NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3508

LAMINAR FREE CONVECTION ON A VERTICAL PLATE WITH PRESCRIBED NONUNIFORM WALL HEAT FLUX OR PRESCRIBED NONUNIFORM WALL TEMPERATURE

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3430

CX-1

SUMMARY

An analysis is made for laminar free convection on a vertical plate with nonuniform thermal conditions at the surface. Prescribed variations are considered for the wall heat flux and for the wall temperature.

For the situation where the wall-heat-flux variation is prescribed, graphs are presented from which the resulting wall-temperature variation may be obtained. Local heat-transfer coefficients may be readily determined using the information given on the graphs. Results for the important special case of uniform wall heat flux are also given.

For the situation where the wall-temperature variation is prescribed, graphs are presented from which the over-all rate of heat transfer from any length of the plate may be obtained. Another set of graphs is presented for obtaining local heat-transfer coefficients.

All the aforementioned results are given for fluids having Prandtl numbers in the range 0.01 to 1000.

The flow is taken to be of the boundary-layer type, and the problem is formulated by the Kármán-Pohlhausen method. The solution of the resulting equations is achieved by series expansion. The first term of the series corresponds to the result for uniform thermal conditions on the wall. The succeeding terms give the influence of the nonuniform thermal conditions. The first five terms of the series have been calculated.

INTRODUCTION

Laminar free convection on a vertical plate has been a subject of study since 1881. Most of the analytical work has been done for the situation where the wall temperature is uniform over the entire surface. An exact solution of the boundary-layer differential equations for free convection on a vertical flat plate with uniform wall temperature is given in reference 1 for several Prandtl numbers in the range 0.01 to 1000. Reference 1 also lists other work on the uniform-wall-temperature problem. The Kármán-Pohlhausen method is applied to the uniform-wall-temperature case in references 2 (pp. 671-673) and 3 (pp. 158-163). The results reported in these references agree well with those from the exact solution of reference 1.

Considerably less analytical work has been done for the situation where the heat transfer is uniform over the surface. An exact solution of the boundary-layer differential equations for the uniform-heat-flux case is given in reference 4, which also contains results calculated by the Kármán-Pohlhausen method. The results of the exact solution and those from the Kármán-Pohlhausen method are in good agreement. 3430

Accounts of experimental investigations of free convection on vertical surfaces are given in references 1 to 6.

In a large number of technical applications the thermal conditions on the surface are nonuniform. These nonuniformities in thermal conditions may be grouped into two categories:

(1) The heat flux may be prescribed to vary over the surface. It is then of interest to calculate the resulting variation of the surface temperature.

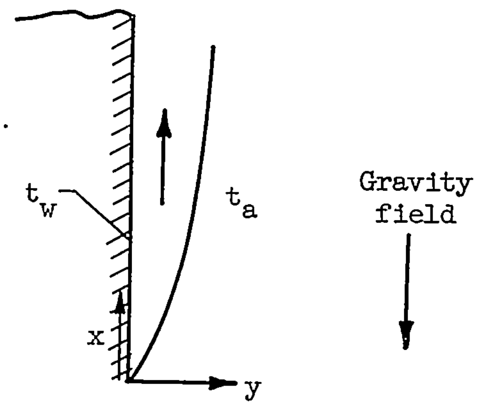
(2) The variation of the temperature on the surface may be prescribed. It is then of interest to calculate either the local rate of heat transfer at various locations on the surface, or the over-all rate of heat transfer from the surface, or both.

This report presents a first attempt at solution of the free-convection problem on a flat plate for these two categories of nonuniform thermal conditions at the surface.

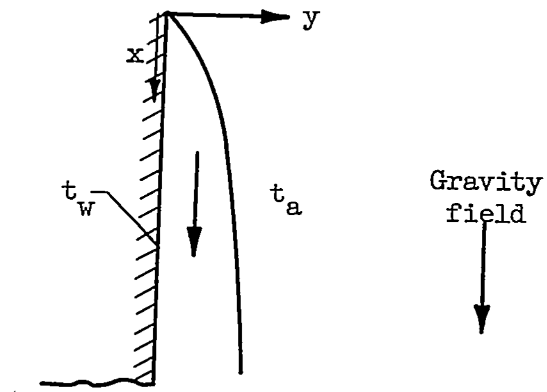
This analysis was made at the NACA Lewis laboratory.

GENERAL CONSIDERATIONS

The physical problem and the coordinate system are indicated in the following sketches, which show a vertical surface that may represent a flat plate or a vertical cylinder of large diameter:



Sketch (a). - Heat transfer from wall to fluid and $t_w > t_a$ everywhere.



Sketch (b). - Heat transfer from fluid to wall and $t_w < t_a$ everywhere

Sketch (a) depicts a situation in which the heat transfer is from the wall to the fluid at all points on the surface and the local wall temperature t_w everywhere exceeds the ambient temperature t_a. On the wall, either the heat flux may be designated to vary with x, or the wall temperature may be prescribed to vary with x. (No variations are considered in the direction of the z-coordinate, normal to the page; the problem is thus taken to be two dimensional.) The fluid in the neighborhood of the wall has a higher temperature and a lower density[1] than the fluid far from the wall. Thus, because of buoyancy, there will be established an upward flow of fluid in the neighborhood of the wall. The region of space in which the upward flow primarily occurs is called the velocity boundary layer. A thermal boundary layer is defined as that region of space where the temperature t deviates markedly from the ambient temperature t_a. In general, the velocity and thermal boundary layers have different thicknesses, the relative magnitudes depending upon the fluid properties. Both boundary layers are assumed to have zero thickness at the leading edge (x = 0). The velocity boundary layer is shown schematically in the sketch.

Sketch (b) shows a situation in which the heat transfer is from the fluid to the wall at all points of the surface and the local wall temperature t_w is everywhere less than t_a. Again, on the wall, either the heat flux or the temperature may be prescribed to vary with x. Here, the flow of fluid in the boundary layer is downward as shown.

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[1]This refers to fluids showing the usual trend of density decreasing with increasing temperature.

If the coordinate systems are taken as shown in the sketches, the method of analysis and the results for the heat-transfer parameters are the same for these two situations; and there will be no need to treat them separately. So, the analysis will be carried out for the case of heat transfer from the wall to the fluid ($t_w > t_a$), but it is to be remembered that the results apply to both situations depicted in the sketches.

A definite class of variations of the wall heat flux will be prescribed. Suppose that the region of interest on the vertical plate lies between $x = 0$ and another location $x = x_L$ (x_L must lie in the region of laminar flow over the plate). The wall heat fluxes to be considered here have a finite nonzero value at $x = 0$ and either increase steadily from $x = 0$ to $x = x_L$ or else decrease steadily from $x = 0$ to $x = x_L$.

The form of the wall-temperature variations considered here is similar to that outlined for the wall-heat-flux variations. The wall temperature relative to ambient has a finite nonzero value at $x = 0$ and either rises steadily from $x = 0$ to $x = x_L$ or else decreases steadily from $x = 0$ to $x = x_L$.

Although the analysis made here is for free convection in a gravity field, it may easily be generalized to include other force fields. It is only necessary to replace the gravitational force per unit mass g by the body force per unit mass of the other force field under consideration.

BASIC EQUATIONS

The equations expressing conservation of mass, momentum, and energy for steady laminar flow in a boundary layer on a vertical flat plate are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta(t - t_a) + \nu \frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2} \tag{3}$$

(All symbols are defined in appendix A.) In accordance with the usual practice in free convection, the density is considered a variable in formulating the buoyancy term $g\beta(t - t_a)$. Aside from this, the fluid properties are taken constant. Viscous dissipation and work against the gravity field are neglected.

Following reference 3, it is assumed that a common boundary-layer thickness δ can be used for both velocity and thermal boundary layers. This assumption has its justification in the fact that the results of calculations performed with it are in good agreement with those from exact solutions of the boundary-layer differential equations for the cases of uniform wall temperature and uniform heat flux.

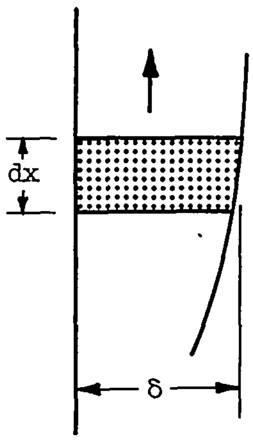
Then, equations (1) to (3) are integrated across the boundary layer to give

$$\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = g\beta \int_0^\delta (t - t_a)dy - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (2a)$$

$$\frac{d}{dx}\left[\int_0^\delta u(t - t_a)dy\right] = -\alpha\left(\frac{\partial t}{\partial y}\right)_{y=0} \qquad (3a)$$

The integrated form of equation (1) has been absorbed into equations (2a) and (3a).

These equations have a definite physical meaning. They are, in fact, expressions of the conservation laws for the element of boundary layer shown in the following sketch:



Sketch (c)

Solutions for the velocity and temperature in the boundary layer will be obtained which satisfy these conservation equations and the boundary conditions. These solutions will in turn be used to calculate the important heat-transfer parameters.

ANALYSIS

Prescribed Nonuniform Wall Heat Flux

The analysis proceeds with the use of the Kármán-Pohlhausen method, according to which the velocity and temperature distributions in the boundary layer are written as polynomials in y whose coefficients are functions of x. The coefficients are found from the boundary conditions of the problem and by using the integrated momentum and energy equations (eqs. (2a) and (3a)).

3430

The following polynomials are chosen:

$$t - t_a = \frac{q\delta}{2k}\left(1 - \frac{y}{\delta}\right)^2 \tag{4}$$

$$u = \omega \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \tag{5}$$

Equation (4) satisfies the conditions: $\frac{\partial t}{\partial y} = -\frac{q}{k}$ when $y = 0$; $t = t_a$ and $\frac{\partial t}{\partial y} = 0$ (smooth-fit condition) when $y = \delta$. The wall heat flux q is to be regarded as a specified function of x. The conditions satisfied by equation (5) are: $u = 0$ when $y = 0$; $u = 0$ and $\frac{\partial u}{\partial y} = 0$ (smooth-fit condition) when $y = \delta$. The functions $\omega(x)$ and $\delta(x)$ still remain to be determined.

The polynomials representing the velocity and temperature distributions are introduced into equations (2a) and (3a), and after the integration is carried out, there results a pair of first order, ordinary differential equations for ω and δ. In dimensionless form, these equations are

$$\frac{1}{105}\frac{d}{dX}\left(\Omega^2\Delta\right) = \frac{\Delta^2}{6}\frac{q}{q_o} - \frac{\Omega}{\Delta} \tag{6}$$

$$\frac{1}{30}\frac{d}{dX}\left(\Omega\Delta^2\frac{q}{q_o}\right) = \frac{2}{Pr}\frac{q}{q_o} \tag{7}$$

X, Ω, and Δ are the dimensionless counterparts of x, ω, and δ and are defined in the symbol list (appendix A). The symbol q_o, which represents the heat flux at the leading edge $(x = 0)$, is used as a reference heat flux in the rest of the analysis.

Next, the variation of q/q_o is to be prescribed. Suppose that the region of interest on the vertical plate lies between $x = 0$ and another location $x = x_L$. The surface heat fluxes to be considered here have finite nonzero values at $x = 0$ and either increase steadily from $x = 0$ to $x = x_L$ or else decrease steadily from $x = 0$ to $x = x_L$. Explicitly, the class of surface-heat-flux variations to be considered is written as

3430

$$\frac{q}{q_o} = 1 \pm \varepsilon\left(\frac{x}{x_L}\right)^r \tag{8}$$

The exponent r, which gives the shape of the variation, is required to be a positive number, integral or nonintegral. From equation (8), it may be seen that ε represents the maximum (percentage) deviation of q from q_o in the region $x = 0$ to $x = x_L$.

