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ASYMPTOTIC FORMULAE FOR THE DISTRIBUTION OF
HOTELLING'S GENERALIZED T_0^2 STATISTIC

by

Koichi Ito
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ASYMPTOTIC FORMULAE FOR THE DISTRIBUTION OF
HOTELLING'S GENERALIZED T_0^2 STATISTIC

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1. Summary.

In this paper the asymptotic expansion of a percentage point of Hotelling's generalized T_0^2 distribution is derived in terms of the corresponding percentage point of a χ^2 distribution. Our result generalizes Hotelling's and Frankel's asymptotic expansion for the generalized Student T [2], [3]. The technique used in this paper for obtaining the asymptotic expansion of T_0^2 is an extension of the previous methods of Welch [7] and of James [4], [5], who used them to solve the distribution problem of various statistics in connection with the Behrens-Fisher problem. An asymptotic formula for the cumulative distribution function (c.d.f.) of T_0^2 is also given together with an upper bound for the error committed when all but the first few terms are omitted in the series. This formula is a sort of multivariate analogue of Hartley's formula of "Studentization" [1].

2. Introduction.

In the multivariate analysis of variance we use the following canonical probability law:

$$(2.1) \quad P(X_0, X_1) = \text{const.} \exp \left[-\frac{1}{2} \text{tr} \Lambda (X_1 - \xi)(X_1 - \xi)' - \frac{1}{2} \text{tr} \Lambda X_0 X_0' \right] dX_0 dX_1,$$

where X_1 and X_0 are $p \times m$ and $p \times n$ matrices respectively, and $\frac{1}{m} X_1 X_1' = S_1$ is the sample "between" dispersion matrix and $\frac{1}{n} X_0 X_0' = S_0$ is the sample "within" dispersion matrix, the prime denoting the transpose of a matrix. ξ is a $p \times m$ matrix, $\frac{1}{m} \xi \xi'$ being the population "between" dispersion matrix, and Λ is a $p \times p$ symmetric positive definite matrix. It is assumed that m may be $\geq p$ or $< p$, but $n \geq p$. To test the null

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hypothesis $H_0: \xi = 0$, H. telling [2] proposed a test based on the statistic:

$$(2.2) \quad T_0^2 = m \operatorname{tr} S_1 S_0^{-1}$$

and derived the exact distribution of this statistic when $p = 2$ and $\xi = 0$. For general values of p the exact distribution of T_0^2 is not available at present, even in the null case $\xi = 0$.

3. Derivation of asymptotic formula of T_0^2 .

For general values of p it is known that the statistic

$$(3.1) \quad \chi^2 = m \operatorname{tr} S_1 \wedge$$

has a χ^2 distribution with mp degrees of freedom. That is to say, we have

$$(3.2) \quad \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} = G_\rho(\theta),$$

where 2θ denotes the tabled value of χ^2 for a particular level of significance, $\rho = mp/2$, and

$$(3.3) \quad G_\rho(\theta) = [\Gamma(\rho)]^{-1} \int_0^\theta t^{\rho-1} e^{-t} dt.$$

Hence, if \wedge is known, the statistic χ^2 given by (3.1) may be used to test H_0 exactly, and if \wedge is unknown but if S_0 is based on a large number of degrees of freedom, i.e., if n is large, we may use as an approximation the result:

$$(3.4) \quad \Pr \{ m \operatorname{tr} S_1 S_0^{-1} \leq 2\theta \} = G_\rho(\theta).$$

This suggests that in the general case we try to find a function $h(S_0)$ of the elements of S_0 such that

$$(3.5) \quad \Pr \{ m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) \} = G_\rho(\theta).$$

When n is large, $2h(S_0)$ will approach $2\theta \equiv \chi^2$, and we now expect to write $2h(S_0)$ as a series with χ^2 as its first term and successive terms of decreasing order of magnitude.

Now

$$(3.6) \quad \Pr \left\{ m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) \right\} = \int_R \Pr \left\{ m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) | S_0 \right\} \Pr \left\{ d S_0 \right\} ,$$

where the first expression on the right denotes the conditional probability of the relation indicated for fixed values of the elements of S_0 , and the second denotes the probability element of S_0 , which has a Wishart distribution with n degrees of freedom, and the domain of integration R is over all possible values of the elements of S_0 . Now we may expand $\Pr \left\{ m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) | S_0 \right\}$ about an origin $(\sigma_{11}, \sigma_{22}, \dots, \sigma_{pp}, \sigma_{12}, \dots, \sigma_{p-1,p})$ in a Taylor series, where

$$\Lambda^{-1} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \cdot & \cdot & \dots & \cdot \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{pmatrix} .$$

