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UNITED STATES AIR FORCE

INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY

RESIDENT COLLEGE

WRIGHT-PATTERSON AIR FORCE BASE-OHIO

THE I-SPIN AS A NUCLEAR QUANTUM NUMBER

THESIS

Presented to the Faculty of the Resident College of
the United States Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the
Graduate Diploma

By

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GNE-55

March, 1955

Acknowledgements

The author expresses appreciation to Dr. Walter Wessel who suggested the subject of this paper and gave helpful advice during its formation. The author's thanks are also extended to Dr. William Lehmann who read the preliminary draft and gave valuable criticism.

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Abstract

The i-spin was introduced by Wigner on the assumption that nuclear forces are charge-independent. He used it to classify nuclear states in tables of energy splittings of nuclear multiplets. Examination of experimental evidence indicates the validity of Wigner's assumption. Comprehension of the i-spin theory presupposes some knowledge of the matrix theory of the electron, so the latter theory is presented first. The discussion of the matrix theory of the electron begins with a short introduction to matrix mechanics and its postulates. Next the commutation relations involving the angular momentum are derived. Following this the Pauli spin matrices are derived on the assumption that they obey the commutation rules for angular momentum. The splitting of the energy levels of the outer electron of a neutral silver atom in a magnetic field is described using the Pauli spin matrices, and the spin functions are simultaneously introduced. It is shown that the Hamiltonian for the magnetic interaction energy of the foregoing electron is invariant under a rotation of coordinates. Some of the theoretical consequences of this rotational invariance conclude the discussion of the matrix theory of the electron. The i-spin theory is presented at this point. An explanation is given as to why the i-spin theory is similar to the matrix theory of the electron and an important distinction between the two is noted. The generalized Pauli principle for nucleon wave functions is introduced. A two-nucleon system is considered as an example of how the i-spin causes a splitting of nuclear energy levels. A central interaction between the nucleons is postulated. A listing is made of the eigenfunctions permitted for the two-nucleon system on the basis of the generalized Pauli principle. Sample calculations show how the spin quantum numbers and the energy level splittings are obtained. A short discussion explains how the i-spin is responsible for nuclear energy level splittings.

THE I-SPIN AS A NUCLEAR QUANTUM NUMBER

I. Introduction

1.1. In the tables of the energy splittings of nuclear multiplets made by Feenberg, Wigner, and others, the nuclear states are listed according to the usual quantum numbers L and S of the (LS)-coupling scheme and also according to another quantum number T. This quantum number T was introduced by Wigner on the assumption that nuclear forces are charge-independent. (Ref 4: 395) Charge-independence is a modern hypothesis that grew out of the older assumption of charge-symmetry.

1.2. The idea that nuclear forces are charge-symmetric, i.e. the force between two neutrons is the same as the force between two protons, is suggested by the similarity of isobaric sets. Among the odd-numbered isobars there are the mirror nuclei such as He^5 - Li^5 and Be^7 - Li^7 which differ only in the exchange of a neutron for a proton. This means that some proton-proton bonds have been replaced by neutron-neutron bonds leaving the number of neutron-proton bonds unchanged. For example, He^5 with two protons and three neutrons has one proton-proton bond, three neutron-neutron bonds, and six proton-neutron bonds; (each proton has a bond with each of the three neutrons). The number of bonds between like particles is given by the rule for the number of combinations which can be formed with m things taken 2 at a time:

(1.2.1) $$\frac{m!}{(m-2)!\,2!} = \frac{m(m-1)}{2}$$

Thus, Li^5 has three proton-proton bonds, one neutron-neutron bond, and six proton-neutron bonds; (each of the three protons has a bond with each of the two neutrons). The masses of the ground states of such mirror nuclei have been found to be in close agreement after Coulomb effects and the neutron-proton mass difference have been taken into account. This gives an indication that nuclear forces are at least charge symmetric.

1.3. But more fundamental is the question of the charge-independence of nuclear forces. Charge-independence means that the specific nuclear interaction (neglecting Coulomb interaction) between two nucleons is independent of their nature as neutrons or protons. (Ref 4: 401) This idea is a little bit more difficult to test because of the lack of sets of stable isobaric triads. One set which does permit examination is Be^{10} - B^{10} - C^{10}. The outer members differ only in the exchange of neutron pairs for proton pairs and they have corresponding levels as mirror nuclei do. The center member, with equal numbers of neutrons and protons, has more neutron-proton bonds and fewer neutron-neutron or proton-proton bonds than its neighbors. On the basis of charge-independence every corresponding state of the outer members will have an analogue in the center one. The common states should differ in energy only because of the Coulomb interaction and the neutron-proton mass difference. It has been possible to determine this association among states in three cases for the above triad so far.

Among other even light isobars there are several examples where a reasonable association may be made but the total evidence is not completely convincing. However, there is probably enough evidence to make the assumption of charge-independence of nuclear forces as well as the less-restrictive assumption of charge symmetry, tentatively at least. (Ref 2: 322)

1.4. These corresponding states observed in even and odd isobars may be described as members of a multiplet of the previous quantum number T. They are characterized by the difference between the number of protons and neutrons, (p - n). The quantum number denoting each state is defined as

$$M_T = \frac{1}{2}(p - n) \qquad (1.4.1)$$

For the triad above the values of M_T are

	M_T
Be^{10}	-1
B^{10}	0
C^{10}	1

If the corresponding levels in the nuclei above appear in no others, it might be implied that they are triplet members of the state with M_T 1.

1.5. Another triad which can be examined is the one having as members C^{14} - N^{14} - O^{14}. After the correction for the Coulomb interaction and the neutron-proton mass difference has been made, the ground states of C^{14} and O^{14} are well above the ground state of N^{14}. Since M_T=0 for N^{14}, it may be implied that the ground state of N^{14} is a singlet with T=0.

However, the first excited state of N^{14} corresponds closely to the corrected ground states of C^{14} and O^{14} which have M_T value of -1 and 1 respectively. Hence, it may be surmised that these three states are a triplet with the quantum number $T=1$. (Ref 8: 226) T will be called the i-spin quantum number, or simply the i-spin (after Fermi). This overrides the terms isotopic spin and isobaric spin which have formerly been used.

1.6. It follows from the principle of the conservation of charge that M_T should be conserved in any nuclear reaction which does not involve the production or annihilation of electrons. The conservation of T cannot be deduced from any basic principle of physics so Wigner introduced it as a postulate to the i-spin theory. As a consequence of this postulate, certain reactions will be forbidden although they might otherwise be allowed from considerations of intrinsic spin and parity. (Ref 1: 1041) There are instances where this has been verified. One example of this is the reaction $O^{16}(d,\alpha)N^{14}$ where N^{14} is formed in the first excited state. This reaction is forbidden by the i-spin conservation postulate and has never been observed. It is forbidden because the total i-spin of the initial nuclei is 0 and the total i-spin of the products would be 1. (Ref 1: 1042) This fact, that results predicted on the basis of the i-spin theory can be verified, is another argument for the theory but a weaker one than that based on evidence which indicates the charge-independence of nuclear forces.

1.7. In order to familiarize the reader with the mathematical foundation for the i-spin theory, the matrix theory of the electron will be presented first because the i-spin theory is basically the same. The reader is assumed to have had an introduction to wave mechanics such as may be found in (Ref 7) and also some basic knowledge about matrix algebra which may be easily obtained for these purposes from texts on quantum mechanics such as (Ref 9) and (Ref 6).