For the q/q_o given by equation (8), the pair of differential equations (6) and (7) can be solved by expanding Ω and Δ in Maclaurin series in terms of ε as follows:

$$\Omega(X,\varepsilon,Pr,r) = \sum_{n=0}^{\infty} \varepsilon^n \Omega_n(X,Pr,r) = \Omega_0(X,Pr,r) + \varepsilon\Omega_1(X,Pr,r) + \cdots \tag{9}$$

$$\Delta(X,\varepsilon,Pr,r) = \sum_{n=0}^{\infty} \varepsilon^n \Delta_n(X,Pr,r) = \Delta_0(X,Pr,r) + \varepsilon\Delta_1(X,Pr,r) + \cdots \tag{10}$$

The expressions for Ω, Δ and q/q_o are introduced into equations (6) and (7). Terms are grouped according to the power of ε that multiplies them, that is,

$$\varepsilon^0\left[\frac{1}{105}\frac{d}{dX}\left(\Omega_0^2\Delta_0\right) - \frac{\Delta_0^2}{6} + \frac{\Omega_0}{\Delta_0}\right] + \varepsilon^1\left[\quad\right] + \cdots = 0 \tag{11}$$

$$\varepsilon^0\left[\frac{1}{30}\frac{d}{dX}\left(\Omega_0\Delta_0^2\right) - \frac{2}{Pr}\right] + \varepsilon^1\left[\quad\right] + \cdots = 0 \tag{12}$$

In order that equations (11) and (12) be satisfied for any value of ε, each of the brackets must be identically zero. Equating to zero the brackets multiplying ε^0 yields a pair of simultaneous equations for Ω_0 and Δ_0. In a similar fashion, the brackets multiplying ε^n give a pair of equations for Ω_n and Δ_n. It may be noted that the equations for Ω_n and Δ_n will include all the Ω_0 through Ω_{n-1} and the Δ_0 through Δ_{n-1}.

The solution for Ω_0 and Δ_0, which coincides with the results for uniform wall heat flux ($\varepsilon = 0$), is treated more fully in appendix B. The succeeding Ω_n and Δ_n evidently give the effects of the nonuniformity of the heat flux.

The first five functions in the Maclaurin series (Ω_0 through Ω_4 and Δ_0 through Δ_4) have been computed. The results are listed in concise form as follows: 3430

$$\Delta_n = (\pm 1)^n \lambda_n(Pr,r)\left(\frac{X}{X_L}\right)^{nr} X^{1/5} \tag{13}$$

$$\Omega_n = (\pm 1)^n \gamma_n(Pr,r)\left(\frac{X}{X_L}\right)^{nr} X^{3/5} \tag{14}$$

The factors λ_n and γ_n were calculated from linear algebraic equations.

The results thus found are applied in a later section in which the heat-transfer parameters are obtained.

Prescribed Nonuniform Wall Temperature

The analysis for the prescribed-nonuniform-wall-temperature case proceeds in a similar fashion to that for the prescribed heat-flux case. The temperature and velocity profiles in the boundary layer are approximated by the following polynomials:

$$t - t_a = (t_w - t_a)\left(1 - \frac{y}{\delta}\right)^2 \equiv \theta\left(1 - \frac{y}{\delta}\right)^2 \tag{15}$$

$$u = \omega \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \tag{16}$$

Equation (15) satisfies the conditions: $t = t_w$ when $y = 0$; $t = t_a$ and $\frac{\partial t}{\partial y} = 0$ when $y = \delta$. The wall temperature t_w is to be regarded as a specified function of x. Since the ambient temperature t_a is taken to be constant, then θ is regarded as a prescribed function of x. The conditions satisfied by equation (16) are: $u = 0$ when $y = 0$; $u = 0$ and $\frac{\partial u}{\partial y} = 0$ when $y = \delta$.

The variation of the wall temperature t_w is prescribed in a region of interest between $x = 0$ and some other location $x = x_L$. The wall temperature relative to ambient considered here has a finite nonzero value at $x = 0$ and will either increase steadily from $x = 0$ to

3430

$x = x_L$ or else decrease steadily from $x = 0$ to $x = x_L$. The explicit form of the temperature variation considered here is

$$t_w - t_a = (t_w - t_a)_{x=0}\left[1 \pm \varepsilon\left(\frac{x}{x_L}\right)^r\right] \tag{17}$$

or

$$\theta = \theta_0\left[1 \pm \varepsilon\left(\frac{x}{x_L}\right)^r\right] \tag{17a}$$

Again, the exponent r is a positive number, integral or nonintegral. As may be seen from equation (17a), ε is the maximum (percentage) deviation of θ from θ_0 in the region $0 \leq x \leq x_L$.

CX-2

Equations (15) and (16) are introduced into the integrated momentum and energy equations (eqs. (2a) and (3a)). The resulting simultaneous equations for ω and δ are solved subject to the prescribed wall-temperature variation (eq. (17)) using Maclaurin series. The first five functions in the Maclaurin series have been calculated. The results have a form similar to those given in equations (13) and (14). The solution for the uniform-wall-temperature case coincides with the leading term of the Maclaurin series expansion.

HEAT-TRANSFER RESULTS

Prescribed Nonuniform Wall Heat Flux

For the case where the wall heat flux is prescribed, it is of interest to determine the resulting wall-temperature variation and local heat-transfer coefficients.

Resulting wall-temperature variation. - The wall temperature is obtained by setting y equal to 0 in equation (4). Thus,

$$t_w - t_a \doteq \frac{q\delta}{2k} \tag{18}$$

or

$$t_w - t_a = \left(\frac{q}{q_0}\right)\left(\frac{\delta}{\delta_0}\right)\frac{q_0\delta_0}{2k} = \left(\frac{q}{q_0}\right)\left(\frac{\Delta}{\Delta_0}\right)\frac{q_0\delta_0}{2k} \tag{18a}$$

For a uniform wall heat flux $q = q_0$, it has already been noted that $\Delta = \Delta_0$; so,

$$\frac{q_0\delta_0}{2k} \equiv (t_w - t_a)_{q_0} \tag{19}$$

where $(t_w - t_a)_{q_0}$ is the wall-temperature variation corresponding to a uniform heat flux $q = q_0$. In equation (18a), Δ/Δ_0 is evaluated from equations (8), (10), and (13), and $q_0\delta_0/2k$ is evaluated from equation (19), giving the result

$$\frac{t_w - t_a}{(t_w - t_a)_{q_0}} = \left(\frac{q}{q_0}\right)\sum_{n=0}^{\infty}\frac{\lambda_n}{\lambda_0}\left(\frac{q}{q_0} - 1\right)^n \tag{20}$$

3430

This equation gives the ratio of the wall temperature[2] at some location x ($x \leqslant x_L$) having a specified ratio q/q_0 to the wall temperature at the same location on a plate having a uniform heat flux $q = q_0$. Since the λ's depend upon the Prandtl number and r, the temperature ratio given by equation (20) depends on q/q_0, Pr and r.

The ratio $\frac{(t_w - t_a)}{(t_w - t_a)_{q_0}}$ is plotted in figures 1(a) to (e). Each of the plots applies for a specific value of r. The values of $r = 0$, 1/2, 1, 2, and 3 have been used for the five plots. Results for other values of r may be obtained by replotting[3] the information given in figures 1(a) to (e) using r as the abscissa variable. On each of the plots, the temperature ratio is plotted against q/q_0 for the range $0.5 < q/q_0 < 1.5$ with Prandtl numbers between 0.01 and 1000 appearing as parameters on the curves. In cases where curves for different Prandtl numbers fall so close together as to make it impossible to plot them separately, one curve was used for the several Prandtl numbers.

Once the ratio $\frac{t_w - t_a}{(t_w - t_a)_{q_0}}$ has been determined from one of the figures (or by replotting the data given therein), the wall temperature $(t_w - t_a)$ can be found if an expression for $(t_w - t_a)_{q_0}$ is given. The expression for $(t_w - t_a)_{q_0}$, the wall temperature corresponding to the case of uniform heat flux $q = q_0$, is derived in appendix B by the Kármán-Pohlhausen method, and appears on figure 1.

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[2]"Relative to ambient" is to be understood every time the wall temperature is mentioned.

[3]$r = 0$ is a limiting case, which is included here to facilitate the replotting when results for r near zero are required.

3430

Local heat-transfer coefficient. - The definition of the local heat-transfer coefficient is

$$h_x = \frac{q}{t_w - t_a} \tag{21}$$

For the case of a uniform heat flux $q = q_o$, equation (21) is

$$h_{x,q_o} = \frac{q_o}{(t_w - t_a)_{q_o}} \tag{21a}$$

The ratio of equation (21) to equation (21a) is

$$\frac{h_x}{h_{x,q_o}} = \frac{q}{q_o} \frac{(t_w - t_a)_{q_o}}{(t_w - t_a)} \tag{22}$$

CX-2 back

This equation gives the ratio of the local coefficient at some position x having a specified ratio q/q_o to the local coefficient at the same position on a plate having a uniform wall heat flux $q = q_o$. When q/q_o, Pr and r are known, the temperature ratio appearing in equation (22) is known (from fig. 1); hence, $h_x/h_{x,q_o}$ can be calculated. The expression h_{x,q_o} is given in appendix B.

Prescribed Nonuniform Wall Temperature

For the case where the wall temperature is specified, it is of interest to calculate the over-all heat-transfer rate and local heat-transfer coefficients.

Over-all heat-transfer rate. - The over-all rate of heat transfer in a section of plate of width b from $x = 0$ to another location $x(x \leqslant x_L)$ is

$$Q = b \int_0^x q \, dx = -kb \int_0^x \left(\frac{\partial t}{\partial y}\right)_{y=0} dx \tag{23}$$

The derivative $(\partial t/\partial y)_{y=0}$ is evaluated from equation (15). Substitution into equation (23) gives

$$Q = 2kb \int_0^x \frac{(t_w - t_a)}{\delta} dx = 2kb \int_0^x \frac{\theta}{\delta} dx \tag{24}$$

For the case of uniform wall temperature $\theta = \theta_o$, equation (24) becomes

$$Q_{\theta_o} = 2kb\theta_o \int_0^x \frac{dx}{\delta_0} \tag{25}$$

The ratio of equation (24) to equation (25) is

$$\frac{Q}{Q_{\theta_o}} = \frac{\int_0^x \frac{\theta}{\theta_o} \frac{dx}{\delta}}{\int_0^x \frac{dx}{\delta_0}} \tag{26}$$

3430

Equation (26) is the ratio of the over-all heat transfer from $x = 0$ to x $(x < x_L)$ on a plate with variable surface temperature to the heat transferred in the same region on a plate with a uniform wall temperature $\theta = \theta_o$. Since δ corresponding to the prescribed wall-temperature variation (eq. (17)) has been calculated in the Analysis section, it is possible to evaluate the integrals appearing in equation (26).

The results for Q/Q_{θ_o}, which are shown in figures 2(a) to (e), depend upon the value of θ/θ_o at the location x, upon Prandtl number, and upon r. Each plot is for a specific value of r. The values of 0, 1/2, 1, 2, and 3 have been used for the five plots. Results for other values of r may be obtained by replotting the information given in figures 2(a) to (e) with r as the abscissa variable. On each plot, the ratio Q/Q_{θ_o} is plotted against θ/θ_o for the range $0.7 < \theta/\theta_o < 1.3$, with Prandtl numbers between 0.01 and 1000 appearing as a parameter on the curves.

It is to be emphasized that the value of θ/θ_o at x is to be used for determining values from the figure when the over-all heat transfer from $x = 0$ to x is required. The quantity Q_{θ_o}, which is the over-all heat-transfer rate from $x = 0$ to x on a plate having uniform surface temperature $\theta = \theta_o$, is calculated in reference 3 by the Kármán-Pohlhausen method. The expression for Q_{θ_o} is given in figure 2.

