustration of the problem under attack is provided by two studies conducted by the Personnel Research Section of the Personnel Research and Procedures Branch, Office of the Adjutant General, Department of the Army in cooperation with representatives of the other Armed Services. One study

*The author is indebted to a number of staff members in the Department of Statistical Analysis, Educational Testing Service, who have worked on development of the material of this report. Deserving of special mention are Mrs. Gertrude Diederich who assisted in the analysis of the major example and Miss Angela E. Nolan who assisted in the preparation of the computing directions. Portions of the manuscript were read by Dr. William G. Mollenkopf and Dr. Frederic M. Lord, both of the Educational Testing Service Research Department, and by Miss Henrietta L. Gallagher of the Department of Statistical Analysis.

involved Army and Navy classification tests* which were given to a group of Naval Recruits after which a factor analysis was performed including rotation of axes to an orthogonal simple structure given in Table 1. This study will be called Study A. The other study, which will be called Study B, involved Army, Navy, and Air Force tests given to some Airmen and some Soldiers. A factor analysis again was performed with rotation to the orthogonal simple structure also given in Table 1. Ten of the variables were common to the two studies as indicated by the cross references included in the two tables. Some difficulty was experienced in identification of the same factors in the two studies. These two tables will be called matrices $F_{\underline{jmA}}$ and $F_{\underline{jMB}}$. It is the purpose of the present investigation to develop a means for finding a factor space common between such studies as determined by similarity of factor loadings of tests common to the studies. This may be expressed mathematically as attempting to discover transformation matrices $T_{\underline{mrA}}$ and $T_{\underline{MrB}}$ such that when

$$F_{jrA} = F_{jmA}T_{mrA} \tag{I.1A}$$

$$F_{jrB} = F_{jMB}T_{MrB} \tag{I.1B}$$

the differences between matrices $F_{\underline{jrA}}$ and $F_{\underline{jrB}}$ are negligible for tests overlapping the two studies. The subscripts $\underline{m}$ and $\underline{M}$ have been used to designate the reference factors in the two studies while $\underline{r}$ is used to designate what we will term congruent factors. $\underline{A}$ and $\underline{B}$ are used to designate the studies. The number of congruent factors may be less than the number of factors in either study.

Once a set of congruent factors have been determined, further rotation of axes in both studies jointly within the space of the congruent factors can be accomplished to rotated congruent factors. Whenever the congruence has been established as of sufficient strength, these steps should facilitate across study comparisons.

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*PRS Report 778. Comparison of Army and Navy Classification Tests. 29 April 1949.

Table 1

ROTATED FACTOR LOADINGS

STUDY A

Matrix F_{jmA}

Test Code Numbers		Loadings on Rotated Factors* (Decimal points are omitted.)					
Study A	Study B	i	ii	iii	iv	v	vi
9	26	83	04	01	01	08	-01
18	1	59	21	09	-01	07	-01
1	..	87	06	-02	23	01	19
5	23	57	-09	01	53	27	21
12	28	58	07	16	35	11	31
4	22	27	09	03	44	53	35
13	32	63	10	03	27	46	27
14	34	60	04	00	10	30	08
15	33	40	00	01	37	56	00
11	27	78	05	41	33	01	00
3	..	70	16	38	26	00	-08
10	..	72	15	46	10	-02	11
2	..	62	22	46	01	09	04
6	..	59	45	22	09	-09	10
7	..	45	63	00	01	02	00
8	..	33	80	13	05	03	08
16	30	39	59	08	18	-01	03
17		52	50	10	03	03	00

*These rotated factors will be used as reference factors in the following analysis and will be so labeled.

Table 1 (Continued)

ROTATED FACTOR LOADINGS

STUDY B

Matrix F_{BB}

Test Code Numbers		Loadings on Rotated Factors (Decimal points are omitted.)											
Study A	Study B	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
18	1	421	-075	303	246	175	454	080	041	061	-010	048	116
..	2	304	-300	604	002	-007	430	-077	072	015	051	081	125
..	3	230	061	630	430	249	085	-102	154	118	-187	102	015
..	4	273	067	330	236	236	061	-023	244	225	-036	-015	-085
..	5	-004	320	-611	278	044	073	-066	-167	-049	-134	-025	049
..	6	192	-142	748	137	088	234	-143	165	-073	-004	108	186
..	7	-040	-202	-145	-215	-164	-154	070	-065	132	035	-035	-298
..	8	192	-124	849	180	143	-029	-043	225	128	-088	145	-061
..	9	690	-049	109	274	166	230	-002	177	115	-052	285	082
..	10	396	-033	070	257	074	418	022	116	220	120	445	025
..	11	290	-038	134	265	329	500	-033	131	246	085	390	045
..	12	274	017	094	156	181	594	022	078	135	039	256	-043
..	13	590	025	079	370	229	110	-054	257	173	049	216	-003
..	14	668	-040	125	198	160	210	-035	150	120	107	243	075
..	15	306	046	100	578	147	151	-015	284	237	017	283	059
..	16	138	041	-015	456	290	104	-034	220	313	132	206	070
..	17	227	070	020	723	098	175	-014	124	149	038	277	019
..	18	078	074	-011	701	195	016	-009	043	238	089	288	075
..	19	221	020	109	167	484	174	-020	025	291	034	212	-010
..	20	225	055	049	129	327	335	-055	152	139	022	220	024
..	21	157	-019	132	221	370	205	-057	103	355	-105	466	025
4	22	090	032	020	615	228	008	014	135	380	-006	165	164
5	23	182	043	049	532	265	151	-027	188	447	092	202	041
..	24	492	-078	117	160	041	364	102	140	410	000	281	002
..	25	567	030	057	330	130	245	-056	284	183	-082	335	-069
9	26	550	-022	366	230	154	370	-031	245	174	-060	274	-037
11	27	385	-065	107	292	090	481	010	154	296	086	466	088
12	28	231	-004	124	263	359	228	009	080	446	-080	466	020
..	29	224	-042	003	495	180	250	-007	200	435	086	129	-056
16,17	30	315	001	099	120	341	550	-002	065	180	-008	129	-001
..	31	100	-007	095	207	366	293	-071	244	001	020	215	-030
13	32	320	052	061	588	186	236	016	180	366	033	208	015
15	33	073	105	-014	765	-007	153	-042	147	230	-009	173	-015
14	34	147	-045	044	415	035	172	-055	587	171	022	270	-010
..	35	001	-057	120	178	044	049	-064	617	033	-054	130	025
..	36	221	113	-005	-076	151	140	-407	-091	160	007	-198	-074
..	37	-142	599	-150	314	-099	-092	020	-115	093	-220	080	-065
..	38	032	584	013	123	093	015	045	008	160	200	084	075
..	39	-007	557	-011	410	-016	-035	459	-024	-025	-099	095	-102
..	40	134	016	227	154	009	105	282	464	220	-050	023	316
..	41	055	-657	000	-170	-059	-105	022	-075	-127	029	-194	-109
..	42	014	-668	-070	-217	-017	204	050	002	-085	-094	064	224
..	43	070	095	118	307	130	-015	439	222	-018	099	112	-060
..	44	-021	-354	-123	-303	058	-235	-136	-275	-184	164	164	-264

II. Results of a Search of the Literature

As the study of the literature progressed it became apparent that publications to date could be classified under two headings. First, there has been some controversy on invariance of results from factorial studies. Second, several problems similar or related to the one we are studying have been treated and solutions have been found. Evaluation of the material presented in the literature became guided, as more material was covered, by a greater precision in definition of the general nature of the attack we were going to make on the solution of the problem.

A. Controversy on Invariance of Factorial Results When Several Studies Are Analyzed in Accordance with Thurstone's Multiple Factor Analysis Theory.

Thurstone has suggested (25, 26, 28) that rotation of axes in factor analyses to simple structure will yield invariant results for common factors under broad conditions of change from one study to another. Several limiting conditions in which invariance could not be expected were also noted. Specific changes for which invariance was claimed were:

1. The battery of tests could be altered by addition of other variables or by deletion of variables, provided that:

   (a) The new variables did not have loadings on a factor specific to one of the original variables, in which case this previous specific factor would become a new common factor; however, the previous common factors were invariant.

   (b) The new variables did not have additional common factors among themselves, in which case a new common factor or factors would be added; however, the previous common factors were invariant.

(c) The variables deleted did not eliminate one of the previous common factors; however, the other common factors remained invariant.

(Thurstone was cognizant of the fact that deletion of tests could leave the configuration of vectors such that the simple structure was so indeterminate that the rotation of axes would not result in the same factors. Also recognized was that addition of new tests could help determine a simple structure previously indeterminate.)

2. The battery of tests could be administered to different groups of individuals, provided that:

(a) The groups were sufficiently similar so that the psychological nature of tasks involved in the tests did not change.

(b) Partial special selection had not occurred between groups on two or more variables. In this case a new "incidental" common factor would be added, but the simple structure would remain for the common factors.

(c) Complete special selection had not occurred on one or more variables.

(Thurstone, in this area of change, considered it permissible for the factors to change in correlation and for the exact values of the non-vanishing factor loadings to change. The factor loading changes, however, are approximately by a constant of proportionality for each factor under theoretical conditions of selection.)

Meyer (16) has experimentally demonstrated factorial invariance when tests are deleted from a battery. He dealt explicitly with cases in which the test deletions left the simple structure indeterminate. In one of three small batteries formed by a selection of the tests used by the Thurstones in a study at the eighth grade level (27), one factor which should have been present could not be identified after separate analysis of the small battery and rotation to what appeared to be its simple structure. Meyer used Thurstone's correlations so that only invariance under test selection was being tested. It seems, however, that data from

Meyer's third battery as compared with Thurstone's results for a larger battery would be ideal for a tryout of any methods developed in the present project.

Godfrey Thomson (23), applying Pearson's and Aitken's formulation of effects of selection, has pointed out that univariate and multivariate selection affected factorial solutions. Thurstone's analysis of the results of these effects has been previously noted.

Cyril Burt (1) has used the correlation between two sets of factor loadings as an indication of agreement. R. B. Cattell (7) called this a "shape correlation coefficient." William Stephenson (22) has used the rank correlation coefficient in his Q-technique as a measure of similarity of profiles. Cattell (7) also suggested a coefficient of pattern similarity:

$$r_p = \frac{2k - \Sigma d^2}{2k + \Sigma d^2}$$

in which $\underline{k}$ is the median for χ^2 on a sample of size $\underline{n}$, the number of tests involved. (Each $\underline{d}$ would be the difference in factor loadings for one test in two analyses.) Cattell was particularly interested in similarity of profiles for two individuals and the $\underline{d}$'s would be the differences in their scores; but he suggested that the coefficient could be applied to similarity of factor loadings of tests in two studies.

Two other suggestions by Cattell were reviewed. In one (2) he suggested a different method for determining rotation of axes than the principle of simple structure. His suggestion was to make two studies, using the same tests in both studies, but so operating on the groups of subjects examined that the variability of the tests attributable to one psychological function would differ between groups. Rotation would be guided to that position where loadings on all factors but one would be equal between two studies and loadings on the one factor would be proportional between studies. Mathematics for accomplishing the desired result were presented only for the two dimensional case with no minor inconsistencies arising from such sources as sampling errors. In another article (4) Cattell used bivariate frequency counts and probabilities to support matching of factors after two studies had been rotated to simple structure.

H. S. Reyburn with J. G. Taylor in 1943 (18) and with M. J. Raath in 1949 (19) has criticized simple structure as a basis for rotation of axes in factor analysis. Young and Householder (29) suggested pivoting rotations in successive studies on particular tests by which they hoped a large body of knowledge about relations could be built.

Points of particular interest in the foregoing review of literature on factorial invariance of simple structure solutions to the present project are:

1. Disagreement exists as to whether or not to expect simple structure factorial solutions to be invariant.
2. Several indices for identity of factors in two studies have been put forward. Of most importance to the present project are:
   (a) The correlation coefficient between loadings on two factors.
   (b) Cattell's coefficient of pattern similarity.
3. Possible effects of change in battery analyzed by test addition or deletion have been noted. Common factors may be added when tests are added or common factors may be deleted when tests are deleted. Addition of tests might help determine a simple structure while test deletion might leave the structure indeterminate.
4. Effects of selection of the group of subjects used in a study can change the extent of correlations between factors and the relative magnitude of loadings of the tests on the factors. In some extreme cases, factors may be added or deleted. The factorial composition of tests may change when groups of different levels of ability are being examined.

B. Solutions to Similar or Related Problems

Two particularly relevent developments have been reported in the literature. In one, Mosier (17) has treated the case where loadings on a factor are assumed to be known and it is desired to locate an axis which has these loadings within small differences. A least-squares method of fitting actual loadings to the theoretical loadings were used. His solution resulted in the following matrix equation:

$$(A'A + \beta I)\Lambda = A'V, \qquad \text{(II.}$$

where $\underline{A}$ is the factor matrix on reference factors, $\underline{\beta}$ is a constant to be determined, $\underline{\Lambda}$ is the column vector with direction cosines of the desired axis, and $\underline{V}$ is a column vector with the theoretical loadings to be approximated. Since the labor of solution would be great, Mosier suggested as approximation obtained when $\underline{\beta}$ was set at zero.

$$\Lambda = (A'A)^{-1}A'V \qquad \text{(II.}$$

This would result in direction numbers instead of direction cosines, and it would be necessary to normalize the solution. Justification of approximation can be made in either of two ways: one, that $\underline{\beta}$ should be small; and two, that the restriction that the resulting vector be of unit length be discarded. This latter method of justification alters the problem to that of finding an axis for which the loadings are as nearly proportional to the theoretical loadings as possible. Equation II.2 is a solution to this problem.

Harold Hotelling (13) has dealt with the situation where there are not only several predictor variables to be combined in a regression equation but also several criteria which are to be combined so that optimum prediction can be obtained.

The matrix equations are:

$$Z = X_{\alpha}A_{\alpha}; \qquad \text{(II}$$

$$(R'_{i\beta}R^{-1}_{ij}R_{j\alpha} - \lambda R_{\alpha\beta})A_{\alpha} = 0 \; ; \qquad \text{(II}$$

$$A'_{\alpha}R_{\alpha\beta}A_{\beta} = 1; \qquad \text{(II}$$

where: X_α is the matrix of standard scores on the criteria,

$A_\alpha = A_\beta$ is a column vector of optimum weights for the criteria,

Z is a column vector of optimally predictable criterion standard scores,

R_{ij} is the matrix of predictor intercorrelations,

$R_{\alpha\beta}$ is the matrix of criterion intercorrelations,

$R_{i\alpha} = R_{j\beta}$ is the matrix of correlations of predictors with criteria,

λ is an undetermined multiplier.

Once equations II.4 and II.5 are solved, the appropriate regression of the predictors for the criterion Z can be obtained by usual methods where:

$$R_{iz} = R_{i\alpha} A_\alpha \, , \qquad \text{(II.6}$$

R_{iz} being a column vector of correlations of predictors with Z. Hotelling notes that equation II.4 may be simplified by treating the original criteria so as to obtain a derived uncorrelated set to be used in this solution. The matrix $R_{\alpha\beta}$ is then the identity matrix and may be dropped. Equation II.4 is then in the form for solution for principal components.

Hotelling's solution is of interest in the context of the present project in that he was matching optimally two separate sets of observations. We will be interested in matching factor loadings of tests rather than scores of individuals.

III. General Conditions and Assumptions

A. Assumption of Thurstone's General Theory of Factor Analysis

Since considerable controversy exists between several systems of factor analysis it was felt necessary to limit consideration of synthesis of factor analysis studies to within a particular factorial system, that developed by L. L. Thurstone (28). Of particular importance are the following basic assumptions made in Thurstone's theory of multiple factor analysis.

1. Variability of scores on a test among members of a population can be accounted for by variability of underlying abilities among members of the population, plus errors of observation.
2. For any particular battery of tests, some of the abilities are common in that they contribute variability to scores on several tests and some abilities are unique to each test in the battery.
3. A linear combination of ability scores is an adequate approximation to the actual mode of combination in producing test scores.

As a consequence of the choice of framework it would be expected that the method of synthesis would be more likely to be satisfactory for studies made within Thurstone's theory. Additional values might accrue if the method of synthesis were found to be usable for other factorial theories, but the reasonableness of the application would have to be separately evaluated.

B. Effects of the Groups Tested.

One of the problems in synthesizing factorial studies is that of abstracting common meaning in spite of the fact that different groups of individuals have been used as the base for the several studies. Since these base groups are not usually selected as unbiased samples from the same population, it seems wise to consider that they may have different parameters such as test and factor variances and intercorrelations. The base groups often are composed of the people most readily available for testing. Some restrictions may be imposed, but no real attempt is made to obtain an unbiased sample of a previously defined population. When several such catch-as-catch-can groups are involved, the statistics for these groups may well differ greatly. It is obvious that an adequate method of synthesis must be able to cope with these inter-group differences. One restriction that must be made in order to develop a method of synthesis is that the base groups do not differ so widely that the factorial pattern of tests change markedly. If performance on a particular test is more dependent on reasoning ability for grade school children but more dependent on perceptual speed for college students, the factor patterns would be different and use of results for this test would be misleading when attempting to synthesize the two studies. Wide differences between the groups must, consequently, be considered with skepticism.

C. Defined Restrictions.

In the light of the foregoing discussion and the material in Appendix A, it seems necessary that three areas of possible restrictions be considered and that necessary definitions be made.

1. Since the test variances are a function of the group of people who happen to take the tests, and the rows of the factor matrix are affected proportionately if the units of measurements of the tests are changed, it seems desirable to establish a common unit of measurement for each test taken by the several groups. This requires a factor analysis of covariances rather than of correlations. When correlations are analyzed, a set of units are implied which yield unit variances for one particular group of people. When two groups are considered with different test variances, two different sets of test units of measurement are necessary to yield correlations for each of the groups. In Section III of the Appendix A, it is shown that a change in test unit of measurement results in proportionate changes in the factor loadings of the tests (See equation A.20).

As a consequence it will be defined throughout the present development that the factor matrices for the several groups will have been adjusted so that a common unit of measurement will have been used for each test across the several groups. The method for making the adjustment for units of measurement is outlined below. In this outline it is assumed that the factor analyses have been carried out on correlations.

a. Determine for each group a desired standard deviation on one test. These standard deviations are to be proportional to the raw score standard deviations on the test for the groups. The average of the standard deviations over the several groups is to be considered as a weighting factor for the test indicating the importance of that test in determining the synthesis of the factor analyses. If one of the tests is not to be relied on very heavily in synthesizing the studies, the average standard deviation of that test can be made low. Conversely, an important test can be given a high weight by using a large average standard deviation. In the illustrative pair of studies, for which the factor loadings were given in Table 1, all tests were assigned unit average standard deviations. Table 2 presents the raw score standard deviations in the two studies for each test. For the first test, the raw score standard deviations were 1.73 in Study A and 1.884 in Study B. The mean of these two standard deviations was 1.807. The adjusted standard deviations were obtained by dividing the raw score standard deviations by the mean:

$$.9574 = 1.73/1.807,$$
$$1.0429 = 1.884/1.807.$$

In case this test were to have a weight different from unity, the adjusted standard deviations would have been multiplied by this weight.

b. Multiply the factor loadings for the test in the reference factor matrix for each group by the desired standard deviation of the test for the group. Table 3 presents adjusted factor loadings for the example. Only the tests common to the two studies are included in this table. In obtaining these values, the factor loadings for each test were multiplied by the corresponding adjusted standard deviation. The loadings in Table 1 for Study A test 18 were multiplied by .9574, the corresponding adjusted standard deviation given in Table 2. Similarly, the loadings in Study B for this same test were multiplied by 1.0429. These steps can be stated in matrix algebra as:

$$F_{JmA} = D_{JA} F_{jmA}, \qquad \text{(III.1A}$$

$$F_{JMB} = D_{JB} F_{jMB}. \qquad \text{(III.1B}$$

The entries in the diagonal matrices are the adjusted standard deviations. The subscript j is used to designate the tests with unit standard deviations in the separate studies and the subscript J is used to designate the tests after adjustment of the standard deviations.

c. Repeat steps a and b for each of the remaining tests.

d. The resulting factor matrices with adjusted standard deviations will be used in all following steps.

2. In accordance with the results of section 4 and equation A.28 of Appendix A, the variances of scores on the factors will be permitted to vary between the two studies.

3. Since biased selection of the groups is anticipated as possible and the results of section 4 of Appendix A indicate for this case that the correlations between the factors may differ for the several groups, no restriction of similar correlations between factors for the several groups will be imposed.

D. Synthesis of Studies Two at a Time.

In order to simplify the problem to some manageable size, consideration will be given only to the case in which there are two studies to be synthesized. When there are more studies than two, it is hoped that complete synthesization can be accomplished by progressive synthesis of pairs of studies.

E. Some Tests Common to Both Batteries, Other Tests in Either Battery.

It will be assumed that the batteries of tests in the two factor studies contain some overlap tests and some tests that appear in either of the batteries but not in both. A test will be considered as an overlap test only if 1) the test has not undergone editorial changes, 2) the time limits have not been changed, 3) instructions have not been changed, 4) test administration conditions are similar, and 5) scoring method is the same. Synthesis of the studies will depend on the overlap tests only.

Table 2

ADJUSTED TEST STANDARD DEVIATIONS

Test Code Numbers		Raw Score Standard Deviation			Adjusted Standard Deviation	
Study A	Study B	Study A	Study B	Average	Study A	Study B
18	1	1.73	1.884	1.807	.9574	1.0429
9	26	8.04	9.487	8.764	.9174	1.0825
5	25	8.05	8.920	8.485	.9487	1.0513
12	28	12.39	14.942	13.666	.9066	1.0934
4	22	8.30	8.497	8.398	.9883	1.0118
13	32	6.49	6.971	6.730	.9643	1.0358
14	34	5.91	6.529	6.220	.9502	1.0497
15	33	7.22	8.695	7.958	.9073	1.0926
11	27	10.62	12.076	11.348	.9358	1.0642
16+17	30	25.67	25.038	25.354	1.0125	.9875

Table 5

LOADINGS ON REFERENCE FACTORS FOR OVERLAP TESTS WITH ADJUSTED STANDARD DEVIATIONS

STUDY A

Matrix F_{JmA}

Test Code Numbers		Loadings on Reference Factors (Decimal points are omitted.)					
Study A	Study B	i	ii	iii	iv	v	vi
18	1	565	201	086	-010	067	-010
9	26	761	037	009	009	073	-009
5	23	541	-086	009	503	256	199
12	28	526	063	145	317	100	281
4	22	267	089	030	435	524	346
13	32	608	096	029	260	444	260
14	34	570	038	000	095	285	076
15	33	363	000	009	336	508	000
11	27	730	047	384	309	009	000
16+17	30	528	633	104	122	011	017

STUDY B

Matrix F_{JMB}

Test Code Numbers		Loadings on Reference Factors (Decimal points are omitted.)											
Study A	Study B	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
18	1	439	-078	316	256	182	473	083	043	064	-010	050	121
9	26	595	-024	071	249	167	401	-034	265	188	-065	297	-040
5	23	191	045	052	559	279	159	-028	198	470	097	212	043
12	28	253	-004	136	288	393	249	010	087	488	-087	510	022
4	22	091	033	020	622	231	008	014	137	384	-006	167	166
13	32	331	054	063	609	193	238	017	186	379	034	215	016
14	34	154	-047	046	436	037	181	-058	616	179	023	283	-010
15	33	080	125	-015	836	-008	167	-046	161	251	-010	189	-016
11	27	410	-069	114	311	075	512	011	164	315	092	496	094
16+17	30	311	001	098	119	357	543	-002	064	178	-008	127	-001

IV. Congruence of Factor Matrices

A. Definition of Congruence.

The term congruence is used in this development to indicate a lower level of precision of coincidence than is associated with its use in geometry. Rather than meaning that an exact fit of one matrix to the other has been obtained, an approximate fit is to be indicated by the term. Two matrices will be considered as congruent if they are generally similar, with only relatively small random differences.