II. The Matrix Theory of the Electron

Matrix Mechanics

2.1. The matrix theory of quantum mechanics antedates Schroedinger's wave mechanics by about a year. At first the relation between the two was uncertain but in a short time Schroedinger and Eckart independently showed that the two are equivalent. (Ref 7: 417)

2.2. In the first paper written on quantum mechanics Heisenberg formulated and successfully attacked the problem of calculating values of the frequencies and intensities of the spectral lines which a system could emit or absorb. He did not use wave functions or wave equations but rather developed a formal mathematical method for calculating values of these quantities. Born and Jordan quickly pointed out that Heisenberg was making use of quantities which were previously known to mathematicians, and that his newly invented operations were those of matrix algebra. Heisenberg's matrix mechanics was rapidly developed and extended to other problems. (Ref 7: 417)

2.3. As do all theories explaining physical phenomena, matrix mechanics has a set of postulates. There may be many ways of stating these, all equally good. One set of postulates is the following:

1. Every physical quantity can be represented by an Hermitian matrix and the eigenvalues of this matrix are the observable values of this quantity. Hermiteity insures that the eigenvalues are always real.

2. Heisenberg's commutation relationships: The non-commutative character of canonically conjugate dynamical quantities. These relationships are:

(2.3.1)
$$p_j q_j - q_j p_j = \frac{\hbar}{i} I$$
$$p_j q_k - q_k p_j = 0 \qquad k \neq j$$
$$q_j q_k - q_k q_j = 0$$
$$p_j p_k - p_k p_j = 0$$

where p and q refer to the matrices of the particular component of momentum or displacement, respectively, and I is the unit matrix. The commutation rules of equations (2.3.1) together with the rules for converting the Hamiltonian equations of motion into matrix form constitute matrix mechanics. (Ref 7: 420)

2.4. It is possible to show that there is some basis for the first commutation relationship if it is assumed that px is not quite equal to xp but differs from it by the smallest quantity of action, Planck's constant. (The subscript has been omitted from p_x.) That is, assume

(2.4.1)
$$p x - x p \approx h$$

or in matrix form

(2.4.2)
$$p x - x p \approx h I$$

where x and p are now matrices.

2.5. The unit matrix is diagonal and has real eigenvalues; hence it is Hermitian, but the left side of equation (2.4.2) is not Hermitian. A factor i is needed to fulfill this condition on the left: ($\dagger$) denotes the adjoint, ($*$) the complex conjugate.

$$(ipx - ixp)^{\dagger}_{mn} = (ipx - ixp)^{*}_{nm} \tag{2.5.1}$$

$$= \sum_{k} \left(-i p^{*}_{nk} x^{*}_{km} + i x^{*}_{nk} p^{*}_{km}\right) \tag{2.5.2}$$

Since p and x are Hermitian the last expression on the right becomes

$$\sum_{k} \left(-i x_{mk} p_{kn} + i p_{mk} x_{kn}\right) = (ipx - ixp)_{mn} \tag{2.5.3}$$

So in consequence of the assumption

$$i(px - xp) \approx h I \tag{2.5.4}$$

This differs from the standard commutation relation by the factor 2π.

Angular Momentum

2.6. The commutation relations for angular momentum are introduced here. They will be used later in the discussion of the spin matrices. It is possible to derive the commutation rules for angular momentum from the postulates above. As usual angular momentum is defined by

$$\vec{M} = \vec{r} \times \vec{p} \tag{2.6.1}$$

It is necessary that the components of $\vec{r}$ and $\vec{p}$ satisfy Heisenberg's commutation relations. Then

$$[M_x, M_y] \equiv M_x M_y - M_y M_x \tag{2.6.2}$$

$$= (y p_z - z p_y)(z p_x - x p_z) - (z p_x - x p_z)(y p_z - z p_y) \tag{2.6.3}$$

$$= y p_x (p_z z - z p_z) + x p_y (z p_z - p_z z) \tag{2.6.4}$$

$$= \frac{\hbar}{i}(y p_x - x p_y) \tag{2.6.5}$$

$$= i\hbar M_z \tag{2.6.6}$$

So it is found that

$$[M_x, M_y] = i\hbar M_z, \quad [M_y, M_z] = i\hbar M_x, \quad [M_z, M_x] = i\hbar M_y \tag{2.6.7}$$

These equations apply to the angular momentum of a system of particles also, since the $\vec{r}$ and $\vec{p}$ matrices for the separate particles commute and so do their momenta. (Ref 9: 140)

2.7. Hermitian matrices may be brought into diagonal form by a unitary transformation. This transformation maintains the commutation relationships. Hence two non-commutative matrices do not commute regardless of how many unitary transformations are made on them. Therefore, since no two of the components of $\vec{M}$ commute with each other, it is impossible to find a unitary transformation which diagonalizes more than one of them because diagonal matrices always commute. This means that it is never possible to measure more than one component of $\vec{M}$ at the same time. However, all three components commute with

$$\vec{M}^2 = M_x^2 + M_y^2 + M_z^2 \tag{2.7.1}$$

For example,

$$[\vec{M}^2, M_z] = M_x^2 M_z + M_y^2 M_z - M_z M_x^2 - M_z M_y^2 \tag{2.7.2}$$

The commutation relationships may be used to eliminate M_z :

$$M_z M_x - M_x M_z = i\hbar M_y \tag{2.7.3}$$

$$M_z M_x = i\hbar M_y + M_x M_z \tag{2.7.4}$$

$$M_z M_x^2 = i\hbar M_y M_x + M_x M_z M_x \tag{2.7.5}$$

Similar equations can be found in a similar manner for the other terms in $[\vec{M}^2, M_z]$. Using them to eliminate M_z gives

$$[\vec{M}^2, M_z] = i\hbar(M_y M_x + M_x M_y) - i\hbar(M_x M_y + M_y M_x) = 0 \tag{2.7.6}$$

Therefore, it is possible to measure simultaneously the value of $\vec{M}^2$ and the value of one component, here M_z, of $\vec{M}$.

2.8. It is convenient to work with M_z and the non-Hermitian matrix

$$L = M_x + i M_y \tag{2.8.1}$$

Then

$$\vec{M}^2 = M_z^2 + \frac{1}{2}(LL^+ + L^+L) \tag{2.8.2}$$

The commutation relations for L are

$$[\vec{M}^2, L] = 0 \qquad [M_z, L] = \hbar L \qquad [L, L^+] = 2\hbar M_z \tag{2.8.3}$$

The proof for the first is similar to that for $[\vec{M}^2, M_z]$ and is omitted.

$$[M_z, L] = M_z(M_x + iM_y) - (M_x + iM_y)M_z \tag{2.8.4}$$

$$= (M_z M_x - M_x M_z) - i(M_y M_z - M_z M_y) \tag{2.8.5}$$

$$= i\hbar M_y - i(i\hbar M_x) \tag{2.8.6}$$

$$= \hbar(iM_y + M_x) = \hbar L \tag{2.8.7}$$

$$[L, L^+] = (M_x + iM_y)(M_x - iM_y) - (M_x - iM_y)(M_x + iM_y) \tag{2.8.8}$$

$$= -iM_x M_y + iM_y M_x + iM_y M_x - iM_x M_y \tag{2.8.9}$$

$$= -2i(M_x M_y - M_y M_x) \tag{2.8.10}$$

$$= -2i(i\hbar M_z) = 2\hbar M_z \tag{2.8.11}$$

It is possible to use the commutation relations for $\vec{M}$ and L and find a representation in which M^2 and M_z are diagonal, but this will not be done. For the solution the reader is referred to (Ref 9: 141-4) from which the author has obtained the previous material on $\overrightarrow{M}$ and L. A representation of a physical quantity is an Hermitian matrix whose eigenvalues are the observable values of the quantity, and which satisfies the commutation relations involving the quantity.