Local heat-transfer coefficient. - From the defining equation (eq. (21)), the local coefficient is found to be

$$h_x = \frac{q}{(t_w - t_a)} = \frac{-k\left(\frac{\partial t}{\partial y}\right)_{y=0}}{t_w - t_a} = \frac{2k}{\delta} \tag{27}$$

The derivative is obtained from equation (15). For uniform wall temperature $\theta = \theta_o$, $\delta = \delta_0$; therefore

$$\frac{2k}{\delta_0} \equiv h_{x,\theta_o} \tag{28}$$

where h_{x,θ_o} is the local coefficient for the case of a uniform wall temperature $\theta = \theta_o$. Combining equations (27) and (28) gives

3430

$$\frac{h_x}{h_{x,\theta_o}} = \frac{\delta_0}{\delta} \tag{29}$$

This ratio is plotted in figures 3(a) to (e) in the manner already outlined for the preceding results of the analysis. The expression for h_{x,θ_o} (from ref. 3) is given on the figure.

DISCUSSION

The following generalizations can be made from inspection of the results:

(1) At a location x with a fixed q/q_o, the deviation of $\frac{t_w - t_a}{(t_w - t_a)_{q_o}}$ from 1 increases with decreasing r for a fixed Prandtl number. Also, the deviation from 1 is larger for larger Prandtl numbers at a fixed r and q/q_o.

(2) In a length of plate between $x = 0$ and x, having a fixed θ/θ_o at x, it is seen that the deviation of Q/Q_{θ_o} from 1 increases with decreasing r at a fixed Prandtl number. Also, the deviation from 1 is smaller for the larger Prandtl numbers at a fixed r and θ/θ_o.

(3) The deviation of $h_x/h_{x,q_o}$ from 1 increases with increasing r at a fixed Prandtl number and a fixed q/q_o at x. Also, the deviation from 1 is smaller for the larger Prandtl numbers at a fixed r and q/q_o.

(4) The deviation of $h_x/h_{x,\theta_o}$ from 1 increases with increasing r at a fixed Prandtl number and a fixed θ/θ_o at x. Also, the deviation from 1 is smaller for the larger Prandtl numbers at a fixed r and θ/θ_o.

It has already been noted that the temperature ratio $\frac{t_w - t_a}{(t_w - t_a)_{q_o}}$ was plotted in figure 1 for $0.5 \leq q/q_o \leq 1.5$. This range was decided upon by studying the extent of the errors due to truncation of the infinite series in equation (20) after $n = 4$. The range $0.7 \leq \theta/\theta_o \leq 1.3$ used in figures 2 and 3 was decided upon by studying the truncation errors of series associated with equations (26) and (29).

3430

CONCLUDING REMARKS

When the nonuniform wall heat flux can be written in the form of equation (8), the resulting wall temperatures may be found directly from figure 1, and the local heat-transfer coefficient can be calculated from equation (22). All that is needed to use the graphs is the value of q/q_o at the point of interest, the exponent r which gives the shape of the q/q_o variation, and the Prandtl number. It is to be noted that ε is not needed.

When the wall temperature is specified by a relation of the form of equation (17), the over-all heat-transfer rate and local coefficients are found from figures 2 and 3, respectively. The over-all heat-transfer rate for a section of plate from $x = 0$ to x $(x \leq x_L)$ may be found from the graphs when the following are known: θ/θ_o at x, the exponent r which gives the shape of the θ/θ_o variation, and the Prandtl number. The same quantities are needed to find the local heat-transfer coefficient at x.

It is recognized that whenever a new application of the Kármán-Pohlhausen method is made, it is desirable to confirm the results by checking with those of experiment or of a less approximate analysis.

The author is not acquainted with any experimental data that may be used to check the results derived here for nonuniform thermal conditions at the surface. Nor is there now available any other analysis with which the present results may be compared.

For the special cases of uniform wall temperature and uniform heat flux, there are exact solutions of the laminar-boundary-layer equations as well as experimental data. For these cases, the heat-transfer results derived from the Kármán-Pohlhausen method agree well with those from the exact solutions and those of experiment.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, May 4, 1955

APPENDIX A

SYMBOLS

The following symbols are used in this report:

3430

b	width of plate, ft
c_p	specific heat at constant pressure, Btu/(lb)(°F)
g	acceleration due to gravity, ft/sec^2
Gr_x^*	modified Grashof number based on x, $\frac{g\beta q_o x^4}{k\nu^2}$, dimensionless
Gr_x	Grashof number based on x, $\frac{g\beta\theta_o x^3}{\nu^2}$, dimensionless
h_x	local heat-transfer coefficient, Btu/(sec)(sq ft)(°F)
k	thermal conductivity, Btu/(sec)(ft)(°F)
n	index for naming terms of a series, dimensionless
Pr	Prandtl number, $\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$, dimensionless
Q	over-all heat-transfer rate on a plate of width b between $x = 0$ and x $(x \leqslant x_L)$, Btu/sec
q	local heat-transfer rate per unit area, Btu/(sec)(sq ft)
q_o	heat-transfer rate per unit area at $x = 0$, Btu/(sec)(sq ft)
r	exponent defined by eqs. (8) and (17), dimensionless
t	static temperature, °F
u	velocity component in x-direction, ft/sec
v	velocity component in y-direction, ft/sec
x	coordinate measuring distance along plate from leading edge, ft
x_L	coordinate defining region of interest $x = 0$ to $x = x_L$, ft
X	dimensionless coordinate, $\left(\frac{g\beta q_o}{k\nu^2}\right)^{1/4} x$

3430

y	coordinate measuring normal distance from plate, ft
α	thermal diffusivity, $\frac{k}{\rho c_p}$, sq ft/sec
β	coefficient of thermal expansion, $(^{\circ}F)^{-1}$
$\gamma_0,\gamma_1,\cdots\gamma_n$	factors in solutions for $\Omega_0,\Omega_1,\cdots\Omega_n$ defined by eq. (14), dimensionless
Δ	dimensionless boundary-layer thickness for prescribed-heat-flux case, $\left(\frac{g\beta q_o}{k\nu^2}\right)^{1/4}\delta$
$\Delta_0,\Delta_1,\cdots\Delta_n$	coefficients in Maclaurin series expansion of Δ in terms of ε, dimensionless
δ	boundary-layer thickness, ft
δ_0	boundary-layer thickness for uniform heat flux or uniform wall temperature, ft
ε	number giving percentage deviation of q at $x = x_L$ from q_o, or number giving percentage deviation of θ at $x = x_L$ from θ_o, dimensionless
θ	wall- to ambient-temperature difference, $t_w - t_a$, $^{\circ}F$
θ_o	wall-to ambient-temperature difference at $x = 0$, $^{\circ}F$
$\lambda_0,\lambda_1,\cdots\lambda_n$	factors in solutions for $\Delta_0,\Delta_1,\cdots\Delta_n$ defined by eq. (13), dimensionless
μ	absolute viscosity, lb/(sec)(ft)
ν	kinematic viscosity, sq ft/sec
ρ	density, lb/cu ft
ω	velocity function defined by eqs. (5) and (16), ft/sec
Ω	dimensionless velocity function, $\left(\frac{g\beta q_o\nu^2}{k}\right)^{-1/4}\omega$
$\Omega_0,\Omega_1,\cdots\Omega_n$	coefficients in Maclaurin series expansion of Ω in terms of ε, dimensionless

Subscripts:

a	ambient
q_o	on a plate having uniform heat flux q_o
w	wall
θ_o	on a plate having uniform wall- to ambient-temperature difference θ_o

APPENDIX B

RESULTS FOR UNIFORM WALL HEAT FLUX

When $q' = q_0$ for all values of x, equations (6) and (7) become

$$\frac{1}{105} \frac{d}{dX} \left(\Omega_0^2 \Delta_0\right) = \frac{\Delta_0^2}{6} - \frac{\Omega_0}{\Delta_0} \tag{B1}$$

3430

$$\frac{1}{30} \frac{d}{dX} \left(\Omega_0 \Delta_0^2\right) = \frac{2}{Pr} \tag{B2}$$

The solutions for Ω_0 and Δ_0 are

$$\Omega_0 = (6000)^{1/5} \, Pr^{-1/5} \left(\frac{4}{5} + Pr\right)^{-2/5} X^{3/5} \tag{B3}$$

$$\Delta_0 = (360)^{1/5} \left(\frac{4}{5} + Pr\right)^{1/5} Pr^{-2/5} \, X^{1/5} \tag{B4}$$

Equation (B4) can be rewritten as

$$\frac{\Delta_0}{X} = \frac{\delta_0}{x} = (360)^{1/5} \left[\frac{\left(\frac{4}{5} + Pr\right)}{Pr^2 \, Gr_x^*}\right]^{1/5} \tag{B5}$$

where Gr_x^* is a modified Grashof number based on x and defined by

$$Gr_x^* = \frac{g\beta q_0 x^4}{k\nu^2} \tag{B6}$$

The surface-temperature distribution found by introducing equation (B5) into equation (18) is

$$\left(t_w - t_a\right)_{q_0} = 1.622 \, \frac{q_0 x}{k} \left(\frac{0.8 + Pr}{Pr^2 \, Gr_x^*}\right)^{1/5} \tag{B7}$$

The local heat-transfer coefficient is obtained by the following rearrangement of equation (B7):

$$h_{x,q_0} = \frac{q_0}{\left(t_w - t_a\right)_{q_0}} = \frac{0.62 \, k}{x} \left(\frac{Pr^2 \, Gr_x^*}{0.8 + Pr}\right)^{1/5} \tag{B8}$$