Thus,

$$(3.7) \quad \Pr \left\{ m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) | S_0 \right\} \\ = \left\{ \exp \left[- \frac{1}{2} \sum_{i \leq j=1}^p (s_{0ij} - \sigma_{ij}) \frac{\partial}{\partial \sigma_{ij}} \right] \right\} \Pr \left\{ m \operatorname{tr} S_1 \Lambda \leq 2h(\Lambda^{-1}) \right\} \\ = \left\{ \exp \left[- \operatorname{tr} (S_0 - \Lambda^{-1}) \partial \right] \right\} \Pr \left\{ m \operatorname{tr} S_1 \Lambda \leq 2h(\Lambda^{-1}) \right\} ,$$

where s_{0ij} is the i -th row, j -th column element of S_0 , and ∂ denotes the matrix of derivative operators:

$$(3.8) \quad \partial = \begin{pmatrix} \frac{\partial}{\partial \sigma_{11}} & \frac{1}{2} \frac{\partial}{\partial \sigma_{12}} & \dots & \frac{1}{2} \frac{\partial}{\partial \sigma_{1p}} \\ \frac{1}{2} \frac{\partial}{\partial \sigma_{21}} & \frac{\partial}{\partial \sigma_{22}} & \dots & \frac{1}{2} \frac{\partial}{\partial \sigma_{2p}} \\ \cdot & \cdot & \dots & \cdot \\ \frac{1}{2} \frac{\partial}{\partial \sigma_{p1}} & \frac{1}{2} \frac{\partial}{\partial \sigma_{p2}} & \dots & \frac{\partial}{\partial \sigma_{pp}} \end{pmatrix}$$

its typical element being $\partial_{ij} = \frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial \sigma_{ij}}$, where δ_{ij} is the Kronecker delta.

Whether uniformly convergent or not, the right hand side of (3.7) is an asymptotic representation of $\Pr \{m \operatorname{tr} S_1 S_0^{-1} \leq 2h(S_0) | S_0\}$, for sufficiently large values of n . Hence, substitution of (3.7) into (3.6) and term by term integration which may be done legitimately, yields:

$$(3.9) \quad G_p(\theta) = \int_R \exp[-\operatorname{tr}(S_0^{-1} \partial)] \Pr \{m \operatorname{tr} S_1 \wedge \leq 2h(\wedge^{-1})\} \Pr \{d S_0\} \\ = \Theta \Pr \{m \operatorname{tr} S_1 \wedge \leq 2h(\wedge^{-1})\},$$

where

$$\Theta = \int_R \exp[-\operatorname{tr}(S_0^{-1} \partial)] \Pr \{d S_0\}.$$

Since S_0 has a Wishart distribution with n degrees of freedom, we have

$$\Theta = \exp[-\operatorname{tr} \wedge^{-1} \partial] \cdot \operatorname{const.} |\wedge|^{\frac{n}{2}} \int_R |S_0|^{\frac{n-p-1}{2}} \exp[-\operatorname{tr} (S_0 \partial - \frac{n}{2} \wedge S_0)] d S_0 \\ = \exp[-\operatorname{tr} \wedge^{-1} \partial] \cdot \operatorname{const.} |\wedge|^{\frac{n}{2}} \int_R |S_0|^{\frac{n-p-1}{2}} \exp[-\frac{n}{2} \operatorname{tr} (\wedge - \frac{2}{n} \partial) S_0] d S_0 \\ = \exp[-\operatorname{tr} \wedge^{-1} \partial] \cdot |\wedge|^{\frac{n}{2}} |\wedge - \frac{2}{n} \partial|^{-\frac{n}{2}} \\ = \exp[-\operatorname{tr} \wedge^{-1} \partial] \cdot |I - \frac{2}{n} \wedge^{-1} \partial|^{-\frac{n}{2}},$$

where I is the p x p identity matrix. Now using [5],

$$(3.10) \quad -\log |I - Y| = \text{tr } Y + \frac{1}{2} \text{tr } Y^2 + \frac{1}{3} \text{tr } Y^3 + \dots,$$

we obtain

$$(3.11) \quad \begin{aligned} \Theta &= \exp \left[-\text{tr } \Lambda^{-1} \partial - \frac{n}{2} \log \left| I - \frac{\varepsilon}{n} \Lambda^{-1} \partial \right| \right] \\ &= \exp \left[-\text{tr } \Lambda^{-1} \partial + \frac{n}{2} \left\{ \text{tr} \left(\frac{\varepsilon}{n} \Lambda^{-1} \partial \right) + \frac{1}{2} \text{tr} \left(\frac{\varepsilon}{n} \Lambda^{-1} \partial \right)^2 + \frac{1}{3} \text{tr} \left(\frac{\varepsilon}{n} \Lambda^{-1} \partial \right)^3 + \dots \right\} \right] \\ &= \exp \left[-\frac{1}{n} \text{tr} (\Lambda^{-1} \partial)^2 + \frac{4}{3n^2} \text{tr} (\Lambda^{-1} \partial)^3 + \dots \right] \\ &= 1 + \frac{1}{n} \text{tr} (\Lambda^{-1} \partial)^2 + \frac{1}{n^2} \left\{ \frac{4}{3} \text{tr} (\Lambda^{-1} \partial)^3 + \frac{1}{2} (\text{tr} (\Lambda^{-1} \partial)^2)^2 \right\} + o(n^{-3}). \end{aligned}$$

It is to be noted here that in (3.11) the operator ∂ does not act on Λ^{-1} present in Θ itself, and it is more useful for our purpose to write (3.11) in suffix form:

$$(3.12) \quad \begin{aligned} \Theta &= 1 + \frac{1}{n} \sum \sigma_{rs} \sigma_{tu} \partial_{st} \partial_{ur} \\ &\quad + \frac{1}{n^2} \left\{ \frac{4}{3} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \partial_{st} \partial_{uv} \partial_{wr} + \frac{1}{2} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} \partial_{st} \partial_{ur} \partial_{wx} \partial_{yv} \right\} \\ &\quad + o(n^{-3}), \end{aligned}$$

where \sum denotes the summation over all suffixes r, s, ... each of which ranges from 1 to p.