Spin Matrices

2.9. Now that the reader has gained some familiarity with matrix algebra it will be possible to introduce the spin matrices. Experiment has shown that the component of the electron spin angular momentum along a preferred direction (usually designated by a magnetic field) can take on only two values, $\pm \frac{1}{2}\hbar$. Since there are just two eigenvalues, the components of the spin are represented by two-row, two-column matrices. These matrices were originally introduced by Pauli on the assumption that they obeyed the commutation rules for angular momentum derived previously. Let σ_x, σ_y, and σ_z be the spin matrices referring to the components of the electron spin $\overrightarrow{S}$ along the X, Y, and Z axes respectively. The following relations are assumed to hold among them:

$$(2.9.1)\quad [\sigma_x, \sigma_y] = i\hbar\sigma_z\,, \quad [\sigma_y, \sigma_z] = i\hbar\sigma_x\,, \quad [\sigma_z, \sigma_x] = i\hbar\sigma_y$$

Compare these equations to equations (2.6.7).

2.10. Since the spin is a measurable quantity, the spin matrices are Hermitian and can be written as follows:

$$(2.10.1)\quad \sigma_x = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \quad \sigma_y = \begin{pmatrix} m & n \\ n^* & p \end{pmatrix} \quad \sigma_z = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

Writing σ_z diagonal means that the Z-axis is the one along which the component of the electron spin will be measured.

2.11. To solve for the elements of the spin matrices it is first necessary to write out the foregoing commutation relationships in matrix form. For each relationship this leads to an equality between two matrices and for two matrices to be equal their corresponding elements must be equal.

2.12. For the first relationship

$$(2.12.1)\quad \sigma_x \sigma_y = \begin{pmatrix} am + bn^* & an + bp \\ b^*m + cn^* & b^*n + cp \end{pmatrix}$$

$$(2.12.2)\quad \sigma_y \sigma_x = \begin{pmatrix} ma + nb^* & mb + nc \\ n^*a + pb^* & n^*b + pc \end{pmatrix}$$

$$(2.12.3)\quad i\hbar\sigma_z = i\hbar \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$(\sigma_x \sigma_y - \sigma_y \sigma_x)$ is a two-row, two-column matrix as is $i\hbar\sigma_z$ to which it is equal. They are equal if and only if the corresponding matrix elements are equal. This condition of equality gives the following four equations.

(2.12.4) $am + bn^* - (ma + nb^*) = i\hbar\sigma_1$

(2.12.5) $an + bp - (mb + nc) = 0$

(2.12.6) $b^*n + cn^* - (n^*a + pb^*) = 0$

(2.12.7) $b^*n + cp - (n^*b + pc) = i\hbar\sigma_2$

2.13. Performing the same operations with the other commutation relationships gives eight more equations. In writing these eight equations, use will be made of the fact that equations (2.12.4) and (2.12.7) give

(2.13.1) $\sigma_2 = -\sigma_1$

(2.13.2) $\sigma_1(m-m) = i\hbar a$

(2.13.3) $\sigma_1(-n-n) = i\hbar b$

(2.13.4) $\sigma_1[n^* - (-n^*)] = i\hbar b^*$

(2.13.5) $\sigma_1[-p - (-p)] = i\hbar c$

(2.13.6) $\sigma_1(a-a) = i\hbar m$

(2.13.7) $\sigma_1(b+b) = i\hbar n$

(2.13.8) $\sigma_1(-b^*-b^*) = i\hbar n^*$

(2.13.9) $\sigma_1(-c+c) = i\hbar m$

2.14. Equations (2.13.2), (2.13.5), and (2.13.6) give immediately that a, c, and m are 0. Using these results in equation (2.12.5) gives that either b or p is 0, but p must be 0 for if b were 0, then equation (2.12.4) would give that σ_1 and hence σ_2 are also 0. Equation (2.13.8) can be solved for b^* and this can be substituted into equation (2.13.4). From this it is found that σ_1 is $\pm\frac{1}{2}h$. The (+) sign is chosen and σ_2 becomes $-\frac{1}{2}h$. Solving equation (2.13.8) for n^* and equation (2.13.3) for n and substituting these along with the value of σ_1 into equation (2.12.4) gives the following information about b:

(2.14.1) $$bb^* = |b|^2 = \frac{\hbar^2}{4}$$

The phase of b is undetermined but there is no loss of generality in assuming it to be such that

(2.14.2) $$b = \frac{1}{2}\hbar$$

However, this arbitrariness in the phase of b is what is to be expected on the basis of Heisenberg's generalized uncertainty principle: Since the angular momentum is known with complete certainty, nothing can be said about the canonically conjugate coordinate and so one choice is just as good as any other. The phases of the non-diagonal elements of the spin matrices would be the angles which a coordinate system fixed in the electron makes with the reference system.

2.15. When this value of b is substituted into equation (2.13.7), it is found that

(2.15.1) $$n = -\tfrac{1}{2} i \hbar$$

Thus the spin matrices are

(2.15.2) $$\sigma_x = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This representation is consistent with the general rule that

(2.15.3) $$\vec{S}^2 \hbar^2 = S(S+1)\hbar^2$$

for an angular momentum with absolute value S.

(2.15.4) $$\vec{S}^2 = \vec{S}\cdot\vec{S} = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$$

(2.15.5) $$= \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I$$

(2.15.6) $$= \frac{3}{4}\hbar^2 I$$

Since $S = \frac{1}{2}$

(2.15.7) $$S(S+1)\hbar^2 = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 = \tfrac{3}{4}\hbar^2$$

Spin Functions

2.16. One phenomenon which can be described in terms of spin matrices is the splitting of the energy levels of an electron in a magnetic field. This treatment will be given now and the discontinuous spin functions will be introduced. The electron considered for this discussion will be in the outer shell of a neutral silver atom and will be in the ground state ($\ell = 0$). Then any magnetic moment associated with the atom will be due to the intrinsic spin of the outer electron because the closed shells are known to possess no resultant angular momentum or magnetic moment and $\ell = 0$ means that there is no magnetic moment associated with the orbital motion of the outer electron. It is the interaction of the magnetic moment with the magnetic field which produces the energy level splittings.

2.17. It has been stated before that matrix mechanics and wave mechanics are equivalent ways of treating the same problem. In wave mechanics the equation of the silver atom would be

$$H \Psi = E \Psi \tag{2.17.1}$$

The Hamiltonian, H, is made up of three terms:

$$H = H_{int}(\vec{r}_1, \vec{r}_2, \ldots) + H_{kin}(\vec{R}) + H_{mag}(\vec{S}) \tag{2.17.2}$$

$H_{int}(\vec{r}_1, \vec{r}_2, \ldots)$ represents the internal energy of the atom. $\vec{r}_i$ is the position vector of the i^{th} electron with respect to the nucleus. $H_{kin}(\vec{R})$ is the term representing the kinetic energy of the atom. $\vec{R}$ is the position vector of the nucleus (the center of mass) with respect to the reference system. $H_{mag}(\vec{S})$ represents the magnetic interaction energy.

The operator for H_{mag} is obtained by taking the classical expression for the magnetic interaction energy and replacing the dynamic variables with the corresponding operators. The magnetic interaction energy is given classically by the scalar product of the magnetic field strength $\vec{B}$ and the magnetic moment associated with the electron. The absolute value of the magnetic moment of the electron is equal to the Bohr magneton μ ; its direction is that of a unit vector along the axis of spin. This unit vector is given by $\frac{2}{\hbar}\vec{S}$. The classical expression for the magnetic interaction energy is then $\frac{2}{\hbar}\mu(\vec{B}\cdot\vec{S})$.

2.18. The wave equation with the Hamiltonian given in equation (2.17.2) is

$$(H_{int} + H_{kin} + H_{mag})\Psi = E\Psi \tag{2.18.1}$$

This equation can be solved by the familiar separation of variables method. The first step in this method is to assume

$$\Psi = u(\vec{r}_1, \vec{r}_2, \ldots)\,\mathcal{U}(\vec{R})\,\psi(s_z) \tag{2.18.2}$$

where s_z is the spin coordinate.