In devising an index of extent of congruence, the concept of being able to replace the two matrices by an average matrix is convenient. Let the two matrices in which we are interested be the factor matrices $\underline{F_{JrA}}$ and $\underline{F_{JrB}}$. Their average can be designated $\underline{F_{Jrc}}$, and is obtained by the following equation in which lower case letters designate cell entries.

$$f_{Jrc} = (1/2)\ (f_{JrA} + f_{JrB}). \qquad \text{(IV.1}$$

In the index, the differences between each element of one matrix and the corresponding element in the average matrix is squared and these squared differences summed separately for each factor. This is done for both of the original matrices. Squares of the entries for each factor in the average matrix are then summed. Since it is important that each factor is congruent, a separate index of congruence, $\underline{g_r}$, is defined for each factor as the ratio of the sum of squares of differences for the factor to twice the sum of squares of average loadings for the factors:

$$g_r = \frac{\sum_J (f_{JrA} - f_{Jrc})^2 + \sum_J (f_{JrB} - f_{Jrc})^2}{2 \sum_J f_{Jrc}^2}\ . \qquad \text{(IV.2}$$

The denominator is doubled so as to balance the number of values summed in the numerator and the denominator. This index might best be described as a measure of the extent to which it would be sensible to replace the factor from the two studies by the average. A low value would indicate that this would be possible.

When equation IV.1 is substituted into equation IV.2, it is found that

$$g_r = \frac{\sum_J (f_{JrA} - f_{JrB})^2}{\sum_J (f_{JrA} + f_{JrB})^2}\ . \qquad \text{(IV.3}$$

The index of congruence for each factor r is then the ratio of the sum of squares of the differences between loadings for the tests on the factor in the two studies and the sum of squares of the sum of loadings for the tests on the factor in the two studies.

It is to be noted that the closest agreement between factors in the two studies, and thus maximum congruence, occurs when the index of congruence, g_r, is a minimum. A coefficient of congruence, ϕ_r, for each factor is developed in Appendix B:

$$\phi_r = \frac{\sum_J f_{JrA} f_{JrB}}{\sqrt{(\sum_J f^2_{JrA})\ (\sum_J f^2_{JrB})}} \qquad \text{(B.19)}$$

The relation between g_r and ϕ_r was found to be

$$g_r = \frac{1 - \phi_r}{1 + \phi_r} . \qquad \text{(B.18)}$$

The coefficient of congruence, ϕ_r, has similar properties to a coefficient of correlation approaching a maximum of unity for the most precise congruence and a lower limit of zero for the least precise congruence.

Table 4 gives the factor loadings of the overlap tests in the two studies in the example for the six factors corresponding to the six maxima of the coefficient of congruence. Inspection of the loadings on factor B reveals only small differences between the two studies. The coefficient of congruence, ϕ_r, for factor B is .999984 while the index for congruence, g_r, is .000008. High congruence also occurs for factors A, D, and E. The congruence for factor C is moderately high but distinctly below that for those noted above. The congruence for factor F is definitely low so that this factor will not be considered as a congruent factor. The congruent space between the two studies is therefore defined by the five factors A-E.

Table 5 gives the transformation matrices T_{MrA} and T_{MrB} used in rotation from the reference axes of F_{JmA} and F_{JMB} of Table 3 to the congruent factors of Table 4 by equations B.1A and B.1B. The columns of these transformations have been determined so as to maximize the coefficients of congruence (or, it may be stated, to minimize the indices of congruence). Once the reference frame of congruent factors has been determined, joint rotations within the congruent space are possible. These rotated factors will also have high coefficients of congruence. Appendix B gives the mathematical development of the solution and Appendix C gives the computational procedures.

B. The Non-Congruent Spaces

In the example there is a five dimensional congruent space. Study A has six factors which leaves one dimension not included in the congruent space. Table 6 gives the direction cosines for the dimension in this study which is uncorrelated with the congruent space. This dimension constitutes the Study A non-congruent space. Study B has a seven dimensional non-congruent space. The axes given in this space are mutually orthogonal and, as a set, orthogonal to the congruent space. In general, it would be expected that only a portion of the space in each of a pair of studies would be congruent, leaving a remainder of the space in each study non-congruent with the other study. This situation can arise from each study involving as a common factor some mental function not included in the common factor space of the other study. A second possibility is that the same factor might be involved in both studies, but the overlap tests in the two studies would not include tests adequately loaded with this factor to establish the congruence.

C. Meaning of an Observed Congruence

A basic assumption underlies the general attempt to coalesce results from several factor analysis studies. This assumption is: if a mental function is represented by a factor in each of several studies, the factor loadings of the tests should be the same in these studies. The invariance of factor loadings, then, becomes a necessary condition for identity of factors in two studies. It has been found, however, to be both reasonable and necessary to qualify this condition when there is likelihood of between group changes in the mental functions underlying differences in test performances within the groups of people on which the factorial studies are based. Even so, the invariance of factor loadings remains as a basic principle. Congruent factors between two studies satisfy the necessary condition of invariance of factor loadings. These two factors may represent the same mental function.

Before concluding, however, that congruent factors always represent the same mental function in two studies, it is important to ask whether factorial invariance is a sufficient condition. The answer is no, invariance of factor loadings is not a sufficient condition to identity of mental functions. Consider, for example, a test composed of verbally stated computation problems. This test, in a comprehensive study, might have loadings on a verbal factor and a computation factor. Suppose that this test is common to two smaller studies and is the only overlap test in the verbal and computation domains. Suppose, further, that in Study A there are several other verbal tests but no other computation tests, and that in Study B the reverse is true, there are other computation tests but no verbal tests. In Study A the common factor space will include the verbal factor but not the computation factor. The reverse will be true for Study B for which the common factor space will include the computation factor but not the verbal factor. Remember that our verbally stated computing test had loadings on both of these factors but was the only overlap test in these domains between the two studies. In this situation it would appear that the verbal factor of Study A was congruent with the computation factor of Study B. The mental functions, however, would not be identical. While this example may seem extreme, it demonstrates the proposition that invariance of factor loadings for a limited set of tests is not a sufficient condition for identity of factors between two studies. A number of other situations may also lead to congruence of factors without identity of mental functions.

The congruence of factors is, then, a necessary but not a sufficient condition for identity of factors between two studies. The credence, as to identity of factors, one can place on an observed congruence of factors between two studies depends on the extent of data on which the congruence is based. If the overlap tests in two studies are few in number and of limited variety for each mental function, very little confidence as to identity of factors can be placed on an observed congruence. In the example there are only ten overlap tests while there are six common factors and twelve common factors in the two studies. If the ten tests covered a ten dimensional sub-space out of the twelve dimensions of Study B, perfect congruence would be found for all six factors in Study A. Mathematics indicates this would be necessarily true. Suppose, even, that the ten tests were not identical between the two studies but were artificially matched. Perfect identity of factor loadings still could have been obtained. The congruent factors would be artificial.

In order to avoid this situation, in so far as possible in the method for obtaining congruent factors, the principal axes for the overlap tests are found in each study separately and minor axes are eliminated until the dimensionalities are reduced somewhat below the number of overlap tests. The extent that this is possible depends on the set-up of the two studies and choice of overlap tests. In the illustration, the dimensionalities used were six and eight principal axes in the two studies. This represented no reduction for Study A and a moderate one for Study B. Artificial congruence will still be a major problem in interpretation of the results.

Several rotations of axes within the congruent space were tried in order to find what dimensions might be more strongly determined than other dimensions. Table 7 gives the resulting factor loadings. Table 8 gives the corresponding transformations from the reference axes. Loadings between .15 and .24 have been singly underlined, loadings of .25 or more have been doubly underlined. Factors d and e have only one test each with underlines. In consequence very little confidence can be placed in the meaning of the congruence of these factors between the two studies. Factors a and b each have four tests with underlines and much more confidence can be placed in their congruence. Factor c occupies an intermediate position. As a result of our judgments of confidence in an observed congruence, some of the observed congruent factors in which we have low confidence can be reclassified to the non-congruent space. Factors d and e might be so transferred. The remaining factors, then, constitute the space on which confidence of the meaning of the congruence can be placed. Table 9 presents the factor loadings of the non-overlap tests in the two studies on factors a, b, and c. These loadings and those of the overlap tests may be inspected for interpretation of the two studies combined.

Whenever strong indications of identity of factors is desired by the use of overlap tests, we may conclude, it is necessary to include in the study plans provision for an adequate number of overlap tests. These tests should be varied as to form while depending on the same factors. Several parallel forms of each of a few tests will not be an improvement over including only one form of each test. These parallel forms will not provide distinctive data on which to base confidence in an observed congruence. Basically, then, confidence in observed similarities between results of two studies depends on adequate experimental design. Methods of analysis can then be of assistance.

Table 4

LOADINGS OF OVERLAP TESTS ON CONGRUENT FACTORS

Matrices F_{JrA} and F_{JrB}

Factor Loadings on Congruent Factors

Test Code Numbers		Factor A		Factor B		Factor C		Factor D		Factor E		Factor F	
Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B
18	1	.30	.30	.14	.14	-.04	-.03	-.04	-.04	-.02	-.02	.04	.17
9	26	.38	.38	.07	.07	-.14	-.18	-.05	-.05	.00	.00	-.07	-.08
5	23	.14	.14	-.29	-.29	.03	.04	.14	.14	.05	.05	-.16	-.07
12	28	.08	.07	-.06	-.06	-.06	-.04	.28	.28	.04	.03	.02	.08
4	22	.01	.01	-.39	-.39	.04	-.03	.09	.09	-.01	-.01	.10	.05
13	32	.21	.22	-.23	-.23	-.08	-.01	.02	-.02	-.02	-.02	.08	-.05
14	34	.26	.26	-.09	-.09	-.09	-.09	-.08	-.08	-.05	-.05	.02	.01
15	33	.18	.17	-.26	-.26	.13	.14	-.17	-.17	-.15	-.15	-.03	.03
11	27	.15	.15	.26	.26	.07	.06	.22	.21	-.22	-.21	.00	-.04
16+17	30	.43	.43	.14	.13	.17	.16	.01	.01	.17	.17	.04	-.02
ϕ_r*		.999833		.999984		.939811		.999875		.999670		.459717	
δ_r*		.000058		.000008		.031028		.000062		.000165		.370129	

*ϕ_r and δ_r were computed from factor loadings carried to four decimal places.

Table 5

TRANSFORMATION TO CONGRUENT FACTORS

STUDY A

Matrix T_{mrA}

Reference Factors	Congruent Factors A	B	C	D	E	F*
i	.4786	.1156	-.2175	-.0188	.0372	-.1249
ii	.3765	.0829	.3400	-.0972	.3290	.1473
iii	-.5678	.7978	.1197	.4222	-.8345	.6199
iv	-.0015	-.4349	.5502	.2500	.2159	-.5081
v	.0148	-.3283	.0456	-.4994	-.4410	.3549
vi	-.4190	-.2634	-.5835	.7162	.3301	.3955

STUDY B

Matrix T_{MrB}

Reference Factors	Congruent Factors A	B	C	D	E	F*
I	.1584	-.0729	-.6344	-.0656	-.0529	-.3057
II	.1545	-.1961	.1053	-.1769	.1405	-.1439
III	-.1594	.1217	-.2053	.0816	-.1206	.6394
IV	.1097	-.3825	.1095	-.3203	-.1741	.1678
V	.3365	-.3786	.0097	.0739	.6545	.1293
VI	.6015	.5972	.6999	-.1570	.0429	-.0269
VII	-.1792	.0149	-.1208	.1086	-.1020	.1910
VIII	.3478	-.1368	-.2637	-.1773	.1952	-.0268
IX	-.1983	-.2664	.0524	.5074	.0942	-.2988
X	-.1803	.1450	.3323	.4671	-.1605	-.5283
XI	-.3897	.4004	-.0623	.3630	-.4435	.1459
XII	-.4391	.0301	-.1371	.4019	-.2581	.2066

*Congruent Factor F had such a low coefficient of congruence that it was eliminated from the congruent space.

Table 6

DIRECTION COSINES OF NON-CONGRUENT FACTORS

STUDY A

Matrix A_{muA}

Reference Factors	Non-Congruent Factor
i	-.0827
ii	.7591
iii	.1610
iv	-.1653
v	.4628
vi	.3865

STUDY B

Matrix A_{MUB}

Reference Factors	Non-Congruent Factors 1	2	3	4	5	6	7
I	-.0043	-.0592	-.0244	-.0399	-.1537	-.2042	-.4464
II	.1890	-.2934	.8158	-.2423	-.1728	.0619	-.1633
III	-.5234	-.0476	.2507	-.3179	.6857	0	0
IV	-.0229	-.1110	-.1946	.0522	.3070	.1871	-.0515
V	.1235	-.4013	-.2307	.0494	.2892	.2273	-.1741
VI	.1315	.1839	.0813	.0437	.2250	.0289	-.1630
VII	-.0064	-.0644	.3262	.8911	.2215	-.0776	-.0782
VIII	.0861	.0407	.1676	.0511	.0332	.1205	.8075
IX	-.2141	.5799	.1935	-.0095	-.0942	.2317	-.1008
X	-.2465	-.4920	-.0268	-.0168	-.1399	-.4656	.2105
XI	.0662	-.3271	-.0238	.0006	-.1280	.7266	0
XII	.7532	.0981	.0179	-.1858	.3907	-.2234	.0424

Table 7

LOADINGS OF OVERLAP TESTS ON ROTATED CONGRUENT FACTORS

Matrices F_{JzA} and F_{JzB}

Test Code Numbers		Loadings on Rotated Congruent Factors									
		Factor a		Factor b		Factor c		Factor d		Factor e	
Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B	Study A	Study B
18	1	.19	.18	-.10	-.10	.01	.01	.03	.03	.03	.03
9	26	.32	.35	-.09	-.09	.00	-.02	-.03	-.04	-.02	-.04
5	23	.02	.01	.32	.32	.11	.12	.03	.03	.09	.10
12	28	.06	.04	.31	.31	-.13	-.12	.13	.14	.04	.05
4	22	-.03	.03	.32	.31	.20	.16	.01	-.01	.02	-.02
13	32	.19	.13	.15	.16	.13	.17	-.01	.00	-.03	.02
14	34	.23	.23	-.03	-.03	.13	.13	-.03	-.03	-.05	-.05
15	33	.00	.00	.00	.00	.44	.44	.00	.00	.00	.00
11	27	-.01	.01	.07	.07	.00	.00	.35	.35	.00	.00
16+17	30	.00	.00	.00	.00	.00	.00	.00	.00	.32	.31

Table 8

TRANSFORMATION TO ROTATED CONGRUENT FACTORS
FROM REFERENCE FACTORS

STUDY A Matrix T_{mzA}

Reference Factors	Rotated Congruent Factors a	b	c	d	e
i	.4368	-.0732	-.0863	-.0342	-.0173
ii	-.2085	-.0435	.0441	-.1374	.4972
iii	-.3343	-.1220	-.0731	.9301	-.4369
iv	-.6100	.5790	.2769	.0906	.5285
v	.1048	-.3297	.7422	-.0567	-.3316
vi	.2558	.7629	-.8453	-.0184	-.1750

STUDY B Matrix T_{MzB}'

Reference Factors	Rotated Congruent Factors a	b	c	d	e
I	.7338	-.1017	-.2149	-.1783	-.4411
II	-.0272	-.0270	.1768	-.1925	.1504
III	.1374	-.0370	-.1681	.0799	-.2273
IV	.0210	-.0886	.5585	-.1060	-.0779
V	.0466	.3592	-.2186	-.4190	.4706
VI	-.4292	-.3465	.2638	.2205	.6306
VII	.0395	.0608	-.0935	.0880	-.1629
VIII	.4256	-.0925	-.0560	-.3036	-.0216
IX	-.2188	.6453	-.1669	.2038	.1270
X	-.4600	.3954	-.0200	.4952	.1780
XI	-.1090	.0604	-.1797	.5336	-.3135
XII	-.0799	.3062	-.1889	.3479	-.2725

Table 9

LOADINGS OF NON-OVERLAP TESTS ON ROTATED CONGRUENT FACTORS

STUDY A

MATRIX F_{JxA}

Test Code Numbers	Loadings on Rotated Congruent Factors a	b	c
1	.2836	.2093	-.1605
3	-.0337	-.0165	.0585
10	.0945	.0316	-.1694
2	.0847	-.1056	-.0417
6	.0516	.0672	-.1735
7	.0612	-.0620	.0067
8	-.0730	.0046	-.0341

STUDY B

MATRIX F_{JxB}

Test Code Numbers	Loadings on Rotated Congruent Factors a	b	c
2	.1115	-.1353	-.1424
3	.3472	-.0339	.0275
4	.3123	.0809	-.0355
5	-.1302	-.0865	.3519
6	.2100	-.1141	-.1112
7	-.0406	.0462	-.0895
8	.3794	.0124	-.1869
9	.4738	-.0378	-.0925
10	.0269	.0315	.0026
11	-.0461	.1036	-.0092
12	-.0676	-.0875	.0650
13	.4293	.0657	-.0264
14	.3587	.0361	-.1263
15	.2163	.0807	.1397
16	.0024	.2867	.0842
17	.0825	.0062	.3041
18	-.0418	.2174	.2423
19	.0362	.2791	-.1138
20	.0406	.0653	-.0291
21	.0349	.2309	-.1105
24	.1714	.0975	-.0994
25	.4162	.0729	.0077
29	.0179	.1987	.1495
31	.0528	-.0166	.0337
35	.2651	-.0655	.0111
36	.0382	.0422	.0007
37	-.0716	-.0548	.3249
38	-.1311	.1797	.1006
39	.0582	-.0939	.2909
40	.2435	.1254	-.1043
41	.1107	-.0617	-.1515
42	-.0351	-.0563	-.2138
43	.1468	.0328	.0594
44	-.0799	.0378	-.2053

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APPENDIX A

THEORY OF MULTIPLE FACTOR ANALYSIS OF COVARIANCE

Multiple factor analysis has been developed using a restrictive definition concerning the unit of measurement for each variable, be it a test or a factor, such that unit variance is obtained for the group of people on whom the study is based. The initial equation relating test scores to factor loadings and factor scores is commonly written in terms of standard scores for tests and factors. The units of measurement are treated as "floating", taking such values as yield unit test or factor variances for the group concerned. It is the purpose of these notes to derive the theory of multiple factor analysis when the variances are permitted to have any values, thus permitting the units of measurement to be otherwise defined. This is a necessary step when several groups of people are used for different factor analyses of the same tests, since these different groups are likely to have different variances of scores when the units of measurement of the variables are held constant for the several groups.

1. General Factorial Equation for Covariances.

Equation A.1 is the usual initial linear equation used in factor analysis. In it and subsequent equations, capital letters are used to designate matrices, and lower case letters are used to designate cell entries in the corresponding matrices. Subscripts are attached both to matrix designations and to cell entry designations for convenience in keeping track of the variables involved. Particular notation used in this section is as follows:

N = number of people in the group,
X = matrix of scores,
C = matrix of variances and covariances,
F = matrix of factor loadings,
i = subscript designating individual person,
j and k = an alternate pair of subscripts designating tests, and
p and q = an alternate pair of subscripts designating factors.

Equation A.1 is then:

$$X_{ji} = F_{jp}X_{pi} \qquad \text{(A.1}$$

These scores are deviation scores, but no restriction is placed on their variances. The restriction of deviation scores is of no importance since corrections for means on the factor scores leads to correction for means of the test scores without requiring a change in the factor matrix. As a consequence we shall ignore any difference in means between groups, assuming that within-group deviation scores are used in all cases. (This action would not be warranted if groups were to be combined.)

Equation A.2 and A.3 are the usual ones for tables of variances and covariances:

$$C_{jk} = (1/N)X_{ji}X'_{ki} \; ; \tag{A.2}$$

$$C_{pq} = (1/N)X_{pi}X'_{qi} \tag{A.3}$$

When equation A.1 is substituted into equation A.2,

$$C_{jk} = (1/N)F_{jp}X_{pi}X'_{qi}F'_{kq} \tag{A.4}$$

Eliminating the factor-score matrices of equation A.4 by noting that the right-hand side of equation A.3 is involved in equation A.4,

$$C_{jk} = F_{jp}C_{pq}F'_{kq} \tag{A.5}$$

Equation A.5 is the general factorial equation for covariances.

2. Transformation of Factors.

In this section an inductive process in which it is assumed that transformations are possible seems appropriate, and the proof consists of a demonstration that this assumption yields consistent results.