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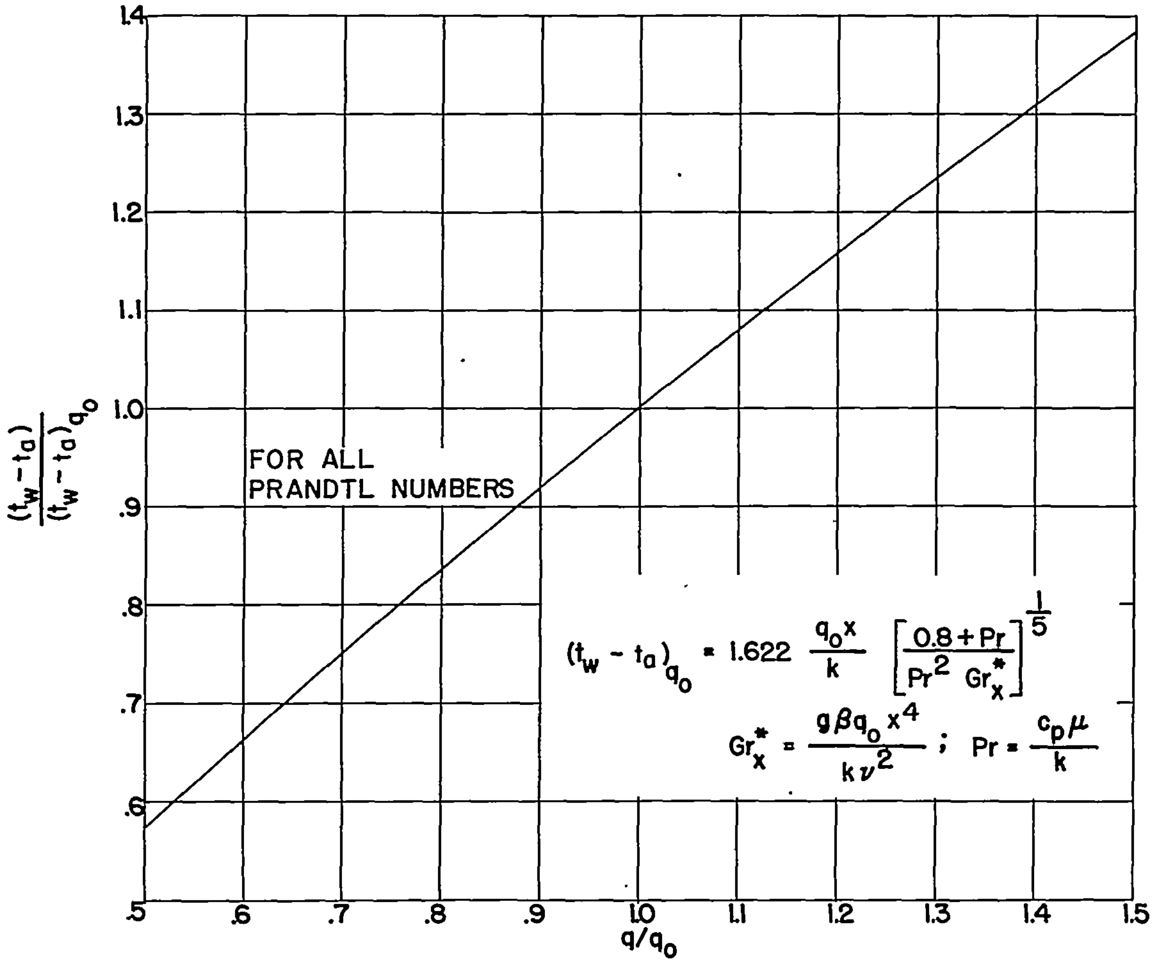
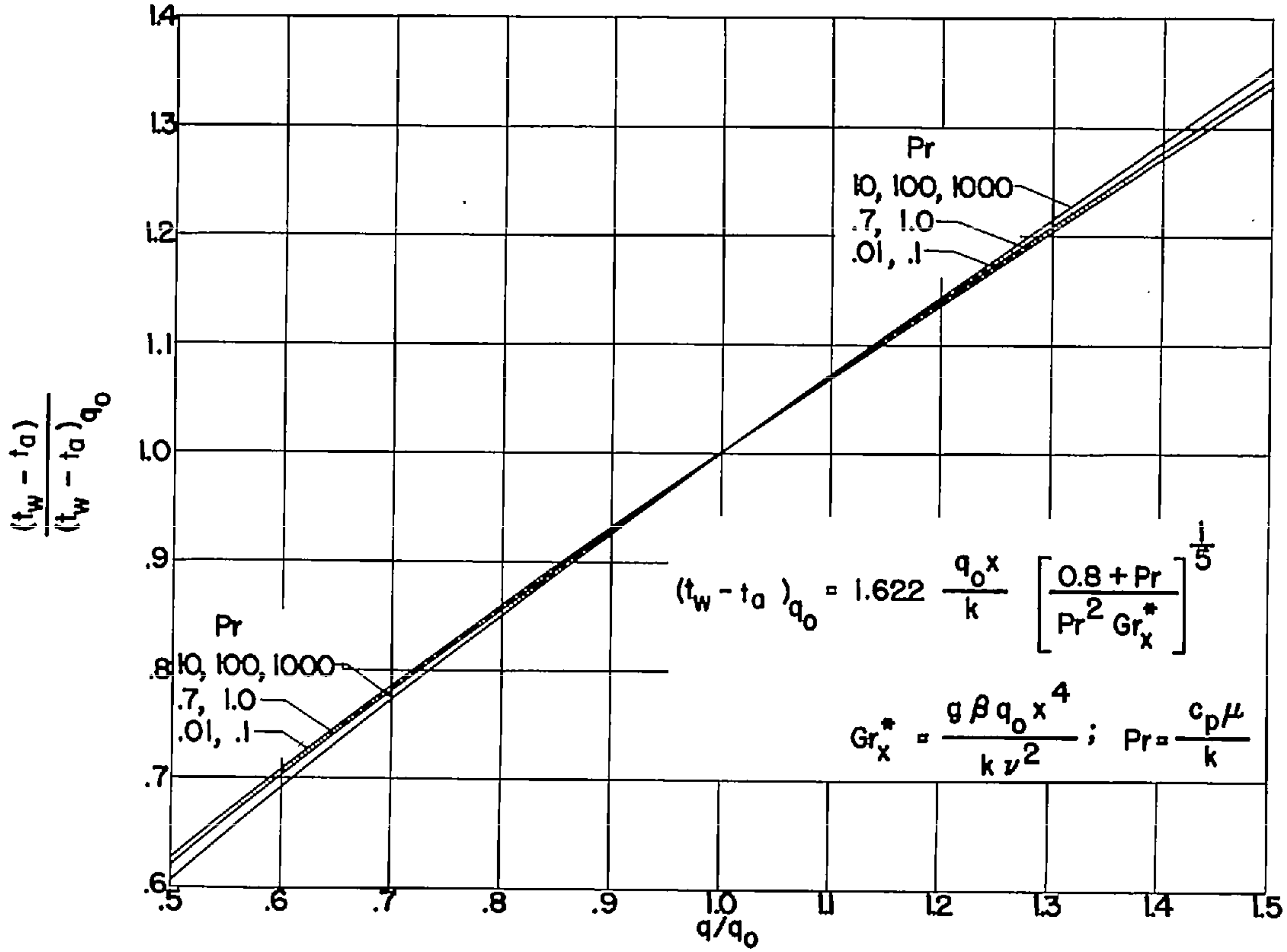


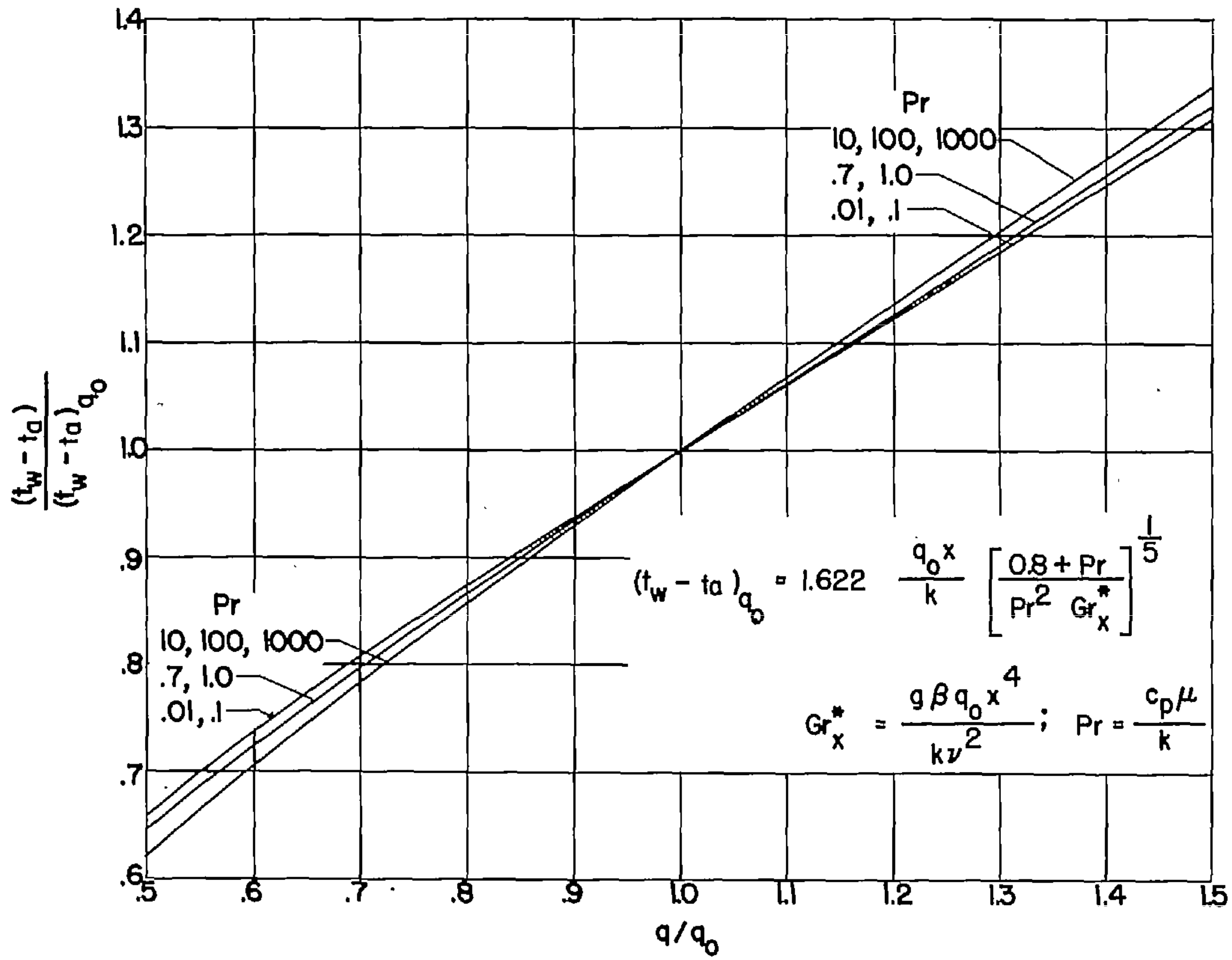
(a) r = 0.

Figure 1. - Ratio of wall temperature at some location x on plate with variable heat flux to wall temperature at same location on plate with uniform heat flux $q = q_o$. Abscissa is value of q/q_o at point of interest; r is an exponent in equation (8)(relation by which heat-flux variation is prescribed); Prandtl number is parameter on curves.



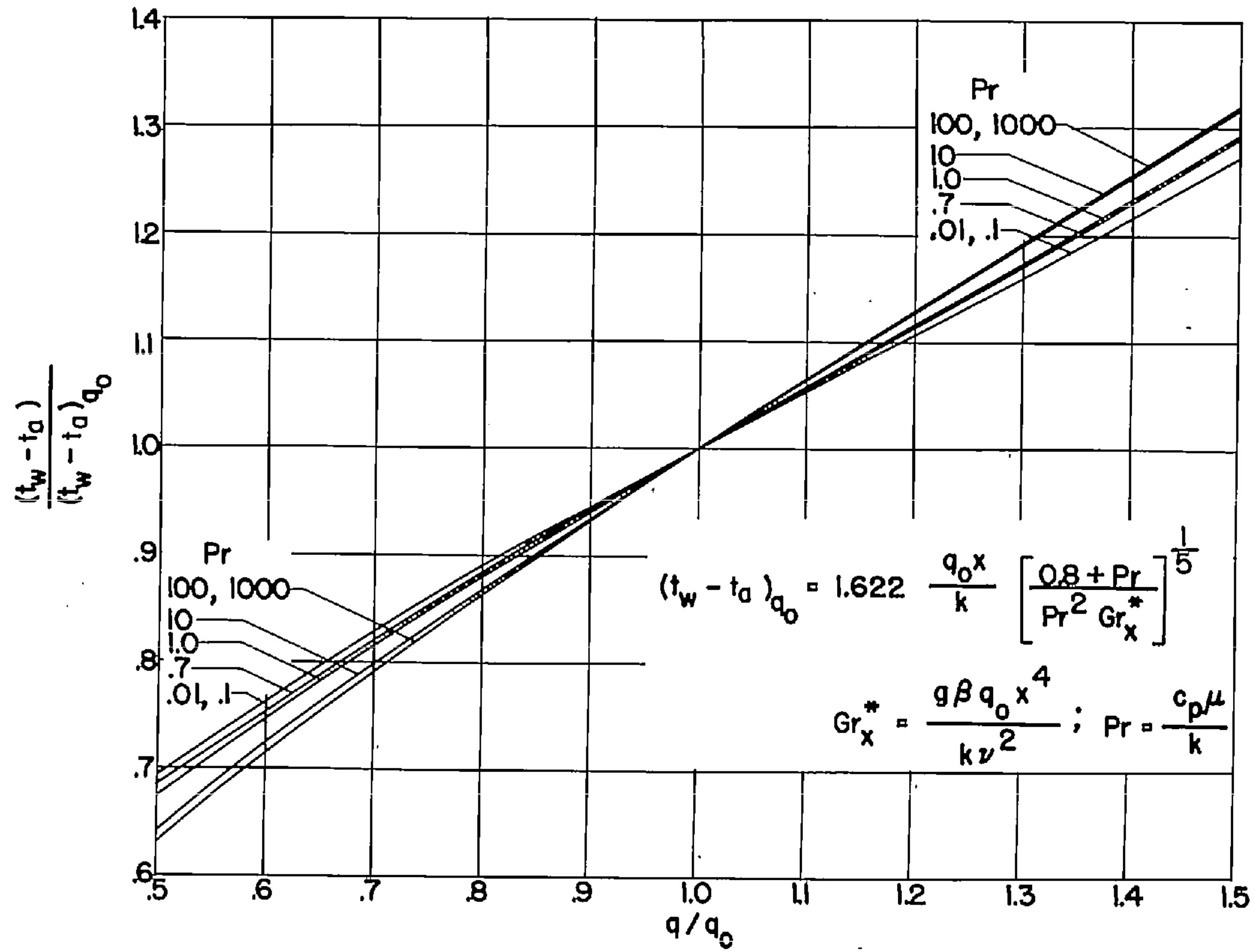
(b) $r = 1/2$.