Now we represent $h(S_0)$ as

$$(3.13) \quad h(S_0) = \theta + h_1(S_0) + h_2(S_0) + \dots,$$

$h_s(S_0)$ being of order n^{-s} , i.e., we write $h(S_0)$ as an asymptotic series such that

$$|n^s \{h(S_0) - \theta - h_1(S_0) - \dots - h_s(S_0)\}|$$

is made arbitrarily small for sufficiently large values of n. Then (3.13) may be substituted into $\text{Pr} \left\{ m \text{tr } S_1 \wedge \leq 2h(\Lambda^{-1}) \right\}$, and by Taylor's expansion we have:

$$\begin{aligned}
 (3.14) \quad & \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2h(\wedge^{-1}) \} \\
 &= \exp \left[- \left\{ h_1(\wedge^{-1}) + h_2(\wedge^{-1}) + \dots \right\} D \right] \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} \\
 &= \left[1 + \left\{ h_1(\wedge^{-1}) + h_2(\wedge^{-1}) + \dots \right\} D + \frac{1}{2} \left\{ h_1(\wedge^{-1}) + h_2(\wedge^{-1}) + \dots \right\}^2 D^2 + \dots \right] \\
 &\quad \times \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} ,
 \end{aligned}$$

where $D \equiv \frac{\partial}{\partial \theta}$. By substituting (3.12) and (3.14) into (3.9) we obtain

$$\begin{aligned}
 (3.15) \quad G_p(\theta) &= \left[1 + \frac{1}{n} \sum \sigma_{rs} \sigma_{tu} \partial_{st} \partial_{ur} + \frac{1}{n^2} \left\{ \frac{4}{3} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \partial_{st} \partial_{uv} \partial_{wr} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} \partial_{st} \partial_{ur} \partial_{wx} \partial_{yv} \right\} + O(n^{-3}) \right] \\
 &\quad \times \left[1 + h_1(\wedge^{-1}) D + \left\{ h_2(\wedge^{-1}) D + \frac{1}{2} h_1^2(\wedge^{-1}) D^2 \right\} + O(n^{-3}) \right] \\
 &\quad \times \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} .
 \end{aligned}$$

By equating terms of successive order in (3.15) we obtain

$$(3.16) \quad \left\{ h_1(\wedge^{-1}) D + \frac{1}{n} \sum \sigma_{rs} \sigma_{tu} \partial_{st} \partial_{ur} \right\} \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} = 0,$$

$$\begin{aligned}
 (3.17) \quad & \left[h_2(\wedge^{-1}) D + \frac{1}{2} h_1^2(\wedge^{-1}) D^2 + \frac{1}{n} \sum \sigma_{rs} \sigma_{tu} \left\{ h_1^{(st,ur)}(\wedge^{-1}) D \right. \right. \\
 &\quad \left. \left. + 2h_1^{(st)}(\wedge^{-1}) \partial_{ur} D + h_1(\wedge^{-1}) \partial_{st} \partial_{ur} D \right\} \right. \\
 &\quad \left. + \frac{4}{3n^2} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \partial_{st} \partial_{uv} \partial_{wr} + \frac{1}{2n^2} \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} \partial_{st} \partial_{ur} \partial_{wx} \partial_{yv} \right] \\
 &\quad \times \Pr \{ m \operatorname{tr} S_1 \wedge \leq 2\theta \} = 0,
 \end{aligned}$$

and so on, where $h_1^{(st)}(\wedge^{-1}) = \partial_{st} h_1(\wedge^{-1})$ and $h_1^{(st,ur)}(\wedge^{-1}) = \partial_{ur} \partial_{st} h_1(\wedge^{-1})$.

It now remains to carry out the operations ∂ and D indicated in (3.16) and (3.17) in order to obtain $h_1(\Lambda^{-1})$, $h_2(\Lambda^{-1})$ and hence $h_1(S_0)$, $h_2(S_0)$. These operators will operate on $\Pr\{m \text{ tr } S_1 \wedge \leq 2\theta\}$ which is a $p \times m$ fold integral, and the operations may be thought of as differentiations, with respect to the boundary only, of the integral of the probability density function of the X_1 throughout a region in the space of X_1 . The method used to evaluate $\partial_{st} \partial_{ur} \Pr\{m \text{ tr } S_1 \wedge \leq 2\theta\}$, $\partial_{st} \partial_{uv} \partial_{wr} \Pr\{m \text{ tr } S_1 \wedge \leq 2\theta\}$, ..., is to change the boundary slightly, expand the integral in powers of the quantities specifying this change, and obtain the derivatives by comparison with Taylor's expansion. We consider

$$(3.18) \quad J = \Pr\{m \text{ tr } S_1 (\Lambda^{-1} + \epsilon)^{-1} \leq 2\theta\},$$

where ϵ is a $p \times p$ symmetric matrix. Then by Taylor expansion we have

$$(3.19) \quad J = \left\{ 1 + \sum \epsilon_{rs} \partial_{rs} + \frac{1}{2!} \sum \epsilon_{rs} \epsilon_{tu} \partial_{rs} \partial_{tu} + \frac{1}{3!} \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} \partial_{rs} \partial_{tu} \partial_{vw} + \frac{1}{4!} \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} \epsilon_{xy} \partial_{rs} \partial_{tu} \partial_{vw} \partial_{xy} + \dots \right\} \Pr\{m \text{ tr } S_1 \wedge \leq 2\theta\}.$$