Let T_{pr} be any square matrix of an order equal to the number of factors and for which an inverse exists. It is our assumption that this matrix can transform one factorial matrix, with factors p (alternate subscript q), into another factorial matrix, with factors r (alternate of subscript s), by equation A.6a.

$$F_{jr} = F_{jp}T_{pr} \; . \tag{A.6a}$$

$$F_{jp} = F_{jr}T^{-1}_{pr}. \quad \text{(A.6b}$$

Substitution of equation A.6b into equation A.1 yields:

$$X_{ji} = F_{jr}T^{-1}_{pr}X_{pi} \quad \text{(A.7}$$

Let the following definition of transformation of scores be made:

$$X_{ri} = T^{-1}_{pr}X_{pi} \,. \quad \text{(A.8a}$$

$$X_{pi} = T_{pr}X_{ri} \,. \quad \text{(A.8b}$$

Equation A.7 then becomes:

$$X_{ji} = F_{jr}X_{ri} \,. \quad \text{(A.9}$$

Equation A.9 reproduces equation A.1 with factors r replacing factors p. Equation A.5 can be immediatedly re-written for factors r:

$$C_{jk} = F_{jr}C_{rs}F'_{ks} \,, \quad \text{(A.10}$$

Similarity of equation A.8b to equation A.9, if matrix T_{pr} is considered to be a factorial matrix for factors p in terms of factors r, permits equation A.10 to be re-written.

$$C_{pq} = T_{pr}C_{rs}T'_{qs} \,, \quad \text{(A.11b}$$

$$C_{rs} = T^{-1}_{pr}C_{pq}T'^{-1}_{qs} \,, \quad \text{(A.11a}$$

(The first of these equations is labeled A.11b and the second A.11a so as to be consistent with equations A.6 and A.8. The a equations relate factors r to factors p and the b equations relate factors p to factors r.) Substitution of equation A.11a into equation A.10 yields:

$$C_{jk} = F_{jr}T^{-1}_{pr}C_{pq}T'^{-1}_{qs}F'_{ks} \quad \text{(A.12}$$

Noting that the first two matrices on the right of equation A.12 reproduce the right-hand side of equation A.6b and that the last two matrices of equation A.12 are the transpose of the right-hand side of equation A.6b, the corresponding substitutions yield equation A.5. Thus the system is internally consistent and the transformation of factors is possible.

Since there are an infinitely large number of matrices which satisfy the restrictions that they are of the order equal to the number of factors and possess an inverse, there are as many possible sets of factors which satisfy equation A.5, or A.10. This is the same problem as encountered in the normal factor analysis of correlations, and the solution proposed by Thurstone, that transformation be to a simple structure, is appropriate for the factor analysis of covariances. Actually, factor analysis of correlation is a special case of factor analysis of covariances in which the additional definition is imposed that the variances (diagonal entries in the C matrices) be unity. Existence of a simple structure, however, will not completely solve the problem for factor analysis of covariances. There remains a problem of changes in the size of the units of measurement for the factors.

Let there be a change in size of unit of measurement from factors $\underline{r}$ to factors $\underline{R}$. This results in a proportional change of all scores on each factor. Equation A.13 accomplishes these proportional changes where the diagonal entries in $D_{\underline{rR}}$ are the constants of proportionally.

$$X_{ri} = D_{rR}X_{Ri} \quad . \qquad \text{(A.13}$$

In order to simplify the algebra, the scores on factors $\underline{r}$ were considered as proportional to the scores on factors $\underline{R}$. Equation A.13 is similar to equation A.8b, in that in A.13 factors $\underline{r}$ are being transformed to factors $\underline{R}$ by the matrix D_{rR} just as factors $\underline{p}$ were transformed to factors $\underline{r}$ by the matrix $T_{\underline{pr}}$ in equation A.8. Equation A.6a is then re-written:

$$F_{jR} = F_{jr}D_{rR} \quad . \qquad \text{(A.14}$$

In this case, then, the columns of factor loadings on factors $\underline{R}$ are proportional to the loadings on factors $\underline{r}$. The configuration of zero factor loadings is unchanged by this transformation and the simple-structure remains. Equation A.11a may be re-written.

$$C_{RS} = D^{-1}_{rR} C_{rs} D'^{-1}_{sS} \quad . \tag{A.15}$$

Since the factors are not directly observed, no experimentally determined set of units of measurement exists. This poses a dilemma, for any set can be used and the simple structure will remain. For any single factor analysis the factor units of measurement can be left as unknown and a restriction placed on the variances of the factor scores. Thus the diagonal entires in the covariance matrix $\underline{C_{RS}}$ can be defined as some constant such as unity. When several factor analyses on different groups are being considered, this simple solution is inappropriate, for the resulting units of measurement may be of different sizes for the several groups.

3. Transformation of Tests

Two types of transformations of the tests are of interest. In theory, the type in which weighted sums of the tests are taken as new variables is the more general and includes as a special case the second type in which the units of measurement of the tests are changed so as to change the scores proportionally. Let it be desired to obtain variables $\underline{h}$ from tests $\underline{j}$ by a weighting matrix $\underline{W_{hj}}$ in accordance with the equation:

$$X_{hi} = W_{hj} X_{ji} \quad . \tag{A.16}$$

If equation A.1 is pre-multiplied by W_{hj}:

$$W_{hj} X_{ji} = W_{hj} F_{jp} X_{pi} \quad . \tag{A.17}$$

The right-hand side of this equation can be simplified by the following definition:

$$F_{hp} = W_{hj} F_{jp} . \tag{A.18}$$

When equations A.16 and A.18 are substituted into equation A17:

$$X_{hi} = F_{hp} X_{pi} \tag{A.19}$$

Equation A.19 is similar to equation A.1 and all of the derivations of sections 1 and 2 apply to the variables **h**. It is to be noted that there was no change in the factors or factor scores.

If a change is to be made in the units of measurement of the tests so that test j becomes test J, the weight matrix of equation A.16 becomes a diagonal matrix so that the scores on each test are changed proportionally. The weight matrix can be designated by D_{Jj} for this case. Then, if the subscript J is substituted for h and the matrix D_{Jj} is substituted for W_{hj}, equations A.16-A.19 give the relations for the tests with new units of measurement. A result of this is that the analysis may be carried through with one set of units for the tests and the factor matrix can be transformed for a new set of units. For sake of later convenience equation A.18 is re-written with the necessary changes indicated above:

$$F_{Jp} = D_{Jj}F_{jp}. \tag{A.20}$$

4. Effects of Between-Group Differences

In this section it is assumed that the same battery of tests and set of factors are involved for two or more groups. Equation A.1 is assumed to hold for each group individually. It is relatively obvious that within-group means scores can be ignored provided that the groups are not to be combined. Deviation scores within each group can be obtained without changing the factorial matrix, and therefore will be used. Let there be two groups, A and B. Equation A.1 then expands to:

$$X_{jiA} = F_{jr}X_{riA}, \tag{A.21A}$$

$$X_{jiB} = F_{jr}X_{riB}. \tag{A.21B}$$

(Capital letters are used at the end of equation numbers to indicate the groups, A or B, to which the equation applies.) It is assumed that the groups are sufficiently similar that the factorial nature of the tests remain unaltered. If this is true, then for any particular individual it should not matter within which group he is considered so far as the factorial equation for his scores is concerned. The factorial equations should therefore be parallel, with the same factor matrix F_{jr}, as shown in equation A.21. The matrices X_{jiA}, X_{jiB}, X_{riA}, and X_{riB} differ only with respect to which people are included in the groups A and B. By equation A.5:

$$C_{jkA} = F_{jr}C_{rsA}F'_{ks}, \tag{A.22A}$$

$$C_{jkB} = F_{jr}C_{rsB}F'_{ks}. \tag{A.22B}$$

An implicit assumption in equation A.21 and A.22 is that a single unit of measurement exists for each factor and is common for the two groups. As previously noted, the unit of measurement cannot be observed for factors. Consequently it is necessary to derive the relations between factor matrices for the two groups when the group factor variances are defined to be some constant. Employing the transformations of equations A.13-A.15 and noting that the matrices D_{rR} are particular to the groups:

$$X_{RiA} = D^{-1}_{rRA}\, X_{riA}; \tag{A.23A}$$

$$X_{RiB} = D^{-1}_{rRB}X_{riB}\ ; \tag{A.23B}$$

$$F_{jRA} = F_{jr}D_{rRA}\ ; \tag{A.24A}$$

$$F_{jRB} = F_{jr}D_{rRB}\ ; \tag{A.24B}$$

$$C_{RSA} = D^{-1}_{rRA}C_{rsA}D'^{-1}_{sSA}\ ; \tag{A.25A}$$

$$C_{RSB} = D^{-1}_{rRB}C_{rsB}D'^{-1}_{sSB}\ . \tag{A.25B}$$

It is to be noted in equations A.24 that the factor matrices for factors R now have a subscript designating group. When equations A.24 are solved simultaneously so as to eliminate F_{jr},

$$F_{jRB} = F_{jRA}D^{-1}_{rRA}D_{rRB}. \tag{A.26}$$

It is possible to combine the two diagonal matrices by defining:

$$D_{RAB} = D_{rRA}^{-1} D_{rRB}. \tag{A.27}$$

Then

$$F_{jRB} = F_{jRA} D_{RAB}. \tag{A.28}$$

Thus the two factor matrices are proportional by columns. It is of interest that the diagonal entries in D_{RAB} are the ratios of the variances on factors r for group B to those for group A.

Even though the covariance matrices C_{RSA} and C_{RSB} have the same diagonal values, the off-diagonal entries will differ due to sampling effects, either from random sampling or selective sampling. If, as might be usual, the factor variances were to be defined as unity, these covariance matrices would become the correlation matrices between the factors for the two different groups. Under the assumption that the factors have some reality (which is necessary for any of this development and any hope that the same factors are operative for the two groups), it would be expected that the correlation matrices for the two groups would differ.

APPENDIX B

DETAILS OF MINIMUM SOLUTION FOR INDEX OF CONGRUENCE

In equation IV.3 the formula for the index of congruence for each factor was given as:

$$g_r = \frac{\sum_J (f_{JrA} - f_{JrB})^2}{\sum_J (f_{JrA} + f_{JrB})^2}. \qquad \text{(IV.3}$$

It is desired to determine the factors so as to minimize $\underline{g_r}$.

Consider that the analyses for the two studies have been factored to uncorrelated reference factors for the groups involved. Any set of orthogonal factors may be used whether they are the original factors obtained or an orthogonal rotation from the original factors. Let $\underline{m}$ represent the reference factors for study A and $\underline{M}$ represent the reference factors for study B. Then by equation A.6a:

$$F_{JrA} = F_{JmA}T_{mrA}, \qquad \text{(B.1A}$$

$$F_{JrB} = F_{JMB}T_{MrB}. \qquad \text{(B.1B}$$

(The last letter of equation number indicates to which study the equation applies. When both studies are involved, no letter will be used.) Writing these equations in summational notation:

$$f_{JrA} = \sum_m f_{JmA}t_{mrA}, \qquad \text{(B.2A}$$

$$f_{JrB} = \sum_M f_{JMB}t_{MrB}. \qquad \text{(B.2B}$$

When equation B.2 are substituted into the index of congruence in equation IV.3,

$$g_r = \frac{\sum_J (\sum_m f_{JmA}t_{mrA} - \sum_M f_{JMB}t_{MrB})^2}{\sum_J (\sum_m f_{JmA}t_{mrA} + \sum_M f_{JMB}t_{MrB})^2}. \qquad \text{(B.3}$$

Inspection of equation B.3 reveals that all $\underline{t}$'s can be multiplied by some constant without altering the value of $\underline{\varepsilon}_r$. This is true because the constant enters into each term in a similar manner, can be factored out, and then cancelled from numerator and denominator. Therefore, it is possible to define a condition that the denominator equals some constant K without limiting the generality of the solution. The equation giving this condition is

$$\theta_r = \sum_J \left(\sum_m f_{JmA} t_{mrA} + \sum_M f_{JMB} t_{MrB}\right)^2 - K = 0, \qquad \text{(B.4}$$

and B.3 becomes

$$\varepsilon_r = \frac{1}{K} \sum_J \left(\sum_m f_{JmA} t_{mrA} - \sum_M f_{JMB} t_{MrB}\right)^2 \qquad \text{(B.5}$$

Using LaGrange's system of undetermined multiplers, the minimum can be obtained when

$$\frac{\partial \varepsilon_r}{\partial t_{mrA}} + \beta_r \frac{\partial \theta_r}{\partial t_{mrA}} = 0. \qquad \text{(B.6A}$$

Substituting the indicated partial derivatives into B.6A,

$$\frac{2}{K} \sum_J \left(\sum_m f_{JmA} t_{mrA} - \sum_M f_{JMB} t_{MrB}\right) f_{JmA} + 2\beta_r \sum_J \left(\sum_m f_{JmA} t_{mrA} + \sum_M f_{JMB} t_{MrB}\right) f_{JmA} = 0. \qquad \text{(B.7A}$$

When the equation is expanded, and terms are regrouped,

$$(1 + K\beta_r) \sum_J f_{JmA} \left(\sum_m f_{JmA} t_{mrA}\right) = (1 - K\beta_r) \sum_J f_{JmA} \left(\sum_M f_{JMB} t_{MrB}\right). \qquad \text{(B.8A}$$

Writing this equation in matrix form,

$$(1 + K\beta_r)\, F'_{JmA} F_{JmA} T_{mrA} = (1 - K\beta_r) F'_{JmA} F_{JMB} T_{MrB}. \qquad \text{(B.9A}$$

Define

$$\phi_r = \frac{(1 + K\beta_r)}{(1 - K\beta_r)} \qquad \text{(B.10A}$$

Then

$$F'_{JmA}F_{JMB}T_{MrB} = \phi_r F'_{JmA}F_{JmA}T_{mrA}. \quad (B.11A$$

Similarly, finding derivatives with respect to t_{MrB} leads to

$$F'_{JMB}F_{JmA}T_{mrA} = \phi_r F'_{JMB}F_{JMB}T_{MrB}. \quad (B.11B$$

(Equation B.11B can be written from B.11A by interchanging m's and M's, and A's and B's.)

The meaning of ϕ_r can be obtained by premultiplying equation B.11A by T'_{mrA} which gives

$$T'_{mrA}F'_{JmA}F_{JMB}T_{MrB} = \phi_r T'_{mrA}F'_{JmA}F_{JmA}T_{prA}. \quad (B.12A$$

Substituting from equations B.1A and B.1B yields

$$F'_{JrA}F_{JrB} = \phi_r F'_{JrA}F_{JrA}. \quad (B.13A$$

Writing equation B.13 in summational rotation,

$$\sum_J f_{JrA}f_{JrB} = \phi_r \sum_J f^2_{JrA}; \quad (B.14A$$

or,

$$\phi_r = \frac{\sum_J f_{JrA}f_{JrB}}{\sum_J f^2_{JrA}} \quad (B.15A$$

Similarly from equation B.11B it can be shown that

$$\phi_r = \frac{\sum_J f_{JrA}F_{JrB}}{\sum_J f^2_{JrB}}, \quad (B.15B$$

Noting that the numerators of equations B.15A and B.15B are equal, these equations solved simultaneously give

$$\sum_J f^2_{JrA} = \sum_J f^2_{JrB} \quad (B.16$$

Therefore, the sum of squares of loadings in one study on a congruent factor is equal to the sum of squares of loadings in the other study on this factor.

In order to relate ϕ_r and g_r, equation IV.3 is expanded to

$$g_r = \frac{\sum_J f^2_{JrA} + \sum_J f^2_{JrB} - 2\sum_J f_{JrA}f_{JrB}}{\sum_J f^2_{JrA} + \sum_J f^2_{JrB} + 2\sum_J f_{JrA}f_{JrB}} . \tag{B.17}$$

Substitutions from equations B.15A, B.15B, and B.16 yield

$$g_r = \frac{1 - \phi_r}{1 + \phi_r} . \tag{B.18}$$

From equation B.18 it is to be noted that g_r is a minimum when ϕ_r is a maximum. (Only the positive range of ϕ_r need be considered.) Making use of equation B.16, equation B.15A may be rewritten:

$$\phi_r = \frac{\sum_J f_{JrA}f_{JrB}}{\sqrt{(\sum_J f^2_{JrA})(\sum_J f^2_{JrB})}} , \tag{B.19}$$

which is similar to the formula for the product-moment correlation between the loadings on factor r for studies A and B. The difference from the equation for a correlation is that no corrections are made in this case for means of the factor loadings. Thus, ϕ_r might be called a <u>coefficient of congruence</u>. This is in contrast with g_r being called an <u>index of congruence</u>.

One consequence of equation B.19 is that ϕ_r can never be greater than unity. A value of unity would indicate perfect congruence of the factor in the two studies. Values of ϕ_r less than unity indicate various degrees of congruence down to no congruence at a value of zero. For practical purposes it may be desirable to set up some value of ϕ_r less than unity which will be regarded as acceptable for indicating the identity of the factors in the two studies. However, no guiding values have yet been developed, and it seems proper to delay specifying any minimally acceptable value of the coefficient of congruence until adequate experience in the application of the method has been gained.

In solving equations B.11A and B.11B it seems advisable to obtain the latent roots and latent vectors of the matrices $(F'_{JmA}F_{JmA})$ and $(F'_{JMB}F_{JMB})$. Let Λ_{mpA} be an orthogonal transformation containing the latent vectors and β_{pA} be a diagonal matrix containing the latent roots of $(F'_{JmA}F_{JmA})$. Then,

$$(F'_{JmA}F_{JmA}) = \Lambda_{mpA}\,\beta_{pA}\,\Lambda'_{mpA}; \tag{B.20A}$$

$$(F'_{JMB}F_{JMB}) = \Lambda_{MPB}\,\beta_{PB}\,\Lambda'_{MPB}. \tag{B.20B}$$

See Tables 10 and 11.

The latent vectors are frequently called principal axes in factor analysis. Let the matrix of factor loadings on the principal axes be F_{JpA}. Then,

$$F_{JpA} = F_{JmA}\,\Lambda_{mpA}; \tag{B.21A}$$

$$F_{JPB} = F_{JMB}\,\Lambda_{MPB}. \tag{B.21B}$$

See Table 12.

Since Λ_{mpA} is an orthogonal matrix,

$$F_{JmA} = F_{JpA}\Lambda'_{mpA}; \tag{B.22A}$$

$$F_{JMB} = F_{JPB}\Lambda'_{MPB}. \tag{B.22B}$$

Substitution of equation B.22A in equation B.20A and simplification yields

$$\beta_{pA} = F'_{JpA}F_{JpA} \tag{B.23A}$$

$$\beta_{PB} = F'_{JPB}F_{JPB} \tag{B.23B}$$

Each latent root, located in the diagonal of β_{pA}, is the sum of squares of the corresponding column of F_{JpA}. The principal axis with the smallest latent root has the property of being the factor in the space defined by F_{JmA} with the minimum sum of squares of loadings. The principal axis with the next to smallest latent root has the minimum sum of squares in the space orthogonal to the principal axis with the smallest latent root. Similar statements concerning minimum sum of squares of loadings can be made concerning the remaining principal axes, taking them in order from smallest to largest, in each case considering the space orthogonal to the preceeding axes. Since the latent roots are these sums of squares of the loadings on the principal axes, these roots represent indices of the extent to which the tests project into the dimensions represented by the principal axes. Whenever a latent root is zero, all tests must have zero loadings on that principal axis. As indicated in detail later, this condition of a zero latent root causes the solution to equations B.11A and B.11B to be non-unique. A small, but non-zero latent root indicates small loadings. This condition is likely to occur when the overlap tests represent some of the factors in the study but have only small random loadings on the other factors. Such principal axes with small latent roots may be delegated to a non-congruent space of the study by limiting the congruent factors to the space defined by the principal axes with significant latent roots. Study B, the four principal axes with smallest latent roots were placed in the non-congruent space. No precise rule has been developed for dividing between those principal axes to be delegated to the non-congruent space and the axes to be used in determining the congruent space. It is important, however, to exclude from the congruent space those dimensions into which the overlap tests have small projections.

Consider, for the present, that no latent vectors have been dropped from A_{mpA} (or A_{MPB}) and substitute equation B.22A into equation B.1A. Then:

$$F_{JrA} = F_{JpA} A'_{mpA} T_{mrA}; \qquad (B.24A$$

$$F_{JrB} = F_{JPB} A'_{MPB} T_{MrB}. \qquad (B.24B$$

Define:

$$T_{prA} = A'_{mpA} T_{mrA}; \qquad (B.25A$$

$$T_{PrB} = A'_{MPB} T_{MrB}. \qquad (B.25B$$

Then, from equation B.24A:

$$F_{JrA} = F_{JpA} T_{prA}; \qquad (B.26A$$

$$F_{JrB} = F_{JPB} T_{PrB} \qquad (B.26B$$

When equation B.25A is solved for T_{mrA},

$$T_{mrA} = A_{mpA} T_{prA};$$ (B.27A

$$T_{MrB} = A_{MPB} T_{PrB}.$$ (B.27B

Substitution of equations B.20A, B.22B, and B.23A into equation B.11A yields:

$$A_{mpA} F'_{JpA} F_{JPB} A'_{MPB} T_{MrB} = \phi_r A_{mpA} \beta_{pA} A'_{mpA} T_{mrA}.$$ (B.28A

Simplification by means of equation B.25 yields:

$$F'_{JpA} F_{JPB} T_{prB} = \phi_r \beta_{pA} T_{prA}.$$ (B.29A

Similarly:

$$F'_{JPB} F_{JpA} T_{prA} = \phi_r \beta_{PB} T_{PrB}.$$ (B.29B

The matrix product $F'_{JpA} F_{JPB}$ for the example is given in Table 13.

Consider now the case when one of the latent roots in one of the studies is zero. Without loss of generality of the development, the last latent root for study A can be taken as the particular zero latent root. Any other latent root could have been chosen, it is merely a matter of convenience The last diagonal entry in β_{pA} is then zero. Since β_{pA} is a diagonal matrix, the product $\beta_{pA} T_{prA}$ in equation B.21A results in the rows of T_{prA} being multiplied by the corresponding latent roots. The last entry in T_{prA} is, thus, multiplied by zero. When a latent root is zero, however, the loadings on the corresponding principal axis are zero. The last row of F'_{JpA}, therefore, has zero entries and the last entry in the product $F'_{JpA} F_{JPB} T_{PrB}$ is zero. Therefore,

$$0 = \phi_r \cdot 0 \cdot t'_{qrA},$$

where t_{qrA} is used to denote the last entry in T_{prA}. This equation is true no matter what value is given to t_{qrA}. In equation B.29B, the last column of F_{JpA} is zero; and, therefore, the value of t_{qrA} does not affect the equation. The conclusion is then that the value of t_{qrA} is not determined by equation B.29A and B.29B when the corresponding latent root is zero. Thus, in this case, the solution is not unique. It is reasonable, however, to assign a value of zero to t_{qrA}. The congruent factors, then, will not involve this principal axis which can, thus, be delegated to the non-congruent space.

Other principal axes can be delegated to the non-congruent space by defining the corresponding entry in T_{prA} (or T_{PrB}) to be zero. Whenever any of the entries in T_{prA} are defined as zero, these entries, the corresponding columns of F_{JPB} and A_{mpA} (or F_{JPB} and A_{MPB}), and corresponding rows and columns of β_{pA} (or β_{pA}) can be dropped without affecting equations B.26A, B.27A, or B.29A (or the corresponding B equations.) In the following developments, it will be considered that these matrices have been so reduced.

