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PROBABILITY OF DETECTION FOR FLUCTUATING TARGETS

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SUMMARY

This report considers the probability of detection of a target by a pulsed search radar, when the target has a fluctuating cross section.

Formulas for detection probability are derived, and curves of detection probability vs. range are given, for four different target fluctuation models.

The investigation shows that, for these fluctuation models, the probability of detection for a fluctuating target is less than that for a non-fluctuating target if the range is sufficiently short, and is greater if the range is sufficiently long.

The amount by which the fluctuating and non-fluctuating cases differ depends on the rapidity of fluctuation and on the statistical distribution of the fluctuations. Figure 18, p. 36, shows a comparison between the non-fluctuating case and the four fluctuating cases considered.

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SYMBOLS

$f(n,N)$	scale factor, Case 3
$g(n,N)$	scale factor, Case 1
I	incomplete gamma function
N	number of hits integrated
n	false alarm number
P_D	probability of detection
R	range
R_0	range for which average input signal-to-noise ratio equals unity
x	input signal-to-noise power ratio for a single pulse
$\bar{x}$	average of x over all target fluctuations
Y_0	normalized threshold

I. INTRODUCTION

The probability of detection of a target by a pulsed search radar has been treated in considerable detail by J. I. Marcum^(1,2) for the case in which the amplitude of the returned signal pulses is not fluctuating. The purpose of this paper is to extend some of Marcum's results - mainly the computation of probability of detection* vs. range curves - to several kinds of fluctuating targets. (No claim to originality is made. Several of the equations given below have been derived by Marcum in hitherto unpublished work; some of these equations have also appeared elsewhere in the literature. This report is for the purpose of making the results more readily available than heretofore.) A general familiarity, on the part of the reader, with Marcum's paper will be assumed. Marcum's notation will be used throughout.

Four different models of target fluctuation will be considered. The four models chosen for consideration are felt to be typical of situations which are likely to be met in practice, or, at least, to bracket a wide range of practical cases.

In applying probability of detection computations to actual cases, one should first attempt to analyze the fluctuations of the actual target under consideration, and then choose whichever model (including Marcum's non-fluctuating model) appears to most closely approximate the actual target fluctuations. Or, one could consider the actual target to be intermediate between two of the theoretical models.

* Use is made of the term 'probability of detection' in order to conform to Marcum's terminology. Actually 'blip-scan ratio' is a more common term to use for the quantity which is computed. (The curves presented here all correspond to the cases which in Marcum's paper are labeled $Y = N$.)

One of the uses of the ensuing results is, that by comparing the results for different models, one can make some judgment as to the errors introduced by choosing the wrong fluctuation model.

The four fluctuation models considered are as follows:

CASE 1

The returned signal power per pulse is assumed to be constant for the time on target during a single scan, but to fluctuate independently from scan to scan. (This ignores factors such as beam shape effect.) Expressed in statistical terms, the normalized autocorrelation function of target cross section is approximately one for the time in which the beam is on target during a single scan, and is approximately zero for a time as long as the interval between scans. This type of fluctuation will henceforth be referred to as scan-to-scan fluctuation.

The fluctuations of target cross section are evidenced as fluctuations of signal to noise ratio in the receiver. The probability density for the input signal-to-noise power ratio is assumed to be:

$$w(x, \bar{x}) = \frac{1}{\bar{x}} e^{-x/\bar{x}} \quad (x \geq 0) \quad * \quad (I.1)$$

where x = input signal-to-noise power ratio
 $\bar{x}$ = average of x over all target fluctuations

CASE 2

The fluctuations are independent from pulse to pulse. This type of fluctuation will be referred to as pulse-to-pulse fluctuation.

The probability density function is given by (I.1)

* This formula also represents the probability density for target cross section Σ if x is replaced by Σ and $\bar{x}$ by $\bar{\Sigma}$.

CASE 3

Scan-to-scan fluctuation according to the probability density

$$w(x, \bar{x}) = \frac{4x}{\bar{x}^2} e^{-2x/\bar{x}} \quad (x \geq 0) \quad (I.2)$$

CASE 4

Pulse-to-pulse fluctuation according to (I.2).

It would be well at this point to indicate in which actual situations the various models would be likely to apply.

As to the choice of probability density function for the fluctuations:

Theoretically, for a target which can be represented as several independently fluctuating reflectors of approximately equal echoing area, the density function should be close to exponential, even if the number of reflectors is as small as four or five. Thus one would expect objects which are large compared with wavelength (and which are not shaped too much like a sphere) to fluctuate approximately according to the exponential density (I.1).

On the other hand, a target which can be represented as one large reflector together with other small reflectors, or as one large reflector subject to fairly small changes in orientation, would be expected to behave more like (I.2).

The non-fluctuating model would apply to spherical or nearly spherical objects (e.g. balloons, meteors) at fairly large wavelengths.

Most available observational data on aircraft targets indicates agreement with the exponential density (I.1). More definite statements as to actual targets for which (I.2) or the non-fluctuating model apply must await further experimental data.

As to the choice between scan-to-scan and pulse-to-pulse fluctuation:

The scan-to-scan model would apply to targets such as jet aircraft or missiles, for radars having fairly high pulse repetition rate and scan rate.

Pulse-to-pulse fluctuation would apply to propeller-driven craft if the propellers contribute a large portion of the echoing area; or to targets for which very small changes in orientation would mean large changes in echoing area, such as long, thin objects at high frequency; or to targets viewed by a radar with sufficiently low repetition rate.

Most actual targets would probably be intermediate between the various cases considered.

A comparison between Cases 1, 2, 3, 4, and the non-fluctuating case is given in Fig. 18 (for typical false alarm time and number of hits integrated).

In all cases, it is assumed that there are on each scan N hits; after passage through a square law second detector, the resulting N pulses are added and required to exceed a threshold Y_0 in order for detection of a target to occur.* The second detector output is, for mathematical convenience, assumed to be normalized as follows: detector output equals input envelope squared divided by twice the mean square input noise voltage.

Formulas for probability of detection P_D as a function of $\bar{x}$ are given for each case in Section II. Curves (corresponding to Marcum's) for P_D vs range are given in Figs. 1-18. (Marcum's full range of parameters is not duplicated, but the means for doing so are given by the formulas in Section II.) Since the formulas in Cases 1 and 3 lend themselves to very convenient approximations, these cases are further discussed in Section II. The derivations of the formulas in Section II are given in Section III.

* The actual beam shape is in effect being approximated by a beam having uniform gain over a finite sector, and zero gain outside this sector. In principle it is possible to take account exactly of beam shape in computing probability of detection. This is not done here, however, since it is thought that the effect of the aforementioned approximation is small provided the effective number N of hits per scan and effective $\bar{x}$ are properly chosen.

II. FORMULAS FOR P_D

CASE 1

The exact formula is

$$\begin{aligned} N = 1: P_D &= \exp \left[\frac{-Y_b}{1 + \bar{N}} \right] \\ N > 1: P_D &= 1 - I \left[\frac{Y_b}{\sqrt{N-1}}, N-2 \right] \\ &+ \left(1 + \frac{1}{N\bar{N}} \right)^{N-1} I \left[\frac{Y_b}{\left(1 + \frac{1}{N\bar{N}} \right) \sqrt{N-1}}, N-2 \right] \exp \left[\frac{-Y_b}{1+N\bar{N}} \right] \quad (II.1) \end{aligned}$$

where I is the incomplete gamma function. (3)

In most cases, this can be closely approximated by

$$P_D \approx \left(1 + \frac{1}{N\bar{N}} \right)^{N-1} \exp \left[\frac{-Y_b}{1+N\bar{N}} \right] \quad (II.2)$$

For $N = 1$, (II.2) is exact.

For $N > 1$, in most cases of interest, $N\bar{N}$ is several times greater than one. If this is true, and if the false alarm probability is sufficiently small, then, for $N > 1$, the gamma function factors are very nearly unity. It turns out that in most cases of interest, the values of $N\bar{N}$ and false alarm probability are such that (II.2) can be regarded as practically exact. (One must keep in mind, of course, that the assumptions $N\bar{N} > 1$ and negligible false alarm probability mean that one cannot use (II.2) for indefinitely small values of P_D . In most cases of interest, the applicability of (II.2) extends at least down to $P_D =$ one per cent.)

Now, for $N\bar{N} > 1$ the expansion of $\ln P_D$ as given by (II-2) is

$$\ln P_D = -\frac{1}{N\bar{N}} (Y_b - N+1) + \frac{1}{(N\bar{N})^2} \left(Y_b - \frac{N-1}{2} \right) - \frac{1}{(N\bar{N})^3} \left(Y_b - \frac{N-1}{3} \right) + \dots \quad (II.3)$$

This is a convenient way in which to compute P_D . Curves of P_D vs $\frac{R}{R_0}$ based on (II.2) are given in Figs. 1 - 3.

A good approximation to P_D can be obtained by using only the first term of this expansion, giving

$$P_D \approx \exp \left[\frac{-h(n,N)}{\bar{x}} \right] \quad (\text{II.4})$$

where

$$h(n,N) = \frac{Y_b - N + 1}{N} \quad (\text{II.5})$$

Here n is the false alarm number; Y_b is a function of both N and n .*

Since $\frac{1}{\bar{x}} = \left(\frac{R}{R_0} \right)^4$, this leads to the result

$$P_D = e^{-u^4} \quad (\text{II.6})$$

$$\frac{R}{R_0} = \frac{u}{g(n,N)}$$

and

$$g(n,N) = \left(\frac{Y_b - N + 1}{N} \right)^{1/4} \quad (\text{II.7})$$

In other words, the P_D vs $\frac{R}{R_0}$ curves are all, to a good approximation, simply $\exp(-u^4)$, with $\frac{R}{R_0} = u \cdot \text{scale factor } 1/g(n,N)$. Curves of $\exp(-u^4)$ and $g(n,N)$ are given in Figs. 4 and 5.

To see approximately the error introduced in going from (II.2) to (II.6): for each P_D let R_1 be the range computed by using just the first term of the expansion (II.3); let R_2 be the range computed by using the first two terms. Then**

* Curves of Y_b vs N and n are to be found in Ref. 2. In most cases, n is approximately equal to the false alarm time divided by the pulse width.

** See Appendix A for derivation.

$$\frac{R_2}{R_1} \approx 1 - \mathcal{E}(n, N) \ln P_D \quad (\text{II.8})$$

where

$$\mathcal{E}(n, N) = \frac{Y_b - \frac{N-1}{2}}{4(Y_b - N + 1)^2} \quad (\text{II.9})$$

Curves of $\mathcal{E}(n, N)$ are given in Fig. 6.

This is valid for $-\ln \mathcal{E}(n, N) \ln P_D$ less than about .5. Referring to the curves of $\mathcal{E}(n, N)$ in Fig. 6, this means for P_D greater than .01 - .10, depending on the values of n and N . In almost all such cases, the first two terms of (II.3) are the only significant ones, so that (II.8) can be regarded as a correction between (II.6) and (II.2).

The curve $P_D = \exp(-u^4)$ is not necessarily the best curve to use in (II.6) to give the best numerical results in approximating (II.2). It is possible to give this curve an average correction so as to obtain better results over the interesting range of the parameters n and N .

Such a corrected curve, to be used with the scale factor $g(n, N)$ to approximate (II.2), is given in Fig. 4. The use of this curve with $g(n, N)$ gives agreement with (II.2) to within about five per cent in $\frac{R}{R_0}$ for $P_D \geq .01$, $10^6 \leq n \leq 10^{12}$, and $1 \leq N \leq 1000$. The agreement gets better as P_D gets larger, and is practically exact for $P_D \geq .50$.