On the other hand, J is, by definition, written as

$$(3.20) \quad J = \frac{|\Lambda|^{-\frac{m}{2}}}{(2\pi)^{\frac{pm}{2}}} \int_{R'} \exp \left[-\frac{1}{2} \text{tr } \Lambda X_1 X_1' \right] dX_1,$$

where $X_1 X_1' = m S_1$, and domain of integration R' ranges over all possible values of the elements of X_1 such that $m \text{ tr } S_1 (\Lambda^{-1} + \epsilon)^{-1} \leq 2\theta$. It is now easy to show that integration of (3.20) yields

$$(3.21) \quad J = \left(\frac{|I-D|}{|I-D|} \right)^{-\frac{m}{2}} G_p(\theta),$$

where D_η is a diagonal matrix which satisfies

$$(3.22) \quad X_1(p \times m) = \Gamma(p \times p) Z(p \times m),$$

$$\frac{1}{2} \Gamma' (\Lambda^{-1} + \epsilon)^{-1} \Gamma = I(p),$$

and
$$\frac{1}{2} \Gamma' \wedge \Gamma = I(p) - D_\eta,$$

Γ being a non-singular matrix, and E is an operator such that

$$E G_\rho(\theta) = G_{\rho+1}(\theta).$$

Now letting $\Delta = E - I$ and using (3.22) we have

$$\begin{aligned} \frac{|I - D_\eta E|}{|I - D_\eta|} &= \frac{|I - D_\eta - D_\eta \Delta|}{|I - D_\eta|} \\ &= \frac{|\frac{1}{2} \Gamma' \wedge \Gamma - \{\frac{1}{2} \Gamma' (\Lambda^{-1} + \epsilon)^{-1} \Gamma - \frac{1}{2} \Gamma' \wedge \Gamma\} \Delta|}{|\frac{1}{2} \Gamma' \wedge \Gamma|} \\ &= \frac{|\Lambda - \{(\Lambda^{-1} + \epsilon)^{-1} - \Lambda\} \Delta|}{|\Lambda|} = |I - \{\Lambda^{-1} (\Lambda^{-1} + \epsilon)^{-1} - I\} \Delta| \\ &= |I - X \Delta| \end{aligned}$$

where $X = \Lambda^{-1} (\Lambda^{-1} + \epsilon)^{-1} - I$. Hence (3.21) becomes

$$(3.23) \quad J = |I - X \Delta|^{-\frac{m}{2}} G_\rho(\theta)$$

Now using (3.10) again, we rewrite (3.23) as

$$(3.24) \quad J = \exp \left\{ -\frac{m}{2} \log |I - X \Delta| \right\} G_\rho(\theta) \\ = \exp \left\{ \frac{m}{2} \text{tr} X \Delta + \frac{m}{4} \text{tr} X^2 \Delta^2 + \frac{m}{6} \text{tr} X^3 \Delta^3 + \frac{m}{8} \text{tr} X^4 \Delta^4 + \dots \right\} G_\rho(\theta)$$

$$\begin{aligned}
 &= \Delta^{-1} + \frac{m}{2} \text{tr } X \Delta + \left\{ \frac{m}{4} \text{tr } X^2 + \frac{m^2}{8} (\text{tr } X)^2 \right\} \Delta^2 + \left\{ \frac{m}{6} \text{tr } X^3 + \frac{m^2}{8} (\text{tr } X)(\text{tr } X^2) + \frac{m^3}{48} (\text{tr } X)^3 \right\} \Delta^3 \\
 &\quad + \left\{ \frac{m}{8} \text{tr } X^4 + \frac{m^2}{12} (\text{tr } X)(\text{tr } X^3) + \frac{m^2}{32} (\text{tr } X^2)^2 \right. \\
 &\quad \left. + \frac{m^3}{32} (\text{tr } X)^2 (\text{tr } X^2) + \frac{m^4}{384} (\text{tr } X)^4 \right\} \Delta^4 + \dots \} G_p(\theta).
 \end{aligned}$$

X can be represented as

$$\begin{aligned}
 (3.25) \quad X &= \Lambda^{-1} (\Lambda^{-1} + \epsilon)^{-1} - I = \Lambda^{-1} (\Lambda^{-1} + \sum \epsilon_{rs} \Lambda_{rs}^{-1})^{-1} - I = (I + \sum \epsilon_{rs} \Lambda_{rs}^{-1} \Lambda)^{-1} - I \\
 &= -\sum \epsilon_{rs} (\Lambda_{rs}^{-1} \Lambda) + \sum \epsilon_{rs} \epsilon_{tu} (\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) - \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} (\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) (\Lambda_{vw}^{-1} \Lambda) \\
 &\quad + \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} \epsilon_{xy} (\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) (\Lambda_{vw}^{-1} \Lambda) (\Lambda_{xy}^{-1} \Lambda) - \dots,
 \end{aligned}$$

where Λ_{rs}^{-1} is a $p \times p$ matrix obtained by operating ∂_{rs} on Λ , i.e., Λ_{rs}^{-1} has its i -th row, j -th column element, $\frac{1}{2}(\delta_{ri} \delta_{sj} + \delta_{si} \delta_{rj})$. Writing

$$\text{tr}(\Lambda_{rs}^{-1} \Lambda) = (rs),$$

$$\text{tr}(\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) = (rs|tu),$$

$$\text{tr}(\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) (\Lambda_{vw}^{-1} \Lambda) = (rs|tu|vw),$$