Consider the case when there are as many significant latent roots in one study as overlap tests. F_{JpA} (or F_{JPB}) is square. This matrix then will possess an inverse. Equation B.29A can be solved for T_{prA}:

$$T_{prA} = F^{-1}_{JpA} F_{JPB} T_{prB}. \qquad \text{(B.30A}$$

ϕ_r is unity for all T_{PrB}. Any values can be assigned to T_{PrB} and a T_{prA} can be obtained. Thus, perfect congruence has been obtained as a mathematical necessity irrespective of the characteristics of the tests in the other study. No confidence can be placed in the observance of such a congruence of factors.

Since only those principal axes have been retained that have significant latent roots the β matrices are non-singular and possess inverses. It is now assumed that there are more overlap tests than significant latent roots in either study. It is convenient to define the column vectors:

$$M_{rA} = \beta^{\frac{1}{2}}_{pA} T_{prA}; \qquad \text{(B.31A}$$

$$M_{rB} = \beta^{\frac{1}{2}}_{PB} T_{PrB}; \qquad \text{(B.31B}$$

$$G = \beta^{-\frac{1}{2}}_{pA} F'_{JpA} F_{JPB} \beta^{-\frac{1}{2}}_{PB}. \qquad \text{(B.32}$$

The matrix G for the example is given in Table 14.

Solution of equation B.31A for T_{prA} yields:

$$T_{prA} = \beta_{pA}^{-\frac{1}{2}} M_{rA};$$ (B.33A

$$T_{PrB} = \beta_{PB}^{-\frac{1}{2}} M_{rB}.$$ (B.33B

Substitution of these equations into equation B.29A yields:

$$G\, M_{rB} = \phi_r M_{rA}$$ (B.34A

$$G' M_{rA} = \phi_r M_{rB}$$ (B.34B

Solution of equation B.34 B for M_{rB}, substitution in equation B.34A, and simplification yields:

$$GG' M_{rA} = \phi_r^2 M_{rA}.$$ (B.35A

Similarly:

$$G'G M_{rB} = \phi_r^2 M_{rB}.$$ (B.35B

Define:

$$H_A = GG';$$ (B.36A

$$H_B = G'G.$$ (B.36B

See Table 14.

Then:

$$H_A M_{rA} = \phi_r^2 M_{rA};$$ (B.37A

$$H_B M_{rB} = \phi^2 M_{rB}.$$ (B.37B

The matrices H_A and H_B are Gramian with latent roots ϕ_r^2 and latent vectors A_{rA} and A_{rB}. The non-vanishing latent roots are identical for the two matrices. For each latent root there is a latent vector A_{rA} for H_A and a latent vector A_{rB} for H_B. Table 15 gives the results for the example

When M_{rA} and M_{rB} are defined as proportional to the latent vectors, equations B.37A and B.37B are solved since these equations are in a standard form for latent roots and vectors. Then:

$$M_{rA} = \Lambda_{rA}\, d_r; \tag{B.38A}$$

$$M_{rB} = \Lambda_{rB}\, d_r. \tag{B.38B}$$

Note that the same constant of proportionality, d_r, is used for both studies.

Substitution of equations B.38A and B.38B into equations B.33A and B.33B yields:

$$\dot{T}_{prA} = \beta_{pA}^{-\frac{1}{2}}\, \Lambda_{rA} d_r; \tag{B.39A}$$

$$T_{PrB} = \beta_{PB}^{-\frac{1}{2}} \Lambda_{rB} d_r. \tag{B.39B}$$

Substitution of these equations in B.27A and B.27B yields:

$$T_{mrA} = \Lambda_{mpA}\, \beta_{pA}^{-\frac{1}{2}}\, \Lambda_{rA} d_r; \tag{B.40A}$$

$$T_{MrB} = \Lambda_{MPB}\, \beta_{PB}^{-\frac{1}{2}}\, \Lambda_{rB} d_r. \tag{B.40B}$$

It is convenient to define:

$$\mathcal{T}_{mrA} = \Lambda_{mpA}\, \beta_{pA}^{-\frac{1}{2}}\, \Lambda_{rA} \tag{B.41A}$$

$$\mathcal{T}_{MrB} = \Lambda_{MPB}\, \beta_{PB}^{-\frac{1}{2}}\, \Lambda_{rB} \tag{B.41B}$$

See Table 15.

Then:

$$T_{mrA} = \mathcal{T}_{mrA}\; d_r \tag{B.42A}$$

$$T_{MrB} - \mathcal{T}_{MrB}\; d_r \tag{B.42B}$$

The constant, d_r, (one for each congruent factor) can be determined such that the average sum of squares of the entries in T_{mrA} and T_{MrB} is unity:

$$\tfrac{1}{2}\left(\sum_M t^2_{mrA} + \sum_M t^2_{MrB}\right) = 1. \qquad \text{(B.43}$$

This is similar to the usual practice in factor analysis of making the sums of squares of entries in a factor transformation vector unity. In this case, the transformation vectors for the two studies can not be normalized separately. It is, then, reasonable to normalize the two vectors on the average. In order to accomplish this step:

$$d_r = \frac{1}{\sqrt{\frac{1}{2}\left(\sum_m \tau^2_{mrA} + \sum_M \tau^2_{MrB}\right)}}, \qquad \text{(B.44}$$

where the τ's are the entries in the τ vectors. Table 16 contains the d_r's for the example. The transformation matrices T_{mrA} and T_{MrB} are in Table 5.

When the transformations to congruent factors, T_{mrA} and T_{MrB} have been determined, the matrices F_{JrA} and F_{JrB} of loadings on the congruent factors can be obtained by equations B.1A and B.1B. Table 4 contains these matrices of loadings on congruent factors for the example.

The computing procedure given in Appendix C is based on the foregoing equations. A simplification of steps was obtained by defining

$$T_{mpA} = A_{mpA}\, \beta^{-\frac{1}{2}}_{pA}; \qquad \text{(B.45A}$$

$$T_{MPB} = A_{MPB}\, \beta^{-\frac{1}{2}}_{PB}. \qquad \text{(B.45B}$$

Only those principal axes with significant latent roots are included in these equations. Substitution of equations B.21A, B.21B, B.45A, and B.45B into equation B.32 yields:

$$G = T'_{mpA}(F'_{JmA}F_{JMB})\, T_{MPB}. \qquad \text{(B.46}$$

With the definitions of equations B.45A and B.45B, equations B.41A and B.41B become:

$$\tau_{mrA} = T_{mpA}\, \Lambda_{rA} \qquad \text{(B.47A}$$

$$\tau_{MrB} = T_{MPB}\, \Lambda_{rB} \qquad \text{(B.47B}$$

When the congruent factors and coefficients of congruence have been determined, one or more of these factors may be judged not to be sufficiently similar to be continued in the congruent space. This is indicated by a low coefficient of congruence (or a high index of congruence). In the example, factor F, with a coefficient of congruence of .459717, was eliminated. The remaining factors then define the congruent space between the two studies.

After the congruent space between two studies has been determined by the congruent factors, rotation within this space is possible. Consider that it is desired to find the loadings on a set of factors s which are defined as linear combinations of the congruent factors. Let the coefficients for these linear combinations be included in a matrix T_{rs}. There will be a column for each factor s and a row for each congruent factor r. In the example, after an inspection of the loadings on the congruent space, it was decided to:

1. Define five vectors as the sums of test vectors in the congruent space for five sets of tests indicated in Table 17. The sums of loadings on the congruent factors for the tests in each group are also given in Table 17.

2. Define a set of five factors so that each factor would have zero loadings for four of the summation vectors. Each factor would be defined by the summation vector with a non-zero loading. This could be accomplished by computing the inverse of the matrix of loadings of the summation vectors on the congruent factors. The matrix T_{rs} in Table 18 is this inverse.

The factors s are related to the reference axes by the equations:

$$T_{msA} = T_{mrA} T_{rs} \qquad \text{(B.48A}$$

$$T_{MsB} = T_{MrB} T_{rs} \qquad \text{(B.48B}$$

See Table 18.

The transformation vectors in T_{msA} and T_{MsB} are normalized on the average between the two studies just as the congruent factors were normalized on the average between studies in equations B.42 - B.44. The equations for this step are:

$$d_s = \frac{1}{\sqrt{\frac{1}{2}(\Sigma_m \tau^2_{msA} + \Sigma_m \tau^2_{MsB})}} \qquad \text{(B.49}$$

$$T_{msA} = \tau_{msA}\, d_s; \qquad \text{(B.50A}$$

$$T_{MsB} = \tau_{MsB}\, d_s. \qquad \text{(B.50B}$$

See Table 18.

Loadings on these factors are found by:

$$F_{JsA} = F_{JmA}\, T_{msA} \qquad \text{(B.51A}$$

$$F_{JsB} = F_{JMB}\, T_{MsB} \qquad \text{(B.51B}$$

See Table 19.

An alternative method is to define:

$$T_{rs} = \tau_{rs}\, d_s. \qquad \text{(B.52}$$

See Table 18.

Then:

$$F_{JsA} = F_{JrA}\, T_{rs}; \qquad \text{(B.53A}$$

$$F_{JsB} = F_{JrB}\, T_{rs}. \qquad \text{(B.53B}$$

A point to note is that since the loadings in F_{JrA} and F_{JrB} are similar due to the solution to congruence, the loadings in matrices F_{JsA} and F_{JsB} must also be similar.

If it is desired to rotate the axes to a new position z, a similar cycle to the preceeding is taken. In equations B.48A - B.53B, substitutions of s for r and of z for s are made. In the example, a rotation as indicated in Table 20 was decided on. The results of this rotation are given in Tables 7-9.

The loadings of the non-overlap tests may be obtained at any time in this rotational procedure by extending equations B.51A and B.51B to these areas of the reference factor matrices.

Consider now the non-congruent factors. There will be a set for each study. Let u and U designate the non-congruent factors in studies A and B. The transformations to these factors can be defined by the equations:

$$T'_{mrA} A_{muA} = 0; \qquad \text{(B.54A}$$

$$T'_{MrB} A_{MUB} = 0; \qquad \text{(B.54B}$$

$$A'_{muA} A_{muA} = I\ ; \qquad \text{(B.55A}$$

$$A'_{MUB} A_{MUB} = I. \qquad \text{(B.55B}$$

These transformations are sections of orthogonal transformations with as many columns as there are non-congruent factors in the studies. A computing procedure for solution of these equations is given in Appendix C. Results for the example are given in Table 6.

Table 10

MATRICES ($F'_{JmA}F_{JmA}$) AND ($F'_{JMB}F_{JMB}$)

STUDY A MATRIX ($F'_{JmA}F_{JmA}$)

Reference Factors	Reference Factors i	ii	iii	iv	v	vi
i	3.177	.601	.501	1.180	1.054	.546
ii	.601	.475	.115	.134	.108	.068
iii	.501	.115	.189	.205	.061	.061
iv	1.180	.134	.205	.843	.706	.417
v	1.054	.108	.061	.706	.897	.396
vi	.546	.068	.061	.417	.396	.312

STUDY B MATRIX ($F'_{JMB}F_{JMB}$)

Reference Factors	Reference Factors I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
I	1.060	-.042	.331	.997	.566	1.039	.012	.505	.746	.002	.756	.099
II	-.042	.059	-.031	.137	.013	-.051	-.008	-.001	.061	.000	-.004	.000
III	.331	-.031	.155	.272	.200	.353	.024	.120	.214	-.002	.213	.055
IV	.997	.137	.272	2.281	.698	.974	-.041	.873	1.343	.057	1.038	.176
V	.566	.013	.200	.698	.508	.581	.011	.268	.623	-.008	.498	.086
VI	1.039	-.051	.353	.974	.581	1.146	.014	.482	.727	.016	.762	.102
VII	.012	-.008	.024	-.041	.011	.014	.015	-.046	-.016	-.002	-.021	.015
VIII	.505	-.001	.120	.873	.268	.482	-.046	-.607	.525	.027	.525	.037
IX	.746	.061	.214	1.343	.623	.727	-.016	.525	1.016	.030	.830	.125
X	.002	.000	-.002	.057	-.008	.016	-.002	.027	.030	.032	.012	.012
XI	.756	-.004	.213	1.038	.498	.762	-.021	.525	.830	.012	.848	.086
XII	.099	.000	.055	.176	.086	.102	.015	.037	.125	.012	.086	.056

Table 11

LATENT VECTORS AND ROOTS FOR OVERLAP TESTS

STUDY A MATRIX $\dot{A}_{MPA}$

Reference Factors	Principal Axes p-1	p-2	p-3	p-4	p-5	p-6
i	.8208	.4129	-.2910	-.2189	.1289	-.0800
ii	.1524	.3925	.8975	.1030	-.0617	-.0561
iii	.1258	.1799	-.1373	.5118	-.3608	.7355
iv	.3689	-.4269	.0059	.6032	-.2549	-.5026
v	.3431	-.6006	.2603	-.4972	-.3147	.3282
vi	.1826	-.3164	.1529	.2618	.8280	.2987
β_{pA}	4.4565	.7939	.3062	.2104	.0864	.0378
$\beta_{pA}^{\frac{1}{2}}$	2.1110	.8910	.5534	.4587	.2939	.1945
$\beta_{pA}^{-\frac{1}{2}}$	.4737	1.1223	1.8072	2.1800	3.4029	5.1407

STUDY B MATRIX $\dot{A}_{MPB}$

Reference Factors	Principal Axes P-1	P-2	P-3	P-4	P-5	P-6	P-7	P-8	P-9	P-10	P-11	P-12
I	.3713	-.4491	-.1522	-.1588	-.0168	.5476	-.4832	-.2697	-.0043	-.0592	-.0244	-.0399
II	.0109	.1588	.1100	-.0937	.0187	.2726	.1817	-.0062	.1890	-.2934	.8158	-.2423
III	.1119	-.2074	.0494	-.1777	-.1454	-.4346	-.3493	.3719	-.5234	-.0476	.2507	-.3179
IV	.5615	.6085	-.0852	-.4637	.1194	.0223	-.0002	.1536	-.0288	-.1110	-.1946	.0522
V	.2451	-.1450	.5084	.0837	-.6084	.0372	.1856	.1203	.1235	-.4013	-.2307	.0494
VI	.3737	-.5286	-.1354	-.2626	.1696	-.2437	.5881	.0102	.1315	.1839	.0313	.0437
VII	-.0055	-.0557	.0941	-.1046	+.0001	-.1405	-.2216	.0604	-.0064	-.0344	.3262	.8911
VIII	.2492	.1390	-.6732	.3787	-.5151	-.1317	.0125	-.0162	.0851	.0407	.1676	.0511
IX	.3832	.1777	.4420	.2884	-.0446	-.0156	-.0035	-.3425	-.2141	.5799	.1935	-.0095
X	.0101	.0282	-.0468	-.0275	.0834	-.3657	.0567	-.7395	-.2485	-.4920	-.0268	-.0163
XI	.3452	-.0697	.0417	.6249	.5391	-.0707	-.0919	.2602	.0662	-.3271	-.0238	.0006
XII	.0492	.0052	.1302	-.1166	-.0050	-.4468	-.4128	-.1224	.7532	.0581	.0179	-.1858
β_{PB}	5.5207	.9522	.3425	.2638	.1059	.0675	.0615	.0357	.0123	.0000	.0005	.0010
$\beta_{PB}^{\frac{1}{2}}$	2.4532	.9758	.5853	.5136	.3254	.2598	.2480	.1890	.....	.....	.....	.....
$\beta_{PB}^{-\frac{1}{2}}$	.4110	1.0248	1.7087	1.9469	3.0729	3.8496	4.0330	5.2918	.....	.....	.....	.....

Table 12

FACTOR LOADINGS ON PRINCIPAL AXES

STUDY A MATRIX F_{JpA}

Test Code Numbers Study A	Study B	Principal Axes p-1	p-2	p-3	p-4	p-5	p-6
18	1	.5227	.2948	.0200	-.1009	.0026	.0308
9	26	.6581	.2855	-.1718	-.1914	.0598	-.0396
5	23	.7418	-.2402	-.1358	.1055	.0278	-.1412
12	28	.6621	-.0163	-.0456	.1806	.1320	.0185
4	22	.6399	-.4593	.1899	.0565	.0288	.0524
13	32	.8131	-.1660	.0621	-.1042	.0713	.0600
14	34	.6204	.0145	-.0454	-.1854	.0202	.0208
15	33	.5973	-.2970	.0273	-.1248	-.2020	-.0246
11	27	.7717	.2510	-.2188	.2235	-.1289	.0690
16+17	30	.5948	.4211	.4064	.0754	-.0290	-.0539

STUDY B MATRIX F_{JPB}

Test Code Numbers Study A	Study B	P-1	P-2	P-3	P-4	P-5	P-6	P-7	P-8	P-9	P-10	P-11	P-12
18	1	.6203	-.3864	-.0277	-.3019	-.0542	-.0933	-.0980	.0536	-.0207	.0013	-.0006	-.0028
9	26	.7983	-.3227	-.1686	.0394	-.0146	.1901	-.0542	-.0120	.0112	.0004	-.0070	-.0116
5	23	.8271	.2273	.1328	.0183	-.0868	-.0208	.0182	-.0991	-.0422	-.0022	-.0070	-.0139
12	28	.8475	-.[illegible]367	.2956	.2631	.0144	.0042	-.0149	.1028	-.0256	-.0001	.0009	-.0006
4	22	.6947	.3840	.1673	-.0495	-.0668	-.0255	-.0754	.0005	.0802	.0010	-.0022	-.0007
13	32	.8775	.1415	.0260	-.0371	-.0121	.0638	-.0237	-.0499	-.0343	.0011	.0124	.0230
14	34	.7041	.2774	-.4008	.1937	-.1207	-.0812	.0136	.0322	.0030	.0003	.0022	.0029
15	33	.7613	.4623	-.0937	-.1945	.1430	.0381	.0840	.0685	-.0102	.0000	-.0031	-.0076
11	27	.8936	-.2670	-.0529	.0592	.2169	-.0969	-.0123	-.0636	.0213	.0020	.0007	.0039
16+17	30	.6076	-.3921	.0862	-.0793	-.0913	-.0057	.1864	-.0061	.0342	-.0037	.0026	.0030

Table 13

THE MATRIX ($F'_{JpA}F_{JPB}$)

Study A Principal Axes	Study B Principal Axes P-1	P-2	P-3	P-4	P-5	P-6	P-7	P-8
p-1	5.1251	-.0021	.0028	-.0133	-.0218	-.0065	.0092	-.0174
p-2	-.0015	-.8259	-.1292	.0029	.0051	-.0065	.0407	.0041
p-3	-.0491	.0066	.0925	-.1101	-.0761	-.0092	.0699	.0207
p-4	-.0059	-.0648	.1902	.0831	.0451	-.0483	.0159	-.0190
p-5	-.0056	-.0496	.0544	.0619	-.0609	.0182	-.0293	.0019
p-6	.0003	-.0094	-.0115	.0035	.0211	-.0139	-.0218	.0100

Table 14

THE MATRICES G, H_A, H_B

MATRIX G

Study A Principal Axes	Study B Principal Axes P-1	P-2	P-3	P-4	P-5	P-6	P-7	P-8
p-1	.9977	-.0010	.0023	-.0123	-.0317	-.0119	.0176	-.0436
p-2	-.0007	-.9499	-.2478	.0063	.0176	-.0281	.1842	.0243
p-3	-.0365	.0122	.2856	-.3874	-.4226	-.0640	.5094	.1980
p-4	-.0053	-.1448	.7085	.3527	.3021	-.4053	.1398	-.2192
p-5	-.0078	-.1730	.3163	.4101	-.6368	.2384	-.4021	.0342
p-6	.0006	-.0495	-.1010	.0350	.3333	-.2751	-.4520	.2721

MATRIX H_A

Study A Principal Axes	Study A Principal Axes p-1	p-2	p-3	p-4	p-5	p-6
p-1	.9989	.0016	-.0165	-.0006	.0031	-.0271
p-2	.0016	.9994	.0082	.0013	-.0026	.0092
p-3	-.0165	.0082	.7145	-.0098	-.0146	-.3426
p-4	-.0006	.0013	-.0098	.9705	.0411	.0373
p-5	.0031	-.0026	-.0146	.0411	.9234	-.0958
p-6	-.0271	.0092	-.3426	.0373	-.0958	.4790

MATRIX H_B

Study B Principal Axes	Study B Principal Axes P-1	P-2	P-3	P-4	P-5	P-6	P-7	P-8
P-1	.9968	.0013	-.0142	-.0032	-.0126	-.0094	.0010	-.0497
P-2	.0013	.9558	.0866	-.1344	.0281	.0570	-.0971	-.0083
P-3	-.0142	.0866	.7552	.2638	-.1462	-.1953	.1174	-.1215
P-4	-.0032	-.1344	.2638	.4441	.0213	-.0300	-.3278	-.1298
P-5	-.0126	.0281	-.1462	.0213	.7878	-.3390	-.0649	-.0792
P-6	-.0094	.0570	-.1953	-.0300	-.3390	.3018	-.0662	.0093
P-7	.0010	-.0971	.1174	-.3278	-.0649	-.0662	.6793	-.0628
P-8	-.0497	-.0083	-.1215	-.1298	-.0792	.0093	-.0628	.1650

Table 15

LATENT VECTORS AND ROOTS FOR MATRICES H_A AND H_B

STUDY A MATRIX A_{rA}

Study A Principal Axes	Congruent Factors A	B	C	D	E	F
p-1	.8341	-.3459	.0031	.3569	-.2356	.0408
p-2	.4218	.8889	-.0367	-.0073	.1741	-.0151
p-3	.1541	-.1579	.5028	-.1999	.5797	.5679
p-4	-.2803	.1481	.4512	.8318	.0492	-.0397
p-5	-.0408	-.1198	-.7019	.3595	.5885	.1250
p-6	-.1494	.1702	-.2225	.1071	-.4789	.8114

STUDY B MATRIX A_{rB}

Study B Principal Axes	Congruent Factors A	B	C	D	E	F
P-1	.8273	-.3398	-.0131	.3571	-.2627	.0419
P-2	-.3443	-.8552	.1154	-.1842	-.2444	-.0740
P-3	-.2548	-.2164	.2914	.6393	.3930	.2030
P-4	-.1880	.0801	-.3540	.5188	.0214	-.3295
P-5	-.1925	.2711	.3160	.1314	-.7574	-.1334
P-6	.1132	-.1462	-.3419	-.2728	.2139	-.4546
P-7	.2154	.0692	.7425	-.1740	.3113	-.2878
P-8	.0238	.0152	-.0908	-.1967	.0084	.7318