Then

$$\mathcal{U}\psi H_{int}u + u\psi H_{kin}\mathcal{U} + u\mathcal{U}H_{mag}\psi = Eu\mathcal{U}\psi \tag{2.18.3}$$

Dividing by Ψ gives

$$\frac{H_{int}u}{u} + \frac{H_{kin}\mathcal{U}}{\mathcal{U}} + \frac{H_{mag}\psi}{\psi} = E \tag{2.18.4}$$

Since the first term is a function of the $\vec{r}_1$ only, the second a function of $\vec{R}$ only, and the third a function of s_z only, and the sum is equal to a constant E, each must be a constant.

Let

(2.18.5) $$\frac{H_{int}\,u}{u} = E_{int}$$

(2.18.6) $$\frac{H_{kin}\,U}{U} = E_{kin}$$

(2.18.7) $$\frac{H_{mag}\,\psi}{\psi} = E_{sp}$$

Therefore

(2.18.8) $$E_{int} + E_{kin} + E_{sp} = E$$

E_{int} is the internal energy of the atom, E_{kin} is the kinetic energy of the center of mass with respect to the reference system, and E_{sp} is the magnetic interaction energy. Equation (2.18.7) is the one of interest here. Multiplying both sides of the equation by ψ gives

(2.18.9) $$H_{mag}\,\psi = E\psi$$

where the magnetic interaction energy will now be denoted by E to simplify the notation.

2.19. First the case will be worked out in which the magnetic field is along the Z-axis. In this case

(2.19.1) $$H_{mag} = \frac{2}{\hbar}\mu(\vec{B}\cdot\vec{S}) = \frac{2}{\hbar}\mu B S_z$$

The usual procedure at this point would be to replace S_z by a differential operator and use the expression above in equation (2.18.9) to find the eigenvalues of E and the spin eigenfunctions. This cannot be done with the usual differential operator for the angular momentum, $\left(\frac{\partial}{\partial\phi}\right)$, because one of the requirements that S_z has to satisfy is

(2.19.2) $$S_z \psi = m_s \psi$$

where m_s is the eigenvalue of the Z-component of the spin. This eigenvalue is known to be either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, as was mentioned previously. A differential operator for S_z leads to an infinite number of values of m_s. For example, the solution of the equation

(2.19.3) $$\left(\frac{\partial}{\partial\phi}\right)\Phi = m_s \Phi$$

is

(2.19.4) $$\Phi = e^{i m_s \phi}$$

where m_s can be any number. The periodicity requirement, $\Phi(0) = \Phi(2\pi)$, only restricts m_s to positive and negative integers. Therefore, there is the necessity to find another operator for S_z. Pauli pointed out a suitable solution to this difficulty. He suggested that the operator for S_z be the spin matrix σ_z. Then the wave equation would be

(2.19.5) $$\frac{2}{\hbar}\mu B \sigma_z \psi = E\psi$$

or

(2.19.6) $$\mu B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi = E\psi$$

Unless ψ is written in matrix form this equation is meaningless. Pauli also showed how this is easily taken care of: Since there are two eigenvalues of the Z-component of the electron spin, there must be two spin eigenfunctions and ψ can be written as a two-element matrix.

(2.19.7) $$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

where ψ_+ and ψ_- are the spin eigenfunctions corresponding to the values of $S_z = +\frac{1}{2}\hbar$ and $S_z = -\frac{1}{2}\hbar$ respectively. To complete the wave equation the unit matrix must appear on the right:

(2.19.8) $$\mu B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

2.20. Equation (2.19.8) is actually a compact representation of the following two wave equations as may be seen by applying the requirements for equality between two matrices.

(2.20.1) $$\mu B \psi_+ = E\psi_+$$

(2.20.2) $$-\mu B \psi_- = E\psi_-$$

It follows from (2.20.1) and (2.20.2) that ψ_+ and ψ_- are functions of a discontinuous coordinate; otherwise there would be two simultaneous values of the energy. Either one chooses $\psi_+ \neq 0$, then $E = \mu B$, (spin up), and necessarily $\psi_- = 0$; or one chooses $\psi_- \neq 0$, then $E = -\mu B$, (spin down), and then ψ_+ is necessarily 0. Hence take

(2.20.3)
$$\psi_+(s_z) = 1, \quad s_z = \tfrac{1}{2}$$
$$= 0, \quad s_z = -\tfrac{1}{2}$$

(2.20.4)
$$\psi_-(s_z) = 0, \quad s_z = \tfrac{1}{2}$$
$$= 1, \quad s_z = -\tfrac{1}{2}$$

It should be noted that the spin functions are orthonormal.

2.21. Both of the spin functions ψ_+ and ψ_- can be written in matrix form:

(2.21.1) $$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The matrix equation describing the magnetic interaction then becomes two:

(2.21.2) $$\mu B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2.21.3) $$\mu B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

corresponding to the two wave equations

(2.21.4) $$\mu B\psi_+ = E\psi_+$$

(2.21.5) $$-\mu B\psi_- = E\psi_-$$

2.22. The general case will now be worked out, i.e. the case in which the magnetic field is not in the Z-direction. It will be shown that the results are consistent with those above and that, therefore, the Hamiltonian for the magnetic interaction energy is invariant under a rotation of coordinates. The wave mechanical expression for the general case is

(2.22.1) $$\frac{2}{\hbar}\mu(\vec{B}\cdot\vec{S})\psi' = E\psi'$$

If the ideas of Pauli are carried to three dimensions, this becomes

(2.22.2) $$\frac{2}{\hbar}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)\psi' = E\psi'$$

or

(2.22.3) $$\mu\left[B_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + B_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + B_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\begin{pmatrix}\psi_+'\\ \psi_-'\end{pmatrix} = EI\begin{pmatrix}\psi_+'\\ \psi_-'\end{pmatrix}$$

2.23. It is more advantageous to use spherical coordinates in which case the components of $\vec{B}$ become

(2.23.1) $$\begin{aligned} B_x &= B\sin\theta\cos\phi \\ B_y &= B\sin\theta\sin\phi \\ B_z &= B\cos\theta \end{aligned}$$

where θ and ϕ are the usual polar angles. Substituting these components of $\vec{B}$ into equation (2.22.3) and performing the matrix addition gives

(2.23.2)
$$\mu B\begin{pmatrix}\cos\theta & \sin\theta[\cos\phi - i\sin\phi] \\ \sin\theta[\cos\phi + i\sin\phi] & -\cos\theta\end{pmatrix}\begin{pmatrix}\psi_+' \\ \psi_-'\end{pmatrix} = E\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi_+' \\ \psi_-'\end{pmatrix}$$

or

(2.23.3)
$$\mu B\underbrace{\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}}_{A}\begin{pmatrix}\psi_+' \\ \psi_-'\end{pmatrix} = E\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi_+' \\ \psi_-'\end{pmatrix}$$

2.24. Equation (2.23.3) is similar to (2.19.8) for the case in which the magnetic field is in the Z-direction except that it has the matrix denoted as A where the other has the matrix

(2.24.1)
$$\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

Matrices appear in classical mechanics in various places as tensors. These in general are not diagonal but can be brought into diagonal form by a linear transformation which corresponds to a rotation of coordinates. Perhaps the most familiar example of this is the transformation to principal axes of the moment of inertia tensor. In matrix mechanics there is an operation analogous to the linear transformation; it is the unitary transformation. A rotation of coordinates (a linear transformation) does not change the eigenvalues of a tensor, e.g. the principal moments of inertia of a solid, and a unitary transformation does not change the eigenvalues of an Hermitian matrix. Hence, let us try to represent the rotation of coordinates in matrix mechanics by a unitary transformation:

(2.24.2) $$\psi' = \mathcal{U}\psi$$

where U is some unitary matrix as yet unknown. If this value of ψ' is substituted into equation (2.23.3), it becomes