$$\text{tr}(\Lambda_{rs}^{-1} \Lambda) (\Lambda_{tu}^{-1} \Lambda) (\Lambda_{vw}^{-1} \Lambda) (\Lambda_{xy}^{-1} \Lambda) = (rs|tu|vw|xy),$$

and substituting (3.25) into (3.24), we obtain

$$\begin{aligned}
 (3.26) \quad J &= \Delta^{-1} + \sum \epsilon_{rs} \left\{ -\frac{m}{2} (rs) \Delta \right\} + \frac{1}{2!} \sum \epsilon_{rs} \epsilon_{tu} \left\{ (rs|tu) (m\Delta + \frac{m}{2} \Delta^2) + \frac{m^2}{4} (rs)(tu) \Delta^2 \right\} \\
 &+ \frac{1}{3!} \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} \left\{ (rs|tu|vw) (-3m\Delta - 3m\Delta^2 - m\Delta^3) + (rs)(tu|vw) (-\frac{3}{2} m^2 \Delta^2 - \frac{3}{4} m^2 \Delta^3) - \frac{m^3}{8} (rs)(tu)(vw) \Delta^3 \right\} \\
 &+ \frac{1}{4!} \sum \epsilon_{rs} \epsilon_{tu} \epsilon_{vw} \epsilon_{xy} \left\{ (rs|tu|vw|xy) (12m\Delta + 18m\Delta^2 + 12m\Delta^3 + 3m\Delta^4) \right. \\
 &\quad \left. + (rs)(tu|vw|xy) (6m^2 \Delta^2 + 6m^2 \Delta^3 + 2m^2 \Delta^4) + (rs|tu)(vw|xy) (3m^2 \Delta^2 + 3m^2 \Delta^3 + \frac{3}{4} m^2 \Delta^4) \right\}
 \end{aligned}$$

$$+(rs)(tu)(vw|xy)\left(\frac{3}{2}m^3\Delta^3+\frac{3}{4}m^3\Delta^4\right) + (rs)(tu)(vw)(xy)\frac{m^4}{16}\Delta^4\} + \dots \int G_p(\theta).$$

Then term by term comparison between two expansions for J, (3.19) and (3.26), gives $\partial_{rs} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\}$, $\partial_{rs} \partial_{tu} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\}$, etc., but in doing so we must take such a care that, for example,

$$\sum a_{ijk} \epsilon_i \epsilon_j \epsilon_k = \sum b_{ijk} \epsilon_i \epsilon_j \epsilon_k$$

implies $a_{ijk} = b_{ijk}$ if both a_{ijk} and b_{ijk} are completely symmetrical in their suffices. With this in mind and using the relation

$$\Delta G_p(\theta) = -E g_p(\theta),$$

where $g_p(\theta) = D G_p(\theta)$, we obtain

$$(3.27) \quad \partial_{rs} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\} = \frac{m}{2}(rs) E g_p(\theta),$$

$$(3.28) \quad \partial_{rs} \partial_{tu} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\} = - \left\{ \frac{m}{2}(rs|tu)(E^2 + E) + \frac{m^2}{4}(rs)(tu)(E^2 - E) \right\} g_p(\theta)$$

$$(3.29) \quad \partial_{rs} \partial_{tu} \partial_{vw} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\} = \left\{ m(rs|tu|vw)(E^3 + E^2 + E) + \frac{m^2}{4} \int^-(rs)(tu|vw) \right. \\ \left. + (tu)(rs|vw) + (vw)(rs|tu) \int^-(E^3 - E) + \frac{m^3}{8}(rs)(tu)(vw)(E^3 - 2E^2 + E) \right\} \\ \cdot g_p(\theta),$$

$$(3.30) \quad \partial_{rs} \partial_{tu} \partial_{vw} \partial_{xy} \Pr \{m \text{ tr } S_1 \wedge \leq 2\theta\} = - \left\{ m \int^-(rs|tu|vw|xy) + (rs|vw|xy|tu) \right. \\ \left. + (rs|xy|tu|vw) \int^-(E^4 + E^3 + E^2 + E) + \frac{m^2}{2} \int^-(rs)(tu|vw|xy) + (xy)(tu|vw|rs) \right. \\ \left. + (vw)(tu|xy|rs) + (tu)(vw|xy|rs) \int^-(E^4 - E) + \frac{m^2}{4} \int^-(rs|tu)(vw|xy) \right. \\ \left. + (rs|vw)(tu|xy) + (rs|xy)(tu|vw) \int^-(E^4 + E^3 - E^2 - E) + \frac{m^3}{8} \int^-(rs)(tu)(vw|xy) \right. \\ \left. + (tu|xy) + (rs)(xy)(tu|vw) + (tu)(vw)(rs|xy) + (tu)(xy)(rs|vw) \right. \\ \left. + (vw)(xy)(rs|tu) \int^-(E^4 - E^3 - E^2 + E) + \frac{m^4}{16}(rs)(tu)(vw)(xy)(E^4 - 3E^3 + 3E^2 - E) \right\} g_p(\theta).$$