LATENT ROOTS ϕ^2

Congruent Factors A	B	C	D	E	F
1.00173	.99965	.87694	.99517	.99092	.22109

Table 16

MATRICES $\mathcal{T}_{mrA}$, $\mathcal{T}_{MrB}$, d_r

STUDY A MATRIX $\mathcal{T}_{mrA}$

Reference Factors	Congruent Factors A	B	C	D	E	F
i	.6161	.1772	-.7118	-.0428	.1160	-.5498
ii	.4847	.1199	1.1126	-.2212	1.0255	.6484
iii	-.7309	1.1538	.3917	.9609	-2.6011	2.7287
iv	-.0019	-.6290	1.8005	.5690	.6730	-2.2366
v	.0191	-.4748	.1492	-1.1368	-1.3746	1.5622
vi	-.5394	-.3809	-1.9095	1.6303	1.0289	1.7409
$\sum_{m}\mathcal{T}^2_{mrA}$	1.440049	2.139747	8.808218	5.248043	11.231917	18.642088

STUDY B MATRIX $\mathcal{T}_{MrB}$

Reference Factors	Congruent Factors A	B	C	D	E	F
I	.2039	-.1054	-2.0761	-.1493	-.1649	-1.3456
II	.1989	-.2836	.3446	-.4026	.4379	-.6334
III	-.2052	.1760	-.6718	.1857	-.3759	2.8145
IV	.1412	-.5532	.3583	-.7290	-.5427	.7386
V	.4332	-.5476	.0317	.1682	2.0401	.5692
VI	.7743	.8637	2.2904	-.3573	.1337	-.1184
VII	-.2307	.0215	-.3953	.2472	-.3179	.8407
VIII	.4477	-.1978	-.8629	-.4035	.6084	-.1180
IX	-.2553	-.3853	.1715	1.1550	.2936	-1.3153
X	-.2321	.2097	1.0874	1.0633	-.5003	-2.3255
XI	-.5017	.5791	-.2039	.8262	-1.3824	.6422
XII	-.5652	.0435	-.4487	.9147	-.8045	.9094
$\Sigma\mathcal{T}^2_{MrB}$	1.874244	2.043654	12.611188	5.114092	8.200618	20.114876
$\frac{1}{2}(\sum_{m}\mathcal{T}^2_{mrA} + \sum_{M}\mathcal{T}^2_{MrB})$	1.657146	2.091700	10.709703	5.181068	9.716268	19.378482
d_r	.776819	.691433	.305570	.439329	.320812	.227164

Table 17

SUMMATION VECTORS FOR FIRST ROTATION IN CONGRUENT SPACE

Summation Vector	Tests Used From Study	Tests Used Test Number	Sum of Loadings on Congruent Factors A	B	C	D	E
a	A	18	1.3562	.4186	-.3866	-.1810	-.0415
	A	9					
	B	1					
	B	26					
b	A	4	.0133	-.7811	.0071	.1880	-.0156
	B	22					
c	A	15	.3476	-.5268	.2662	-.3428	-.2937
	B	33					
d	A	11	.3006	.5138	.1349	.4511	-.4279
	B	27					
e	A	16+17	.8516	.2690	.3318	.0193	.3379
	B	30					

Table 18

TRANSFORMATION MATRICES FOR FIRST ROTATION CONGRUENT SPACE

MATRIX $\tilde{T}_{rs}$

Congruent Factors	Rotated Congruent Factors a	b	c	d	e
A	.4488	.3537	.1373	.1280	.3529
B	-.0282	-.9717	-.2159	.2371	.0646
C	-.9686	-.4451	.8229	.3937	1.0745
D	-.1251	1.2953	-1.0145	.8833	.2814
E	-.1503	.2460	-.9253	-.9484	.9478

MATRIX $\tilde{T}_{msA}$

*Reference Factors	Rotated Congruent Factors a	b	c	d	e
i	.4190	.1386	-.1536	-.0489	-.0274
ii	-.2000	-.1437	.1078	-.1962	.7880
iii	-.3207	-.6877	.1921	1.3280	-.6924
iv	-.5851	.5541	.0931	.1294	.8375
v	.1005	-.4515	1.0252	-.0810	-.5255
vi	.2453	1.3765	-1.5130	-.0262	-.2774
$\sum_{m} \tilde{T}^2_{msA}$	.731024	2.918422	3.420988	1.828461	2.155620

MATRIX $\tilde{T}_{MsB}$

Reference Factors	Rotated Congruent Factors a	b	c	d	e
I	.7038	.3112	-.3691	-.2545	-.6991
II	-.0261	.0038	.1997	-.2748	.2384
III	.1318	-.0072	-.1883	.1141	-.3603
IV	.0202	-.0960	.6738	-.1513	-.1235
V	.0447	.7393	-.5447	-.5983	.7458
VI	-.4116	-.8719	.6492	.3148	.9994
VII	.0379	.0915	-.1430	.1257	-.2582
VIII	.4082	.2917	-.1405	-.4335	-.0343
IX	-.2099	.6459	-.5287	.2910	.2012
X	-.4412	.2131	-.1081	.7070	.2821
XI	-.1046	-.1381	-.1491	.7518	-.4969
XII	-.0766	.3336	-.3485	.4967	-.4319
$\sum_{M} \tilde{T}^2_{MsB}$	1.108795	2.349342	1.858819	2.248561	2.867015
$\frac{1}{2}(\sum_{m} \tilde{T}^2_{msA} + \sum_{M} \tilde{T}^2_{MsB})$	.919910	2.633882	2.639904	2.058911	2.511318
d_s	1.0426	.6162	.6155	.7004	.6310

MATRIX T_{rs}

Congruent Factors	Rotated Congruent Factors a	b	c	d	e
A	.4679	.2179	.0045	.0896	.3227
B	-.0294	-.5988	-.1329	.1661	.0408
C	-1.0099	-.2743	.5065	.2757	.6780
D	-.1304	.7932	-.6244	.6187	.1776
E	-.1567	.1516	-.5695	-.6643	.5981

MATRIX T_{msA}

Reference Factors	Rotated Congruent Factors a	b	c	d	e
i	.4368	.0854	-.0945	-.0342	-.0173
ii	-.2085	-.0885	-.0664	-.1374	.4972
iii	-.3343	-.4238	.1182	.9301	-.4369
iv	-.6100	.3414	.0573	.0906	.5285
v	.1048	-.2782	.6310	-.0567	-.3316
vi	.2558	.8482	-.9313	-.0184	-.1750

MATRIX T_{MsB}

Reference Factors	Rotated Congruent Factors a	b	c	d	e
I	.7338	.1918	-.2272	-.1783	-.4411
II	-.0272	.0023	.1229	-.1925	.1504
III	.1374	-.0044	-.1159	.0799	-.2273
IV	.0210	-.0592	.4147	-.1060	-.0779
V	.0466	.4556	-.3353	-.4190	.4706
VI	-.4292	-.5373	.3996	.2205	.6306
VII	.0395	.0564	-.0880	.0880	-.1629
VIII	.4256	.1181	-.0865	-.3036	-.0216
IX	-.2188	.5212	-.3254	.2038	.1270
X	-.4600	.1313	-.0665	.4952	.1780
XI	-.1090	-.0851	-.0918	.5336	-.3135
XII	-.0799	.2056	-.2145	.3479	-.2725

Table 19

FACTOR LOADINGS FOR FIRST ROTATION IN CONGRUENT SPACE

STUDY A MATRIX F_{JsA}

Test Code Numbers		Loadings on Rotated Congruent Factors - s				
Study A	Study B	a	b	c	d	e
18	1	.19	-.04	.02	.03	.03
9	26	.32	.03	-.01	-.03	-.02
5	23	.02	.32	-.05	.03	.09
12	28	.06	.30	-.21	.13	.04
4	22	-.03	.30	.02	.01	.02
13	32	.19	.22	.01	-.01	-.03
14	34	.23	.06	.06	-.03	-.05
15	33	.00	.00	.31	.00	.00
11	27	-.01	.00	.00	.35	.00
16+17	30	.00	.00	.00	.00	.32

STUDY B MATRIX F_{JsB}

Test Code Numbers		Loadings on Rotated Congruent Factors - s				
Study A	Study B	a	b	c	d	e
18	1	.18	-.04	.03	.03	.03
9	26	.36	.04	-.03	-.04	-.04
5	23	.01	.32	-.04	.03	.10
12	28	.04	.29	-.20	.14	.05
4	22	.03	.32	-.02	-.01	-.02
13	32	.13	.20	.04	.00	.02
14	34	.23	.06	.06	-.03	-.05
15	33	.00	.00	.31	.00	.00
11	27	.01	.00	.00	.35	.00
16+17	30	.00	.00	.00	.00	.31

Table 20

TRANSFORMATION MATRICES FOR SECOND ROTATION IN CONGRUENT SPACE

MATRIX

Factors	Factors z a	b	c	d	e
a	1.00	-.35			
b		1.00	.40		
c			1.00		
d		.20		1.00	
e					1.00

MATRIX T_{mzA}

Factors	Factors - z a	b	c	d	e
i	.4368	-.0744	-.0504	-.0342	-.0173
ii	-.2085	-.0431	.0309	.1374	.4972
iii	-.3343	-.1208	-.0512	.9301	-.4369
iv	-.6100	.5731	.1939	.0906	.5285
v	.1048	-.3263	.5197	-.0567	-.3316
vi	.2558	.7551	-.5919	-.0184	-.1750
$\sum_m T^2_{mzA}$	.794540	1.027077	.665255	.896896	.858285

MATRIX T_{MzB}

Factors	Factors - z a	b	c	d	e
I	.7338	-.1007	-.1505	-.1783	-.4411
II	-.0272	-.0267	.1258	-.1925	.1504
III	.1374	-.0366	-.1177	.0799	-.2273
IV	.0210	-.0877	.3911	-.1060	-.0779
V	.0466	.3555	-.1531	-.4190	.4706
VI	-.4292	-.3430	.1847	.2205	.6306
VII	.0395	.0602	-.0655	.0880	-.1629
VIII	.4255	-.0916	-.0592	-.3036	-.0216
IX	-.2188	.6387	-.1159	.2033	.1270
X	-.4600	.3914	-.0140	.4952	.1780
XI	-.1090	.0558	-.1258	.5336	-.3135
XII	-.0799	.3031	-.1323	.3479	-.2725
$\sum_M T^2_{MzB}$	1.205340	.932505	.315360	1.103086	1.141398
$\frac{1}{2}(\sum_m T^2_{mzA} + \sum_M T^2_{MzB})$	.999940	.979791	.490308	.999991	.999842
d_z	1.0000	1.0103	1.4281	1.0000	1.0000

MATRIX T_{sz}

Factors	Factors - z a	b	c	d	e
a	1.0000	-.3536			
b		1.01103	.5712		
c			1.4281		
d		.2021		1.0000	
e					1.0000

MATRIX T_{mzA}

Factors	Factors - z a	b	c	d	e
i	.4368	-.0752	-.0863	-.0342	-.0173
ii	-.2085	-.0435	.0441	-.1374	.4972
iii	-.3343	-.1220	-.0731	.9301	-.4369
iv	-.6100	.5790	.2769	.0906	.5285
v	.1048	-.3297	.7422	-.0567	-.3316
vi	.2558	.7629	-.8453	-.0184	-.1750

MATRIX T_{MzB}

Factors	Factors - z a	b	c	d	e
I	.7338	-.1017	-.2149	-.1783	-.4411
II	-.0272	-.0270	.1768	-.1925	.1504
III	.1374	-.0370	-.1681	.0799	-.2273
IV	.0210	-.0886	.5585	-.1060	-.0779
V	.0466	.3592	-.2186	-.4190	.4706
VI	-.4292	-.3465	.2638	.2205	.6306
VII	.0395	.0608	-.0935	.0880	-.1629
VIII	.4256	-.0925	-.0560	-.3036	-.0216
IX	-.2188	.6453	-.1669	.2038	.1270
X	-.4600	.3954	-.0200	.4952	.1780
XI	-.1090	.0604	-.1797	.5336	-.3135
XII	-.0799	.3062	-.1889	.3479	-.2725

APPENDIX C

COMPUTING PROCEDURE FOR SYNTHESIS OF FACTOR ANALYSIS STUDIES

The following set of notes give detailed computing procedures to implement application of the method presented in this report for synthesis of factor analysis studies. For convenience these notes are divided into five sections:

1. Congruent factor computations.
2. Rotation of axes in the congruent space.
3. Determination of non-congruent axes.
4. Determination of latent roots and vectors.
5. Notes on matrix computations.

Section 1 contains the basic elements of the method for synthesis of factorial studies. Section 2 and 3 pertain to subsequent steps. Sections 4 and 5 are included to facilitate the computations of the preceding three sections.

While it will be assumed that the work will be under the direction of a person competent in factor analysis, only a minimum knowledge will be assumed for the person doing the computations. It will be assumed that the person performing the computations is trained in the operation of a calculating machine and has some knowledge of statistical computations. No knowledge of matrix algebra nor of matrix computations will be assumed. Section 5 of these notes is intended to supply the limited instruction necessary concerning matrices. While specific references will be made in the other sections to relevant portions of section 5, it would be advisable that the person doing the computations to become familiar with the contents of section 5.

1. Congruent Factor Computations

The computational procedure for determining congruent factors will be illustrated by a fictitious example. Table 21 gives the factor matrices F_{jmA} and F_{jMB} and standard deviations σ_{jA} and σ_{jB} for six overlap tests in Studies A and B. Study A has three factors, Study B has four factors. Each row of each factor matrix has been summed with the sum being recorded in the Σ column.

a. Equalize Units of Measurement

A) Compute the constants $d_{jA} = \sigma_{jA} / \frac{1}{2}(\sigma_{jA} + \sigma_{jB})$, and $d_{jB} = \sigma_{jB}/\frac{1}{2}(\sigma_{jA} + \sigma_{jB})$. In order to facilitate computations a worksheet was set up as shown in Table 22. The operations in the successive columns are as follows:

1) Enter the σ_{jA} for each test in Study A in column 1. Total the entries in column 1 and enter results in row Σ column 1.
2) Enter the σ_{jB} for each test in Study B in column 2. Total the entries in column 2 and enter the results in row Σ column 2.
3) Sum the σ_{jA} and σ_{jB} for each test and divide the sum by 2. Record the result in the third column designated $\frac{1}{2}(\sigma_{jA} + \sigma_{jB})$. In order to check the entries in the third column, total the entries in the Σ cells of the first and second column and divide the sum by 2. Enter the result in the Ch cell of the third column. Total the entries in the third column and enter in the Σ cell. The Ch and Σ entries should agree.
4) Divide the σ_{jA} and σ_{jB} in columns 1 and 2 for each test by the $\frac{1}{2}(\sigma_{jA} + \sigma_{jB})$ in column 3 for the test and record in column 4 and 5, d_{jA} and d_{jB}.

5) Sum the d_{jA} and d_{jB} for each test and enter the total in column 6. The sum should equal 2.00.

B) Compute the Reference Factorial Matrices for Tests with Adjusted Units of Measurement, F_{JmA} for Study A and F_{JMB} for Study B. See Table 23.

1) To obtain the matrix F_{JmA}:

a) Multiply each entry in the first row of matrix F_{jmA}, Table 21, by the constant d_{jA} for the first test in column 4 of Table 22:

$$
\begin{aligned}
.35 \times .89 &= .32 \\
-.34 \times .89 &= -.30 \\
.48 \times .89 &= .43
\end{aligned}
$$

b) In order to check the first row of F_{JmA} multiply the sum of the first row of F_{jmA} by the constant d_{jA} for the first test in column 4 of Table 22:

$$.49 \times .89 = .44.$$

Record the result in the Ch column of F_{JmA}. Sum the entries in the first row of F_{JmA} and record in the Σ column. The entries in the Ch and Σ columns should agree within $\pm$ 2 of the last decimal place carried.

c) Compute and check the entries in each row of F_{JmA} in the same way, using the corresponding row of F_{jmA} in Table 21 and the corresponding constant d_{jA} from column 4 of Table 22.

d) Obtain the sum of each column of F_{JmA}.

2) Compute the matrix F_{JMB} following the same procedure as outlined for the computation of F_{JmA} by using the rows of F_{jMB} and the constants d_{jB}.

b. Compute the principal axes for the tests in each study.

A) Compute the matrix product $F'_{JmA}F_{JmA}$ of Table 24 (An explicit following of the formula $F'_{JmA}F_{JmA}$ would involve (1) recording of F'_{JmA}, the transpose of matrix F_{JmA}, see Section 5, Paragraph a, Point 6; and (2) matrix multiplication of matrices F'_{JmA} and F_{JmA}, see Section 5, Paragraph c. It is unnecessary to record the transpose matrix F'_{JmA}, the same results may be obtained by multiplying each column of F_{JmA} by every column of F_{JmA}. The rows of the transpose matrix are implied by their equivalents, the columns of F_{JmA}.)

1) First row of $F'_{JmA}F_{JmA}$

   a) Compute the sum of squares of the entries in the first column in F_{JmA} and enter the result in the first cell of the first row of $F'_{JmA}F_{JmA}$.

   b) Compute the sum of products between the entries in the first column and in each other column of F_{JmA} and enter the result in the corresponding cell of the first row of $F'_{JmA}F_{JmA}$. For example: the sum of products between the entries in the first column and second column of F_{JmA} is entered in the first row, second cell of $F'_{JmA}F_{JmA}$.

   c) Compute the sum of products between the entries in first column and in the Σ column of F_{JmA} and enter in the Ch cell of the first row of $F'_{JmA}F_{JmA}$.

   d) Sum the entires in the first row of $F'_{JmA}F_{JmA}$ (exclusive of the entry in the Ch cell) and enter the total in the Σ cell of the first row of $F'_{JmA}F_{JmA}$. This entry should agree to within $\pm$ 2 of the last decimal place carried.

2) Second row of $F'_{JmA}F_{JmA}$

   Compute the second row of $F'_{JmA}F_{JmA}$ by using the second column of F_{JmA} and repeating preceeding steps a-d. The sum of squares of the second column will be entered in the second cell of the second row.

3) Remaining rows of $F'_{JmA}F_{JmA}$

   Compute the remaining rows of $F'_{JmA}F_{JmA}$ by using the corresponding columns of F_{JmA} and repeating steps a-d. In each case, the sum of squares of the columns of F_{JmA} will be the diagonal of $F'_{JmA}F_{JmA}$. Note that $F'_{JmA}F_{JmA}$ is symmetric and once a row is computed and checked it may be copies into the corresponding column.

B) Compute the matrix product $F'_{JMB}F_{JMB}$ for Study B following the procedure as outlined in the preceeding steps for

$F'_{JmA}F_{JmA}$.

C) Solve for latent roots and vectors Λ_{mpA}, β_A and Λ_{MPB}, β_B by the method outlined in Section 3. The resulting matrices Λ_{mpA}, β_A and Λ_{MPB}, β_B are given in Table 25.

D) Discard principal axes with low diagonal entries in the β matrix. For study A discard the third column of the matrices Λ_{mpA} and β_A. For Study B discard the fourth column of the matrices Λ_{MPB} and β_B.

   1) Compute the square root of each remaining diagonal and record in the $\sqrt{\beta}$ row.

   2) Compute the reciprocal of the $\sqrt{\beta}$ and enter the result in the $1/\sqrt{\beta}$ row.

   3) Check the computations in 1 and 2 above by multiplying each $1/\sqrt{\beta}$ by the corresponding diagonal entry. The product should approximate $\sqrt{\beta}$ within $\pm$ 2 of the last decimal place carried.

   4) The matrices $\beta_A^{-\frac{1}{2}}$ and $\beta_B^{-\frac{1}{2}}$ are given in Table 26 and contain the $1/\sqrt{\beta}$ as diagonal entries.

E) Compute the matrix product $T_{mpA} = \Lambda_{mpA}\ \beta_A^{-\frac{1}{2}}$ for study A, Table 27. (See Section 5, Paragraph c).

   1) Eliminate the column in Λ_{mpA} corresponding to the principal axes discarded in D above. Since the third column of β_A was eliminate the third column of Λ_{mpA} will likewise be eliminated. The Λ_{mpA} matrix will now have three rows and two columns.

   2) Multiply each entry in the first column of Λ_{mpA} by the

diagonal entry of the first column of $\beta_A^{-\frac{1}{2}}$ and record in the corresponding cell of T_{mpA}. (Table 27).

3) Multiply the Σ entry in the first column of A_{mpA} by the diagonal entry of the first column of $\beta_A^{-\frac{1}{2}}$ and record in the Ch row of T_{mpA}. Sum the entries in the first column of T_{mpA} (exclusive of the Ch entry) and enter in Σ cell. This should agree with the Ch entry to within ± 2 of the last decimal place carried.

4) Compute the entries in the second column of T_{mpA} by using the entries in the second column of A_{mpA} and the diagonal entry of the second column of $\beta_A^{-\frac{1}{2}}$. Follow the procedure outlined in 2 and 3 above.

F) Compute the matrix product $T_{MPB} = A_{MPB}\,\beta_B^{-\frac{1}{2}}$ for Study B. (Table 27).

Follow procedure outlined in E above. The matrix A_{MPB} in the example has four rows and three columns after the fourth column has been eliminated.

c. Compute transformations to congruent factors

A) Compute the matrix $G = T'_{mpA}\,(F'_{JmA}F_{JMB})\,T_{MPB}$ (See Table 28).

1) Compute the matrix $F'_{JmA}F_{JMB}$

a) Compute the sum of products between the first column of F_{JmA} and first column of F_{JMB} (Table 23) and enter the result in the first cell of the first row of $F'_{JmA}F_{JMB}$

b) Compute the sum of products between each remaining column of F_{JmA} and the first column of F_{JMB}. Enter the result in the corresponding cell of the first column of $F'_{JmA}F_{JMB}$.

c) Compute the sum of products between the entries in the Σ column of F_{JmA} and the first column of F_{JMB}. Enter result in the Ch cell of the first column of $F'_{JmA}F_{JMB}$.

d) Sum the entries in the first column of $F'_{JmA}F_{JMB}$ (exclusive of the entry in the Ch cell) and enter the total in the Σ cell of the first column of $F'_{JmA}F_{JMB}$. This entry should agree with the Ch entry to within ± 2 of the last decimal carried.

e) Second column of $F'_{JmA}F_{JMB}$

Compute the second column of $F'_{JmA}F_{JMB}$ by using the second column of F_{JMB} and repeating steps a-d.

f) Remaining column of $F'_{JmA}F_{JMB}$

Compute the remaining columns of $F'_{JmA}F_{JMB}$ by using the corresponding columns of F_{JMB} and repeating steps a-d.

2) Compute the matrix product $T'_{mpA}\ (F'_{JmA}F_{JMB})$

Using the columns of T_{mpA} and the columns of $(F'_{JmA}F_{JMB})$ compute the matrix product according to the procedure outlined in the preceding step.