(2.24.3) $$\mu B A \mathcal{U} \psi = E I \mathcal{U} \psi$$

2.25. If both sides of equation (2.24.3) are multiplied from the left by $U^\dagger$, the result is

(2.25.1) $$\mu B \mathcal{U}^\dagger A \mathcal{U} \psi = E I \psi$$

since $U^\dagger$ commutes with scalars and with the unit matrix, and $U^\dagger U$ is I. In order that this equation is equal to the one obtained by assuming the field was in the Z-direction, it is necessary to find a U such that

(2.25.2) $$\mathcal{U}^\dagger A \mathcal{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To solve equation (2.25.2) multiply both sides of the equation from the left by U. Then

$$A\mathcal{U} = \mathcal{U}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.25.3}$$

2.26. U must be a two-row, two-column matrix. Therefore, the matrix equation (2.25.3) gives four algebraic equations for the elements of U. Additional conditions are obtained from the fact that U is unitary:

$$\mathcal{U}^{+}\mathcal{U} = \mathcal{U}\mathcal{U}^{+} = I \tag{2.26.1}$$

U has four elements but it must not be suspected that they will be over-determined by all these conditions because the non-diagonal elements are complex and hence they each have two unknowns in them. It can be shown with the use of these conditions that

$$\mathcal{U} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & -\cos\frac{\theta}{2} \end{pmatrix} \tag{2.26.2}$$

Since the unitary matrix which transforms equation (2.23.3) to the form of equation (2.19.8) has been found, the Hamiltonian for the magnetic interaction energy is invariant under rotation of coordinates.

2.27. It is interesting to use U and note the relations between ψ' and ψ. Since

$$\psi' = \mathcal{U}\psi$$

(2.27.1) $$\begin{pmatrix} \psi_+' \\ \psi_-' \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} e^{i\phi} & -\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Therefore

(2.27.2) $$\psi_+' = \cos\frac{\theta}{2}\psi_+ + \sin\frac{\theta}{2} e^{-i\phi}\psi_-$$

(2.27.3) $$\psi_-' = \sin\frac{\theta}{2} e^{i\phi}\psi_+ - \cos\frac{\theta}{2}\psi_-$$

Since $\psi_+(\frac{1}{2}) = 1,\ \psi_-(\frac{1}{2}) = 0,\ \psi_+(-\frac{1}{2}) = 0,\ \psi_-(-\frac{1}{2}) = 1$

(2.27.4) $$\psi_+'(\tfrac{1}{2}) = \cos\frac{\theta}{2} \qquad \psi_-'(\tfrac{1}{2}) = \sin\frac{\theta}{2} e^{i\phi}$$

(2.27.5) $$\psi_+'(-\tfrac{1}{2}) = \sin\frac{\theta}{2} e^{-i\phi} \qquad \psi_-'(-\tfrac{1}{2}) = -\cos\frac{\theta}{2}$$

2.28. Since in general ψ^2 is proportional to the probability of occurrence, equations (2.27.4-5) tell something about the relative number and orientations of electrons which have passed through a magnetic field at an angle θ if their previous orientations were known. The previous orientation is denoted by either $s_z = \frac{1}{2}$ or $s_z = -\frac{1}{2}$ for the unprimed spin functions. First of all, the equations predict that all the electrons will get through the field; e.g.

(2.28.1) $$\left[\psi_+'(\tfrac{1}{2})\right]^2 + \left[\psi_-'(\tfrac{1}{2})\right]^2 = 1$$

or

(2.28.2) $[\psi_+'(-\frac{1}{2})]^2 + [\psi_-'(-\frac{1}{2})]^2 = 1$

Second, the equations state that the number coming through parallel or antiparallel to the field depends on the original orientation. For example, if the electrons originally had $s_z = \frac{1}{2}$, the relative probabilities are $\cos^2\frac{\theta}{2}$ for coming through parallel, and $\sin^2\frac{\theta}{2}$ for coming through antiparallel. Another fact revealed by the equations is that all the electrons which enter an "upward" magnetic field with "spin down" ($\theta = \pi$) leave it with "spin up".

2.29. As yet there is no direct experimental evidence supporting these predictions but Jordan (Ref 5) has noted that the equations giving the probabilities of the electrons leaving the field either parallel or antiparallel to it if they enter it an angle θ are the same as those determined under the assumption that the values measured in the classical sense correspond to the quantum mechanical average. Classical electromagnetism and mechanics say that the spinning electron which enters the magnetic field at the angle θ should precess about the direction of the field at the angle θ with constant precessional velocity. Then the classical measurement of the component of the angular momentum in the direction of the field would give the result

$$\frac{1}{2}\hbar\cos\theta$$

2.30. The general formula for the quantum mechanical average of a dynamical function, say $f(x,p)$ is:

(2.30.1) $$f_{av} = \int_{\infty}^{\infty} \Psi^{*} f(x, p_x) \Psi \, dx$$

The spin functions are discontinuous so the integral is replaced by a summation which becomes for the average of the spin angular momentum:

(2.30.2) $$S_{av} = \tfrac{1}{2}\hbar \left[\psi_{+}'(\tfrac{1}{2})\right]^{2} + (-\tfrac{1}{2}\hbar)\left[\psi_{-}'(\tfrac{1}{2})\right]^{2}$$

The minus sign appears in the second term because $-\frac{1}{2}\hbar$ is the value of the spin for antiparallel orientation; this is multiplied by the probability of the occurrence of this orientation, $\left[\psi_{-}'(\tfrac{1}{2})\right]^{2}$.

2.31. It is expected from the correspondence argument above that

(2.31.1) $$\tfrac{1}{2}\hbar \left[\psi_{+}'(\tfrac{1}{2})\right]^{2} + (-\tfrac{1}{2}\hbar)\left[\psi_{-}'(\tfrac{1}{2})\right]^{2} = \tfrac{1}{2}\hbar \cos\theta$$

while the spin functions are normalized, i.e. that

(2.31.2) $$\left[\psi_{+}'(\tfrac{1}{2})\right]^{2} + \left[\psi_{-}'(\tfrac{1}{2})\right]^{2} = 1$$

Adding equations (2.31.1) and (2.31.2) gives

(2.31.3) $$2\left[\psi_{+}'(\tfrac{1}{2})\right]^{2} = 1 + \cos\theta$$

or

(2.31.4) $$\left[\psi_{+}'(\tfrac{1}{2})\right]^{2} = \frac{1+\cos\theta}{2} = \cos^{2}\frac{\theta}{2}$$

Subtracting equation (2.31.1) from equation (2.31.2) gives

(2.31.5) $$2\left[\psi_{-}'(\tfrac{1}{2})\right]^{2} = 1 - \cos\theta$$

or

(2.31.6) $$\left[\psi_-'\left(\tfrac{1}{2}\right)\right]^2 = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$$

These probabilities are seen to be the same as those determined from ψ_+ and ψ_- with the unitary transformation.

III. The I-Spin Theory

3.1. The theory developed above provides the foundation for the understanding of the i-spin theory and its effect on nuclear energy levels because i-spin theory closely parallels that of the intrinsic spin of the electron.