Upon substituting (3.28) into (3.16) we obtain

$$h_1(\Lambda^{-1}) = \frac{1}{4n} \sum \sigma_{rs} \sigma_{tu} \left[-2m(st|ur) \left\{ \frac{\theta^2}{\rho(\rho+1)} + \frac{\theta}{\rho} \right\} + m^2(st)(ur) \left\{ \frac{\theta^2}{\rho(\rho+1)} - \frac{\theta}{\rho} \right\} \right].$$

$$\text{Now } (st) = \text{tr } \Lambda_{st}^{-1} \Lambda = \frac{1}{2} \sum_{i,j} (\delta_{si} \delta_{tj} + \delta_{ti} \delta_{sj}) \sigma^{ji} = \frac{1}{2} (\sigma^{ts} + \sigma^{st}) = \sigma^{st}$$

and also,

$$(st|ur) = \text{tr } (\Lambda_{st}^{-1} \Lambda) (\Lambda_{ur}^{-1} \Lambda) = \frac{1}{2} (\sigma^{rs} \sigma_{tu} + \sigma^{su} \sigma_{tr}).$$

Hence we have

$$\sum \sigma_{rs} \sigma_{tu} (st|ur) = \frac{1}{2} p(p+1)$$

and

$$\sum \sigma_{rs} \sigma_{tu} (st)(ur) = p.$$

We also note that $2\theta = x^2$, $\rho = mp/2$. Therefore we finally obtain, after some simplification,

$$(3.31) \quad h_1(\Lambda^{-1}) = \frac{1}{4n} \left\{ \frac{p+m+1}{mp+2} x^4 + (p-m+1) x^2 \right\}.$$

In a similar way we substitute (3.29), (3.30) and (3.31) into (3.17) to evaluate $h_2(\Lambda^{-1})$. We note here that since $h_1(\Lambda^{-1})$ given by (3.31) is independent of Λ^{-1} , the terms involving $h_1^{(st)}(\Lambda^{-1})$ and $h_1^{(st,ur)}(\Lambda^{-1})$ in (3.17) do not appear. As before, it can be easily shown that

$$\sum \sigma_{rs} \sigma_{tu} \sigma_{vw} (st|uv|wr) = \frac{1}{8} p(p^2+3p+4), \quad \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} (st)(uv|wr)$$

$$= \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} (uv)(st|wr) = \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} (wr)(st|uv) = \frac{1}{2} p(p+1),$$

$$\sum \sigma_{rs} \sigma_{tu} \sigma_{vw} (st)(uv)(wr) = p, \quad \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|ur|wx|yv)$$

$$= \sum \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|wx|yv|ur) = \frac{1}{4} p(p+1)^2,$$

$$\begin{aligned}
 \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|yv|ur|wx) &= \frac{1}{4} p(p+3), \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st)(ur|wx|yv) \\
 &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (yv)(ur|wx|st) = \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (wx)(ur|yv|st) \\
 &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (ur)(wx|yv|st) = \frac{1}{2} p(p+1), \\
 \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|ur)(wx|yv) &= \frac{1}{2} p^2 (p+1)^2, \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|wx)(ur|yv) \\
 &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st|yv)(ur|wx) = \frac{1}{2} p(p+1), \\
 \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st)(ur)(wx|yv) &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (wx)(yv)(st|ur) = \frac{1}{2} p^2 (p+1), \\
 \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st)(wx)(ur|yv) &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st)(yv)(ur|wx) \\
 &= \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (ur)(wx)(st|yv) = \Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (ur)(yv)(st|wx) = p.
 \end{aligned}$$

and

$$\Sigma \sigma_{rs} \sigma_{tu} \sigma_{vw} \sigma_{xy} (st)(ur)(wx)(yv) = p^2.$$

Using these results we obtain from (3.17), after some simplification,

$$\begin{aligned}
 (3.32) \quad h_2(\wedge^{-1}) &= \frac{1}{48n^2} \left[\frac{-6(p-1)(p+2)(m-1)(m+2)}{(mp+2)^2 (mp+4)(mp+6)} x^8 \right. \\
 &+ \frac{4mp^3 + 2(3m^2 + 3m + 10)p^2 + 2(2m^3 + 3m^2 + 17m + 18)p + 4(5m^2 + 9m + 2)}{(mp+2)^2 (mp+4)} x^6 \\
 &+ \left. \frac{13p^2 + 24p - 11m^2 + 7}{mp+2} x^4 + \{7p^2 + (-12m+12)p + (7m^2 - 12m + 1)\} x^2 \right]_7,
 \end{aligned}$$

which is independent of \wedge^{-1} just as $h_1(\wedge^{-1})$.

Now we substitute (3.31) and (3.32) into (3.13) to obtain

$$\begin{aligned}
 (3.33) \quad T_0^2 &= 2h(S_0) = 2\theta + 2h_1(S_0) + 2h_2(S_0) + 0(n^{-3}) = x^2 + \frac{1}{2n} \left\{ \frac{p+m+1}{mp+2} x^4 + (p-m+1) x^2 \right\} \\
 &+ \frac{1}{24n^2} \left\{ \frac{6(p-1)(p+2)(m-1)(m+2)}{(mp+2)^2 (mp+4)(mp+6)} x^8 + \frac{4mp^3 + 2(3m^2 + 3m + 10)p^2 + 2(2m^3 + 3m^2 + 17m + 18)p + 4(5m^2 + 9m + 2)}{(mp+2)^2 (mp+4)} x^6 \right. \\
 &+ \left. \frac{13p^2 + 24p - 11m^2 + 7}{mp+2} x^4 + \left[-7p^2 + (-12m+12)p + (7m^2 - 12m + 1) \right] x^2 \right\} + 0(n^{-3}),
 \end{aligned}$$

which is the asymptotic expression of a percentage point of the T_0^2 distribution in terms of the corresponding percentage point of the χ^2 distribution with mp degrees of freedom.