3) Compute the matrix product

$G = T'_{mpA}\ (F'_{JmA}F_{JMB})\ T_{MPB}$ (See Section 5 Paragraph c.)

a) Compute the sum of products between the entries in the first row of the matrix $T'_{mPA}\ (F'_{JmA}F_{JMB})$ and the entries in the first column of T_{MPB} and record in the first cell of the first column of G. For example:

$$.9456 = (.8775)(.8287) + (.3411)(.2814) + (.4263)(.2415) + (.1041)(.1871)$$

b) Compute the sum of products between the entries in the second row of $T'_{mpA}\ (F'_{JmA}F_{JMB})$ and the first column of T_{MPB}. Enter the result in the second cell of the first column of G. For example:

$$-.1941 = (-.2424)(.8287) + (-.1105)(.2814) + (.4702)(.2415) + (-.4207)(.1871)$$

c) Compute the sum of products between the Σ row of $T'_{mpA}\ (F'_{JmA}F_{JMB})$ by the first column of T_{MPB}. Enter the result in the Ch cell of the first column of G. Sum the entries in the first column (exclusive of the Ch cell) and enter the total in the Σ cell. This entry should agree with the Ch entry to within ± 2 of the last decimal place carried.

d) Compute the remaining columns of G by computing the sum of products between each row of $T'_{mpA}\ (F'_{JmA}F_{JMB})$ and each remaining column of T_{MPB} and entering the result in the corresponding cell of G.

B) If the number of principal axes p with significant latent roots β_A for Study A is less than or equal to the number of principal axes P with significant latent roots β_B for Study B, as is true for the example: (For detailed procedure see following paragraph D.)

1) Compute $H_A = GG'$
2) Obtain latent vectors and roots of H_A: Λ_{rA} and ϕ_r^2 See Section 3.
3) Discard vectors with low ϕ_r's.
4) Compute

$$T_{mrA} = T_{mpA}\Lambda_{rA}$$
$$T_{MrB} = T_{MPB}G'\Lambda_{rA}\phi_r^{-1}$$

C) If the number of principal axes p with significant latent roots β_A for Study A is greater than the number of principal axes P with significant latent roots β_B for Study B:

1) Compute $H_B = G'G$
2) Obtain latent vectors and roots of H_B: Λ_{rB} and ϕ_r^2
3) Discard vectors with low ϕ_r's.
4) Compute

$$T_{mrA} = T_{mpA}G\Lambda_{rB}\phi^{-1}$$
$$T_{MrB} = T_{MPB}\Lambda_{rB}$$

D) Since the number of principal axes with significant latent roots β_A for Study A is less than the number of principal axes with significant latent roots β_B for Study B, the equations in B will be solved as follows:

1) Compute the matrix product $H_A = GG'$ (Table 29)
   a) Compute sum of squares of first row of G and record in the first cell of the first row of H_A.
   b) Compute the sum of products between the first and second rows of G and enter in the second cell of the first column of H_A

c) Compute the sum of products between the first and Σ row of G and enter in the Ch cell of the first column of H_A. To check the computations, obtain the sum of the first column of H_A (exclusive of the Ch entry) and record in the Σ row. The Ch entry and Σ entry should agree within ± 2 of the last decimal place carried.

d) Using the second row of G compute the second column of H_A as outlined in a to c above, the sum of squares of the second row being the diagonal of the second column of H_A.

2) Obtain the latent vectors and roots of H_A by the procedure outlined in Section 3. The resulting matrices are given in Table 29 as matrice A_{rA} and ϕ_r^2.

3) Compute the square root of the diagonal entries in ϕ_r^2 and record in the $\sqrt{d}$ row. Compute the reciprocal of $\sqrt{d}$ and enter the result in the $1/\sqrt{d}$ row. Check the computations by multiplying each $1/\sqrt{d}$ by the corresponding diagonal entry. The product should approximate $\sqrt{d}$ to within ± 2 of the last decimal place.

4) The entries in the $1/\sqrt{d}$ row are the diagonal elements of the matrix ϕ_r^{-1} given in Table 29.

5) Compute the matrix product $T_{mrA} = T_{mpA} A_{rA}$ (Table 30)
Using the rows of the matrix T_{mpA} and the columns of the matrix A_{rA}, follow computational procedure for multiplying matrices outlined in Section 5, paragraph c.

6) Compute the matrix product $T_{MrB} = T_{MPB} G' A_{rA} \phi_r^{-1}$ (Table 31)

a) Compute the matrix product $T_{MPB}G'$

(1) Compute the sum of products between the first row of T_{MPB} and first row of G and enter the results in the first cell of the first column of $(T_{MPB}G')$

(2) Compute the sum of products between the remaining rows of T_{MPB} and the first row of G and enter the result in the corresponding cell of the first column of $(T_{MPB}G')$

(3) Compute the sum of products between the Σ row of T_{MPB} and the first row of G and enter in the Ch row of the first column of $(T_{MPB}G')$

(4) Sum the entries in the first column (exclusive of the Ch cell) and enter the result in Σ cell of the column. This should agree with the Ch entry to within ± 2 of the last decimal place carried.

(5) Compute the remaining columns of $(T_{MPB}G')$ by using the rows of T_{MPB} and each of the remaining rows of G and entering the results in the columns of $(T_{MPB}G')$ corresponding to the rows of G.

b) Using the rows of the matrix $(T_{MPB}G')$ and the columns of the matrix Λ_{rA} follow the computational procedure for multiplying matrices outlined in Section 5, Paragraph c, to obtain the matrix $(T_{MPB}G')\ \Lambda_{rA}$

c) Using the rows of the matrix $(T_{MPB}G')\ \Lambda_{rA}$ computed in b,

and the columns of the matrix ϕ_r^{-1} (Table 29) follow the computational procedure for multiplying matrices outlined in Section 5, Paragraph c to obtain the matrix

$$\mathcal{T}_{MrB} = T_{MPB}G'\Lambda_{rA}\phi_r^{-1}$$

7) Adjust columns of $\mathcal{T}_{mrA}$ and $\mathcal{T}_{MrB}$ to congruent factors:

a) For illustrative purposes the matrix $\mathcal{T}_{mrA}$ and the matrix $\mathcal{T}_{MrB}$ were copies in Table 32 from the results of previous computations (Tables 30, 31)

b) For each study compute the sum of squares for each column and enter the results in the row designated $\Sigma\mathcal{T}_{mrA}^2$ and $\Sigma\mathcal{T}_{MrB}^2$.

c) For corresponding columns compute the average and enter the results in row designated $\frac{1}{2}(\Sigma\mathcal{T}_{MrA}^2 + \Sigma\mathcal{T}_{MrB}^2)$. For example the average between 1.356101 and 1.871665 is 1.613883

d) Compute the square root of the values obtained in (c)

e) Compute the reciprocal of the square roots computed in (d) Check the computations by multiplying each $1/\sqrt{\frac{1}{2}(\Sigma\mathcal{T}_{mrA}^2 + \Sigma\mathcal{T}_{MrB}^2}$ by the corresponding $\frac{1}{2}(\Sigma\mathcal{T}_{mrA}^2 + \Sigma\mathcal{T}_{MrB}^2)$. The product should approximate $\sqrt{\frac{1}{2}(\Sigma\tau^2_{mrA} + \Sigma\tau^2_{MrB})}$ to within ±2 of the last decimal place carried. These reciprocals of the square roots are the diagonal elements of the matrix D_r (shown in Table 32)

f) Compute the matrix $T_{mrA} = \tau_{mrA}D_r$ (Table 33)

g) Compute the matrix $T_{MrB} = \tau_{MrB}D_r$ (Table 33)

d. Compute loadings on congruent factors (Table 34)

A) Compute the matrix $F_{JrA} = F_{JmA}T_{mrA}$

Using the rows of the matrix F_{JmA} (Table 23) and the columns of the matrix T_{mrA} (Table 33) follow the computational procedure outlined in Section 5, Paragraph c.

B) Compute the matrix $F_{JrB} = F_{JMB}T_{MrB}$

Using the rows of the matrix F_{JMB} (Table 23) and the columns of T_{MrB} (Table 33) follow the computational procedure outlined in Section 5, Paragraph c.

Table 21

REFERENCE FACTOR MATRICES

STUDY A

F_{jmA}

	i	ii	iii	Σ	σ_{jA}
1	.35	-.34	.48	.49	4
2	.32	-.12	.34	.54	10
3	.54	-.40	.08	.22	8
4	.34	-.23	-.01	.10	4
5	.40	.12	-.26	.26	6
6	.27	-.30	-.28	-.31	8

STUDY B

F_{jMB}

	I	II	III	IV	Σ	σ_{jB}
1	.32	.32	.03	.28	.95	5
2	.61	-.15	-.24	.18	.40	12
3	.49	.25	.29	.01	1.04	7
4	.48	.27	.35	.06	1.16	3
5	.19	.03	.36	-.12	.46	5
6	.12	-.02	.42	-.29	.23	6

Table 22

	σ_{JA}	σ_{JB}	$\frac{1}{2}(\sigma_{JA} + \sigma_{JB})$	$d_{JA} = \sigma_{JA} / \frac{1}{2}(\sigma_{JA} + \sigma_{JB})$	$d_{JB} = \sigma_{JB} / \frac{1}{2}(\sigma_{JA} + \sigma_{JB})$	$d_{JA} + d_{JB}$
	4	5	4.5	.89	1.11	2.00
	10	12	11.0	.91	1.09	2.00
	8	7	7.5	1.07	.93	2.00
	4	3	3.5	1.14	.86	2.00
	6	5	5.5	1.09	.91	2.00
	8	6	7.0	1.14	.86	2.00
Ch			39.0			
Σ	40	38	39.0			

Table 23

REFERENCE FACTOR MATRICES FOR TESTS

WITH ADJUSTED UNITS OF MEASURE

STUDY A

F_{JMA}

	i	ii	iii	Ch	Σ
1	.31	-.30	.43	.44	.44
2	.29	-.11	.31	.49	.49
3	.58	-.43	.09	.24	.24
4	.39	-.26	-.01	.11	.12
5	.44	.13	-.28	.28	.29
6	.31	-.34	-.32	-.35	-.35
Σ	2.32	-1.31	.22		1.23

STUDY B

F_{JMB}

	I	II	III	IV	Ch	Σ
1	.36	.36	.03	.31	1.05	1.06
2	.66	-.16	-.26	.20	.44	.44
3	.46	.23	.27	.01	.97	.97
4	.41	.23	.30	.05	1.00	.99
5	.17	.03	.33	-.11	.42	.42
6	.10	-.02	.36	-.25	.20	.19
Σ	2.16	.67	1.03	.21		4.07

Table 24

STUDY A

$F_{JmA}'F_{JmA}$

	i	ii	iii	Ch	Σ
i	.9584	-.5239	.0491	.4836	.4836
ii	-.5239	.4871	-.1268	-.1636	-.1636
iii	.0491	-.1268	.4700	.3923	.3923

STUDY B

$F_{JMB}'F_{JMB}$

	I	II	III	IV	Ch	Σ
I	.9838	.2272	.1785	.2250	1.6145	1.6145
II	.2272	.2623	.1862	.0951	.7708	.7708
III	.1785	.1862	.4699	-.1513	.6833	.6833
IV	.2250	.0951	-.1513	.2133	.3821	.3821

Table 25

STUDY A

A_{mpA}

	p-1	p-2	p-3
i	.8275	.2407	.5073
ii	-.5449	.1268	.8288
iii	.1352	-.9023	.2362
Σ	.4178	-.5948	1.5723

β_A

	p-1	p-2	p-3
p-1	1.3116	.0036	-.0021
p-2	.0036	.4749	-.0007
p-3	-.0021	-.0007	.1290
$\sqrt{\beta}$	1.1453	.6891	
$1/\sqrt{\beta}$	.8731	1.4512	

STUDY B

A_{MPB}

	P-1	P-2	P-3	P-4
I	.8940	-.2165	.3674	-.1374
II	.3036	.2236	-.7885	-.4859
III	.2605	.8347	.0390	.4836
IV	.2018	-.4542	-.4917	.7150
Σ	1.6599	.3876	-.8738	.5753

β_B

	P-1	P-2	P-3	P-4
P-1	1.1639	.0040	.0030	.0028
P-2	.0041	.5532	.0030	.0019
P-3	.0030	.0030	.2085	.0002
P-4	.0028	.0019	.0001	.0036
$\sqrt{\beta}$	1.0788	.7438	.4566	
$1/\sqrt{\beta}$	.9270	1.3444	2.1901	

Table 26

STUDY A

$\beta_A^{-\frac{1}{2}}$

	p-1	p-2
p-1	.8731	
p-2		1.4512

STUDY B

$\beta_B^{-\frac{1}{2}}$

	P-1	P-2	P-3
P-1	.9270		
P-2		1.3444	
P-3			2.1901

Table 27

STUDY A

$T_{m\mu A} = A_{m\mu A}\,\beta_A^{-\frac{1}{2}}$

	p-1	p-2	Σ
i	.7225	.3493	1.0718
ii	-.4758	.1840	-.2918
iii	.1180	-1.3965	-1.2785
Ch	.3648	-.8632	
Σ	.3647	-.8632	-.4985

STUDY B

$T_{MPB} = A_{MPB}\,\beta_B^{-\frac{1}{2}}$

	P-1	P-2	P-3	Σ
I	.8287	-.2911	.8046	1.3422
II	.2814	.3006	-1.7269	-1.1449
III	.2415	1.1222	.0854	1.4491
IV	.1871	-.6106	-1.0769	-1.5004
Ch	1.5387	.5211	-1.9137	
Σ	1.5387	.5211	-1.9138	.1460

Table 28

$F_{JmA}'F_{JMB}$

	I	II	III	IV
i	.8355	.2953	.4643	.0535
ii	-.4969	-.2384	-.2540	-.0616
iii	3171	.1216	-.2540	.3065
Ch	.6557	.1785	-.0437	.2984
Σ	.6557	.1785	-.0437	.2984

$T_{mpA}'(F_{JmA}'F_{JMB})$

	I	II	III	IV
p-1	.8775	.3411	.4263	.1041
p-2	-.2424	-.1105	.4702	-.4207
Ch	.6351	.2306	.8965	-.3165
Σ	.6351	.2306	.8965	-.3166

$G = T_{mpA}'(F_{JmA}'F_{JMB})T_{MPB}$

	P-1	P-2	P-3
p-1	.9456	.2619	.0413
p-2	-.1971	.8219	.4890
Ch	.7485	1.0838	.5303
Σ	.7485	1.0838	.5303

Table 29

$K_A = 00'$

	p-1	p-2
p-1	.9645	.0491
p-2	.0491	.9535
Ch	1.0135	1.0026
Σ	1.0136	1.0026

A_{rA}

	A	B
p-1	.7470	-.6648
p-2	.6648	.7470

ϕ_r^2

	A	B
A	1.0084	.0003
B	.0002	.9095
$\sqrt{d}$	1.0042	.9537
$1/\sqrt{d}$	.9958	1.0485

ϕ_r^{-1}

	A	B
A	.9958	
B		1.0485

Table 30

STUDY A

$T_{mrA} = T_{mpA}A_{pA}$

	A	B
i	.7719	-.2194
ii	-.2331	.4538
iii	-.8402	-1.1216
Ch	-.3014	-.8873
Σ	-.3014	-.8872

Table 31

STUDY B

$T_{MPB}G'$

	p-1	p-2
I	.7406	-.0091
II	.2735	-.6529
III	.5258	.9165
IV	-.0275	-1.0653
Ch	1.5124	-.8108
Σ	1.5124	-.8108

$(T_{MPB}G')(A_{YA})$

	A	B
I	.5472	-.4991
II	-.2297	-.6695
III	1.0021	.3351
IV	-.7288	-.7775
Ch	.5907	-1.6111
Σ	.5908	-1.6110

$T_{MrB} = (T_{MPB}G')(A_{YA})\phi^{-1}$

	A	B
I	.5449	-.5233
II	-.2287	-.7020
III	.9979	.3514
IV	-.7257	-.8152
Ch	.5883	-1.6891
Σ	.5884	-1.6891

Table 32

	τ_{mrA}			τ_{MrB}	
	A	B		A	B
i	.7719	-.2194	I	.5449	-.5233
ii	-.2331	.4538	II	-.2287	-.7020
iii	-.8402	-1.1216	III	.9979	.3514
			IV	-.7257	-.8152
Σ	-.3014	-.8872	Σ	.5884	-1.6891
$\Sigma\tau_{mrA}^2$	1.356101	1.512057	$\Sigma\tau_{MrB}^2$	1.871665	1.554680

	A	B
$\frac{1}{2}(\Sigma\tau_{mrA}^2+\Sigma\tau_{MrB}^2)$	1.613883	1.533369
$\sqrt{\frac{1}{2}(\Sigma\tau_{mrA}^2+\Sigma\tau_{MrB}^2)}$	1.270387	1.238293
$1/\sqrt{\frac{1}{2}(\Sigma\tau_{mrA}^2+\Sigma\tau_{MrB}^2)}$	.787162	.807563

D_r

	A	B
A	.7872	
B		.8076

Table 33

STUDY A

$T_{mrA} = \mathcal{T}_{mrA} D_r$

	A	B
i	.6076	-.1772
ii	-.1835	.3665
iii	-.6614	-.9058
Ch	-.2373	-.7165
Σ	-.2373	-.7165

STUDY B

$T_{MrB} = \mathcal{T}_{MrB} D_r$

	A	B
I	.4289	-.4226
II	-.1800	-.5669
III	.7855	.2838
IV	-.5713	-.6584
Ch	.4632	-1.3641
Σ	.4631	-1.3641

Table 34

	STUDY A			STUDY B	
	$F_{JrA} = F_{JmA}T_{mrA}$			$F_{JrB} = F_{JmB}T_{mrB}$	
	A	B		A	B
1	-.0410	-.5544	1	-.0639	-.5518
2	-.0086	-.3725	2	-.0066	-.3937
3	.3718	-.3419	3	.3623	-.2547
4	.2913	-.1553	4	.3415	-.2514
5	.4287	.2233	5	.3896	.0772
6	.4624	.1103	6	.4721	.2358
Ch	1.5045	-1.0905		1.4949	-1.1386
Σ	1.5046	-1.0905		1.4950	-1.1386

2. Rotation of Axes in the Congruent Space

The illustration used in the preceding section will be used in this section also. One rotation of the axes in the congruent space will be described. As indicated in Section IV of the body of the report, the computations are of an identical nature each rotation from one set of axes in the congruent space to another set of axes. The forms given in this section can be used for all such rotations of axes. In the particular case illustrated and described, the rotation is from the congruent factors, r, to a rotated set, s. An identical computational procedure would be used in rotating from factors, s, to a set of factors,

Table 35 gives the transformation matrices for the desired rotation. It is assumed that the entries in the T_{rs} matrix have been derived from some procedure for deciding where to rotate the axes. In the present case these entries were determined graphically from a plot of the loadings on the two congruent factors in accordance with usual graphical rotation of axes procedures involving oblique axes. In the example described in Section IV, the entries in the T_{rs} matrix were obtained from solving sets of simultaneous linear equations. Other methods might be employed to obtain the entrees in T_{rs}. The computational procedure to complete the rotation is described in the following steps.

a. Sum the columns of T_{rs}.

b. Compute the matrix product $T_{mrA}\ T_{rs}$ by the procedure described in Section 5, Paragraph c. Record the results in matrix T_{msA}. Table 33 gives matrix T_{mrA} for the example. To obtain the checks, multiply the Σ row of T_{mrA} by the columns of T_{rs}. Sum the columns of T_{msA} (exclusive of the Ch. entries) and record in the Σ row. These sums should agree with the Ch entries within ± 2 in the last decimal place carried.

c. Compute the matrix product $T_{MrB}\ T_{rs}$ by the same procedure as in step b, recording the results in the matrix T_{MsB}.

d. Compute the sum of squares of the entries in each column of matrix $\mathcal{T}_{msA}$ and record the results in the $\sum_m \tau^2_{msA}$ row.

e. Compute the sum of squares of the entries in each column of matrix $\mathcal{T}_{MsB}$ and record the results in the $\sum_M \tau^2_{MsB}$ row.

f. Sum each pair of corresponding entries in the $\sum_m \tau^2_{msA}$ and $\sum_M \tau^2_{MsB}$ rows, divide the sum by 2, and record in the $\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})$ row.

g. Find the square root and reciprocal of the square root of each entry in the $\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})$ row and record the results in the $\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ and $1/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ rows. A check on these computations is to multiply each $1/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ entry by the corresponding $\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})$, the product should equal the $\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ within ± 2 in the last decimal place carried.

h. Using the $1/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ for each column, multiply the entries in matrices $\mathcal{T}_{rs}$, $\mathcal{T}_{msA}$, $\mathcal{T}_{MsB}$ in that column and record the results in the corresponding cells of the matrices T_{rs}, T_{msA}, and T_{MsB}. The Ch entry for each T matrix is obtained by multiplying Σ entry by the $1/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$ for the column. Sum each column in each T matrix (exclusive of the Ch entry) and enter results in the Σ row. These entries in the Σ rows should agree with the corresponding Ch entries within ± 2 in the last decimal place carried.

i. Obtain the sum of squares of entries in each column of the T_{msA} matrix and record in the $\sum_m t^2_{msA}$ row. Repeat for the T_{MsB} matrix.

j. Sum each pair of corresponding entries in the $\sum_m t^2_{msA}$ and $\sum_M t^2_{MsB}$ rows, divide each sum by 2, and record in the $\frac{1}{2}(\sum_m t^2_{msA} + \sum_M t^2_{MsB})$ row. These entries should equal unity within ± 2 in the last decimal place carried in the T matrices.

k. Compute the matrices of loadings of the tests on the rotated factors, F_{JsA} and F_{JsB} (See Table 36). Obtain the matrix products $F_{JmA}T_{msA}$ and $F_{JMB}T_{MsB}$ by the procedure given in Section 5, Paragraph c. A computational short cut which is only slightly less accurate is to obtain the matrix products $F_{mrA}T_{rs}$ and $F_{MrB}T_{rs}$.