3.2. The nucleon may be treated as a single particle having two different states designated as the proton and the neutron which can be distinguished from one another by the application of an electric potential that adds an energy e or 0 multiplied by the potential. The factor e or 0 does not have the symmetry of the factor $\pm\mu$ in the magnetic interaction energy of the electron but it does indicate the same number of eigenvalues. Because of this and the fact that the various electron multiplets can be designated by simple quantum numbers, the i-spin functions and spin matrices are defined similarly to those for the electron spin. (Ref 4: 397) This close analogy is extremely artificial, but it does lead to results which are easy to remember, e.g. triplets and singlets for i-spin multiplets can then be designated by the familiar quantum numbers 1 or 0 respectively, which can be remembered in terms of a vector model concept. I-spin has a very definite meaning nevertheless in terms of states differing in energy and characterized by some sort of proper values of a new dynamic variable which is conserved under appropriate conditions. (Ref 4: 397) In the electronic case with two particles there are two possible states, the one in which the spins of the particles are parallel and the one in which the spins are antiparallel. The first state is characterized by $S=1$ and has a certain energy associated with it. The second state has $S=0$ and a different energy associated with it. The energy of the state with $S=1$ does not depend on the space orientation of the spin, being the same for all of the M_S quantum numbers 1, 0, and -1 (in the absence of a magnetic field). In the nuclear case there is a similar possibility of $T=1$, that gives a triad such as C^{14} - N^{14*} - O^{14} (* denotes the first excited state) all members of which have the same energy after Coulomb and neutron-proton mass difference corrections have been made. There is also the possibility of $T=0$ which gives only one nucleus such as N^{14} in the ground state. However, there is an important distinction between the use of the electron spin functions and the i-spin functions. The electron spin functions are useful in explaining one-body problems such as the interaction energy of a bound electron (the outer electron of a neutral silver atom) in a magnetic field but the usefulness of the i-spin comes mainly in the many-body problem where it is a great aid in classifying nuclear states.

3.3. The i-spin functions will be defined as $\eta^{+}(\tau)$ and $\eta^{-}(\tau)$ where τ is the i-spin coordinate. The convention used will be that in which $\tau=+1$ refers to the proton and $\tau=-1$ refers to the neutron. (Ref 8: 157) This convention is chosen because it gives the i-spin coordinate of the proton the same sign as its charge. The i-spin functions are discontinuous functions of the spin coordinate. Their definitions are

(3.3.1) $\eta^+(1) = 1 \qquad \eta^+(-1) = 0$

(3.3.2) $\eta^-(1) = 0 \qquad \eta^-(-1) = 1$

Therefore, these definitions make η^+ the state in which the nucleon is known to be a proton, and η^- that in which the nucleon is known to be a neutron. (Ref 8: 157)

3.4. Just as in the case of the electron spin functions, the i-spin functions can be written as two-row, one-column matrices:

(3.4.1) $\eta^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \eta^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.5. The i-spin matrix operators are chosen to resemble the Pauli spin matrices σ_x, σ_y, and σ_z (Ref 8: 157):

(3.5.1) $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The subscripts 1, 2, and 3 are used to emphasize that the i-spin vector $\vec{\tau}$ of which τ_1, τ_2, and τ_3 may be regarded as components does not exist in the Euclidean space of the laboratory.

2.6. The matrix operators τ_1 and τ_2 have a very significant physical interpretation. The linear combinations

(3.6.1) $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$

have the matrix form

(3.6.2) $\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Operation on the spin functions with τ_+ and τ_- gives

(3.6.3) $$\tau_+\eta^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(3.6.4) $$\tau_-\eta^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta^-$$

(3.6.5) $$\tau_+\eta^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta^+$$

(3.6.6) $$\tau_-\eta^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the operator τ_+ annihilates the proton state of the nucleon and converts a neutron state into a proton state. Similarly, τ_- converts the proton state of the nucleon to the neutron state and annihilates the neutron. The existence of both positive and negative beta decay implies the applicability of these operations. The possibility of an exchange character of nuclear forces suggests that protons and neutrons are interconverted, and τ_+ and τ_- provide a means of describing this. (Ref 8: 157)

3.7. As suggested by the discussion in paragraph 3.2. the total i-spin is expressed in a manner similar to that for the electronic spin:

(3.7.1) $$\vec{T} = \frac{1}{2}\sum_{i=1}^{n} \vec{\tau}^{(i)}$$

for n nucleons. (Ref 8: 177)

3.8. The Pauli exclusion principle for atoms could be verified experimentally, but the groupings of states in nuclei are too complex for a complete check to be made of the following extension of the principle to nucleon wave functions. (Ref 4: 398) Arguments based on particle statistics can be presented and these lead to the conclusion that the nucleon wave function must be antisymmetric for the interchange of any two nucleons. This result is referred to as the generalized Pauli principle. (Ref 8: 160)

IV. Splitting of Nuclear Energy Levels

Nucleon Interactions

4.1. Enough of the i-spin theory has been presented for some simple caluclations to be made which show how i-spin causes a splitting of nuclear energy levels. First some kind of nucleon interaction must be postulated. Most attempts to formulate the sequence of nuclear states have been based on the assumption of central interactions between two nucleons written as a function of their distance apart, $V(r_{ij})$, multiplied by an exchange operator O_{ij}. This operator is usually written as a linear combination of the space-exchange operator P, the spin-exchange Operator Q, the space-spin-exchange operator PQ, and the Wigner operator I. The operator P exchanges the space coordinates of the two nucleons in a function following it; the operator Q exchanges the intrinsic spin coordinates and the operator PQ exchanges the intrinsic spin and space coordinates. The Wigner operator does nothing. (Ref 4: 391) A simplified and satisfactory version (Ref 4: 391) of O_{ij} is

(4.1.1) $$O_{ij} = 0.8P + 0.2Q$$

Form of the Wave Function

4.2. The nuclear energy level splittings to be described will pertain to a two-nucleon system. It will be assumed that the space-dependent part of the eigenfunction is the product of a function ψ and another function ϕ . The argument of

ψ is the positional vector of either one of the nucleons and the argument of ϕ is the positional vector of the other nucleon. It is assumed that ψ and ϕ are orthonormal functions. The intrinsic spin of the nucleons will be described in exactly the same manner as the intrinsic spin of the electron, but the spin functions will be denoted as α and β rather than as ψ_+ and ψ_- to prevent confusion. The convention shall be used such that in the product of a pair of similar functions, such as ψ as ϕ, the first shall refer to particle 1 and the second to particle 2. (Ref 4: 396) That is

$$\psi(\vec{r}_1)\,\phi(\vec{r}_2) \equiv \psi\phi \tag{4.2.1}$$

4.3. As before, the total eigenfunction will be the product of the space eigenfunction, the intrinsic spin eigenfunction, and the i-spin eigenfunction; e.g. for a one-particle system.

$$\Psi(\vec{r}, s_z, \tau) = \psi(\vec{r})\,\alpha(s_z)\,\eta^+(\tau) \tag{4.3.1}$$

Compare this to equation (2.18.2).

4.4. The complete eigenfunctions permitted for the two-nucleon system by the generalized Pauli principle are listed in Table I (page 39) with their quantum numbers. These eigenfunctions are not normalized. Eigenfunctions with negative M_S and M_T are omitted because they introduce nothing new in the way of energy level splittings that is not demonstrated by the positive values of these quantum numbers. It is necessary to form such terms as $(\psi\phi - \phi\psi)$ or otherwise the generalized Pauli principle would not be satisfied. Another reason for not using the space eigenfunction as $\psi\phi$ is that this type of function says more than can ever be known about the system, namely that particle 1 is in the space state represented by ψ and particle 2 is in the space state represented by ϕ. (Ref 4: 396) It is easy to make the list of eigenfunctions in Table I on the basis of the generalized Pauli principle if the following method is used. First the space eigenfunctions are separated into two groups, the symmetric functions going into one group and the antisymmetric functions into the other. The intrinsic spin eigenfunctions and then the i-spin eigenfunctions are grouped on the same basis as were the space eigenfunctions. This grouping for the functions in Table I is (with the ommission of those with negative M_T and M_S)

GROUP 1 (Symmetric)	GROUP 2 (Antisymmetric)
$(\psi\phi+\phi\psi)$	$(\psi\phi-\phi\psi)$

GROUP 3 (Symmetric)	GROUP 4 (Antisymmetric)	GROUP 5 (Symmetric)	GROUP 6 (Antisymmetric)
$\alpha\alpha$ $(\alpha\beta+\beta\alpha)$	$(\alpha\beta-\beta\alpha)$	$\eta^+\eta^+$ $(\eta^+\eta^- +\eta^-\eta^+)$	$(\eta^+\eta^- -\eta^-\eta^+)$

A function is chosen from GROUPS 1 and 2, then a function is chosen from GROUPS 3 and 4, and finally a function is chosen from GROUPS 5 and 6 according to the rule that either one or all three of these functions must be antisymmetric; multiplying together the three functions chosen according to this rule gives a satisfactory eigenfunction for the system. This process is repeated until all the permissible eigenfunctions have been determined.