If we put $m = 1$ in (3.33), we have

$$(3.34) \quad T^2 = x^2 + \frac{1}{2n} \{x^4 + px^2\} + \frac{1}{24n^2} \{4x^6 + (13p - 2)x^4 + (7p^2 - 4)x^2\} + o(n^{-3}),$$

which is the asymptotic expression of a percentage point of the generalized Student T distribution. This result (3.34) was previously obtained by Hotelling and Frankel [2], [3].

4. Asymptotic formula for the c.d.f. of T_0^2 .

Let $F(2\theta_1)$ be the c.d.f. of T_0^2 , i.e.,

$$(4.1) \quad F(2\theta_1) = \Pr \{m \operatorname{tr} S_1 S_0^{-1} \leq 2\theta_1\}.$$

Then, as (3.6), we can write

$$(4.2) \quad \Pr \{m \operatorname{tr} S_1 S_0^{-1} \leq 2\theta_1\} = \int_R \Pr \{m \operatorname{tr} S_1 S_0^{-1} \leq 2\theta_1 \mid S_0\} \Pr \{dS_0\} \\ = \ominus \Pr \{m \operatorname{tr} S_1 \wedge \leq 2\theta_1\},$$

where \ominus is given by (3.12). Upon substituting (3.28), (3.29) and (3.30) into (4.2) we obtain, after some simplification,

$$(4.3) \quad F(2\theta_1) = G_p(\theta_1) - \frac{1}{2n} \left\{ \frac{2(p+m+1)\theta_1^2}{mp+2} + (p-m+1)\theta_1 \right\} g_p(\theta_1) \\ - \frac{1}{48n^2} \int \frac{24 \{ mp^3 + 2(m^2 + m + 4)p^2 + (m^3 + 2m^2 + 21m + 20)p + 8m^2 + 20m + 20 \} \theta_1^4}{(mp+2)(mp+4)(mp+6)} \\ + \frac{4 \{ 3mp^3 - 2(3m^2 - 3m - 4)p^2 - 3(3m^3 + 2m^2 + 11m - 4)p - 40m^2 - 36m - 4 \} \theta_1^3}{(mp+2)(mp+4)}$$

$$- \frac{2 \{ 3mp^3 + 2(3m^2 + 3m - 4)p^2 - 3(3m^3 - 2m^2 - 5m + 4)p - 8m^2 + 12m + 4 \} \theta_1^2}{mp + 2} - \{ 3mp^3 - 2(3m^2 - 3m + 4)p^2 + 3(m^3 - 2m^2 + 5m - 4)p - 8m^2 + 12m + 4 \} \theta_1^{-1} g_p(\theta_1) + O(n^{-3}),$$

where $G_p(\theta_1) = \int \Gamma(\rho) J^{-1} \int_0^{\theta_1} t^{\rho-1} e^{-t} dt$, $g_p(\theta_1) = \frac{\partial}{\partial \theta_1} G_p(\theta_1)$, and $\rho = mp/2$.

(4.3) is a sort of multivariate analogue of Hartley's formula of "Studentization." In fact it can be shown that when $p = 1$, (4.3) coincides with Hartley's formula for the c.d.f. of the univariate analysis of variance F statistic. (See equation (28), p. 178, [1].)

5. Discussion of the error and remarks.

In view of the methods used in sections 3 and 4, it is rather difficult to set a bound for the error committed by omitting all terms after the first few terms in the asymptotic formula for T_0^2 (3.33) or in the asymptotic formula for the c.d.f. of T_0^2 (4.3). There is, however, a method to find lower and upper bounds to the c.d.f. of T_0^2 which is fairly good for large values of n , and they can be used to set a bound for $O(n^{-3})$, say, in the asymptotic expansion of the c.d.f. of T_0^2 .

We shall first obtain lower and upper bounds for the c.d.f. of T_0^2 . It is well known (e.g. see [6]) that the joint probability law of the characteristic roots e_1, e_2, \dots, e_s of $m \text{ tr } S_1 S_0^{-1}$ under the null hypothesis H_0 is given by

$$(5.1) \quad P(e_1, e_2, \dots, e_s) = C(s, t, p, n) \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} \left(1 + \frac{e_i}{n}\right)^{-\frac{m+n}{2}} de_i \prod_{i < j=1}^{s-1} (e_i - e_j),$$

where $0 \leq e_s \leq e_{s-1} \leq \dots \leq e_1 < \infty$, $s = \min(p, m)$, $t = \max(p, m)$ and

$$C(s, t, p, n) = \frac{\pi^{\frac{s}{2}}}{n^{\frac{st}{2}}} \prod_{i=1}^s \frac{\Gamma\{\frac{1}{2}(n+t-pt+1)\}}{\Gamma\{\frac{1}{2}(t-s+1)\} \Gamma\{\frac{1}{2}(n-pt+1)\} \Gamma(\frac{1}{2})}.$$

The statistic T_0^2 is expressed as

$$(5.2) \quad T_0^2 = m \operatorname{tr} S_1 S_0^{-1} = \sum_{i=1}^s e_i,$$

and the c.d.f. of T_0^2 is given by

$$(5.3) \quad F(2\theta_1) = C(s, t, p, n) \int_{R_1} \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} \left(1 + \frac{e_i}{n}\right)^{-\frac{m+n}{2}} de_1 \prod_{i < j=1}^{s-1} (e_i - e_j),$$

where R_1 is the domain of integration such that $0 \leq e_s \leq e_{s-1} \leq \dots \leq e_1 < \infty$ and