Table 35

	τ_{rs} a	τ_{rs} b		T_{rs} a	T_{rs} b
A	1.00	.37	A	1.0276	.4169
B	-.06	-1.00	B	-.0617	-1.1267
			Ch	.9659	-.7098
Σ	.94	-.63	Σ	.9659	-.7098

	$\tau_{msA} = T_{mrA}\tau_{rs}$ a	$\tau_{msA} = T_{mrA}\tau_{rs}$ b		T_{msA} a	T_{msA} b
i	.6182	.4020	i	.6352	.4529
ii	-.2055	-.4344	ii	-.2112	-.4894
iii	-.6071	.6611	iii	-.6238	.7449
Ch	-.1943	.6287	Ch	-.1998	.7084
Σ	-.1944	.6287	Σ	-.1998	.7084
$\sum_m \tau^2_{msA}$	.792972	.787361	$\sum_m t^2_{msA}$	.837211	.999507

	$\tau_{MsB} = T_{MrB}\tau_{rs}$ a	$\tau_{MsB} = T_{MrB}\tau_{rs}$ b		T_{MsB} a	T_{MsB} b
I	.4543	.5813	I	.4568	.6550
II	-.1460	.5003	II	-.1560	.5637
III	.7685	.0058	III	.7897	.0077
IV	-.5318	.4470	IV	-.5465	.5036
Ch	.5450	1.5354	Ch	.5600	1.7300
Σ	.5450	1.5354	Σ	.5600	1.7300
$\sum_M \tau^2_{MsB}$	1.101108	.783065	$\sum_M t^2_{MsB}$	1.162691	1.000455
$\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})$	.947040	.787713	$\frac{1}{2}(\sum_m t^2_{msA} + \sum_M t^2_{MsB})$	.999951	.999981
$1\Big/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$	.973160	.887532			
$1\Big/\sqrt{\frac{1}{2}(\sum_m \tau^2_{msA} + \sum_M \tau^2_{MsB})}$	1.027580	1.126720			

Table 36

STUDY A

$$F_{JaA} = F_{JmA} T_{maA}$$

	a	b
1	-.0080	.6075
2	.0141	.4161
3	.4031	.5402
4	.3089	.2964
5	.4267	-.0729
6	.4683	.0684
Ch	1.6131	1.8557
Σ	1.6131	1.8557

STUDY B

$$F_{JaB} = F_{JMB} T_{MaB}$$

	a	b
1	-.0317	.5951
2	.0175	.4408
3	.3880	.4381
4	.3665	.4257
5	.3956	.0754
6	.4706	-.0689
Ch	1.6064	1.9062
Σ	1.6065	1.9062

3. Determination of Non-Congruent Axes

The congruent factors established by the preceding procedures will usually be fewer than the total number of factors in either study. A set of non-congruent axes is to be established in each study. The number of congruent factors and non-congruent axes is to equal the total number of factors in the study. In Study B of the pair of studies used in the body of this report there was a total of 12 factors. A set of 5 congruent factors were determined. This left 7 factors to be established as non-congruent axes. The computing procedure for establishing the non-congruent factors follows. These directions will be illustrated by a fictitious example for which there is a total of five factors. Table 37 gives the transformation T_{mrA} to three congruent factors. Two non-congruent factors are to be established.

a. Prepare a work sheet like the one given in Table 38.
   1) Sections A, B, and C are to have a row for each reference factor.
   2) Sections A and D are to be located vertically from each other and are to have as many columns as there are factors in the study. Head the columns 1, 2, 3, etc.
   3) Sections B and E are to be located vertically from each other and are to have as many columns as there are factors. Head the columns 1', 2', 3', etc.
   4) Section C is to have columns for the non-congruent axes. Head the columns 4", 5", etc. The first number in this series is one greater than the number of congruent factors. The last number in the series is the number of reference factors in the study.
   5) Sections D and E are to be located horizontally from each other and are to have as many rows as there are factors in the study. Head the rows 1', 2', 3', etc.
   6) Record unities in the diagonal cells, from upper left to lower right, of Section D. Make dashes in all other cells of Section D.
   7) Make dashes in the diagonal cells, from upper left to lower right, and in all cells to the right of the diagonal of Section E.

b. Copy the matrix T_{mrA} (or T_{MrB}) into the left portion of Section A using as many columns as there are congruent factors. The remaining columns will be left blank for the present. Enter previously determined column totals of T_{mrA} in the Σ row of Section A. Check the copying by summing the columns of Section A, these sums should agree with the previous totals entered in the Σ row.

c. Obtain column 1' of Section B.

1) Copy column 1 of Section A into column 1' of Section B. Enter the sum of column 1 from the Σ cell of column 1 into the Ch cell of column 1'. Sum the column 1'(exclusive of the Ch entry) and record the result in the Σ cell of column 1'. The entries in the Ch cell and Σ cell of column 1' should agree.

2) Find the largest number, ignoring sign, in column 1' of Section B (exclusive of the Ch and Σ row entries). Underline this number and make dashes in the remaining cells of the row of Section B in which the number is located. This number is the 1' pivot entry. The row is the 1' pivot row. In the example the largest number in column 1' of Section B is .6612 in row iv. This is the 1' pivot entry and row iv is the ' pivot row.

d. Compute column 2' of section B:

1) Compute the entry in Section E row 2' column 1' by dividing the entry of Section A column 2 in the 1' pivot row by the 1' pivot entry and recording the result with reverse sign in the Section E row 2' column 1' cell. In the example, the entry of Section A column 2 in the 1' pivot row is .3421. Division of .3421 by .6612, the 1' pivot entry, yields .5174 which is recorded in Section E row 2', column 1' with a negative sign.

2) Compute each entry in Section B column 2' by multiplying each row of Sections A and B by Sections D and E row 2'. For the entry in Section B row i column 2', multiply row i of Section A and B by row 2' of Section D and E. Only columns 2 of Section A and 1' of Section B will be involved since row 2' of Sections D and E has entries in these columns. In the example, row i

$$(.4175)(1.0000) + (.1393)(-.5174) = .3454.$$

Note that the entry in the 1' pivot row should be zero and need not be recorded. If this entry in the 1' pivot row is not zero, the entry in Section E row 2' column 1' is incorrect.

3) Compute product of Σ row of Sections A and B and row 2' of Sections D and E and record the result in the Ch cell of Section B column 2'.

4) Sum the entries in Section B column 2' (exclusive of the Ch entry) and record the result in the Σ row. The Ch and Σ entries should agree within ±2 of the last decimal place carried.

5) Select the largest entry, ignoring sign, in Section B column 2' (exclusive of Ch and Σ entries). This is the 2' pivot entry and is to be underlined. Make dashes in the remaining cells of Section B in the row to the right of the 2' pivot entry. The row containing the 2' pivot entry is the 2' pivot row.

e. Compute column 3' of Section B:

1) Compute the entry in Section E row 3' column 1' by dividing the entry of Section A column 3 in the 1' pivot row by the 1' pivot entry and recording the result with opposite sign in the Section E row 3' column 1' cell. In the example, -.7365 is the entry of Section A column 3 in the 1' pivot row (row iv which contains the underlined 1' pivot entry in Section B column 1'). Then:

$$-(-.7365)/\ .5612 = 1.1139.$$

2) Multiply the 2' pivot row in Sections A and B by the portion of Sections D and E row 3' that has been determined, divide the result by the 2' pivot entry and record the result with reverse sign in the Section E row 3' column 2' cell. For the example, multiplication of Sections A and B row iii (the 2' pivot row) by Sections C and D row 3', division by the 2' pivot entry, and reversal of sign yields:

$$-\left[(.0349)(1.0000) + (-.5023)(1.1139)\right] /\ .9481 = -(-.5246)/\ .9481 = .5533.$$

This result is recorded in Section E row 3' column 2'.

3) Multiply each row of Sections A and B by row 3' of Sections D and E and record the results in Section B column 3'. In the example, multiplication of row i of Sections A and B by row 3' of Sections D and E yields:

$$(.3571)(1.0000) + (.1393)(1.1139) + (.3454)(.5533) = .7034$$

which is recorded in Section B row i column 3'. Note that the entries in the 1' and 2' pivot rows should be zero. If either of these entries is not zero, the entries in Section E row 3' are incorrect.

4) Multiply the Σ row of Sections A and B by the 3' row of Sections D and E and record the result in the Ch cell of Section B column 3'.

5) Sum the entries in Section B column 3' (exclusive of the Ch row) and record the result in the Σ row. The Ch and Σ entries should agree within ± 2 of the last decimal place carried.

6) Select the largest entry, ignoring sign, in Section B column 3' (exclusive of Ch and Σ entries). This is the 3' pivot entry and is to be underlined. Make dashes in the remaining cells of Section B in the row to the right of the 3' pivot entry. The row containing the 3' pivot entry is the 3' pivot row.

f. Compute remaining columns of Section B corresponding to columns of Section A containing congruent factors. In the present example there were three congruent factors and the computations of columns of Section B will stop, therefore, with column 3'. If the matrix $T_{mr\cdot A}$ had seven columns, the computations would continue through column 7' of Section B. (During the process of determining non-congruent axes, corresponding columns of Section B will be determined. In the example the non-congruent factor, later recorded in column 4 of Section A, was used in obtaining column 4' of Section B. These steps are subsequent to the present step.) Follow the procedure outlined in the foregoing step e. For each additional column added to Section B there is an additional entry in the corresponding row of Section E. Step e3 gives the general procedure for determining the entries in the Section E row.

g. Determine the first non-congruent axis.

1) Record unity in the first column of Section C in some row that is not a pivot row. In the example column 4' had not been recorded in Section B and row 1 was not a pivot row. Unity was recorded in row 1 of Section B column 4". (The row selected is likely to become the next pivot row when computations return to Section B.)

2) Record zeros in the first column of Section C in other rows that are not pivot rows. In the example, .0000 was recorded in row 11 of Section C column 4".

3) Multiply the first column of Section C (using those entries already recorded) by the last column recorded in Section B, divide the resulting sum of products by the pivot entry in the column of Section B, and record with reverse sign in the corresponding pivot row in

Section C first column. In the example:

a) **Section C column 4" was multiplied by Section B column 3', yielding:**

(.7034)(1.0000) + (.4984)(.000) = .7034.

b) **This result was divided by the 3' pivot entry:**

$$.7034/(-1.0263) = -.6854$$

c) **The sign of the result was changed and the .6854 was recorded in Section C column 4" in row v, the 3' pivot row.**

4) **Multiply the first column of Section C by the next to last column recorded in Section B, divide the resulting sum of products by the pivot entry in the column of Section B, and record the result with opposite sign in the pivot row in the first column of Section C. In the example, column 4" of Section C was multiplied by column 2' of Section B. The result was divided by the 2' pivot entry (in row iii) and this result was recorded with opposite sign in Section C column 4" row iii, the 2' pivot row.**

$$-\left[(.3454)(1.0000) + (-.6138)(.0000) + (-.4114)(.6854)\right] /.9481$$

$$= -(.0632644)/.9481 = -.0669$$

5) Continue the process described in steps 3 and 4 working back one column of Section B each time and recording the result in the pivot row in the first column of Section C. All entries in the column of Section C will then be determined.

6) Sum the first column in Section C and record the result in the Σ row.

7) Obtain the sum of squares of the entries in the first column of Section C (exclusive of the Σ entry) and record the result in the Σc^2 row.

8) Obtain the square root of the entry in the Σc^2 row, recording in the $\sqrt{\Sigma c^2}$ row, and the reciprocal of the square root, recording in the $1/\sqrt{\Sigma c^2}$ row. These computations may be checked by multiplying the $1/\sqrt{\Sigma c^2}$ entry by the Σc^2 entry, the result should agree with the $\sqrt{\Sigma c^2}$ entry within ± 2 of the last decimal place carried.

9) Multiply each entry in the column of Section C by the $1/\sqrt{\Sigma c^2}$ entry and record in the corresponding column of Section A. Results from column 4" of Section C were recorded in column 4 of Section A. To check these entries, multiply the Σ entry in the column of Section C

by the $1/\sqrt{\Sigma C^2}$ and record in the Ch row of the Section A column. Sum the entries in the Section A column (exclusive of the Ch entry) and record in the Σ row. The Ch and Σ entries should agree within ±2 of the last decimal place carried.

10) Obtain the sum of squares of the entries in the new Section A column and record the result in the Σt^2 row. This entry should equal unity within ±1 in the last decimal place carried.

11) Check that the product between the new column of Section A and and each preceeding column of Section A is zero within ±1 of the last decimal place carried.

h. Compute a new column of Section B for the column added to Section A. Follow the procedure outlined in step e. Also see comments on procedure in Section f.

i. Compute a new column of Section C for the next non-congruent axis. Repeat step g interpreting the directions to indicate the second non-congruent axis werever the first non-congruent axis is mentioned.

j. Repeat steps h and i for subsequent non-congruent axes until all columns of Section A are completed. The columns added to Section A contain the direction cosines of the non-congruent axes. These columns may be copied into a $A_{m:A}$(or A_{MUB}) matrix such as is given in Table 6.

Table 37

TRANSFORMATION TO CONGRUENT
FACTORS FOR A FICTITIOUS EXAMPLE

T_{mrA}

Reference Factors	Congruent Factors A	B	C
i	.1393	.4175	.3571
ii	.4871	-.3618	.2954
iii	-.5023	.6882	.0349
iv	.6612	.3421	-.7365
v	-.2133	-.5218	-.5611
Σ	.5720	.5642	-.6102

Table 38

Section A

	(T_{mrA}) 1	2	3	(A_{mmA}) 4	5
i	.1393	.4175	.3571	.8231	-.1330
ii	.4871	-.3618	.2954	.0000	.7204
iii	-.5023	.6822	.0349	-.0551	.6271
iv	.6612	.3421	-.7365	-.0533	.0571
v	-.2133	-.5218	-.5611	.5642	.2587
Ch				1.2989	1.5303
Σ	.5720	.5642	-.6102	1.2989	1.5303
Σt^2				1.0000	1.0001

Section B

1'	2'	3'	4'	5'
.1393	.3454	.7034	1.2148	------
.4871	-.6138	.4964	.2243	
-.5023	.9481	------	------	------
.6612	------	------	------	------
-.2133	-.4114	-1.0263	------	------
.5720	.2682	.1754	1.4592	
.5720	.2683	.1755	1.4591	

Section C

	4"	5"
	1.0000	-.1846
	.0000	1.0000
	-.0669	.8705
	-.0404	.0793
	.6854	.3591
	1.5781	2.1243
Σc^2	1.4759	1.9271
$\sqrt{\Sigma c^2}$	1.2149	1.3882
$1/\sqrt{\Sigma c^2}$	.8231	.7204

Section D

1'	1.0000	------	------	------	------
2'	------	1.0000	------	------	------
3'	------	------	1.0000	------	------
4'	------	------	------	1.0000	------
5'	------	------	------	------	1.0000

Section E

------	------	------	------	------
-.5174	------	------	------	------
1.1139	.5533	------	------	------
.0504	.0848	.5053	------	------
				------

4. Determination of Latent Roots and Vectors

Given a symmetric matrix A_0, such as in Table 39, it is desired to compute a matrix of latent vectors and the corresponding latent roots. The matrix A_{02} in Table 43 is a close approximation of the matrix of latent vectors for the example. The diagonal entries in matrix A_2 of Table 43 are close approximations to the latent roots for the example. The method to be described is an adaptation of the successive rotations method developed by Truman L. Kelley (14). Each rotation transforms the axes to closer approximations to the latent vectors, with any desired degree of precision being obtained by taking more rotations. It may be necessary to carry more decimal places in later rotations to realize the potential precision. In the example only two rotations were computed to obtain a fair degree of precision. Most salient features of the procedure are illustrated however.

Tables 40 and 41 contain the computations for the first rotation and Tables 42 and 43 contain the computations for the second rotation. Each rotation starts from an A matrix and produces a revised A matrix. Rotation 1 started from the A_0 matrix of Table 39 and produced the A_1 matrix in the lower right of Table 41. Rotation 2 started from the A_1 matrix and produced the A_2 matrix in the lower right of Table 43. Note that each of these A matrices is symmetric (each row of the matrix has idnetical entries with those of the corresponding column). The largest entries are in the diagonal from upper left to lower right and from rotation to rotation the off-diagonal entries are becoming smaller. The solution occurs when the off-diagonal entries become zero. The diagonal entries are then the latent roots. Each rotation has a matrix relating the A matrix produced by that rotation to the original A_0 matrix. The A_{01} matrix for the first rotation is in the upper left of Table 41. The A_{02} matrix for rotation 2 is in the upper left of Table 43. When the off-diagonal entries in the A matrix are zero, the A matrix contains the latent vectors.

In the following directions, only one rotation will be covered explicitly. It is expected that as many such rotations will be taken as necessary to obtain the precision desired for any particular solution.

a. For each rotation prepare a set of work sheets. Tables 40 and 42 illustrate the set up for Work Sheet 1, Tables 41 and 43 illustrate the set up for Work Sheet 2. Each matrix in these work sheets is to be the same size as the matrix A_0.

b. Obtain the entries in the B and C matrices from the preceeding A matrix. For Rotation 1 the A_0 matrix is used, for Rotation 2 the A_1 matrix is used, etc.

1) Find the largest off-diagonal entry irrespective of sign below the diagonal of the preceeding A matrix. In Rotation 1 of the example, the largest entry, ignoring sign, below the diagonal of matrix A_0 in Table 39 is .58 in row 5 and column 2.

a) Subtract the diagonal entry for the row of the selected off-diagonal entry from the diagonal for the column of the selected off-diagonal entry. For the example: the diagonal of row 5 is 1.55, the diagonal of column 2 is 1.21, subtracting 1.55 from 1.21 yields -.34.

b) Divide the difference in diagonal entries by the off-diagonal entry, ignoring the sign of the off-diagonal entry. Record the result in the cell of matrix B corresponding to the selected off-diagonal entry. Note that the sign of the result will depend only on the sign of the difference between the diagonal entries; thus, if the diagonal entry nearer the upper left is larger than the diagonal entry toward the lower right, the sign is plus; if the reverse is true, that is of the two diagonal entries, the second diagonal entry from upper left to lower right is the larger, the sign is negative. For the example:

$$-.34/.58 = -.59$$

which is recorded in the matrix B_{01} row 5 and column 2 cell. The sign is minus because the second diagonal entry of matrix A_0 is less than the fifth diagonal entry. The entry in row 3 and column 1 of matrix B_{01} illustrates the case when the entry is positive. The first diagonal entry of matrix A_0 is larger than the third diagonal entry of matrix A_0. The sign of the -.10 in row 3 and column 1 of matrix A_0 is ignored.

From Table 44 find the c value corresponding to the b entry in the B matrix. The sign of the entry in the B matrix is ignored in finding the corresponding c value. For the example: the .59 from row 5 and column 2 of matrix B_{01} is in the interval of b values of Table 44 of .57 to .59 for which the corresponding c value is .75.

In case the entry in the B matrix is plus, record the c value found in step c in the corresponding off-diagonal cell of the matrix C with the sign of the corresponding entry in the A matrix. Record unities in the corresponding diagonal cells of the C matrix and copy the off-diagonal entry reversing sign, into the symmetrical cell above the diagonal. For an example note the entries in rows 1 and 3 and columns 1 and 3 of matrix C_{01} in Table 40.

In case the entry in the B matrix is minus, record the c value in the two corresponding diagonal cells of the matrix C. Record unity in the corresponding off-diagonal cell of matrix C and assign it the same sign as the off-diagonal entry in matrix A, and record unity with the opposite sign in the symmetric cell above the diagonal of matrix C. This step was followed for the entries in rows 2 and 5 and columns 2 and 5 of the matrix C_{01} in Table 40.

If the off-diagonal entry in a C matrix is .30 or larger, ignoring sign, the rows and columns of the entry are to be excluded from further consideration in determining other entries in the B and C matrices. This is true for the example where the largest off-diagonal entry in matrix A_0 was in row 5 and column 2 which yielded off-diagonal entries of unity in matrix C_{01}. The rows and columns 2 and 5 were then excluded from the following steps in determining other entries in matrix C_{01}. In the case of the second rotation none of the off-diagonal entries in matrix C_{12} of Table 42 were .30 or larger and no rows or columns were excluded from further consideration in determining other entries in the C_{12} matrix. As a consequence, several off-diagonal entries appear in each column of matrix C_{12}. Whenever the C matrix off-diagonal entry is .30 or larger, only this off-diagonal entry should appear in its row or column of matrix C.

2) Select the next to the highest off-diagonal entry, irrespective of sign, below the diagonal of the A matrix. Do not consider any entry in a row or column that has been excluded from further consideration in step 1-f. In the example, in Rotation 1 rows 2 and 5 and columns 2 and 5 were excluded from further consideration in step 1-f. The next highest off-diagonal, irrespective of sign, in matrix A_0 of Table 39 in the rows and columns 1,3, and 4 remaining is -.10 in row 3 and column 1. The -.11 in row 5 and column 3 is not to be considered. When the off-diagonal entry has been selected, follow the procedure of steps 1-a to 1-f in obtaining the entry in matrix C.

In Rotation 2, the highest off-diagonal entry irrespective of sign of matrix A_1 of Table 41 was -.10 in row 4 column 1. This yielded an off-diagonal matrix C_{12}, of Table 32, entry of -.12. Since this entry was not large enough, ignoring sign, to cause the exclusion of rows and columns 1 and 4, the entry of .09 in row 4 and column 2 of matrix A_1 could be selected second. This yielded an entry of .11 in row 4 and column 2 of matrix C_{12}. The selection of entries in the A matrix, from high to low, and subsequent determination of entries in the C matrix is to continue until there are no more off-diagonal entiies in matrix A from which to select.

A special case exists when a C matrix entry of .30 or greater, irrespective of sign, occurs on later selection of off-diagonal entries in matrix A and there is already an entry in the row or column of this entry in the C matrix. In this case,this entry is not to be recorded and the rows and columns in which it is located are to be excluded from further consideration in the selection of additional entries in matrix A.

c. Compute the entries in the T matrix. In Rotation 1, Table 40, matrix T_{01} is identical with matrix C_{01} which is, therefore, to be copied. In all subsequent rotations the T matrix is to be obtained by multiplication of of the preceding A matrix by the present C matrix (see Section 5, paragraph c for the procedure in multiplication of matrices). In Rotation 2, A_{01} for Rotation 1 in Table 41 is multiplied by matrix C_{12} in Table 42 to produce matrix T_{02} of Table 42. In obtaining the Ch row of matrix T, multiply the Σ row of the preceding A matrix by the columns of the C matrix.

(For Rotation 1, the sums of the columns of the C matrix are to be recorded in the Ch row of matrix T_{01}.) The sums of the columns of the T matrix (exclusive of the Ch entries) are to be recorded in the Σ row and should agree with the entries in the Ch row within ±2 of the last decimal place carried. After the matrix T has been determined, the rows are to be summed and the results entered in the Σ column. The sums can be checked by summing both the Σ row and the Σ column. These two sums should agree.

Compute the E matrix. For an example see Table 42.

1) Obtain the first column of the E matrix.

   a) Obtain the sum of squares of the entries in the first column of matrix T and record the result in the first row and first column of matrix E.

   b) Obtain the sum of products between the entries in the first column and second column of matrix T and record the result in the second row and first column of matrix E.

   c) Obtain the sum of products between the entries in the first column and each of the remaining columns of Matrix T and record the results in the corresponding rows and column 1 of matrix E.

   d) Obtain the sum of products between the entries in the first column and the Σ column of matrix T and record the result in the Ch row and column 1 of matrix E.

   e) Sum the entries(exclusive of the Ch entry) in the first column of matrix E and record the result in the Σ row and column 1. The Ch entry and the Σ entry should agree within ±2 in the last decimal place carried.

   f) When the entries in the first column of matrix E have been checked in step e. these entries may be copied into the first row of matrix E.