TABLE I

EIGENFUNCTIONS FOR A TWO-NUCLEON SYSTEM

EIGENFUNCTION	T	S	M_T	M_S	INTERACTION ENERGY
$(\psi\phi-\phi\psi)\alpha\alpha\,\eta^+\eta^+$	1	1	1	1	−3K
$(\psi\phi-\phi\psi)(\alpha\beta+\beta\alpha)\,\eta^+\eta^+$	1	1	1	0	−3K
$(\psi\phi-\phi\psi)\alpha\alpha\,(\eta^+\eta^-+\eta^-\eta^+)$	1	1	0	1	−3K
$(\psi\phi-\phi\psi)(\alpha\beta+\beta\alpha)(\eta^+\eta^-+\eta^-\eta^+)$	1	1	0	0	−3K
$(\psi\phi+\phi\psi)(\alpha\beta-\beta\alpha)\,\eta^+\eta^+$	1	0	1	0	4.2K
$(\psi\phi+\phi\psi)(\alpha\beta-\beta\alpha)(\eta^+\eta^-+\eta^-\eta^+)$	1	0	0	0	4.2K
$(\psi\phi+\phi\psi)\alpha\alpha\,(\eta^+\eta^--\eta^-\eta^+)$	0	1	0	1	7K
$(\psi\phi+\phi\psi)(\alpha\beta+\beta\alpha)(\eta^+\eta^--\eta^-\eta^+)$	0	1	0	0	7K
$(\psi\phi-\phi\psi)(\alpha\beta-\beta\alpha)(\eta^+\eta^--\eta^-\eta^+)$	0	0	0	0	−5K

The Spin Quantum Numbers

4.5. It will now be shown how matrix mechanics gives the spin quantum numbers for the states of a two-nucleon system represented by the eigenfunctions in Table I. The values of these quantum numbers, S and M_S, will be determined for the intrinsic spin function $(\alpha\beta + \beta\alpha)$. ($M_S$ is the Z-component of $\vec{S}$.) It was proved in the discussion on angular momentum that it is possible to diagonalize simultaneously the matrix for the square of the angular momentum, here $\vec{S}^{\,2}$, and the matrix for one of its components, here M_S. If it is assumed that this diagonalization has been performed, operation with $\vec{S}^{\,2}$ on $(\alpha\beta + \beta\alpha)$ will give the eigenvalue of $\vec{S}^{\,2}$ for the system when it is in the spin state described by $(\alpha\beta + \beta\alpha)$; a similar conclusion holds for operation with M_S on $(\alpha\beta + \beta\alpha)$.

4.6. First S will be determined. $\vec{S}$ is a vector given by

(4.6.1) $$\vec{S} = \vec{S}_1 + \vec{S}_2$$

where $\vec{S}_1$ is the vector associated with the spin of particle 1 and $\vec{S}_2$ is the vector associated with the spin of particle 2. Then

(4.6.2) $$\vec{S}^{\,2} = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2(\vec{S}_1 \cdot \vec{S}_2)$$

Now

(4.6.3) $$\begin{aligned} \vec{S}_1^{\,2} &= \sigma_{1x}^2 + \sigma_{1y}^2 + \sigma_{1z}^2 \\ &= \tfrac{1}{4}\hbar^2 I + \tfrac{1}{4}\hbar^2 I + \tfrac{1}{4}\hbar^2 I = \tfrac{3}{4}\hbar^2 I \end{aligned}$$

where the spin matrices of (2.15.2) now refer to the components of $\vec{S}_1$. Similarly

(4.6.4) $$\vec{S}_2^{\,2} = \frac{3}{4}\hbar^2 I$$

$(\vec{S}_1 \cdot \vec{S}_2)$ is given by

(4.6.5) $$(\vec{S}_1 \cdot \vec{S}_2) = \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}$$

(4.6.6) $$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2 + \begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2 + \begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\right]$$

The expression for $\vec{S}^2$ in equation (4.6.2) might be put into diagonal form simultaneously with $\vec{S}_1^{\,2}$ and $\vec{S}_2^{\,2}$ by a unitary transformation because $\vec{S}_1^{\,2}$ and $\vec{S}_2^{\,2}$ commute with all the components of $\vec{S}_1$ and $\vec{S}_2$ and so also with the scalar product $(\vec{S}_1 \cdot \vec{S}_2)$; but for the following application to symmetrized product wave functions, equation (4.6.6) may be used as it stands, showing immediately that the products are eigenfunctions. $\vec{S}_1$ acts only on α_1 and β_1 and $\vec{S}_2$ acts only on α_2 and β_2, so $\vec{S}_2$ commutes with α_1 and β_1 and $\vec{S}_1$ commutes with α_2 and β_2. α and β have the form of equations (2.21.1).

(4.6.7) $$\alpha = \begin{pmatrix}1\\0\end{pmatrix} \qquad \beta = \begin{pmatrix}0\\1\end{pmatrix}$$

It is necessary to find

(4.6.8) $$(\vec{S}_1 \cdot \vec{S}_2)(\alpha\beta + \beta\alpha)$$

Using equations (4.6.6) and (4.6.7)

$$(\vec{S}_1 \cdot \vec{S}_2)(\alpha\beta + \beta\alpha) =$$

(4.6.9)

$$\frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2+\begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2+\begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\right]\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$+\frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2+\begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2+\begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]=$$

(4.6.10) $$\frac{\hbar^2}{4}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$+\frac{\hbar^2}{4}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]=$$

(4.6.11) $$\frac{\hbar^2}{4}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]=\frac{\hbar^2}{4}(\alpha\beta+\beta\alpha)$$

Therefore

(4.6.12) $$\vec{S}^2(\alpha\beta+\beta\alpha)=\left(\frac{3}{4}\hbar^2+\frac{3}{4}\hbar^2+2\frac{\hbar^2}{4}\right)(\alpha\beta+\beta\alpha)$$
$$=2\hbar^2(\alpha\beta+\beta\alpha)$$

but

(4.6.13) $$\vec{S}^2(\alpha\beta+\beta\alpha)=S(S+1)\hbar^2(\alpha\beta+\beta\alpha)$$

So

(4.6.14) $$S(S+1)=2$$

and

(4.6.15) $$S=1$$

4.7. Now the M_S quantum number for the spin state described by $(\alpha\beta+\beta\alpha)$ will be determined. M_S has the same definition here as in atomic physics but the dynamic variables are replaced by matrix operators. Therefore M_S is given by

(4.7.1) $$M_S = \sigma_{1z} + \sigma_{2z}$$

Since it has been assumed that σ_{1z} and σ_{2z} have been diagonalized,

(4.7.2) $$M_S = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_1 + \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_2$$

Then

(4.7.3) $$M_S(\alpha\beta+\beta\alpha) = (\sigma_{1z}+\sigma_{2z})(\alpha\beta+\beta\alpha) =$$

(4.7.4) $$\frac{\hbar}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_1 + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_2\right]\left[\begin{pmatrix}1\\ 0\end{pmatrix}_1\begin{pmatrix}0\\ 1\end{pmatrix}_2 + \begin{pmatrix}0\\ 1\end{pmatrix}_1\begin{pmatrix}1\\ 0\end{pmatrix}_2\right] =$$

(4.7.5) $$\frac{\hbar}{2}\left[\begin{pmatrix}1\\ 0\end{pmatrix}_1\begin{pmatrix}0\\ 1\end{pmatrix}_2 - \begin{pmatrix}1\\ 0\end{pmatrix}_1\begin{pmatrix}0\\ 1\end{pmatrix}_2 - \begin{pmatrix}0\\ 1\end{pmatrix}_1\begin{pmatrix}1\\ 0\end{pmatrix}_2 + \begin{pmatrix}0\\ 1\end{pmatrix}_1\begin{pmatrix}1\\ 0\end{pmatrix}_2\right] = 0$$

Therefore

(4.7.6) $$M_S = 0$$

The Interaction Energy

4.8. The interaction energy which appears in the last column of Table I is calculated with the integral

(4.8.1) $$\int \Psi^* V_{op} \Psi \, dv$$

where Ψ is the normalized eigenfunction. The form V_{op} was stated previously. For two nucleons it is

(4.8.2) $V_{op} = O_{12}V(r_{12}) = (0.8P+0.2Q)V(r_{12})$

It will be assumed that $V(r_{12})$ is a symmetric function. In this case V_{op} can be written:

(4.8.3) $V_{op} = V(r_{12})(0.8P+0.2Q)$

The argument of $V(r_{12})$ will be omitted from here on in writing the potential.