Figure 1. - Continued. Ratio of wall temperature at some location x on plate with variable heat flux to wall temperature at same location on plate with uniform heat flux $q = q_o$. Abscissa is value of q/q_o at point of interest; r is an exponent in equation (8)(relation by which heat-flux variation is prescribed); Prandtl number is parameter on curves.



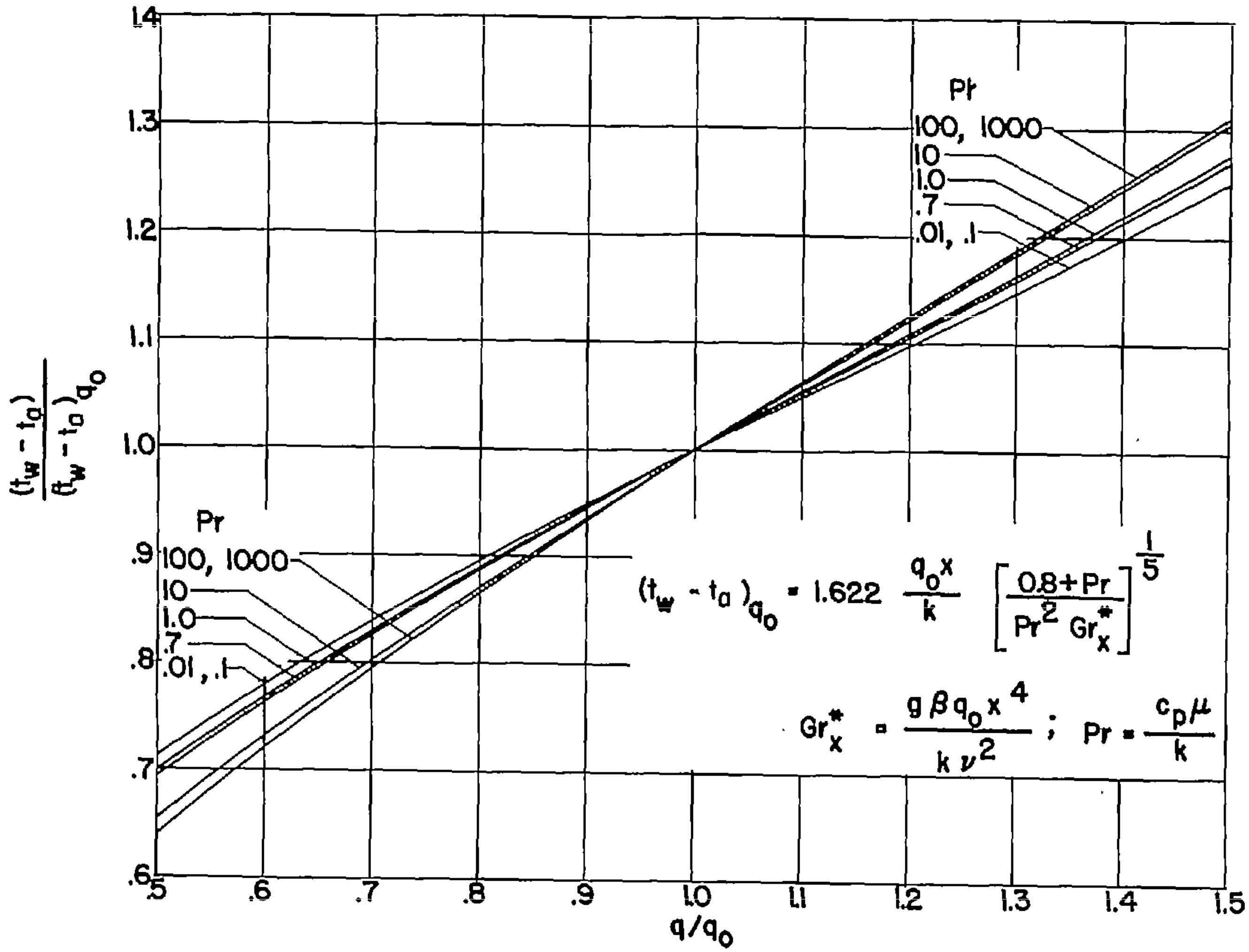
(c) r = 1.

Figure 1. - Continued. Ratio of wall temperature at some location x on plate with variable heat flux to wall temperature at same location on plate with uniform heat flux $q = q_o$. Abscissa is value of q/q_o at point of interest; r is an exponent in equation (8)(relation by which heat-flux variation is prescribed); Prandtl number is parameter on curves.



(d) r = 2.

Figure 1. - Continued. Ratio of wall temperature at some location x on plate with variable heat flux to wall temperature at same location on plate with uniform heat flux $q = q_o$. Abscissa is value of q/q_o at point of interest; r is an exponent in equation (8)(relation by which heat-flux variation is prescribed); Prandtl number is parameter on curves.



(e) r = 3.

Figure 1. - Concluded. Ratio of wall temperature at some location x on plate with variable heat flux to wall temperature at same location on plate with uniform heat flux $q = q_0$. Abscissa is value of q/q_0 at point of interest; r is an exponent in equation (8)(relation by which heat-flux variation is prescribed); Prandtl number is parameter on curves.

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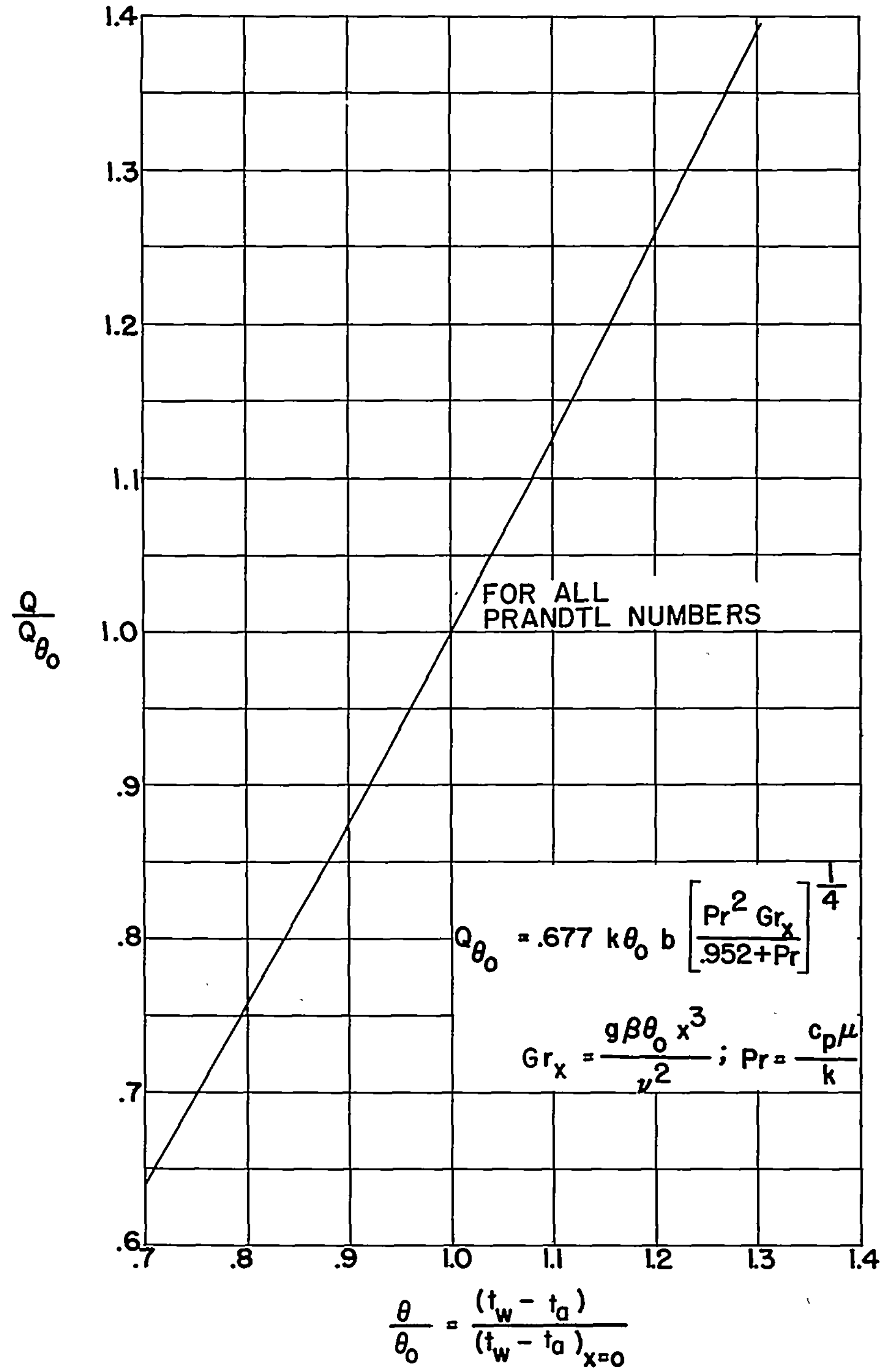
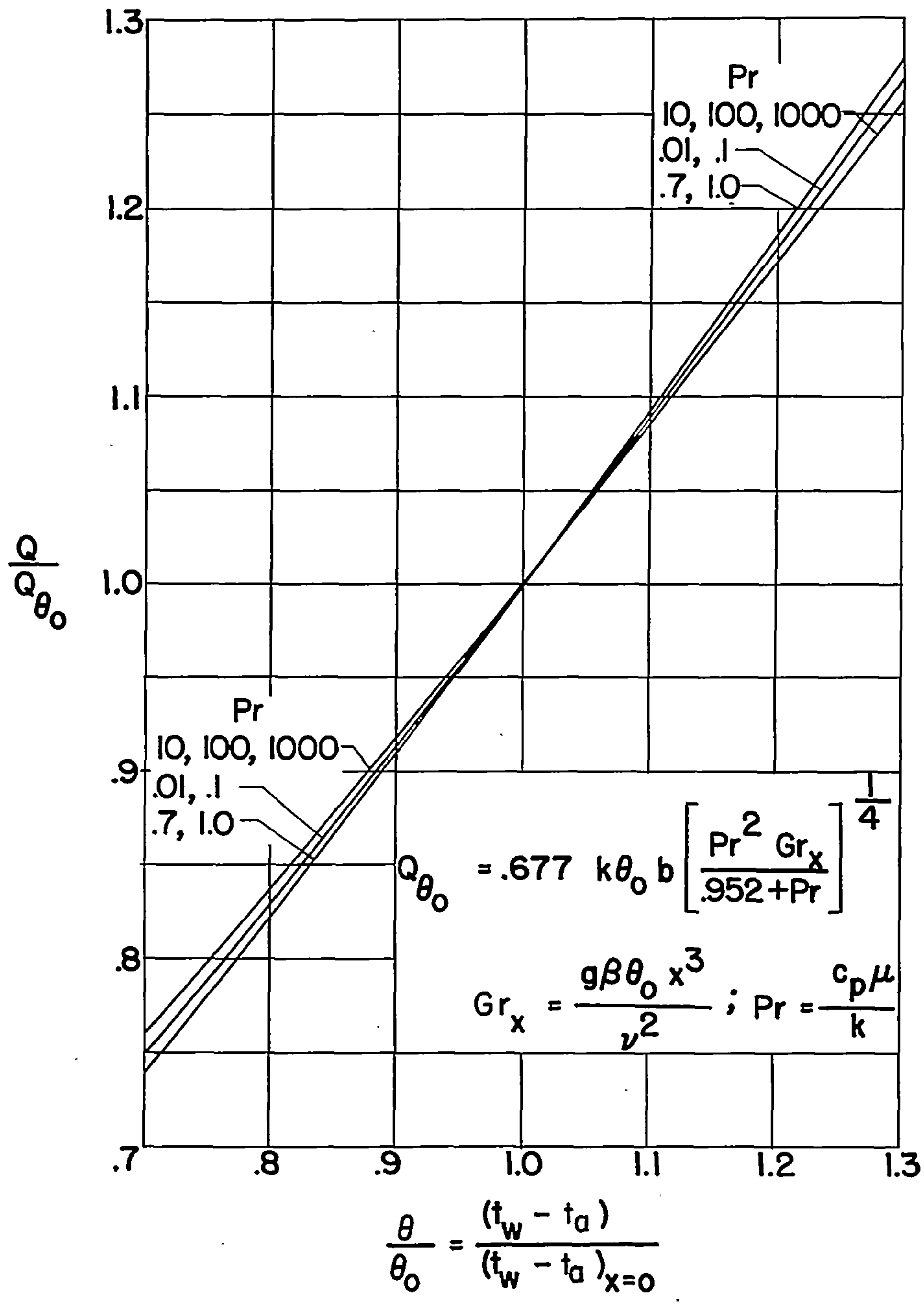
CX-4