CASE 2

The exact formula for P_D is

$$P_D = 1 - I \left[\frac{Y_b}{(1+\bar{x}) \sqrt{N}}, N - 1 \right] \quad (\text{II.10})$$

Reference 3 contains tables of the incomplete gamma function enabling the computation of (II.10) for $N - 1 \leq 50$. Beyond this point an Edgeworth series can be used to compute P_D :

$$P_D \approx \frac{1}{2} \left[1 - \phi^{-1}(T) \right] - c_3 \phi^{(2)}(T) - c_4 \phi^{(3)}(T) - c_6 \phi^{(5)}(T) \quad (\text{II.11})$$

where

$$\begin{aligned} \phi(t) &= \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{t^2}{2} \right] \\ \phi^{-1}(T) &= \frac{1}{\sqrt{2\pi}} \int_{-T}^T \exp \left(-\frac{t^2}{2} \right) dt \\ \phi^{(2)}, \phi^{(3)}, \phi^{(5)} &\text{ are derivatives of } \phi \end{aligned} \quad (\text{II.12})$$

and

$$\begin{aligned} T &= \frac{Y_b - N(1+\bar{x})}{\sqrt{N} (1+\bar{x})} \\ c_3 &= \frac{1}{3\sqrt{N}}; \quad c_4 = \frac{1}{4N}; \quad c_6 = \frac{1}{18N} \end{aligned} \quad (\text{II.13})$$

Curves of P_D vs $\frac{R}{R_0}$ for this case are given in Figs. 7 - 9.

CASE 3

The exact formula in this case is so cumbersome as not to be worth writing out here. (The exact density function is given in Eq. (III.18).)

The formula obtained by assuming the negligibility of false alarm probability, and $\frac{NX}{2} > 1$, is

$$P_D \approx \left(1 + \frac{2}{NX} \right)^{N-2} \left[1 + \frac{Y_b}{1 + \frac{NX}{2}} - \frac{2}{NX} (N-2) \right] \exp \left[\frac{-Y_b}{1 + \frac{NX}{2}} \right] \quad (\text{II.14})$$

This turns out to be the exact formula for $N = 1$ and $N = 2$. Curves of P_D vs $\frac{R}{R_0}$ based on (II.14) are given in Figs. 10 - 12.

A close approximation to (II.14) is given by

$$P_D \approx \left[1 + \frac{2}{\bar{x}} \left(\frac{Y_b - N + 2}{N} \right) \right] \exp \left[- \frac{2}{\bar{x}} \left(\frac{Y_b - N + 2}{N} \right) \right] \quad (\text{II.15})$$

and since $\frac{1}{\bar{x}} = \left(\frac{R}{R_0} \right)^4$, this gives

$$P_D \approx (1 + 2u^4) \exp \left[- 2u^4 \right] \quad (\text{II.16})$$

$$\frac{R}{R_0} = \frac{u}{f(n, N)}$$

where

$$f(n, N) = \left(\frac{Y_b - N + 2}{N} \right)^{1/4} \quad (\text{II.17})$$

The scale factor $f(n, N)$ is almost identical to the factor $g(n, N)$ of Case 1, the maximum difference being about 2 per cent for small N , and negligible for moderately large N . Curves of $(1 + 2u^4) \exp \left[- 2u^4 \right]$ and $f(n, N)$ are given in Figs. 13 and 14.

Actually, over the interesting range of parameters, the curve $P_D = (1 + 2u^4) \exp \left[- 2u^4 \right]$ is not the best curve to use in (II.16) to give the best numerical results in approximating (II.14). It is possible to give this curve an average correction so as to obtain better results over the interesting range of n and N .

Such a corrected curve, to be used with the scale factor $f(n, N)$ to approximate (II.14), is given in Fig. 13. The use of this curve with $f(n, N)$ gives agreement with (II.14) to within about seven per cent in $\frac{R}{R_0}$ for $P_D \geq .01$, $10^6 \leq n \leq 10^{12}$, and $1 \leq N \leq 1000$. The agreement gets better as P_D gets larger, and is practically exact for $P_D \geq .50$.

CASE 4

The exact density function $f(v)$ is a polynomial multiplied by $\exp \left[\frac{-v}{\left(1 + \frac{\bar{x}}{2}\right)} \right]$. Except for small N , the polynomial is too long to be of much use for computation. Hence an Edgeworth series seems to be the best way to compute P_D except for small N .

For $N = 1$, the exact formula is

$$P_D = \left(1 + \frac{2}{\bar{x}}\right)^{-1} \left[1 + \frac{2}{\bar{x}} + \frac{Y_b}{1 + \frac{\bar{x}}{2}}\right] \exp \left[\frac{-Y_b}{1 + \frac{\bar{x}}{2}} \right] \quad (\text{II.18})$$

For N moderately large, the Edgeworth series is given by (II.11) with, letting $\beta = 1 + \frac{\bar{x}}{2}$,

$$\begin{aligned} T &= \frac{Y_b - N(1 + \bar{x})}{\sqrt{N(2\beta^2 - 1)}} \\ C_3 &= \frac{1}{3\sqrt{N}} \frac{(2\beta^3 - 1)}{(2\beta^2 - 1)^{3/2}} \\ C_4 &= \frac{1}{4N} \frac{(2\beta^4 - 1)}{(2\beta^2 - 1)^2} \\ C_6 &= \frac{1}{18N} \frac{(2\beta^3 - 1)^2}{(2\beta^2 - 1)^3} \end{aligned} \quad (\text{II.19})$$

Curves of P_D vs $\frac{R}{R_0}$ for this case are given in Figs. 15 - 17.

III. DERIVATION OF FORMULAS FOR P_D

The characteristic function for the sum, v , of N normalized pulses of signal plus noise, with constant signal-to-noise power ratio x , for a square law detector, is*

$$C(p) = \frac{\exp \left[-Nx \left(\frac{p}{p+1} \right) \right]}{(p+1)^N} \quad (\text{III.1})$$

If x is now a random variable with density function $w(x, \bar{x})$, and x is constant for each group of N pulses but fluctuates independently from group to group, then the characteristic function for the sum of N pulses is, since $w(x, \bar{x}) = 0$ for $x < 0$,

$$\bar{C}(p) = \int_0^{\infty} w(x, \bar{x}) C(p) dx \quad (\text{III.2})$$

CASE 1

$$\text{Here } w(x, \bar{x}) = \frac{1}{\bar{x}} e^{-x/\bar{x}} \quad (x \geq 0)$$

therefore

$$\bar{C}(p) = \frac{1}{(1+p)^{N-1} \left[1+p(1+N\bar{x}) \right]} \quad (\text{III.3})$$

For $N = 1$, the density function is⁽⁴⁾

$$f(v) = \frac{1}{1+\bar{x}} \exp \left[\frac{-v}{1+\bar{x}} \right] \quad (v \geq 0) \quad (\text{III.4})$$

For $N > 1$, the density function can be found from pair 581.7 of Ref. 4:

$$f(v) = \left(1 + \frac{1}{N\bar{x}} \right)^{N-2} \frac{1}{N\bar{x}} I \left[\frac{\bar{x}}{\left(1 + \frac{1}{N\bar{x}} \right) \sqrt{N-1}}, N-2 \right] \exp \left[\frac{-v}{1+N\bar{x}} \right] \quad (\text{III.5})$$

therefore, for $N = 1$,

* In the notation used here, if $f(v)$ is the density function and

$$\zeta(s) = \int f(v) e^{-isv} dv \text{ then } p = is \text{ and } C(p) = \zeta(s).$$

$$P_D = \exp \left[\frac{-Y_b}{1+\bar{x}} \right] \quad (N = 1) \quad (\text{III.6})$$

For $N > 1$, a simple way to get P_D is:

$$\bar{C}(p) - \frac{1}{(1+p)^{N-1}} = -p (1 + N\bar{x}) \bar{C}(p)$$

so

$$f(v) - \frac{v^{N-2} e^{-v}}{(N-2)!} = - (1 + N\bar{x}) f'(v)$$

and therefore

$$\int_0^{Y_b} f(v) dv = I \left[\frac{Y_b}{\sqrt{N-1}}, N-2 \right] - (1 + N\bar{x}) f(Y_b)$$

Since $P_D = 1 - \int_0^{Y_b} f(v) dv$, we have for $N > 1$:

$$P_D = 1 - I \left[\frac{Y_b}{\sqrt{N-1}}, N-2 \right] + \left(1 + \frac{1}{N\bar{x}} \right)^{N-1} I \left[\frac{Y_b}{\left(1 + \frac{1}{N\bar{x}} \right) \sqrt{N-1}}, N-2 \right] \exp \left[\frac{-Y_b}{1 + N\bar{x}} \right] \quad (\text{III.7})$$

CASE 2

The characteristic function for one pulse from a target fluctuating according to (I.1) is (from III.3 for $N = 1$):

$$\bar{C}_1(p) = \frac{1}{1 + p(1 + \bar{x})} \quad (\text{III.8})$$

Therefore the characteristic function for the sum of N pulses, fluctuating independently from pulse to pulse, is

$$\left[\bar{C}_1(p) \right]^N = \frac{1}{[1 + p(1 + \bar{x})]^N} \quad (\text{III.9})$$

The density function is

$$f(v) = \frac{1}{(1+\bar{x})^N (N-1)!} v^{N-1} \exp \left[\frac{-v}{1+\bar{x}} \right] \quad (\text{III.10})$$

and therefore

$$P_D = 1 - I \left[\frac{Y_b}{(1+\bar{x}) \sqrt{N}}, N-1 \right] \quad (\text{III.11})$$

Reference 3 contains tables of this function up to $N-1 = 50$. Beyond this, one can use an Edgeworth series⁽²⁾ to compute P_D . This can be done if one knows the cumulants K_n for the density function $f(v)$ (as well as the mean).⁽²⁾

In this case

$$K_n = N(-1)^{n+1} \left\{ \frac{d^n}{dp^n} \left[\ln(1+p(1+\bar{x})) \right] \right\}_{p=0}$$

or

$$K_n = N(n-1)! (1+\bar{x})^n \quad (\text{III.12})$$

In addition, it is easily found that the mean of the sum of N pulses is $N(1+\bar{x})$; the variance is $N(1+\bar{x})^2$. Therefore the coefficients of the second, third, and fourth terms of the Edgeworth series are

$$C_3 = \frac{1}{3\sqrt{N}}; C_4 = \frac{1}{4N}; C_6 = \frac{1}{18N} \quad (\text{III.13})$$

CASE 3

Here $w(x, \bar{x}) = \frac{4x}{\bar{x}^2} e^{-2x/\bar{x}}$ and therefore

$$\bar{C}(p) = \frac{1}{(1+p)^{N-2} \left[1 + p \left(1 + \frac{N\bar{x}}{2} \right) \right]^2} \quad (\text{III.14})$$

For $N = 1$ and $N = 2$, the density function is obtained from Ref. 4 as:

$$N = 1: f(v) = \frac{1}{\left(1 + \frac{\bar{x}}{2}\right)^2} \left[1 + \frac{v}{1 + \frac{\bar{x}}{2}}\right] \exp\left[\frac{-v}{1 + \frac{\bar{x}}{2}}\right]$$

$$N = 2: f(v) = \frac{1}{(1 + \bar{x})^2} v \exp\left[\frac{-v}{1 + \bar{x}}\right] \quad (\text{III.15})$$

From Ref. 4, pair 581.1, after appropriate transformation, one finds that for $N > 2$,

$$f(v) = \frac{1}{(N-1)! \left(1 + \frac{N\bar{x}}{2}\right)^2} v^{N-1} e^{-v} {}_1F_1\left[2, N, \frac{v}{1 + \frac{N\bar{x}}{2}}\right] \quad (\text{III.16})$$

where ${}_1F_1$ is the confluent hypergeometric function.*

One may now use two identities concerning this hypergeometric function.**

$$a) {}_1F_1(2, N, z) = (z + 2 - N) {}_1F_1(1, N, z) + N - 1$$

$$b) {}_1F_1(1, N, z) = e^z z^{-N+1} (N-1)! I\left[\frac{z}{\sqrt{N-1}}, N-2\right] \quad (\text{III.17})$$

to get

$$f(v) = \frac{\left(1 + \frac{2}{N\bar{x}}\right)^{N-2}}{\left(1 + \frac{N\bar{x}}{2}\right)^2} v I\left[\frac{v}{\left(1 + \frac{2}{N\bar{x}}\right)\sqrt{N-1}}, N-2\right] \exp\left[\frac{-v}{1 + \frac{N\bar{x}}{2}}\right]$$

$$- \frac{(N-2) \left(1 + \frac{2}{N\bar{x}}\right)^{N-1}}{\left(1 + \frac{N\bar{x}}{2}\right)^2} I\left[\frac{v}{\left(1 + \frac{2}{N\bar{x}}\right)\sqrt{N-1}}, N-2\right] \exp\left[\frac{-v}{1 + \frac{N\bar{x}}{2}}\right]$$

$$+ \frac{1}{\left(1 + \frac{N\bar{x}}{2}\right)^2 (N-2)!} v^{N-1} e^{-v} \quad (\text{III.18})$$

* See Refs. 2 and 4.