$0 \leq \sum_{i=1}^s e_i \leq 2\theta_1$. Now for any non-negative values of e_i and n , the following

inequality holds:

$$\log \left(1 + \frac{e_i}{n}\right) \leq \frac{e_i}{n}$$

for $i = 1, \dots, s$, where equality holds when $e_i = 0$ or $n \rightarrow \infty$. Hence we have

$$\prod_{i=1}^s \left(1 + \frac{e_i}{n}\right)^{-\frac{m+n}{2}} \geq e^{-\frac{m+n}{2n} \sum_{i=1}^s e_i}.$$

Therefore the probability law (5.1) is bounded from below as follows:

$$(5.4) \quad P_1(e_1, \dots, e_s) \leq P(e_1, \dots, e_s)$$

where

$$P_1(e_1, \dots, e_s) = C(s, t, p, n) \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} de_1 \exp\left[-\frac{m+n}{2n} \sum_{i=1}^s e_i\right] \prod_{i < j=1}^{s-1} (e_i - e_j).$$

It must be noted here that $P_1(e_1, \dots, e_s)$ is not a probability law although it is non-negative for all e_i such that $0 \leq e_s \leq \dots \leq e_1 < \infty$. Now integrating both sides of (5.4) in R_1 we obtain

$$(5.5) \quad F_1(2\theta_1) \leq F(2\theta_1),$$

where

$$F_1(2\theta_1) = C(s, t, p, n) \int \dots \int_{R_1} \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} de_i \exp\left[-\frac{m+n}{2n} \sum_{i=1}^s e_i\right] \prod_{1 \leq j=1}^{s-1} (e_i - e_j),$$

and also integrating both sides of (5.4) in R_2 where $0 \leq e_s \leq \dots \leq e_1 < \infty$ and

$2\theta_1 \leq \sum_{i=1}^s e_i < \infty$ and subtracting each from 1, we have

$$(5.6) \quad F(2\theta_1) \leq F_2(2\theta_1),$$

where

$$F_2(2\theta_1) = 1 - C(s, t, p, n) \int \dots \int_{R_2} \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} de_i \exp\left[-\frac{m+n}{2n} \sum_{i=1}^s e_i\right] \prod_{1 \leq j=1}^{s-1} (e_i - e_j).$$

In order to evaluate $F_1(2\theta_1)$ and $F_2(2\theta_1)$, we observe that as n tends to ∞ , T_0^2

$= \sum_{i=1}^s e_i$ has a χ^2 distribution with st degrees of freedom in the limit, i.e., we have

$$(5.7) \quad K(s, t, p) \int \dots \int_{R_1} \prod_{i=1}^s e_i^{\frac{t-s-1}{2}} de_i \exp\left[-\frac{1}{2} \sum_{i=1}^s e_i\right] \prod_{1 \leq j=1}^{s-1} (e_i - e_j) = G_{\rho_1}(\theta_1),$$

where

$$K(s, t, p) = \lim_{n \rightarrow \infty} C(s, t, p, n) = \frac{\pi^{\frac{s}{2}}}{2^{\frac{st}{2}}} \frac{1}{\prod_{i=1}^s \Gamma\left\{\frac{1}{2}(t-s+i)\right\} \Gamma\left(\frac{1}{2}\right)}$$

and $\rho_1 = st/2$. Hence integration of (5.5) yields

$$(5.8) \quad F_1(2\theta_1) = L(s, t, p, n) G_{\rho_1}\left(\frac{m+n}{n} \theta_1\right),$$

where

$$L(s, t, p, n) = \frac{C(s, t, p, n)}{K(s, t, p)} \left(\frac{n}{m+n}\right)^{\frac{st}{2}} = \left(\frac{2}{m+n}\right)^{\frac{st}{2}} \prod_{i=1}^s \frac{\Gamma\left(\frac{m+t-p+i}{2}\right)}{\Gamma\left(\frac{n-p+i}{2}\right)}$$

Similarly we obtain from (5.6)

$$(5.9) \quad F_2(2\theta_1) = 1 - L(s, t, p, n) \left\{ 1 - G_{\rho_1}\left(\frac{m+n}{n} \theta_1\right) \right\}.$$

Now if we write (4.3) as

$$(5.10) \quad F(2\theta_1) = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + R_3,$$

where R_3 is the error committed by omitting all terms except the first three terms in the asymptotic series of $F(2\theta_1)$, the absolute value of R_3 has the following upper bound:

$$(5.11) \quad |R_3| \leq \max \left\{ \left| F_1(2\theta_1) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2} \right|, \left| F_2(2\theta_1) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2} \right| \right\},$$

where $F_1(2\theta_1)$ and $F_2(2\theta_1)$ are given by (5.8) and (5.9), respectively.

The actual manner in which (3.33) converges to the true value T_0^2 or (4.3) to the true value $F(2\theta_1)$, is not known, but it is hoped that the use of the first few corrective terms may result in a test which is more accurate than the χ^2 approximation, at any rate for moderately large values of n . In the case of the asymptotic formula for the c.d.f. of T_0^2 (4.3) we may judge the magnitude of the error involved in using the first few terms of the series by (5.11), which turns out to be rather small numerically when n is sufficiently large.

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