(a) $r = 0$.

Figure 2. - Ratio of over-all heat transfer from $x = 0$ to x on plate with variable wall temperature to over-all heat transfer in same region on plate with uniform wall temperature $\theta = \theta_o$. Abscissa is value of θ/θ_o at x; r is an exponent in equation (17)(relation by wich wall-temperature variation is prescribed); Prandtl number is parameter on curves.



(b) $r = 1/2$.

Figure 2. - Continued. Ratio of over-all heat transfer from $x = 0$ to x on plate with variable wall temperature to over-all heat transfer in same region on plate with uniform wall temperature $\theta = \theta_o$. Abscissa is value of θ/θ_o at x; r is an exponent is equation (17) (relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.

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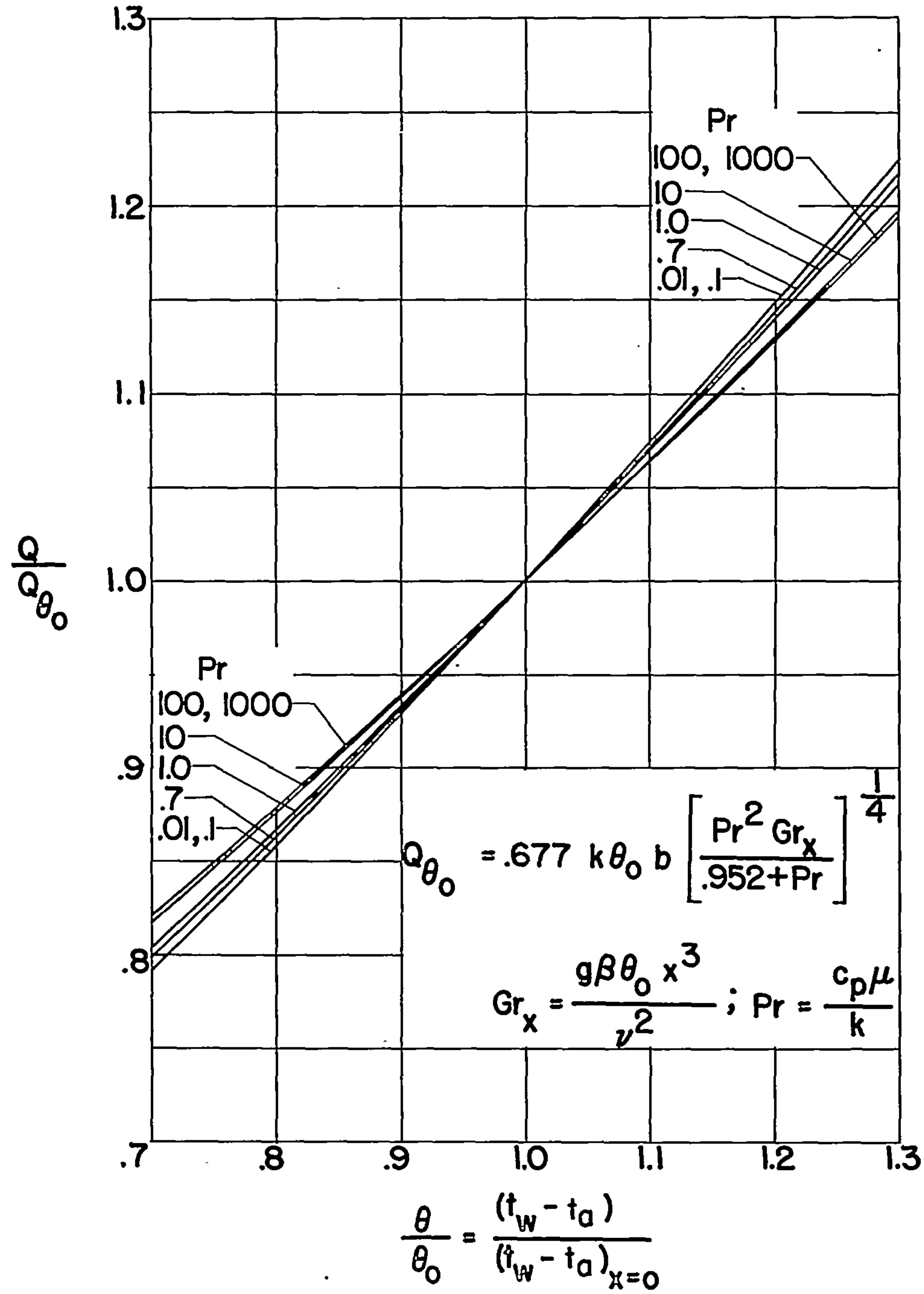
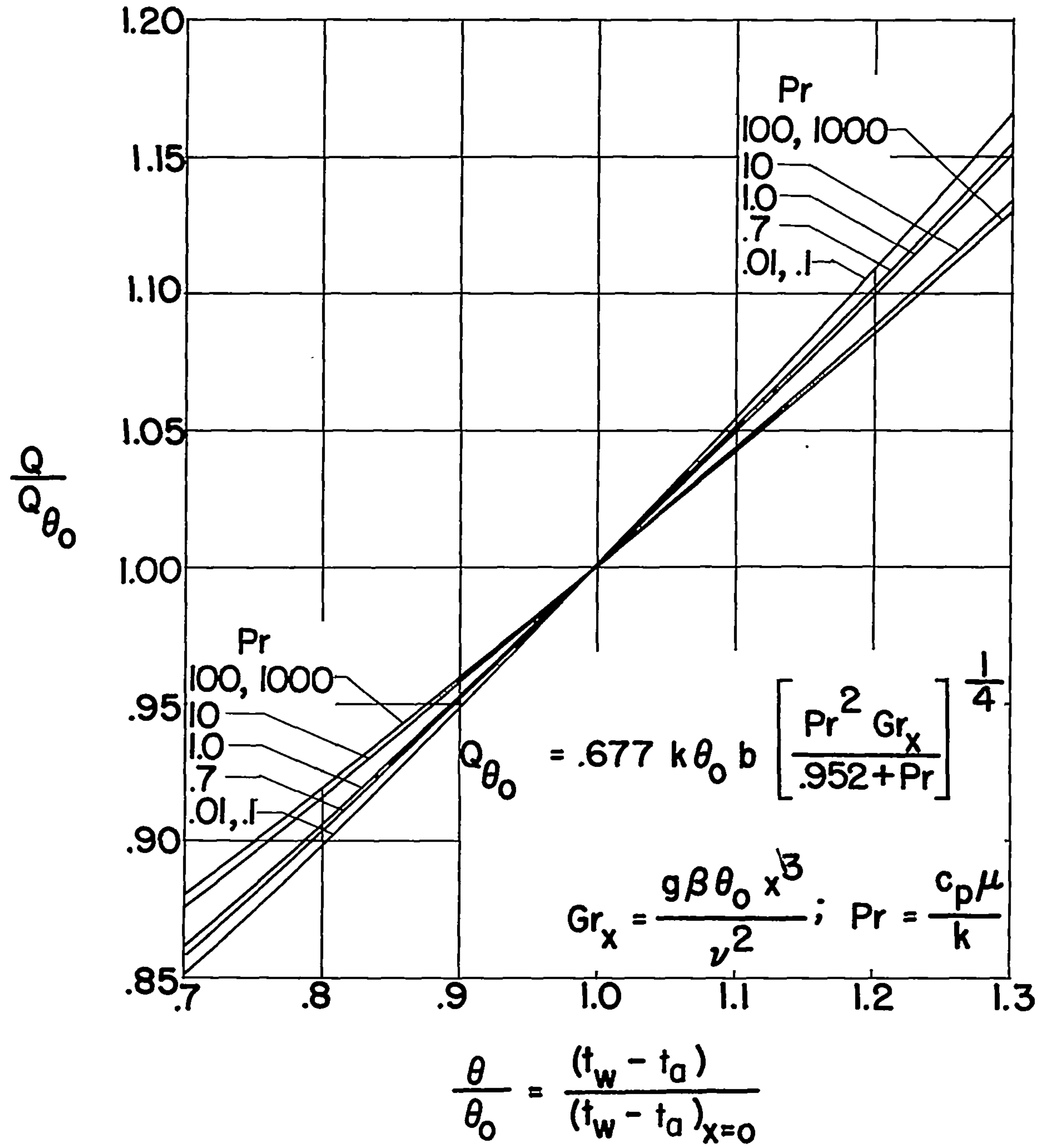
CX-4 back



(c) r = 1.

Figure 2. - Continued. Ratio of over-all heat transfer from $x = 0$ to x on plate with variable wall temperature to over-all heat transfer in same region on plate with uniform wall temperature $\theta = \theta_0$. Abscissa is value of θ/θ_0 at x; r is an exponent in equation (17) (relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.



(d) $r = 2$.

Figure 2. - Continued. Ratio of over-all heat transfer from $x = 0$ to x on plate with variable wall temperature to over-all heat transfer in same region on plate with uniform wall temperature $\theta = \theta_0$. Abscissa is value of θ/θ_0 at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.