** Reference 2, mathematical appendix, pp 20-21. (In this reference there is a misprint in the printing of identity (b)).

If one assumes the false alarm probability to be negligible, and $\frac{N\bar{x}}{2}$ to be >1 , then for $v > Y_b$, the I factors are practically equal to one; also, the integral from Y_b to ∞ of the third term in (III.18) can be neglected, for $N > 2$. Therefore, integrating (III.18) from Y_b to ∞ , one obtains for $N > 2$:

$$P_D \approx \left(1 + \frac{2}{N\bar{x}}\right)^{N-2} \left[1 + \frac{Y_b}{1 + \frac{N\bar{x}}{2}} - \frac{2(N-2)}{N\bar{x}}\right] \exp\left[\frac{-Y_b}{1 + \frac{N\bar{x}}{2}}\right] \quad (\text{III.19})$$

This is the exact formula for $N = 1$ and $N = 2$, as can be found directly from (III.15).

CASE 4

For one pulse (setting $N = 1$ in (III.14))

$$\bar{C}_1(p) = \frac{1 + p}{\left[1 + p \left(1 + \frac{\bar{x}}{2}\right)\right]^2} \quad (\text{III.20})$$

Therefore for the sum of N pulses, fluctuating independently from pulse to pulse, the characteristic function is

$$\left[\bar{C}_1(p)\right]^N = \frac{(1 + p)^N}{\left[1 + p \left(1 + \frac{\bar{x}}{2}\right)\right]^{2N}} \quad (\text{III.21})$$

The exact density function is obtainable and turns out to be a polynomial multiplied by $\exp\left[-v/\left(1 + \frac{\bar{x}}{2}\right)\right]$. Except for small N , this polynomial is too long to be useful for computational purposes. For fairly large N , one can use the Edgeworth series. For $N = 1$, the exact formula is, of course, given by (III.15) with $N = 1$.

The cumulants, obtained in the usual way, are

$$K_n = N(n-1)! \left[2\beta^n - 1\right] \text{ where } \beta = 1 + \frac{\bar{x}}{2} \quad (\text{III.22})$$

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Also, mean = $N(1 + \bar{x})$

variance = $N(2\beta^2 - 1)$

So, the coefficients for the second, third, and fourth terms of the

Edgeworth series are

$$c_3 = \frac{1}{3\sqrt{N}} \frac{(2\beta^3 - 1)}{(2\beta^2 - 1)^{3/2}}$$

$$c_4 = \frac{1}{4N} \frac{(2\beta^4 - 1)}{(2\beta^2 - 1)^2}$$

$$c_6 = \frac{1}{18N} \frac{(2\beta^3 - 1)^2}{(2\beta^2 - 1)^3} \quad (\text{III.23})$$

Appendix A

DERIVATION OF EQUATION II.8

Let

$$f_1 = \frac{Y_b - N + 1}{N} ; \quad f_2 = \frac{Y_b - \frac{N-1}{2}}{N^2}$$

then (referring to page 6 for definition of R_1 and R_2)

$$\left(\frac{R_1}{R_o}\right)^4 = \frac{-\ln P_D}{f_1} \quad (A1)$$

$$\ln P_D = -f_1 \left(\frac{R_2}{R_o}\right)^4 + f_2 \left(\frac{R_2}{R_o}\right)^8 \quad (A2)$$

so

$$\left(\frac{R_2}{R_o}\right)^4 = \frac{f_1}{2f_2} \left[1 - \sqrt{1 + \frac{4f_2 \ln P_D}{f_1^2}} \right] \quad (A3)$$

then, assuming $\frac{4f_2 \ln P_D}{f_1^2} \ll 1$,

$$\frac{R_2}{R_1} \approx 1 - \frac{f_2}{4f_1} \ln P_D = 1 - \xi(n, N) \ln P_D \quad (A4)$$

where

$$\xi(n, N) = \frac{Y_b - \frac{N-1}{2}}{4(Y_b - N + 1)^2} \quad (A5)$$

$$(III.23) \quad c_3 = \frac{1}{3\sqrt{N}} \frac{(2\beta^3-1)}{(2\beta^2-1)^{3/2}}$$

$$c_4 = \frac{1}{4N} \frac{(2\beta^4-1)}{(2\beta^2-1)^2}$$

$$c_6 = \frac{1}{18N} \frac{(2\beta^3-1)^2}{(2\beta^2-1)^3}$$

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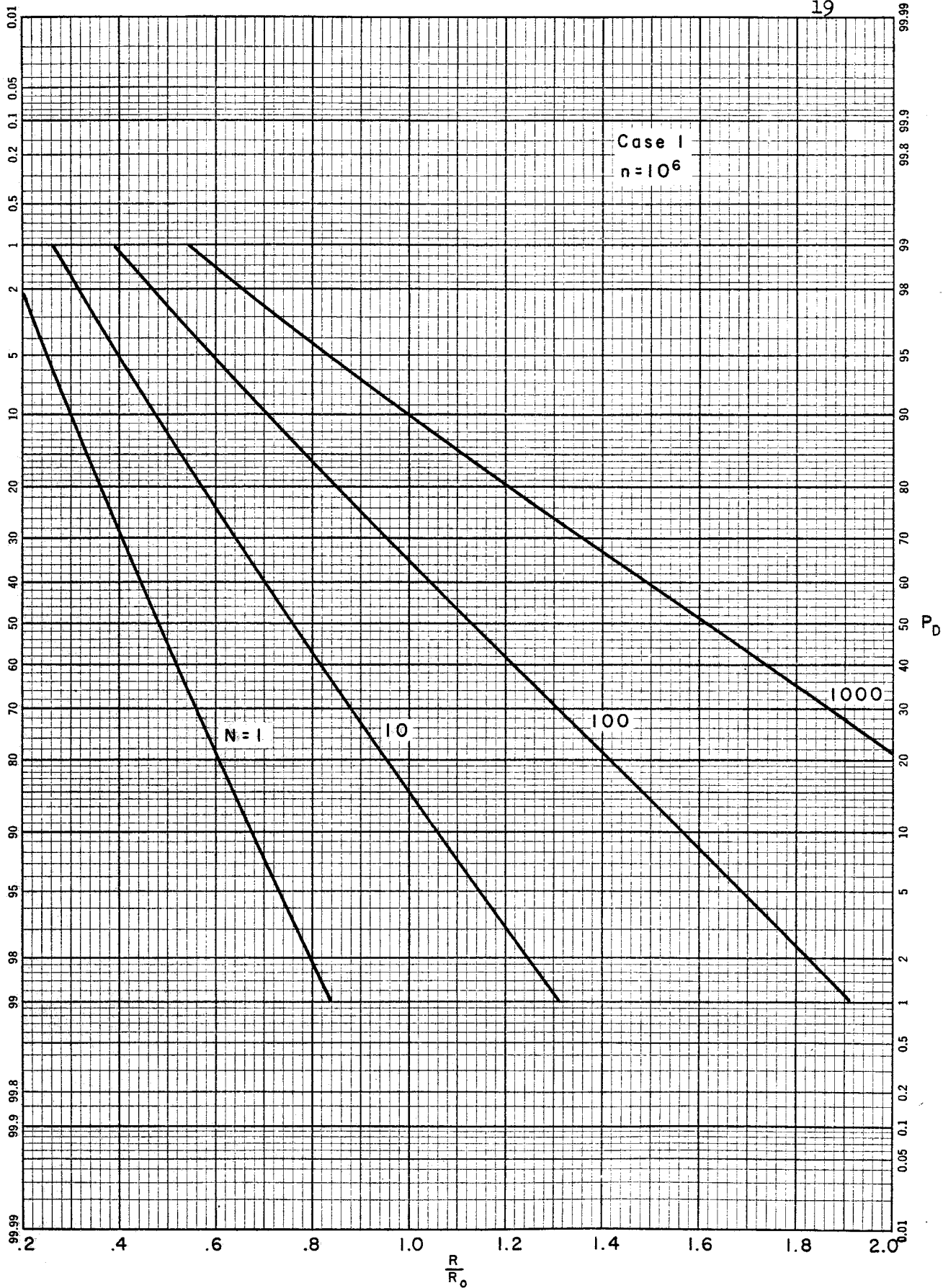


Fig. 1

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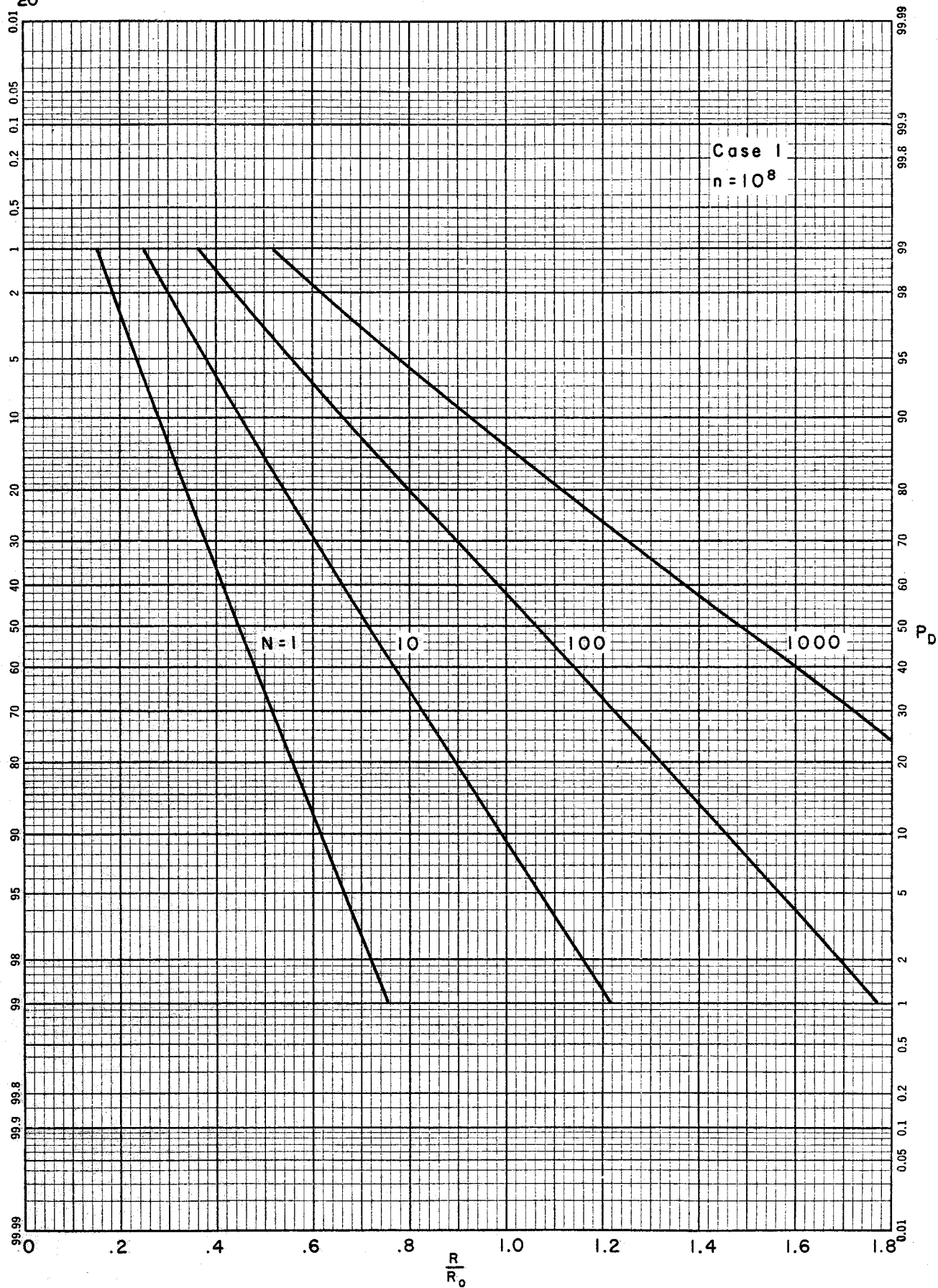