2) Obtain each of the other columns of the matrix E by finding the sums of products between entries in the corresponding column and each of the other columns of matrix T. The sum of squares of entries in each column of matrix T is to be recorded in the corresponding diagonal cell of matrix E.

3) Double each diagonal entry in matrix E and record the result in the corresponding cell of column $2e_{jj}$. To check this work obtain the sum of the diagonal entries, double this sum and record in the Ch cell of

column $2e_{jj}$. Then sum the entries in the column $2e_{jj}$ and record the result in the Σ cell of the column. The entries in the Ch cell and Σ cell should agree.

4) Obtain the square root of each diagonal entry in matrix E and record the results in the $\sqrt{e_{kk}}$ row.

5) Obtain the reciprocals of the square roots determined in step 4 and record in the $1/\sqrt{e_{kk}}$ row. Multiply each diagonal entry by the corresponding $1/\sqrt{e_{kk}}$. This product should agree with the $\sqrt{e_{kk}}$ within ± 2 in the last decimal place carried.

e. Obtain the F matrix. For an example see Table 42.

1) Copy the $1/\sqrt{e_{kk}}$ entries into the diagonal cells of matrix F.

2) Multiply each off-diagonal entry in matrix E by the $1/\sqrt{e_{kk}}$ at the bottom of the column containing the entry, divide this product by the $2e_{jj}$ at the right of the row containing the entry, and record the result with reverse sign in the corresponding cell of matrix F. For example, the entry in row 2 and column 1 of the matrix E_2 in Table 42 is multiplied by the $1/\sqrt{e_{kk}}$ entry at the bottom of column 1:

$$-.0110 \times .9726 = -.01069860,$$

this result is divided by the $2e_{jj}$ in row 2:

$$-.0106980/2.2140 = -.0048,$$

which is recorded with opposite sign in row 2 column 1 of the matrix F_2.

3) Obtain each entry in the Ch column by:

a) Multiplying by 3 the square root of the diagonal entry in the corresponding row of matrix E (this square root can be found in the $\sqrt{e_{kk}}$ row in the column of the diagonal entry),

b) Subtracting from the product of step a, the sum of products between the entries in the row of matrix E and the entries in in the $1/\sqrt{e_{kk}}$ row (including the diagonal entry), and

c) Dividing this result by the $2e_{jj}$ for the row.

The computations for the first Ch entry in matrix F_2 in Table 42 are derived from the first row of matrix E_2.

$$3 \times 1.0282 - \Big[(1.0571)(.9726) + (-.0110)(.9505) + (.0303)(.9621) + (-.0203)(.9775) + (.0080)(.9928)\Big] = 2.04966926$$

$$2.04966926/2.1142 = .9695$$

(The subtractions in step b may be accomplished by adding the products with the signs of the entries in the F matrix row reversed.)

4) Sum the entries (exclusive of the Ch entry) in each row of the matrix F and record the result in the Σ column. The Ch entry and the Σ entry should agree within ±2 in the last decimal place carried.

f. Compute the Λ matrix by multiplying the T matrix by the F matrix. (See Section 5, paragraph c for the procedure in multiplication of matrices.) In the example matrices T_{02} and F_2 are in Table 42. Their product is matrix Λ_{02} of Table 43. The columns of the Λ matrix are checked by multiplying the Σ row matrix T by matrix F and recording the results in the Ch row of the Λ matrix. The entries (exclusive of the Ch row entries) in the columns of the Λ matrix are to be summed and the results entered in the Σ row. The Ch entries and the Σ entries should agree within ±2 in the last decimal place carried. Sum the rows of the Λ matrix and enter the results in the Σ column.

g. Compute the entries in the Λ'Λ matrix following the procedure given in step d for the E matrix. The diagonal entries in the Λ'Λ matrix should be unity within ±2 in the last decimal place carried in the Λ matrix and the off-diagonal entries should be zero with this same degree of accuracy.

h. Compute the matrix product $A_0\Lambda$ by multiplying the A_0 matrix by the Λ matrix. (See Section 5, paragraph c for the procedure in multiplication of matrices.) Check the rows of the product matrix $A_0\Lambda$ by multiplying each row of the matrix A_0 by the Σ column of the Λ matrix and recording the results in the Ch column. Sum the entries (exclusive of the Ch entry) in each row of the product matrix $A_0\Lambda$ and record the result in the Σ column. The Ch and Σ entries should agree within ±2 of the last decimal place carried.

i. Compute the new A matrix.

1) Multiply each column of the product matrix $A_0\Lambda$ by the first column in the Λ matrix and record the result in the cell of the first row of A corresponding to the column of A_0. Also multiply the Σ column of A_0 by the first column of the Λ matrix and record the result in the first row, Ch column of matrix A. Sum the entries (exclusive of the Ch entry) in the first row of the new A matrix and record the result in the Σ column. The Ch entry and the Σ entry should agree to within ±2 of the last decimal place carried.

2) Compute the remaining rows of the new A matrix using the corresponding columns of the A matrix and following the procedure described in step 1. The A matrix should be symmetric if enough decimal places were carried in the product matrix $A_0 A$. It would be preferrable to carry sufficient decimal places in this matrix product to ensure that the matrix A would be symmetric. Then, when one row of matrix A has been computed and checked, the entries in that row may be copied into the corresponding column.

Table 39

AN ILLUSTRATIVE SYMMETRICAL MATRIX

A_o

	1	2	3	4	5
1	1.99	-.02	-.10	-.08	.00
2	-.02	1.21	.01	.02	.58
3	-.10	.01	1.69	.08	-.11
4	-.08	.02	.08	1.20	.10
5	.00	.58	-.11	.10	1.55

Table [illegible]

WORK SHEET 1, ROTATION 1

B_{01} $\left(b_{jk} = \dfrac{a_{kk} - a_{jj}}{|a_{jk}|}\right)$

	1	2	3	4	5
1	----				
2		----			
3	3.00		----		
4				----	
5		-.59			----

C_{01}

	1	2	3	4	5
1	1.00		.30		
2		.75			-1.00
3	-.30		1.00		
4				1.00	
5		1.00			.75

$T_{01} = C_{01}$

	1	2	3	4	5	Σ
1	1.00		.30			1.30
2		.75			-1.00	-.25
3	-.30		1.00			.70
4				1.00		1.00
5		1.00			.75	1.75
Ch	.70	1.75	1.30	1.00	-.25	
Σ	.70	1.75	1.30	1.00	-.25	4.50

$E_1 = T_{01}' T_{01}$

	1	2	3	4	5	$2e_{jj}$
1	1.0900	.0000	.0000	.0000	.0000	2.1800
2	.0000	1.5625	.0000	.0000	.0000	3.1250
3	.0000	.0000	1.0900	.0000	.0000	2.1800
4	.0000	.0000	.0000	1.0000	.0000	2.0000
5	.0000	.0000	.0000	.0000	1.5625	3.1250
Ch	1.0900	1.5625	1.0900	1.0000	1.5625	12.6100 = $2\Sigma a_{jj}$
Σ	1.0900	1.5625	1.0900	1.0000	1.5625	12.6100
$\sqrt{e_{kk}}$	1.0440	1.2500	1.0440	1.0000	1.2500	
$1/\sqrt{e_{kk}}$	.9579	.8000	.9579	1.0000	.8000	

F_1

$(f_{kk} = 1/\sqrt{e_{kk}})$

$(f_{jk} = -e_{jk}/2e_{jj}\sqrt{e_{kk}},\ j \neq k)$

	1	2	3	4	5	Ch	Σ
1	.9579					.9579	.9579
2		.8000				.8000	.8000
3			.9579			.9579	.9579
4				1.0000		1.0000	1.0000
5					.8000	.8000	.8000

Table 41

WORK SHEET 2, ROTATION 1

$\Lambda_{01} = T_{01} F_1$

	1	2	3	4	5	Σ
1	.96		.29			1.25
2		.60			-.80	-.20
3	-.29		.96			.67
4				1.00		1.00
5		.80			.60	1.40
Ch	.67	1.40	1.25	1.00	-.20	
Σ	.67	1.40	1.25	1.00	-.20	

$\lambda_0 \Lambda_{01}$

	1	2	3	4	5	Ch	Σ
1	1.9394	-.0120	.4811	-.0800	.0160	2.3445	2.3445
2	-.0221	1.1900	.0038	.0200	-.6200	.5717	.5717
3	-.5861	-.0820	1.5934	.0800	-.0740	.9313	.9313
4	-.1000	.0920	.0536	1.2000	.0440	1.2896	1 2896
5	.0319	1.5830	-.1056	.1000	.4660	2.0803	2.0803

$\Lambda'_{01} \Lambda_{01}$

	1	2	3	4	5
1	1.0057	.0000	.0000	.0000	.0000
2	.0000	1.0000	.0000	.0000	.0000
3	.0000	.0000	1.0057	.0000	.0000
4	.0000	.0000	.0000	1.0000	.0000
5	.0000	.0000	.0000	.0000	1.0000
Ch	1.0057	1.0000	1.0057	1.0000	1.0000
Σ	1.0057	1.0000	1.0057	1.0000	1.0000

$\Lambda_1 = \Lambda'_{01} \Lambda_0 \Lambda_{01}$

	1	2	3	4	5	Ch	Σ
1	2.03	.01	.00	-.10	.04	1.98	1.98
2	.01	1.98	-.08	.09	.00	2.01	2.00
3	.00	-.08	1.67	.05	-.07	1.57	1.57
4	-.10	.09	.05	1.20	.04	1.29	1.28
5	.04	.00	-.07	.04	.78	.79	.79

Table 42

WORK SHEET 1, ROTATION 2

B_{12} $\left(b_{jk} = \dfrac{a_{kk} - a_{jj}}{|a_{jk}|}\right)$

	1	2	3	4	5
1	----				
2	5.00	----			
3	----	3.88	----		
4	8.30	8.67	9.40	----	
5	31.00	----	12.71	10.50	----

C_{12}

	1	2	3	4	5
1	1.00	-.19	.00	.12	-.03
2	.19	1.00	.24	-.11	.00
3	.00	-.24	1.00	-.11	.07
4	-.12	.11	.11	1.00	-.09
5	.03	.00	-.07	.09	1.00

$T_{02} = A_{01}\ C_{12}$

	1	2	3	4	5	Σ
1	.96	-.25	.29	.08	-.01	1.07
2	.09	.60	.20	-.14	-.80	-.05
3	-.29	-.18	.96	-.14	.08	.43
4	-.12	.11	.11	1.00	-.09	1.01
5	.17	.80	.15	-.03	.60	1.69
Ch	.81	1.08	1.71	.77	-.22	
Σ	.81	1.08	1.71	.77	-.22	4.15

$k_2 = T_{02}'\ T_{02}$

	1	2	3	4	5	$2e_{jj}$
1	1.0571	-.0110	.0303	-.0203	.0080	2.1142
2	-.0110	1.1070	.0068	.0072	-.0218	2.2140
3	.0303	.0068	1.0803	-.0337	-.0060	2.1606
4	-.0203	.0072	-.0337	1.0465	-.0080	2.0930
5	.0080	-.0218	-.0060	-.0080	1.0146	2.0292
Ch	1.0641	1.0882	1.0777	.9917	.9868	$10.6110 = 2\sum_j e_{jj}$
Σ	1.0641	1.0882	1.0777	.9917	.9868	10.6110
$\sqrt{e_{kk}}$	1.0282	1.0521	1.0394	1.0230	1.0073	
$1/\sqrt{e_{kk}}$	.9726	.9505	.9621	.9775	.9928	

F_2

$(f_{kk} = 1/\sqrt{e_{kk}})$

$(f_{jk} = -e_{jk}/2e_{jj}\sqrt{e_{kk}} \qquad j \neq k)$

	1	2	3	4	5	Ch	Σ
1	.9726	.0049	-.0138	.0094	-.0038	.9695	.9693
2	.0048	.9505	-.0030	-.0032	.0098	.9588	.9589
3	-.0136	-.0030	.9621	.0152	.0028	.9635	.9635
4	.0094	-.0033	.0155	.9775	.0038	1.0030	1.0029
5	-.0038	.0102	.0028	.0039	.9928	1.0059	1.0059

Table 43

WORK SHEET 2, ROTATION 2

$A'_{02} = T_{02} F_2$

	1	2	3	4	5	Σ
1	.93	-.23	.27	.09	-.01	1.05
2	.09	.56	.18	-.14	-.79	-.10
3	-.30	-.17	.93	-.12	.08	.42
4	-.11	.10	.12	.98	-.08	1.01
5	.16	.77	.14	-.03	.60	1.64
Ch	.78	1.02	1.64	.78	-.20	
Σ	.77	1.03	1.64	.78	-.20	4.02

$A_0 A_{02}$

	1	2	3	4	5	Ch	Σ
1	1.8877	-.4599	.4311	.1155	-.0057	1.9687	1.9687
2	.1779	1.1291	.3053	-.1702	-.6085	.8336	.8336
3	-.6255	-.3354	1.5407	-.1315	.0559	.5042	.5042
4	-.2126	.2130	.2144	1.1534	-.0446	1.3236	1.3236
5	.3222	1.5470	.2311	-.0165	.4550	2.5388	2.5388

$A'_{02} A_{02}$

	1	2	3	4	5
1	1.0007	-.0003	-.0025	-.0055	.0004
2	-.0003	.9983	.0004	-.0038	.0003
3	-.0025	.0004	1.0042	.0009	.0039
4	-.0055	-.0038	.0009	1.0034	.0037
5	.0004	.0003	.0039	.0037	.9970
Ch	.9928	.9949	1.0069	.9987	1.0053
Σ	.9928	.9949	1.0069	.9987	1.0053

$A_2 = A'_{02} A_0 A_{02}$

	1	2	3	4	5	Ch	Σ
1	2.0342	-.0014	-.0204	.0020	.0009	2.0153	2.0153
2	-.0014	2.0076	.0093	.0031	-.0031	2.0155	2.0155
3	-.0204	.0093	1.6623	.0144	-.0007	1.6648	1.6649
4	.0020	.0031	.0144	1.1808	.0206	1.2209	1.2209
5	.0009	-.0031	-.0007	.0206	.7618	.7795	.7795

Table 44

FASCILITATING TABLE FOR ROTATIONS TO LATENT VECTORS

b From	b To	c	b From	b To	c	b From	b To	c	b From	b To	c	b From	b To	c
.00	.01	1.00												
.02	.03	.99	.47	.48	.79	1.09	1.12	.59	2.14	2.21	.39	4.94	5.22	.19
.04	.05	.98	.49	.51	.78	1.13	1.16	.58	2.22	2.29	.38	5.23	5.53	.18
.06	.07	.97	.52	.54	.77	1.17	1.20	.57	2.30	2.37	.37	5.54	5.89	.17
.08	.09	.96	.55	.56	.76	1.21	1.24	.56	2.38	2.46	.36	5.90	6.29	.16
.10	.11	.95	.57	.59	.75	1.25	1.28	.55	2.47	2.55	.35	6.30	6.75	.15
.12	.13	.94	.60	.62	.74	1.29	1.33	.54	2.56	2.65	.34	6.76	7.27	.14
.14	.15	.93	.63	.65	.73	1.34	1.37	.53	2.66	2.75	.33	7.28	7.87	.13
.16	.17	.92	.66	.68	.72	1.38	1.42	.52	2.76	2.85	.32	7.88	8.58	.12
.18	.19	.91	.69	.71	.71	1.43	1.47	.51	2.86	2.97	.31	8.59	9.41	.11
.20	.22	.90	.72	.74	.70	1.48	1.52	.50	2.98	3.09	.30	9.42	10.43	.10
.23	.24	.89	.75	.77	.69	1.53	1.57	.49	3.10	3.22	.29	10.44	11.67	.09
.25	.26	.88	.78	.80	.68	1.58	1.63	.48	3.23	3.36	.28	11.68	13.25	.08
.27	.29	.87	.81	.83	.67	1.64	1.68	.47	3.37	3.50	.27	13.26	15.31	.07
.30	.31	.86	.84	.87	.66	1.69	1.74	.46	3.51	3.66	.26	15.32	18.12	.06
.32	.33	.85	.88	.90	.65	1.75	1.80	.45	3.67	3.83	.25	18.13	22.17	.05
.34	.36	.84	.91	.93	.64	1.81	1.86	.44	3.84	4.02	.24	22.18	28.53	.04
.37	.38	.83	.94	.97	.63	1.87	1.92	.43	4.03	4.21	.23	28.54	39.97	.03
.39	.41	.82	.98	1.01	.62	1.93	1.99	.42	4.22	4.43	.22	39.98	66.65	.02
.42	.43	.81	1.02	1.04	.61	2.00	2.06	.41	4.44	4.67	.21	66.66	199.99	.01
.44	.46	.80	1.05	1.08	.60	2.07	2.13	.40	4.68	4.93	.20	200.00	------	.00

$c = \frac{1}{2} (\sqrt{b^2+4} - b)$

For b of 10 or more
c is approximately $\frac{1}{b}$

5. Notes on Matrix Computations

The computing procedures in the foregoing sections have been stated in terms of a few standard matrix computations described in this section. Familiarity with the following matrix computational methods will be of distinct assistance in understanding the directions in the first four sections.

a. Definitions:

1) A row of numbers is called a row vector.

Example:

3 2 6 8 5

2) A column of numbers is called a column vector.

Example:

3
2
6
8
5

3) A rectangular table of numbers is called a matrix.

Example:

7	2	1	4
3	4	2	5
8	1	7	5

4) A square matrix with entries in the diagonal from upper left to lower right and zeros elsewhere is called a diagonal matrix.

Example:

5	0	0	0
0	4	0	0
0	0	8	0
0	0	0	9

5) A single letter may be used to designate an entire vector or matrix.

Examples:

Note that the following examples are different than the preceeding examples. The terms vector and matrix can be applied to any set of numbers arranged as a row, a column, and a table.

1) Row vector A equals:

7 5 9

2) Column vector B equals:

1
6
4

3) Matrix C equals:

5 8
3 7
2 8

6) The transpose of a matrix is the matrix with the rows of the original matrix written as columns (or original columns written as rows which gives the same result). The transpose is designated by the letter for the original matrix primed.

Example:

The transpose of matrix C is C' and is:

5 3 2
8 7 8

b. Multiplication of vectors.

Consider the following two row vectors

Vector A	7	5	9
Vector E	8	3	2

Multiply the first number of vector A by the first number of vector E:

$$7 \times 8 = 56$$

Also find the product of the second entries in the two vectors:

$$5 \times 3 = 15$$

Similarly find the products of each piar of corresponding entries in the two vectors. The third terms in the example give:

$$9 \times 2 = 18$$

Sum the products:

$$56 + 15 + 18 = 89.$$

This sum is the result from multiplying the two vectors. Thus:

<u>Two vectors are multiplied by summing products of corresponding entries in the two vectors.</u>

Consider a second e. ·ple; multiplication of row vector **A** and column vector B.

Vector A	7	5	9
Vector B		1	
		6	
		4	

These two vectors are multiplied as follows:

7 x 1 = 7
5 x 6 = 30
9 x 4 = 36
Total 73

The result of multiplying the two vectors is the number 73.

Note: Two vectors must have the same number of entries if they are to be multiplied.

c. Multiplication of Matrices:

Consider the following set of three matrices.

	Matrix C		Matrix F				Total
	5	8	4	6	3	5	18
	3	7	1	9	2	7	19
	2	8					
Total	10	23					

	Product Matrix CF				Total
	28	102	31	81	242
	19	81	23	64	187
	16	84	22	66	188
Total	63	267	76	211	617

Consider the first row of matrix C as a row vector. Consider the first column of matrix F as a column vector. Multiplication of these vectors yields:

5 x 4 = 20
8 x 1 = 8
Total 28

The total 28, is recorded in the first row and first column of the product matrix CF.

Multiplication of the second row of matrix C by the first column of matrix F yields:

$$3 \times 4 = 12$$
$$7 \times 1 = 7$$
Total 19

19 is recorded in the second row and first column of the product matrix CF.

Similarly, the 16 in the third row and first column of product matrix CF results from multiplying the third row of matrix C by the first column of matrix F.

$$2 \times 4 = 8$$
$$8 \times 1 = 8$$
Total 16

When the row of column totals of matrix C is multiplied by the first column of matrix F the result is the total of the first column of the product matrix CF.

$$10 \times 4 = 40$$
$$23 \times 1 = 23$$
total 63

This computation of the total of the column of the product matrix is an efficient check on the computation of the entries in the column.

The second column of the product matrix CF is obtained by multiplying the rows of matrix C by the second column of the matrix F.

First row, second column

$$5 \times 6 = 30$$
$$8 \times 9 = 72$$
Total 102

Second row, second column

$$3 \times 6 = 18$$
$$7 \times 9 = 63$$
Total 81

Third row, second column

$$2 \times 6 = 12$$
$$8 \times 9 = 72$$
Total 84

Total row, second column

$$10 \times 6 = 60$$
$$23 \times 9 = 207$$
Total 267

The third column of the product matrix $\underline{C}$ is obtained similarly by multiplying the rows of matrix $\underline{C}$ by the third column of matrix $\underline{F}$. For the fourth column of the product matrix $\underline{CF}$, the fourth column of matrix $\underline{F}$ is used. A check on the totals of the rows of the product matrix $\underline{CF}$ is obtained by multiplying the rows of matrix $\underline{C}$ by the column of totals of the rows of matrix $\underline{F}$, for example:

$$\begin{array}{r}
5 \times 18 = 90 \\
8 \times 19 = \underline{152} \\
\text{Total} \quad 242
\end{array}$$

Multiplication of the row of totals of columns in matrix $\underline{C}$ by the matrix $\underline{F}$ column of totals of rows yields the grand total of all entries in the product matrix $\underline{CF}$.

$$\begin{array}{r}
10 \times 18 = 180 \\
23 \times 19 = 437 \\
\text{Total} \quad \overline{617}
\end{array}$$

Two matrices are multiplied by multiplying as vectors each row of the first matrix by each column of the second matrix and recording the results in a product matrix with a row for each row of the first matrix and a column for each column of the second matrix.

Note that, for two matrices to be multiplied, the second matrix must have the same number of rows as the first matrix has number of columns.

Note, also, that the order of the matrices makes a difference in the matrix product. With square matrices care must be taken to consider the matrices in the proper order. Multiplication of matrices in the wrong order will produce erroneous results.