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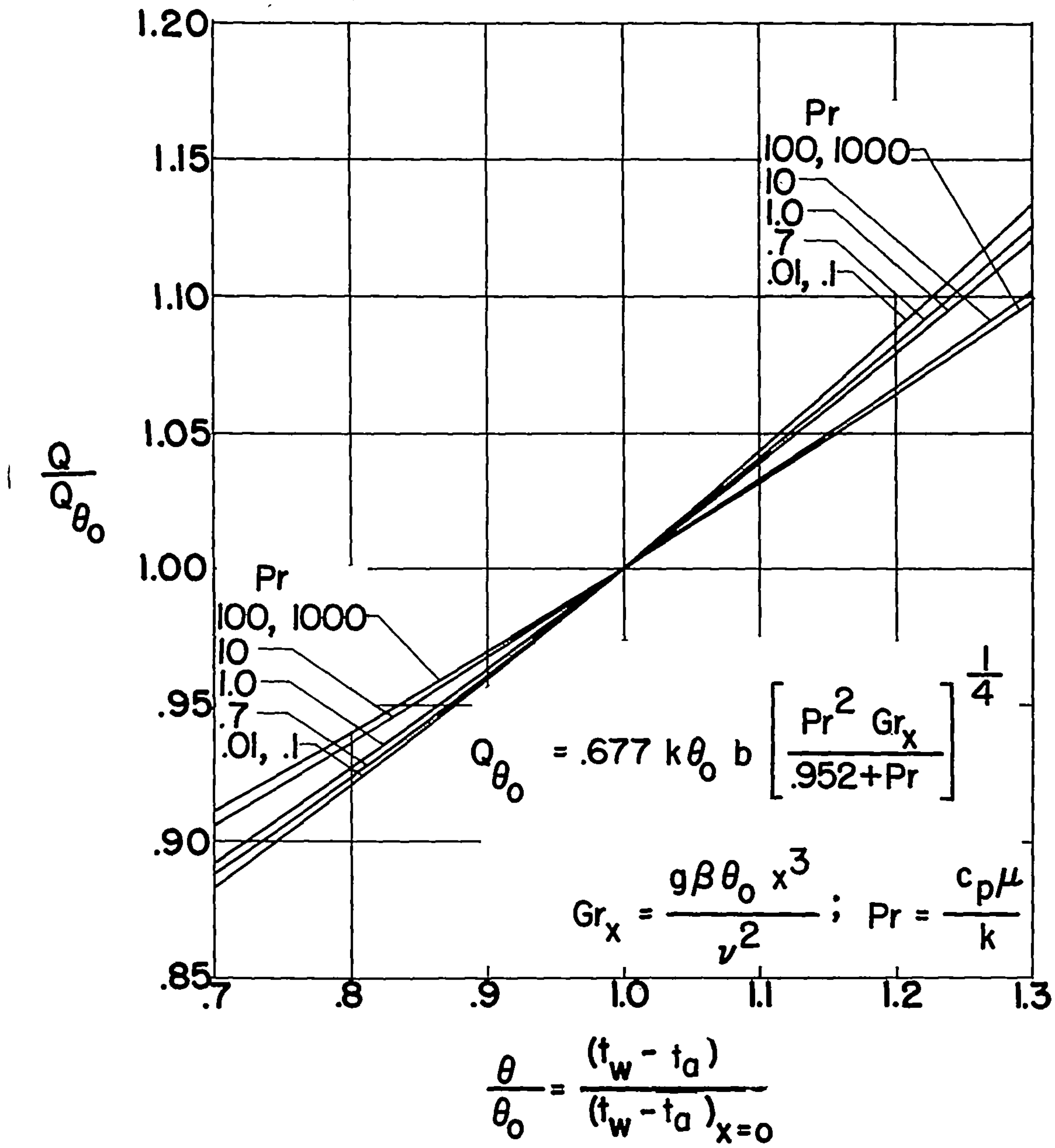
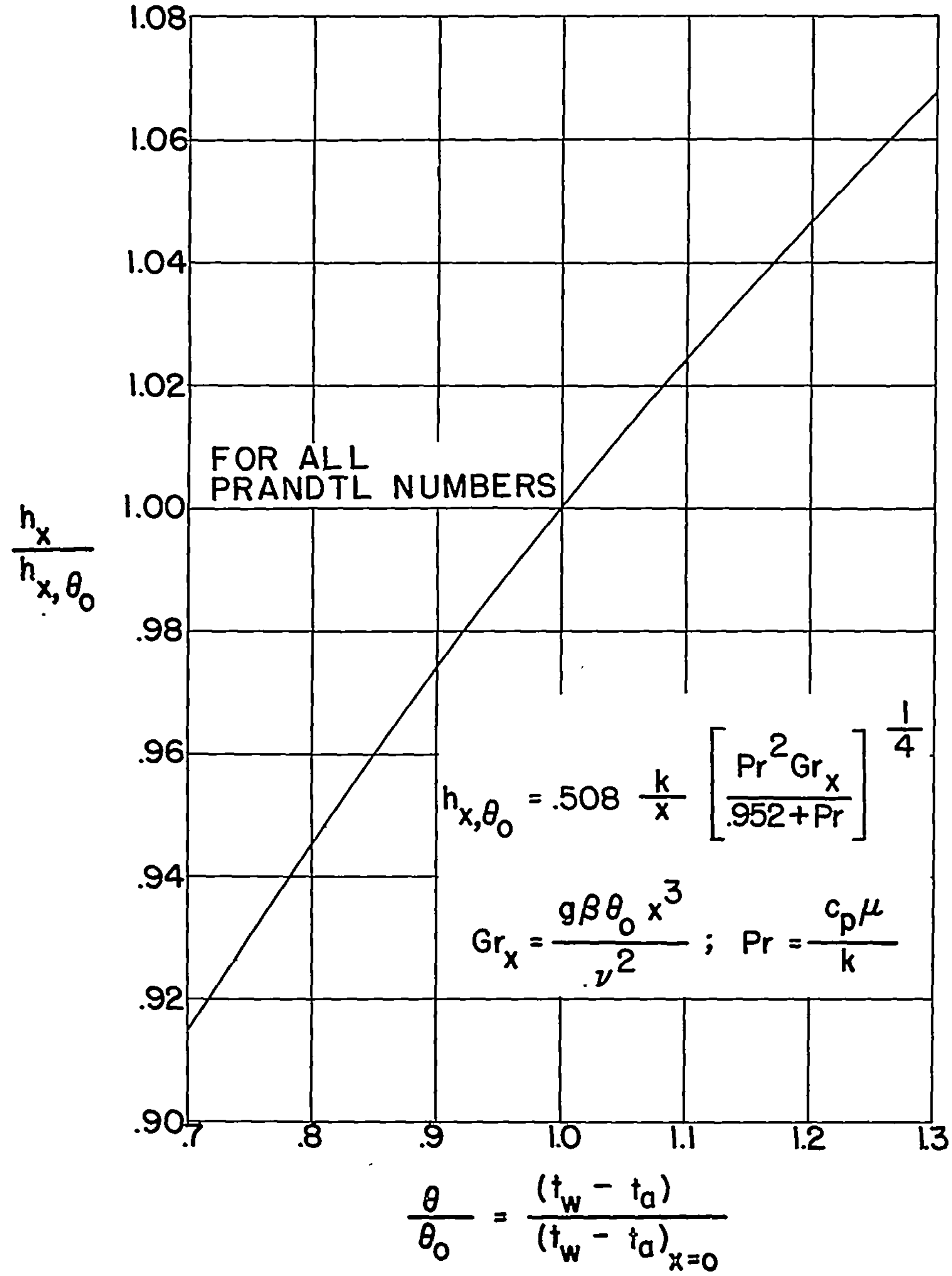


(e) r = 3.

Figure 2. - Concluded. Ratio of over-all heat transfer from x = 0 to x on plate with variable wall temperature to over-all heat transfer in same region on plate with uniform wall temperature ($\theta = \theta_o$). Abscissa is value of θ/θ_o at x; r is an exponent in equation (17) (relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.



(a) $r = 0$.

Figure 3. - Ratio of local heat-transfer coefficient at x for plate with variable wall temperature to local coefficient at same location on plate with uniform wall temperature $\theta = \theta_o$. Abscissa is value of θ/θ_o at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.

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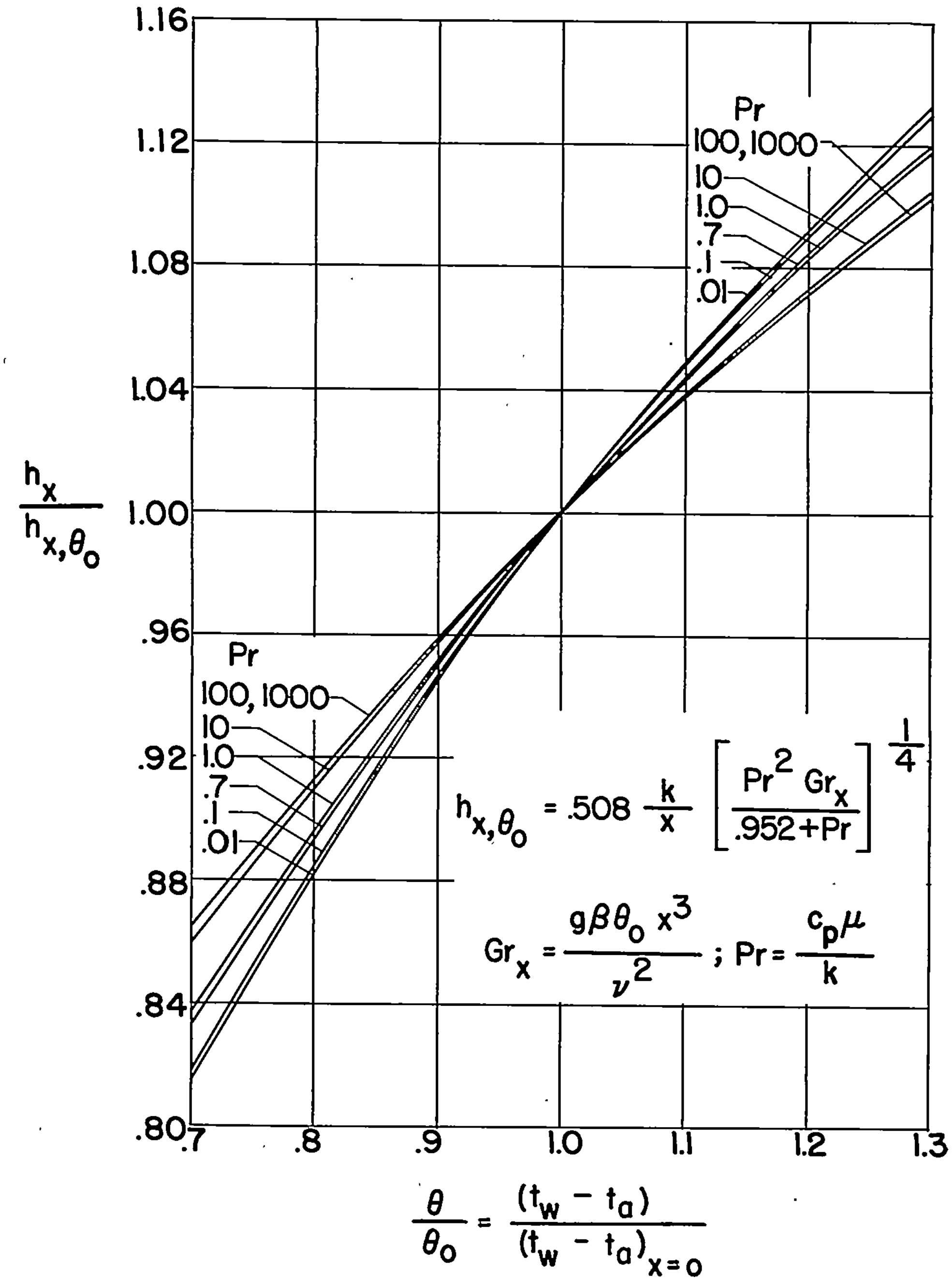
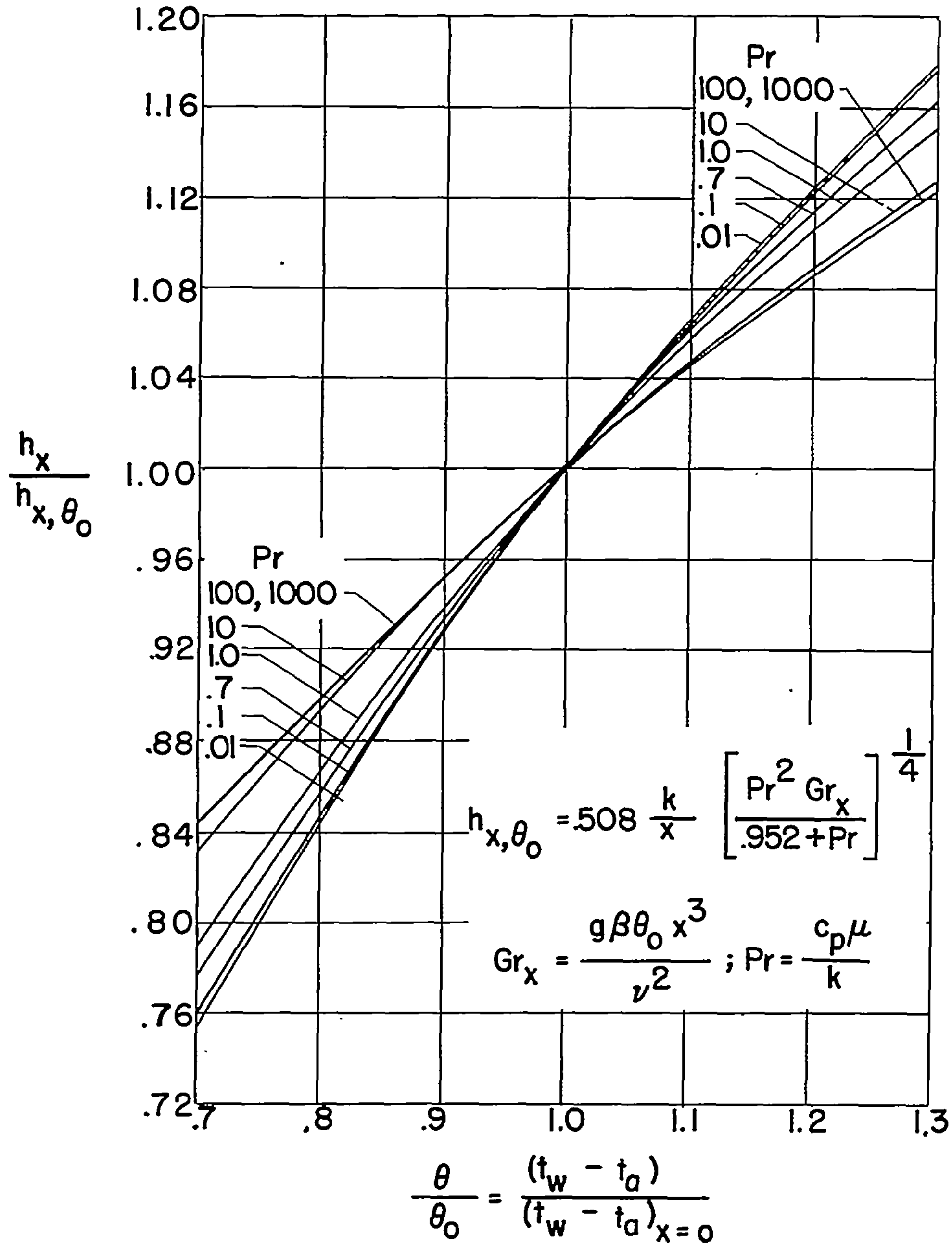