Fig. 2

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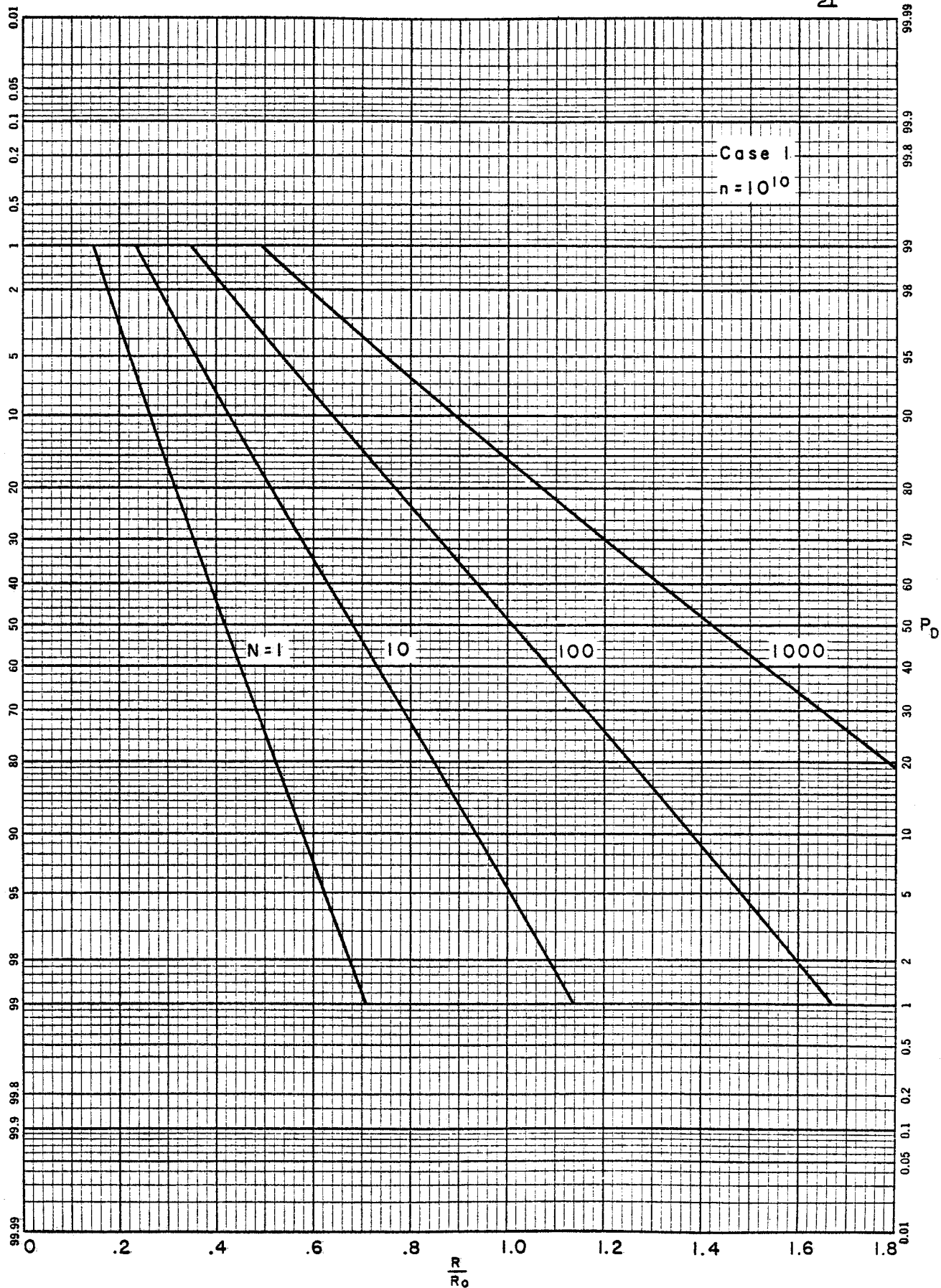
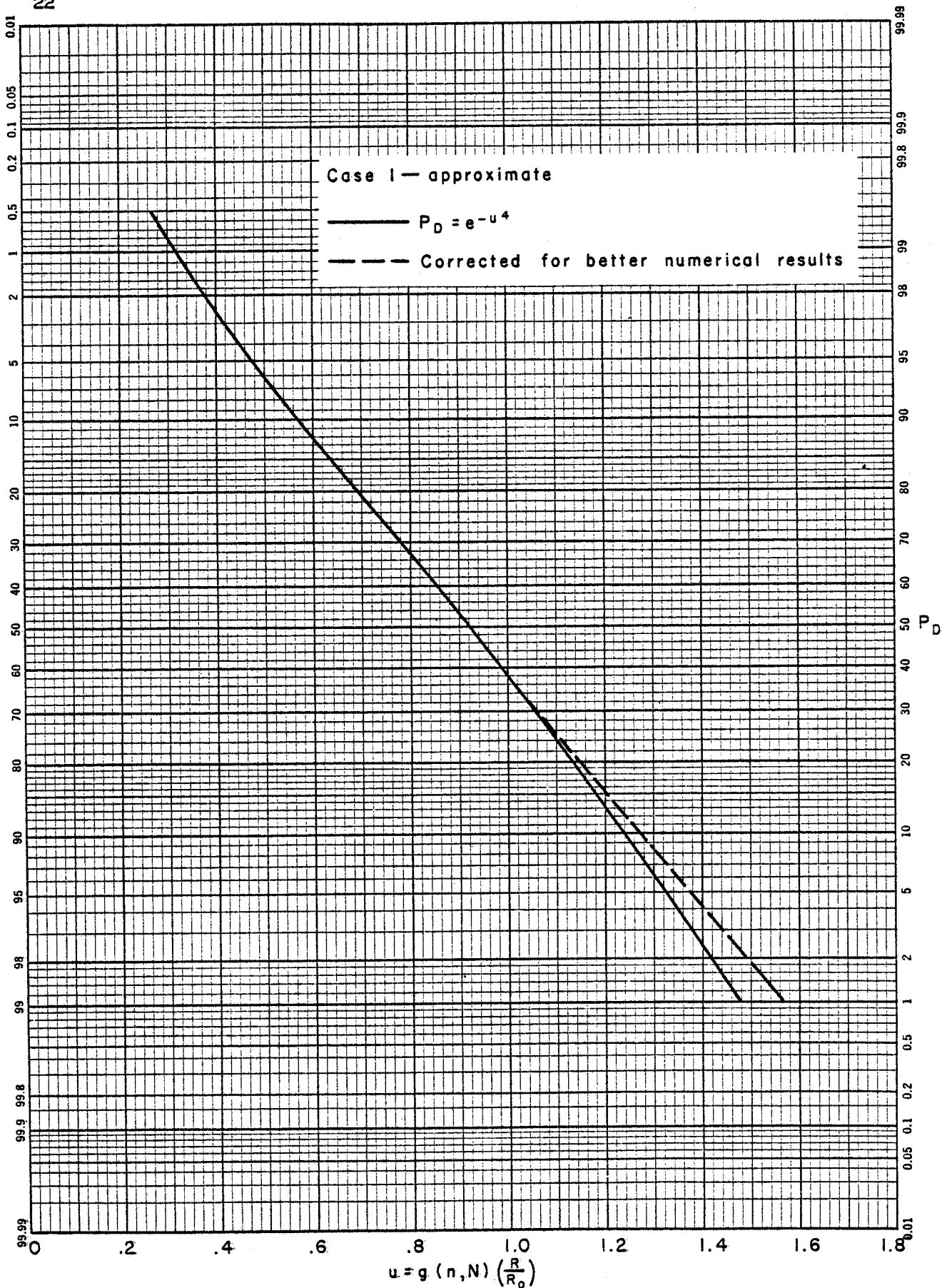
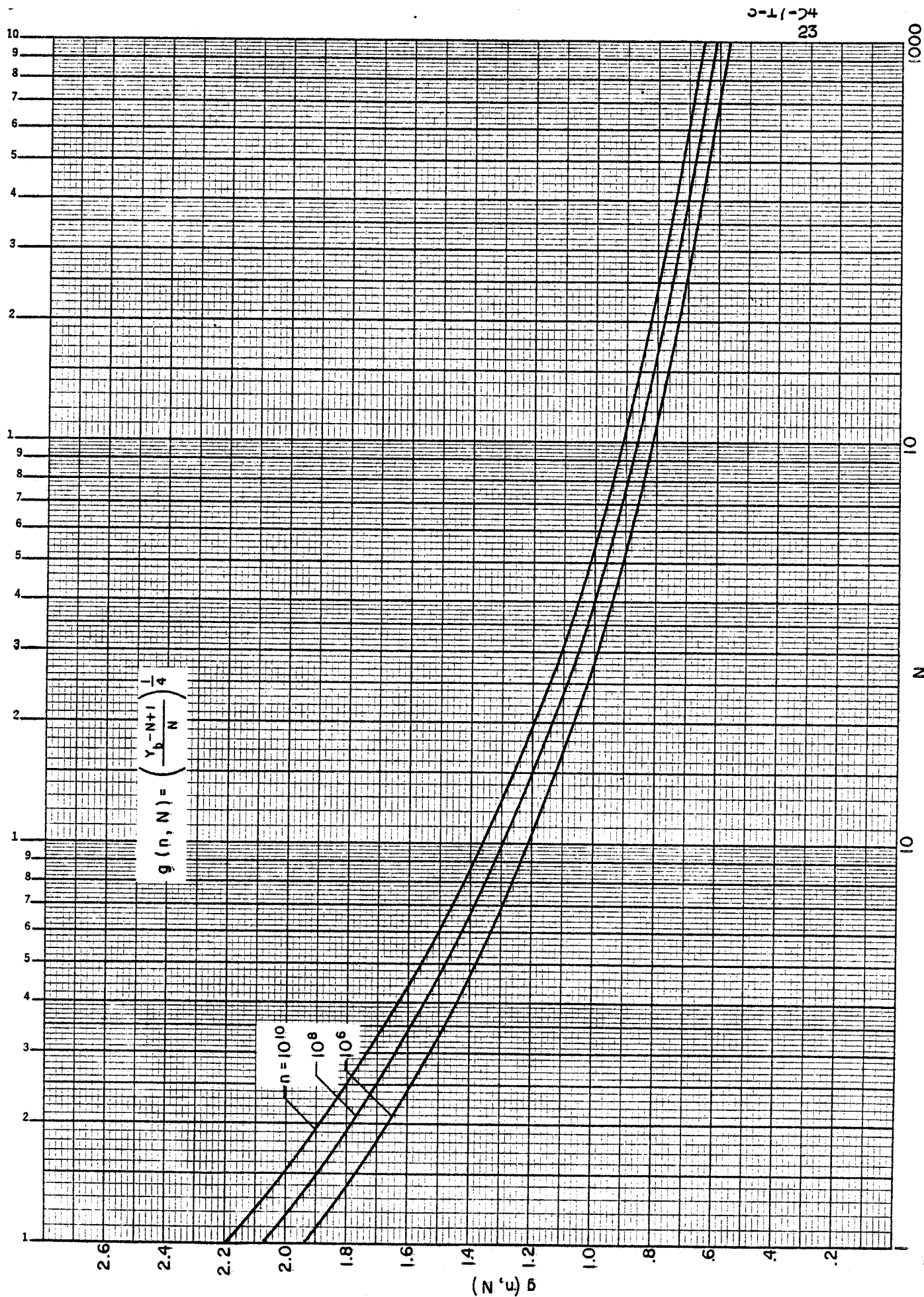