(b) r = 1/2.

Figure 3. - Continued. Ratio of local heat-transfer coefficient at x for plate with variable wall temperature to local coefficient at same location on plate with uniform wall temperature $\theta = \theta_0$. Abscissa is value of θ/θ_0 at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.



(c) r = 1.

Figure 3. - Continued. Ratio of local heat-transfer coefficient at x for plate with variable wall temperature to local coefficient at same location on plate with uniform wall temperature $\theta = \theta_o$. Abscissa is value of θ/θ_o at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.

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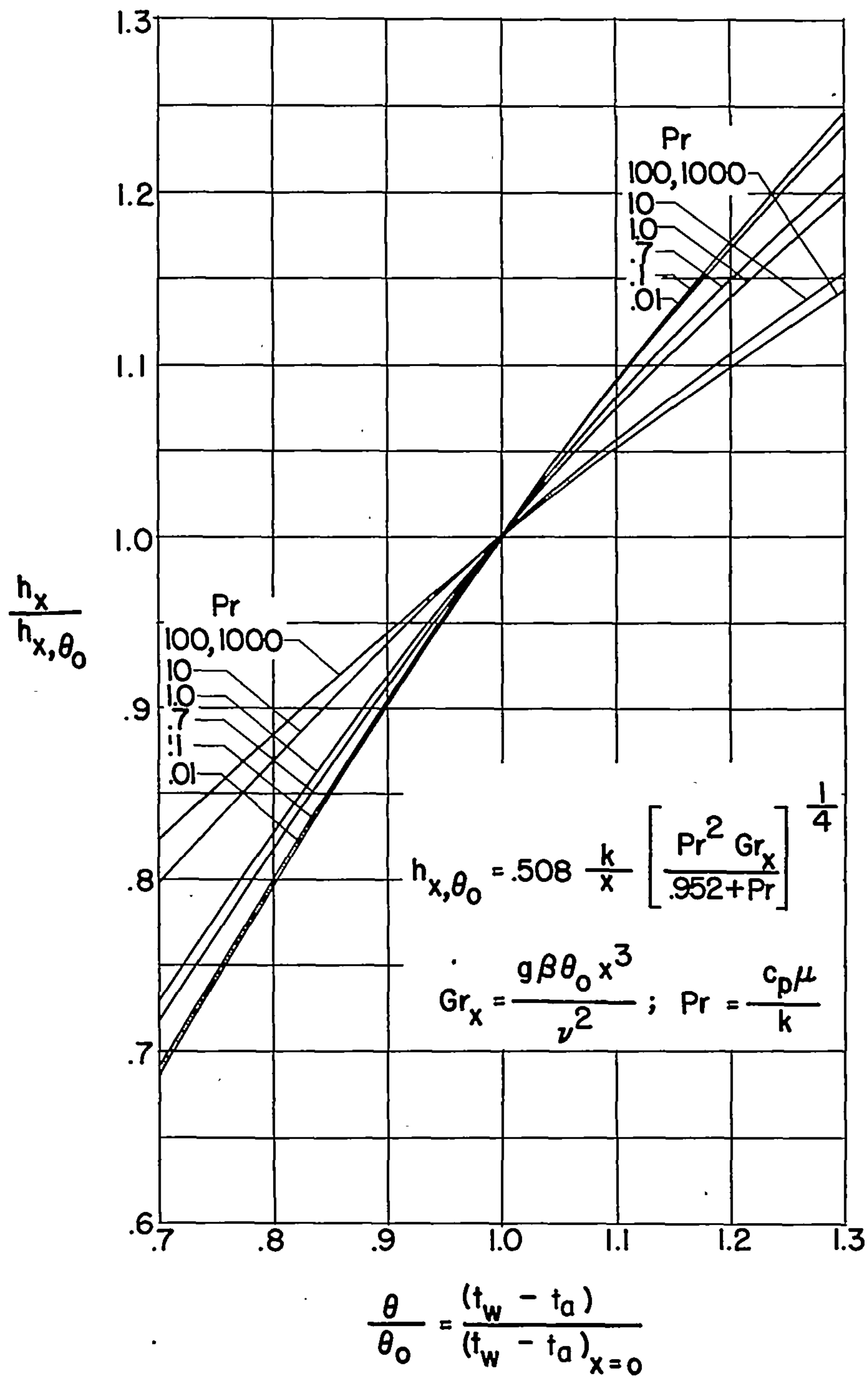
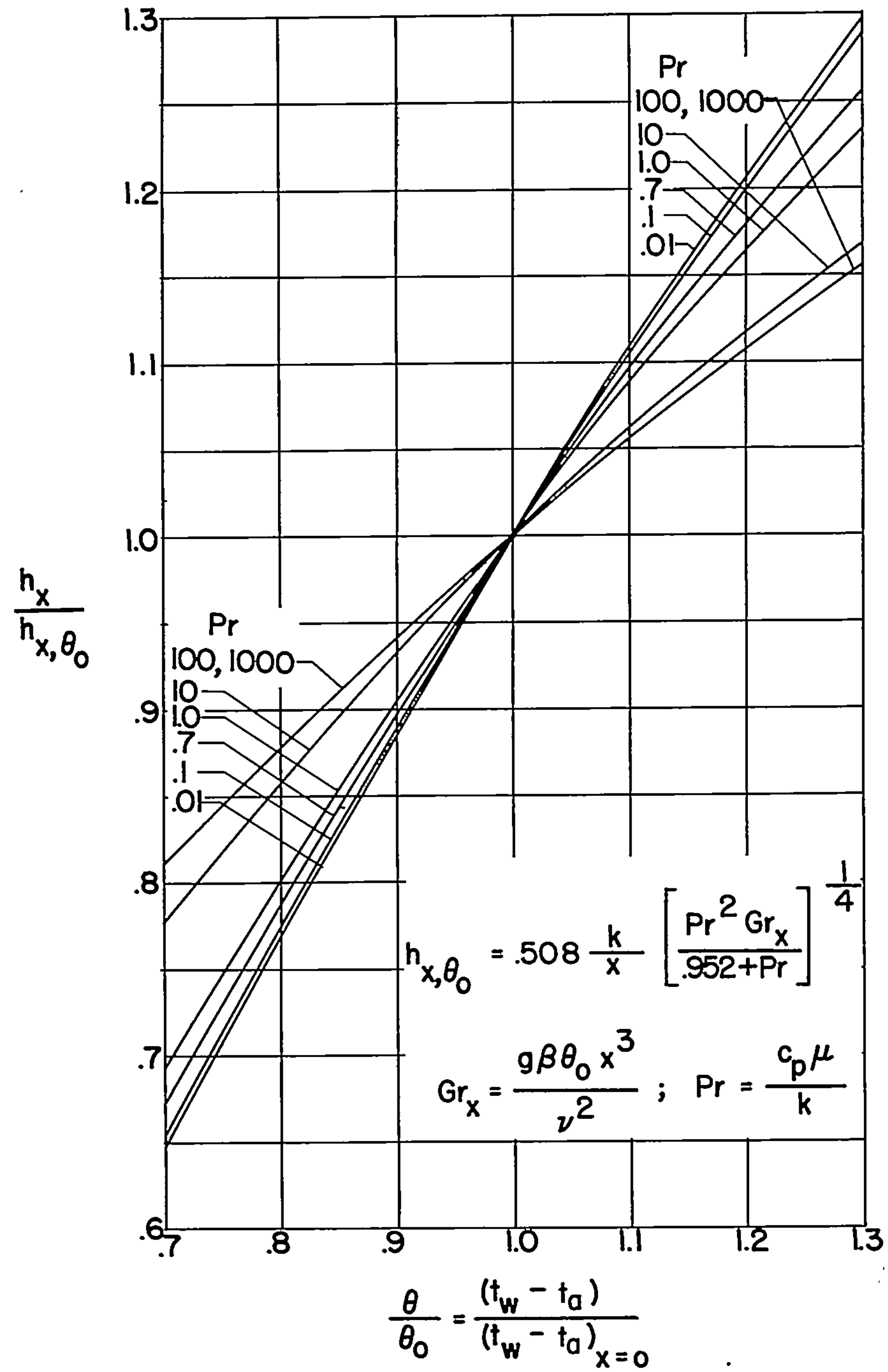


(d) r = 2.

Figure 3. - Continued. Ratio of local heat-transfer coefficient at x for plate with variable wall temperature to local coefficient at same location on plate with uniform wall temperature $\theta = \theta_0$. Abscissa is value of θ/θ_0 at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.



(e) r = 3.

Figure 3. - Concluded. Ratio of local heat-transfer coefficient at x for plate with variable wall temperature to local coefficient at same location on plate with uniform wall temperature $\theta = \theta_o$. Abscissa is value of θ/θ_o at x; r is an exponent in equation (17)(relation by which wall-temperature variation is prescribed); Prandtl number is parameter on curves.

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