Fig 3

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Fig. 5

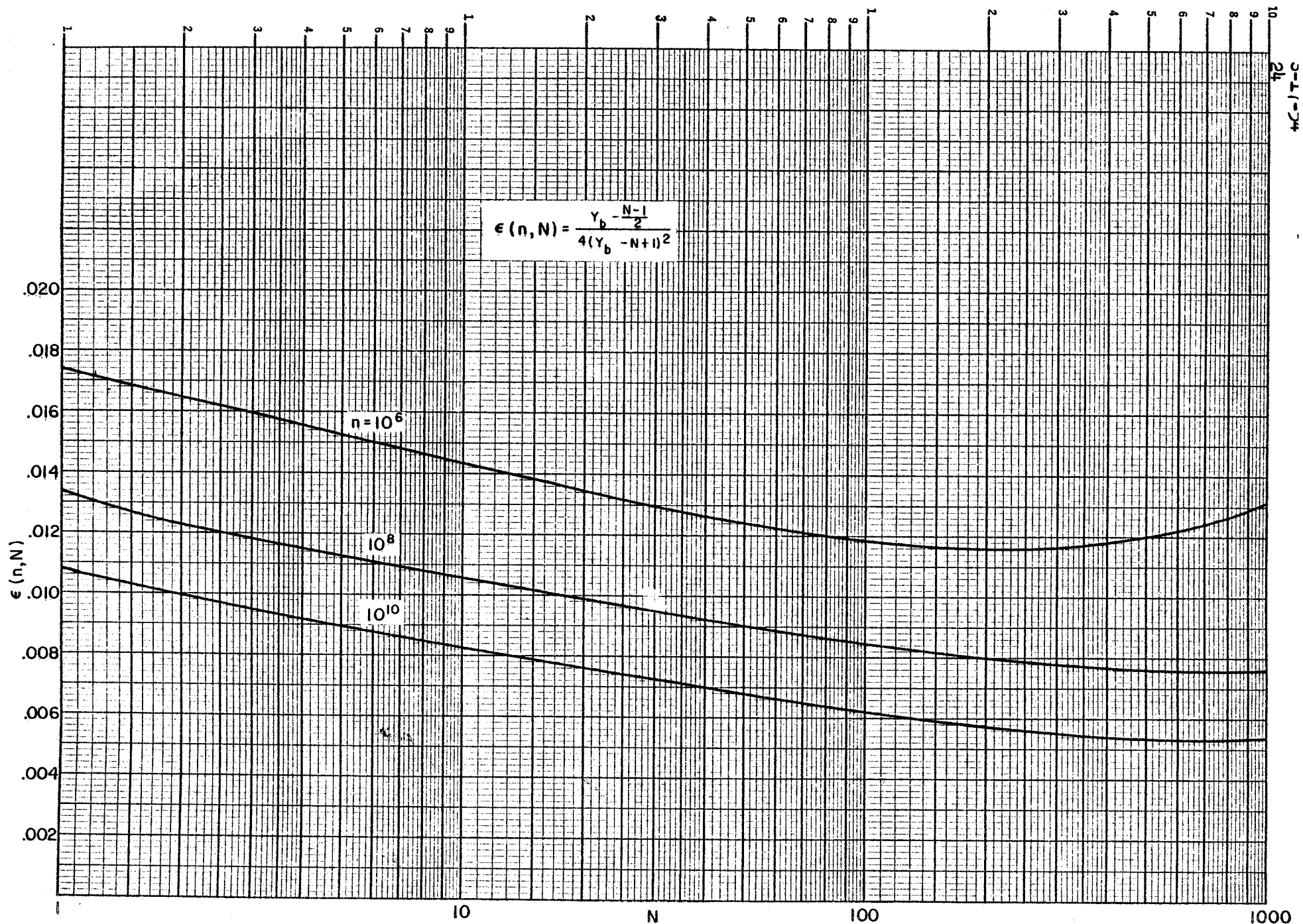


Fig. 6

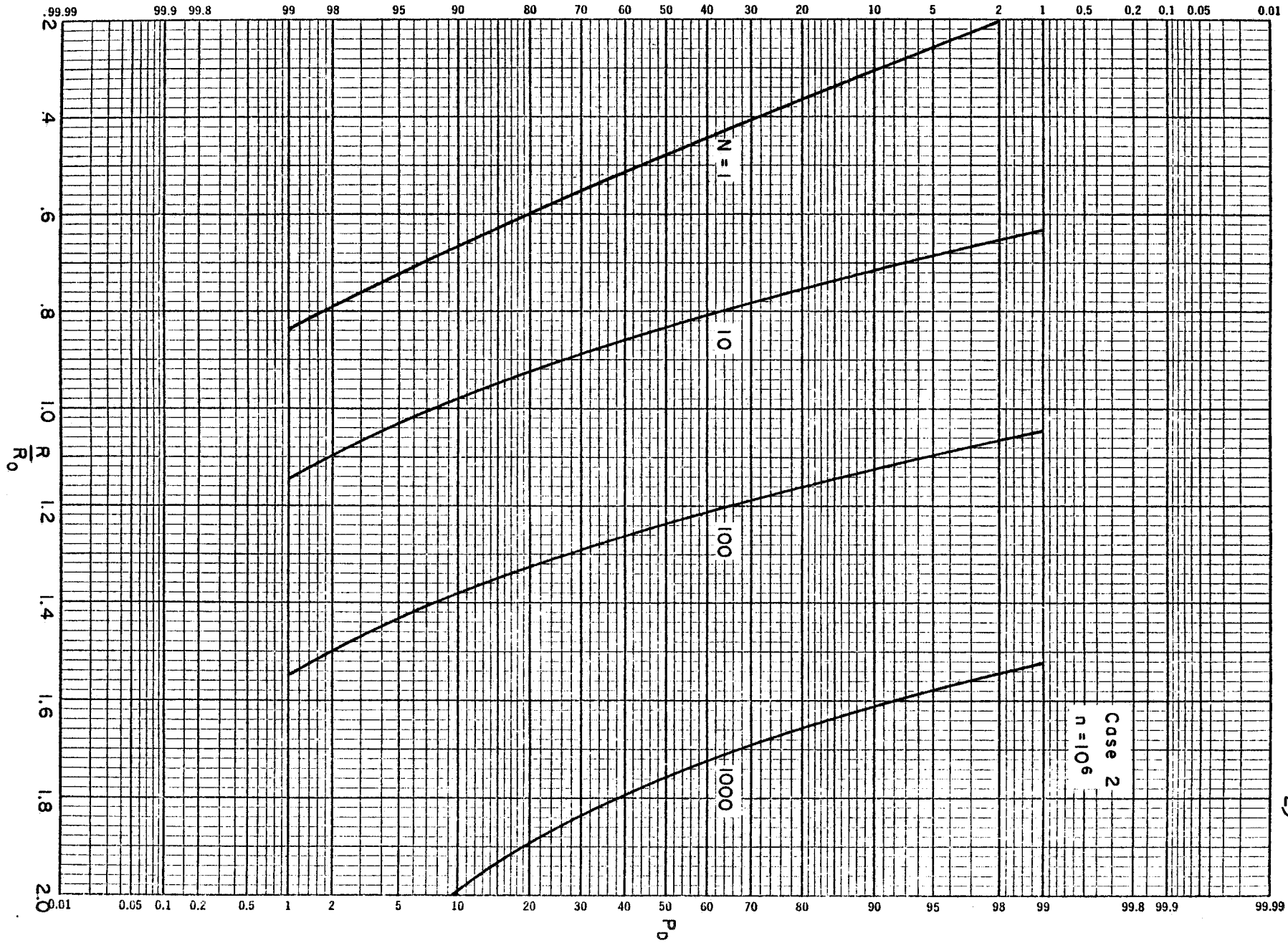


Fig. 7

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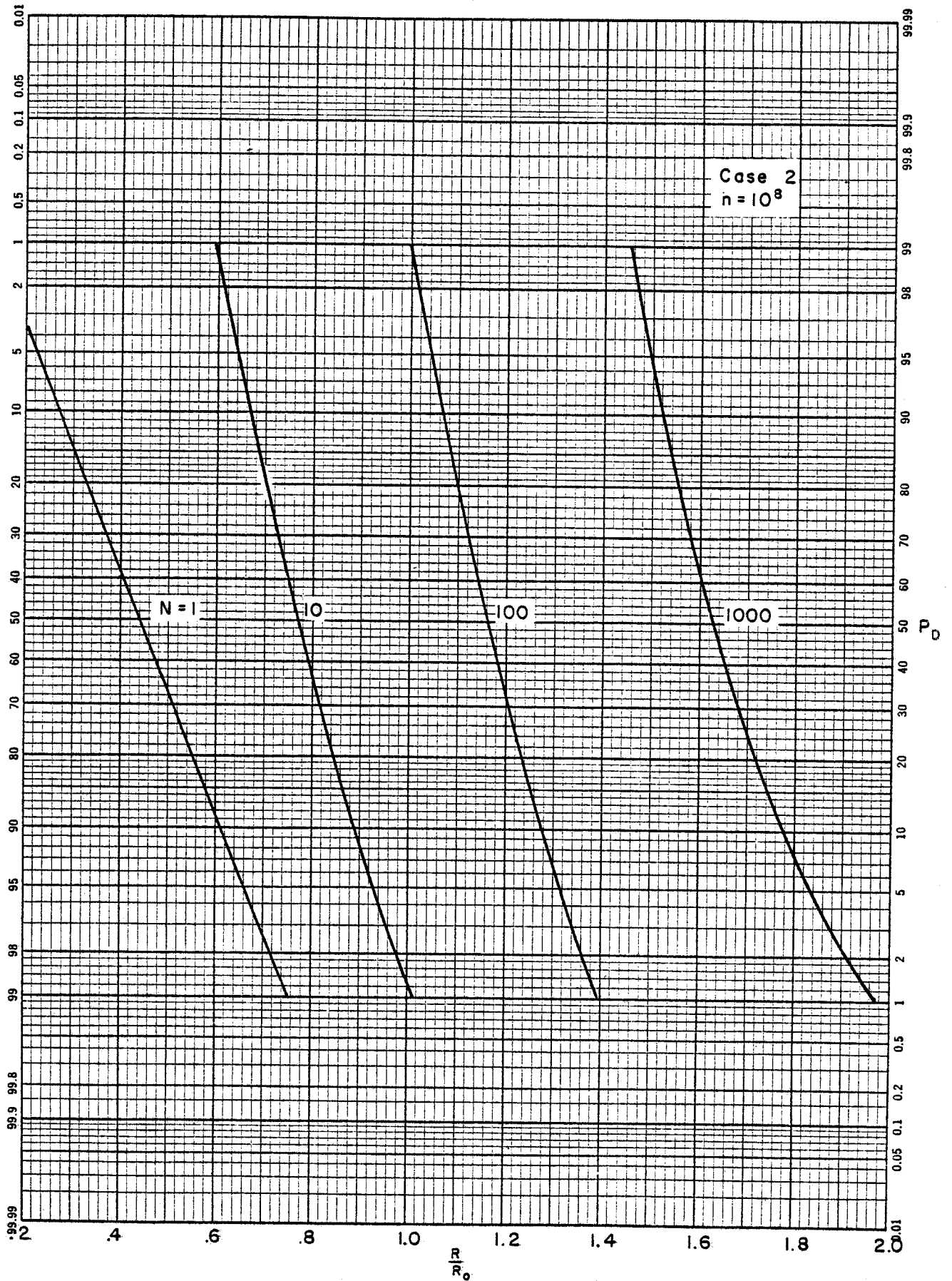


Fig. 8

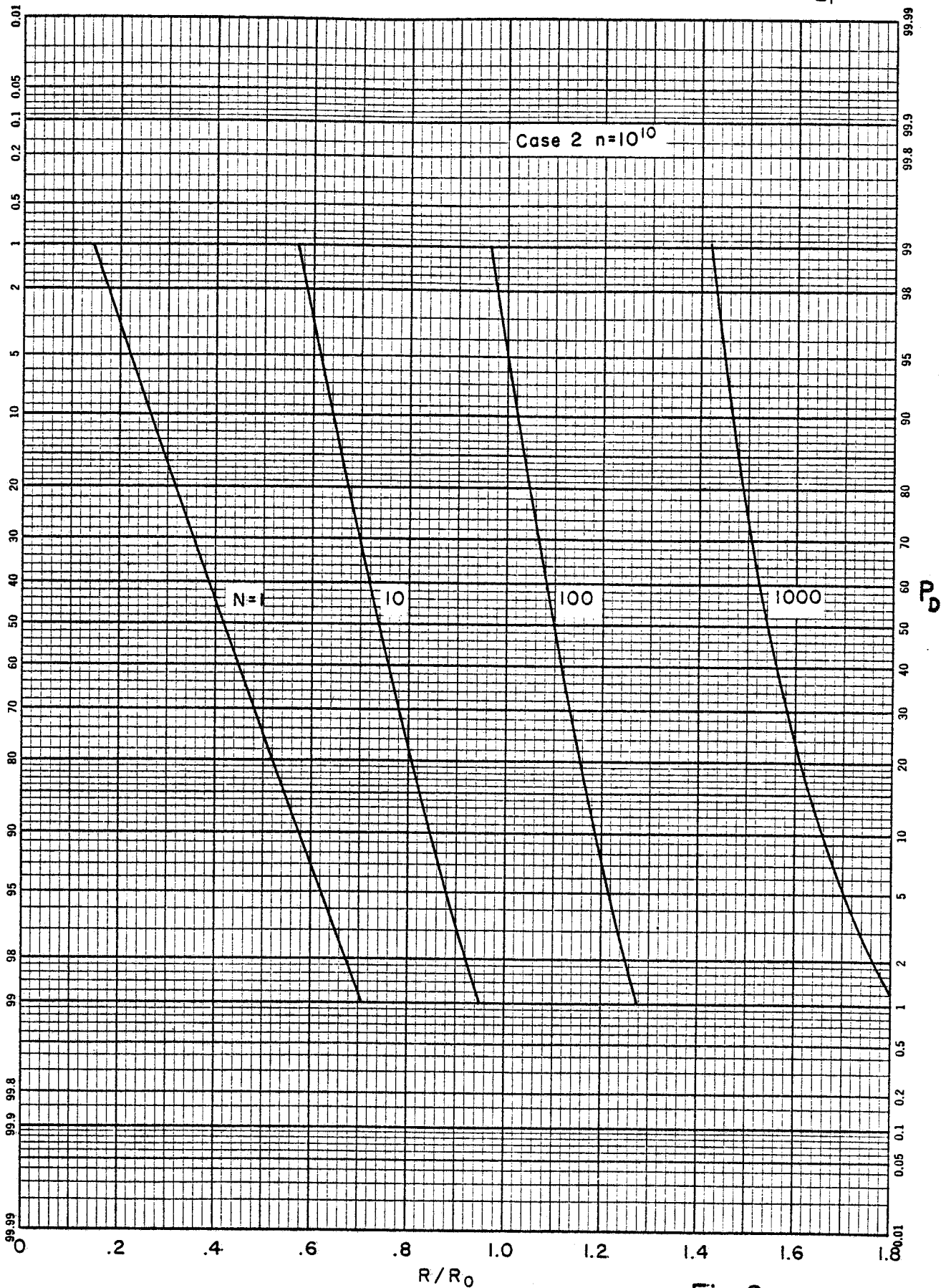


Fig. 9

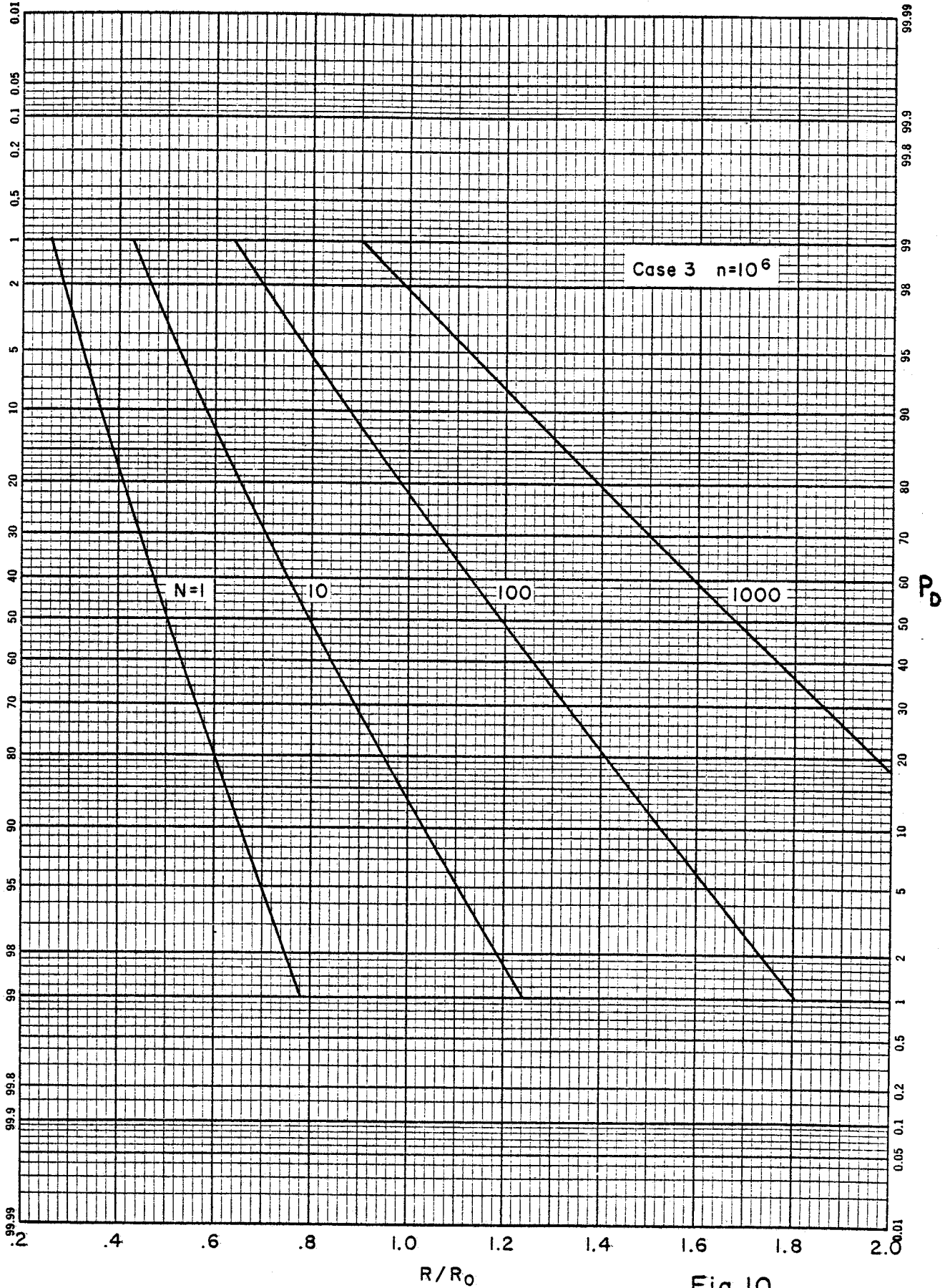


Fig. 10

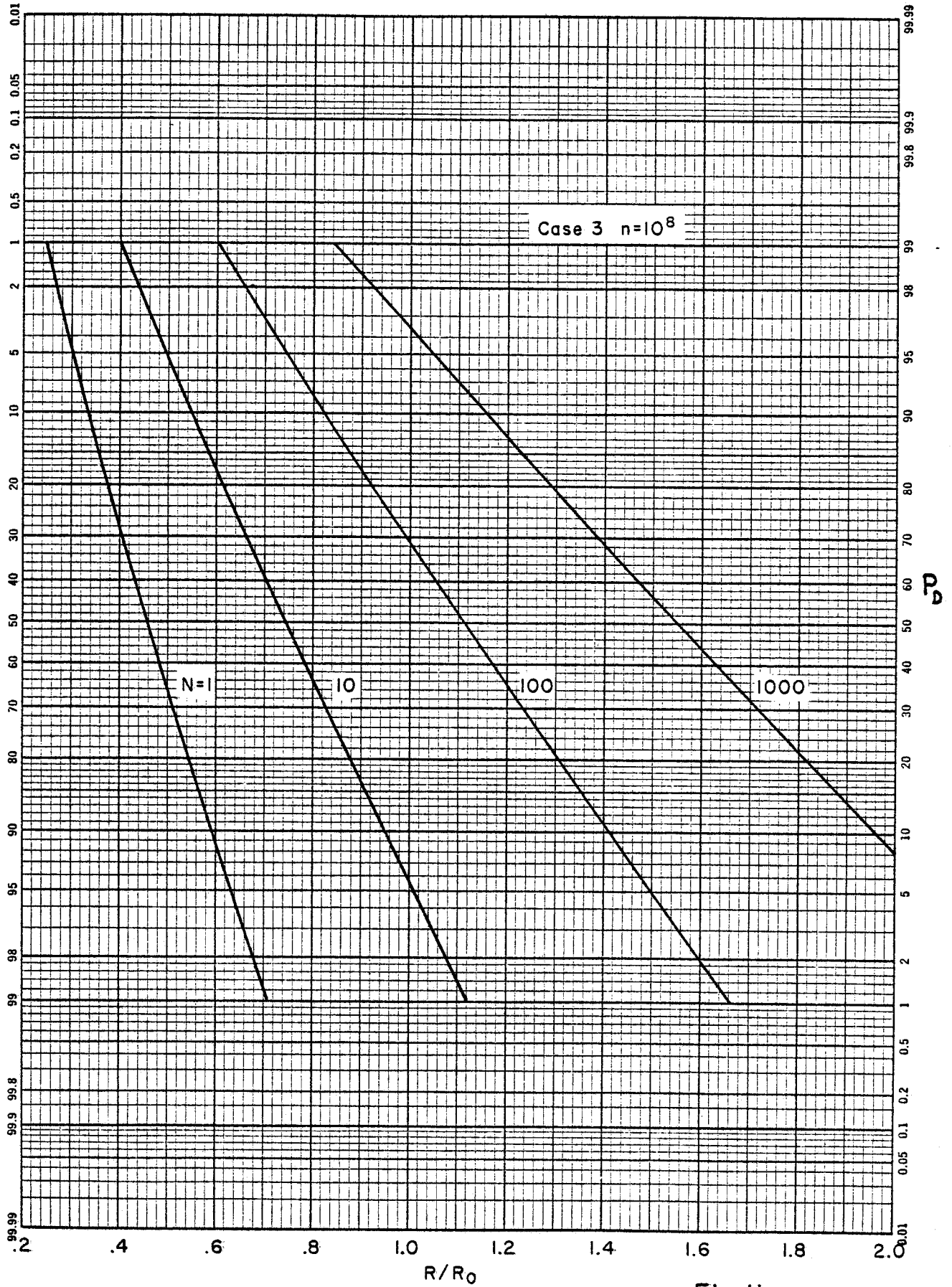


Fig. 11

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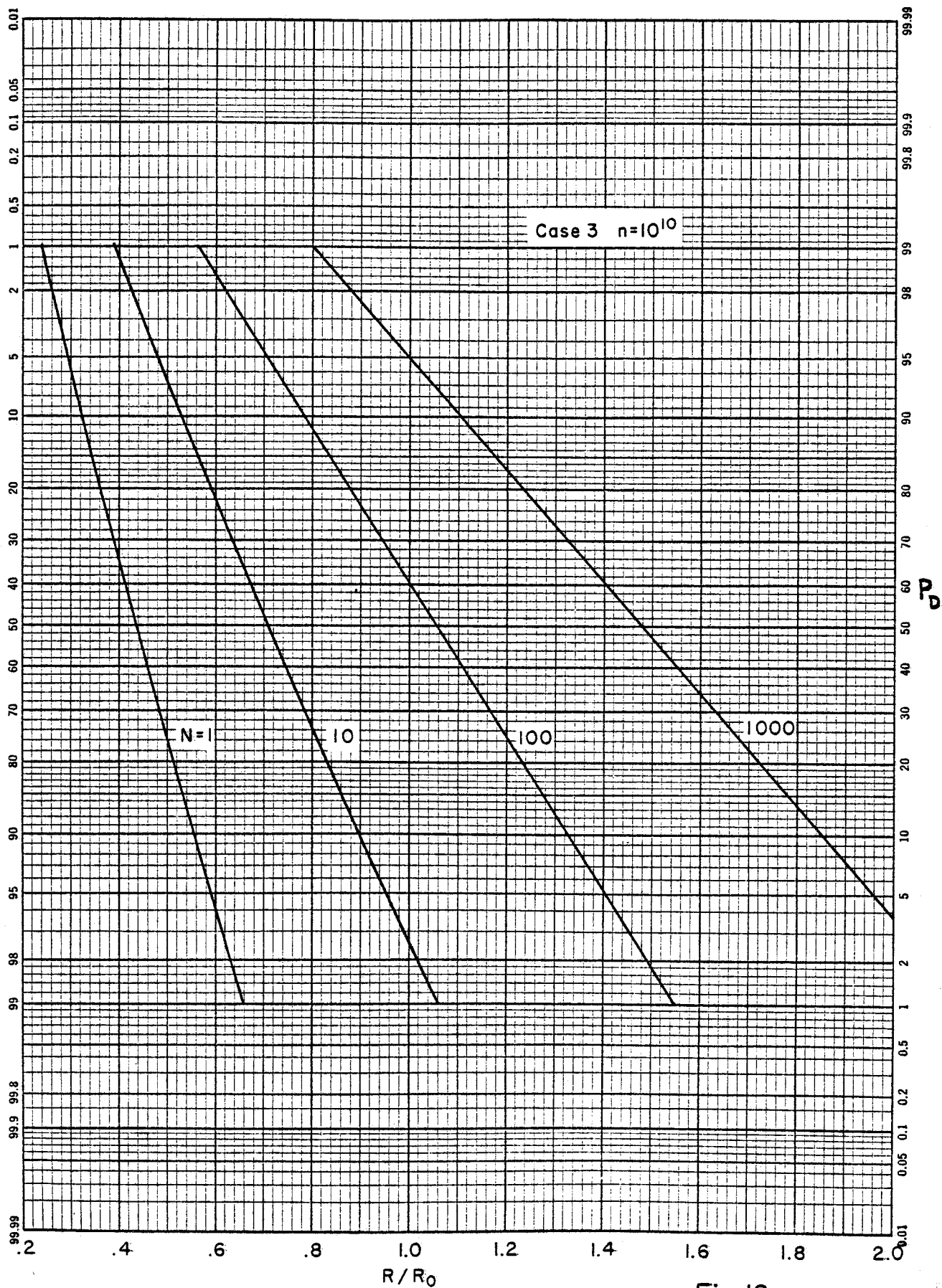


Fig.12

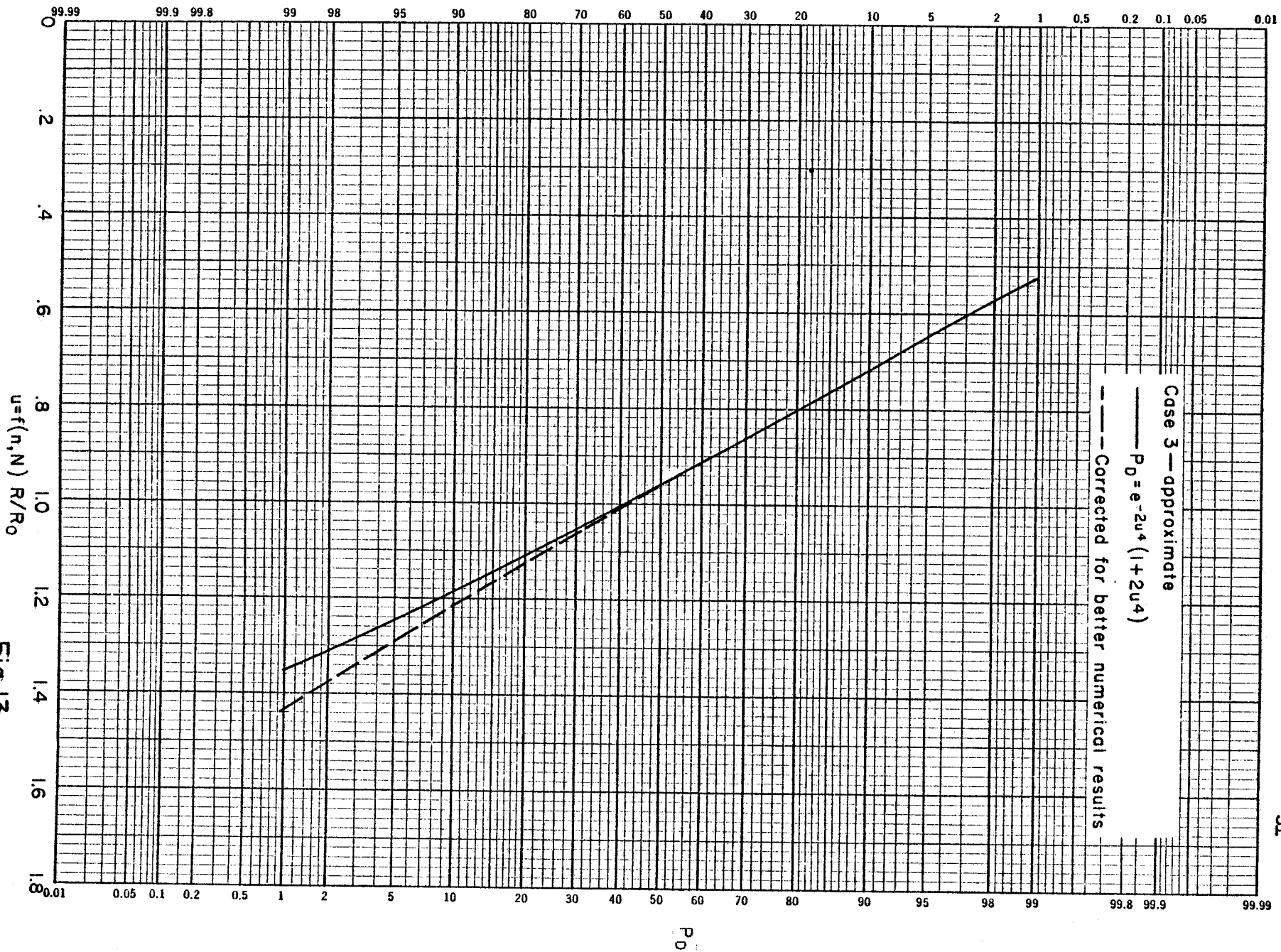


Fig. 13

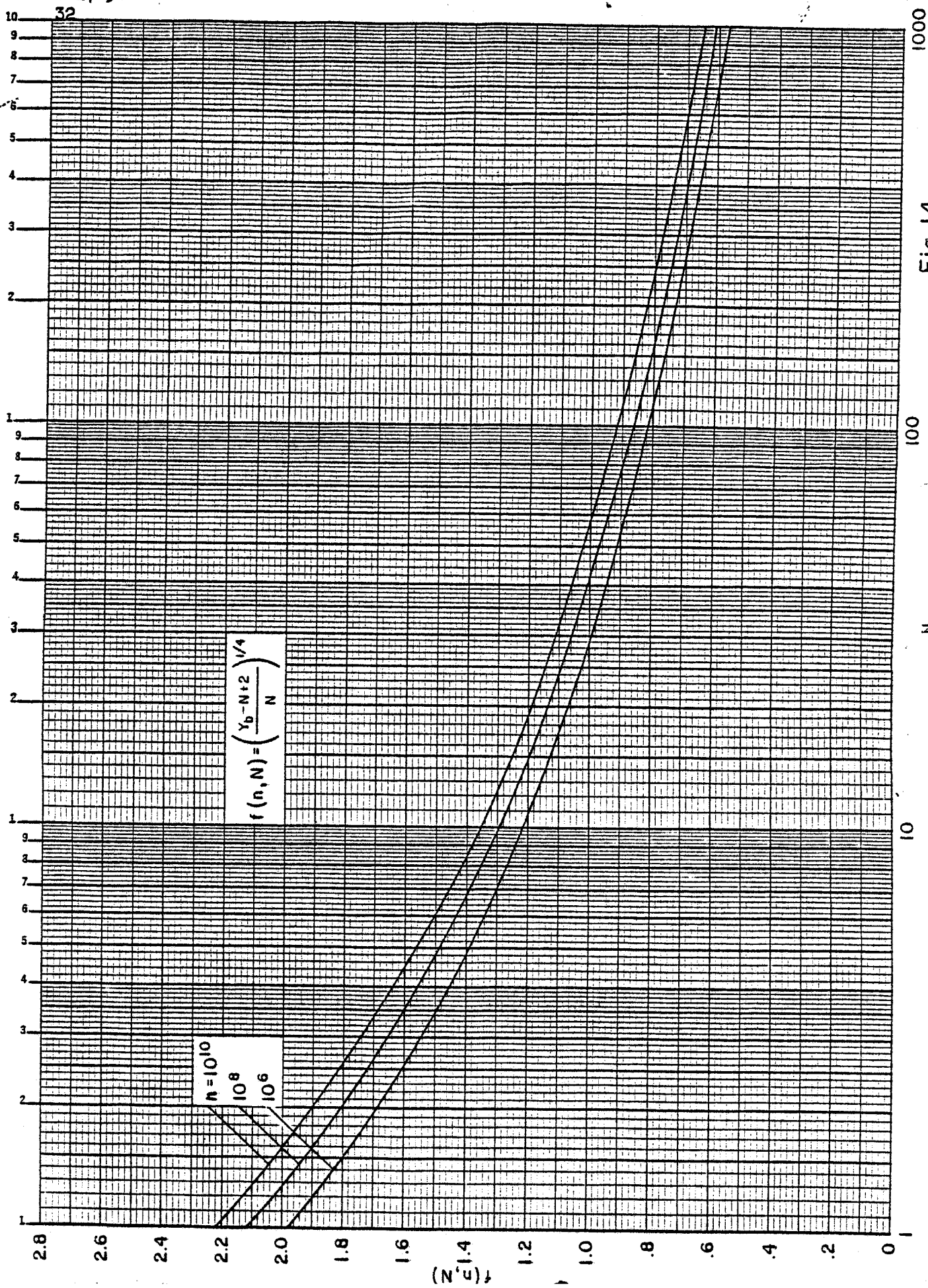


Fig. 14

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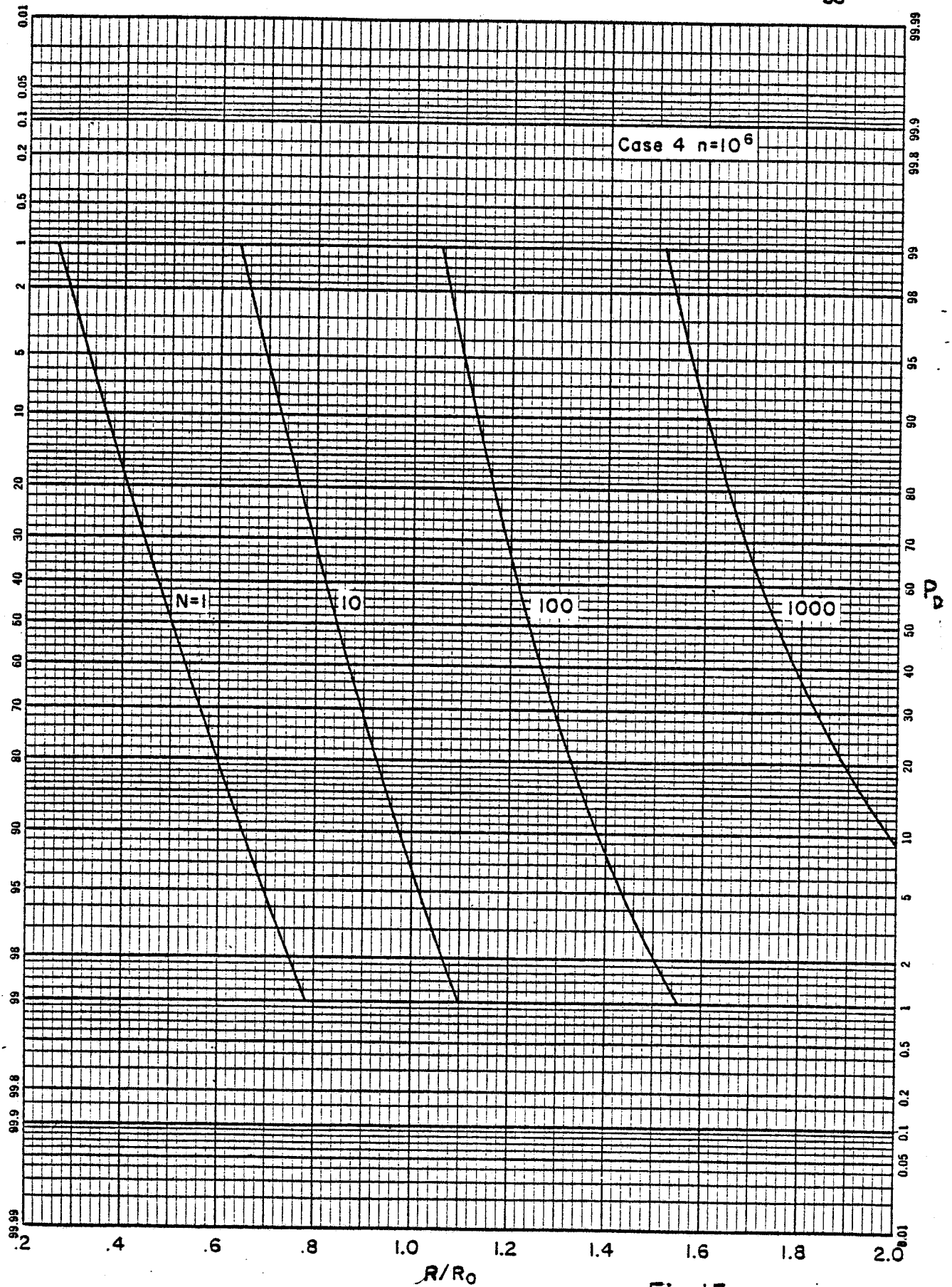
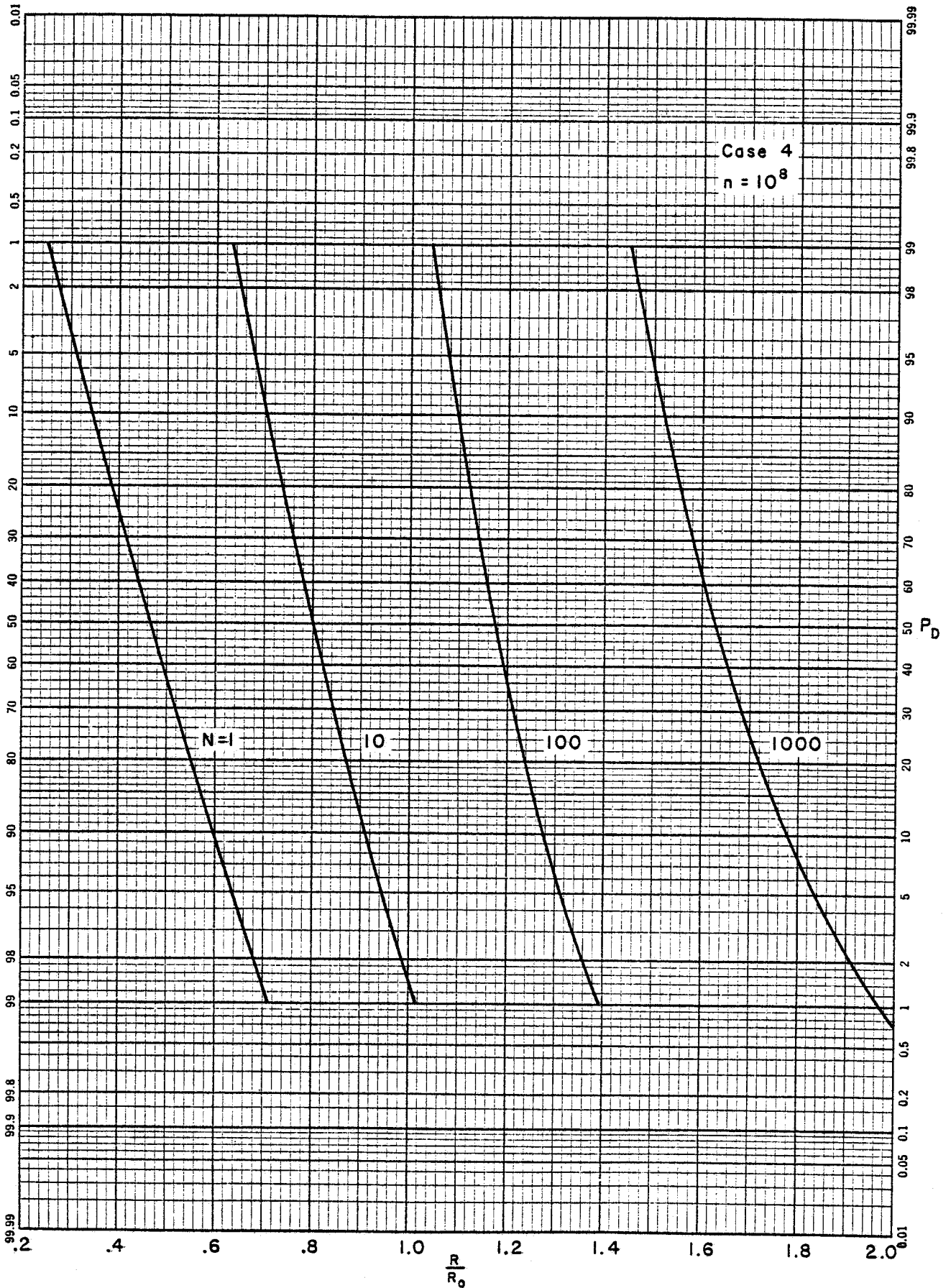


Fig.15

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Fig. 16

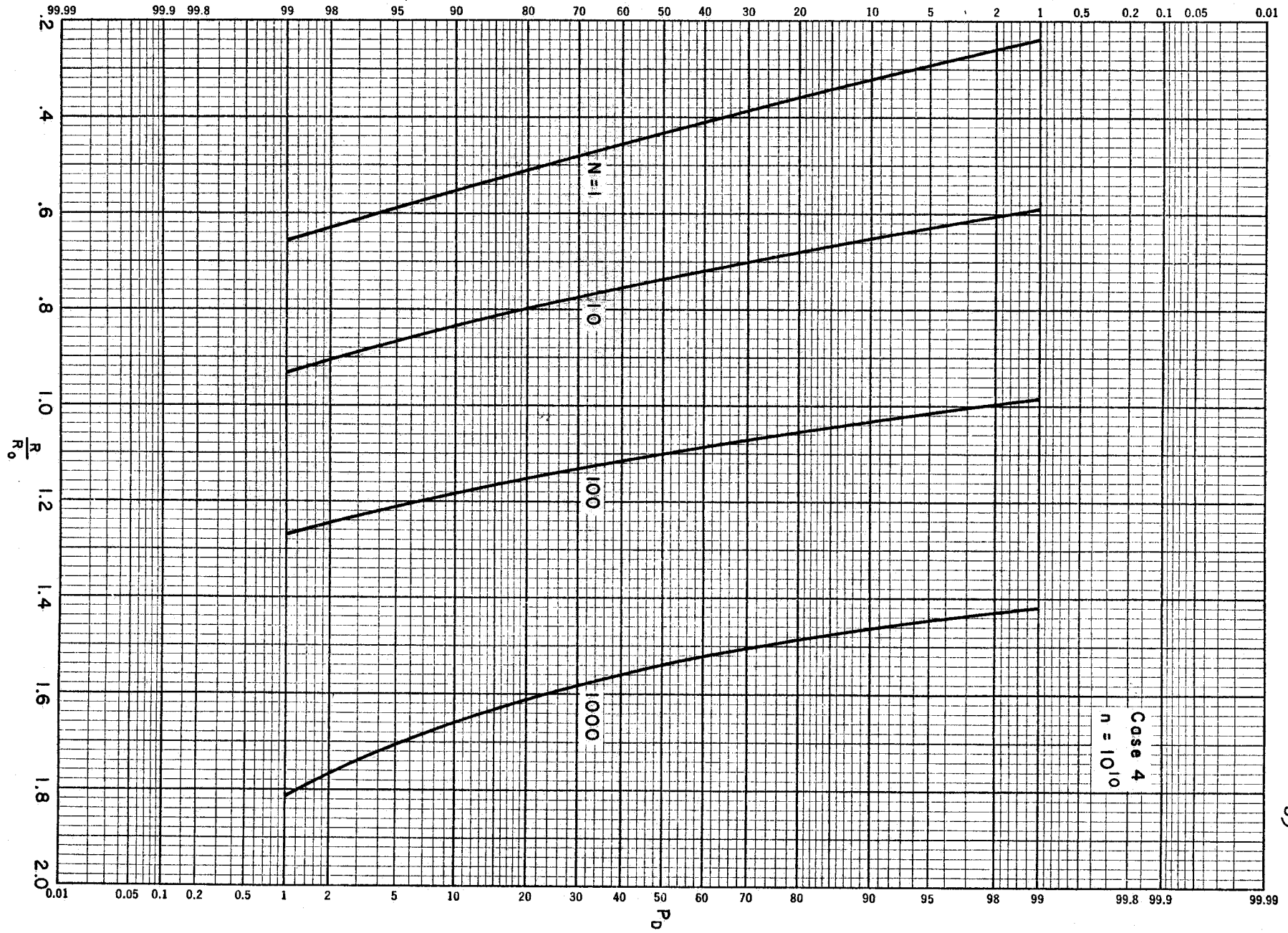


Fig. 17

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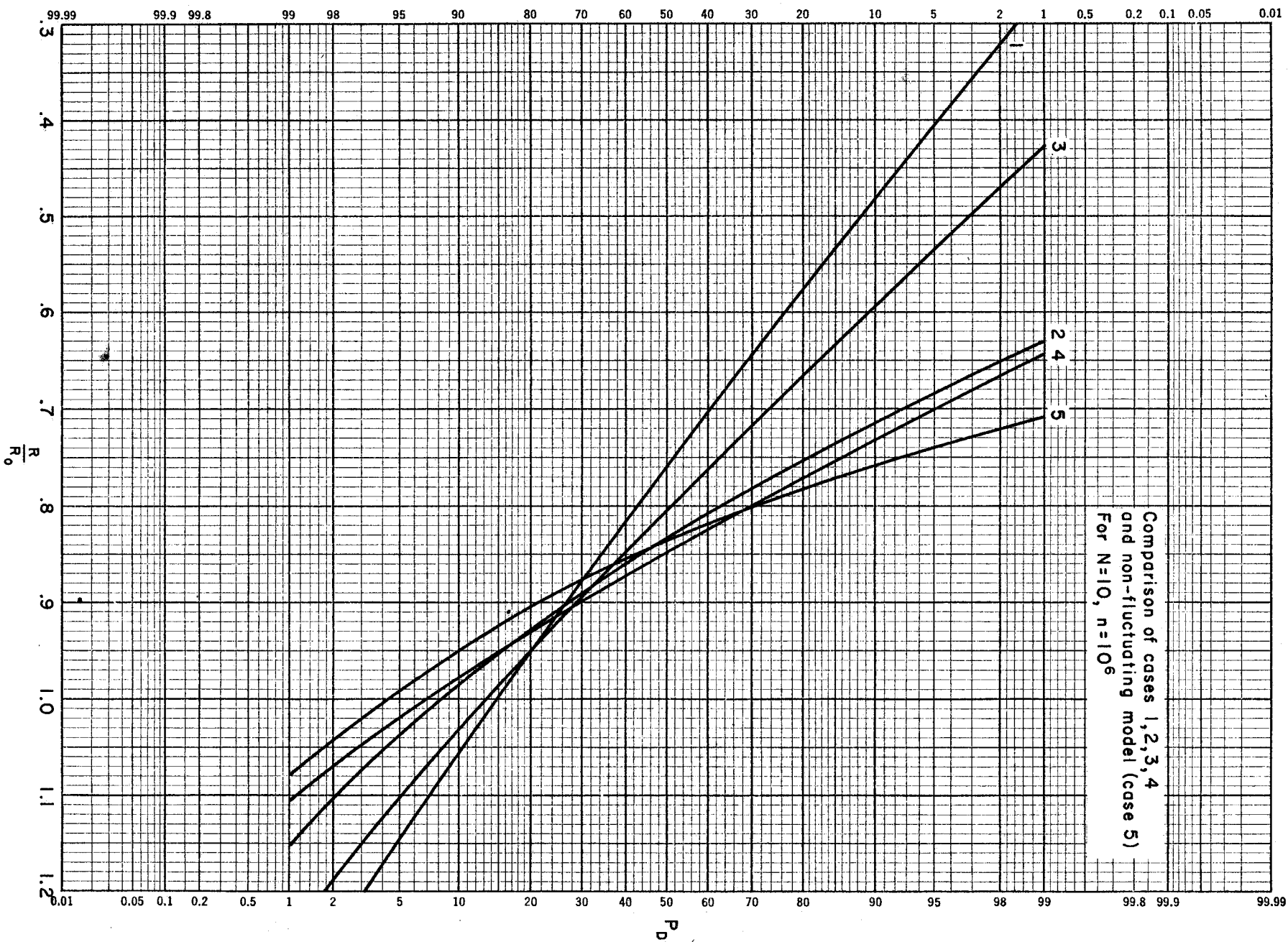


Fig. 18

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