4.9. The following symbols will be used in the calculations for the interaction energy.

(4.9.1) $L = \int \psi^* \phi^* V \psi \phi \, dv_1 \, dv_2$

(4.9.2) $K = \int \psi^* \phi^* V \phi \psi \, dv_1 \, dv_2$

Experimental evidence indicates that the range of V is less than size of the nucleus. Evaluation of L and K based on this fact shows that K is smaller than L; how much smaller it is depends on the shape of V. (Ref 4: 395) If ψ and ϕ are wave functions of the isotropic oscillator and V is given a Gaussian radial dependence, with a reasonable estimate on its depth and width the following relation holds approximately between L and K. (Ref 3: 911)

(4.9.3) $L = 6K$

This relation was assumed in calculating the interaction energies for Table I.

4.10. A few calculations will be made to show how the interaction energies are obtained. The first calculation will be for

(4.10.1) $$\Psi = \frac{(\psi\phi - \phi\psi)(\alpha\beta + \beta\alpha)\eta^{+}\eta^{+}}{\sqrt{2}\quad\sqrt{2}}$$

(The two factors $\frac{1}{\sqrt{2}}$ normalize the eigenfunction.)
For this eigenfunction the interaction energy is given by

(4.10.2)
$$E_{int} = \int \frac{(\psi\phi-\phi\psi)^{*}(\alpha\beta+\beta\alpha)\eta^{+}\eta^{+}}{\sqrt{2}\quad\sqrt{2}} V[0.8P+0.2Q] \frac{(\psi\phi-\phi\psi)(\alpha\beta+\beta\alpha)\eta^{+}\eta^{+}}{\sqrt{2}\quad\sqrt{2}}\, dv$$

The integral sign represents an integral over all space and also a summation over the discontinuous intrinsic spin and i-spin coordinates. After this summation has been carried out the interaction energy is given by the integral

(4.10.3) $$E_{int} = \int \frac{(\psi\phi-\phi\psi)^{*} V[0.8P+0.2](\psi\phi-\phi\psi)}{2}\, dv_1\, dv_2$$

since

(4.10.4) $$Q\,(\alpha\beta+\beta\alpha) = (\beta\alpha+\alpha\beta) = (\alpha\beta+\beta\alpha)$$

From (4.10.3)

(4.10.5) $$E_{int} = \frac{0.8}{2}\int(\psi\phi-\phi\psi)^{*}\, V\,(\phi\psi-\psi\phi)\, dv_1\, dv_2 + \frac{0.2}{2}\int(\psi\phi-\phi\psi)^{*}\, V\,(\psi\phi-\phi\psi)\, dv_1\, dv_2$$

$$(4.10.6) = \frac{0.8}{2}\left[K - L - L + K\right] + \frac{0.2}{2}\left[L - K - K + L\right]$$

$$(4.10.7) \quad = 0.8\left[K - L\right] + 0.2\left[L - K\right]$$

Since $L = 6K$

$$(4.10.8) \quad E_{int} = 0.8\left[K - 6K\right] + 0.2\left[6K - K\right]$$

$$(4.10.9) \quad = -4K + K = -3K$$

4.11. The next calculation will be for the eigenfunction

$$(4.11.1) \quad \Psi = \frac{(\psi\phi - \phi\psi)}{\sqrt{2}}\frac{(\alpha\beta - \beta\alpha)}{\sqrt{2}}\frac{(\eta^+\eta^- - \eta^-\eta^+)}{\sqrt{2}}$$

The exchange operator does not affect the i-spin functions so in writing the interaction energy integral it will be assumed that the summation over the i-spin coordinate has been made.

$$(4.11.2) \quad E_{int} = \int \frac{(\psi\phi - \phi\psi)^*(\alpha\beta - \beta\alpha)}{2} V\left[0.8P + 0.2Q\right] \frac{(\psi\phi - \phi\psi)(\alpha\beta - \beta\alpha)}{2}\, dv$$

$$(4.11.3) \quad = \int \frac{(\psi\phi - \phi\psi)^*(\alpha\beta - \beta\alpha)}{2} V(0.8) \frac{(\phi\psi - \psi\phi)(\alpha\beta - \beta\alpha)}{2}\, dv + \int \frac{(\psi\phi - \phi\psi)^*(\alpha\beta - \beta\alpha)}{2} V(0.2) \frac{(\psi\phi - \phi\psi)\left[-(\alpha\beta - \beta\alpha)\right]}{2}\, dv$$

The operator Q has acted on the intrinsic spin function in the second integral above. After the summation over the spin coordinate has been made

$$E_{int} = \frac{0.8}{2}\int (\psi\phi - \phi\psi)^{*}\, V\, (\phi\psi - \psi\phi)\, dv_1\, dv_2$$

(4.11.4)
$$-\frac{0.2}{2}\int (\psi\phi - \phi\psi)^{*}\, V\, (\psi\phi - \phi\psi)\, dv_1\, dv_2$$

(4.11.5) $$= \frac{0.8}{2}\left[K - L - L + K\right] - \frac{0.2}{2}\left[L - K - K + L\right]$$

(4.11.6) $$= 0.8\left[K - L\right] - 0.2\left[L - K\right]$$

Since $$L = 6K$$

(4.11.7) $$E_{int} = 0.8\left[K - 6K\right] - 0.2\left[6K - K\right]$$

(4.11.8) $$= -4K - K = -5K$$

4.12. The point of making such an arrangement as Table I is not to present a quantitative analysis of nuclear energy level splittings but to show how the quantum numbers T=1 or 0 determine the symmetry that the generalized Pauli principle will allow for the space and spin factors and through them have an influence on the energy of the system. (Ref 4: 400) As an example of this, consider the deuteron which has $M_T=0$. Examination of Table I shows that the state of the deuteron with $M_S=S=1$ has a splitting of 10K. The state with S=1, $M_S=0$ has a splitting of 10K, and the state with $S=M_S=0$ has a splitting of 9.2K. This splitting of the energy levels of the various deuteron states occurs because the i-spin permits both the symmetric and the antisymmetric space eigenfunction to be associated with the same intrinsic spin function regardless of whether the latter is symmetric or not. In consequence of the generalized Pauli principle the eigenfunction for the deuteron could not have both its space-dependent and intrinsic-spin dependent parts symmetric without the i-spin. However, the deuteron is a relatively simple example of how the i-spin aids in classifying nuclear energy levels. The i-spin is of much greater value in the treatment of complex nuclei (the many-body problem) because of the varying degrees of symmetry which may exist among the functions which make up the eigenfunction of the system. The interaction energy is dependent upon the degree of symmetry of the space-dependent part of the eigenfunction; (note that the positive interaction energies of Table I are associated with the symmetric space-dependent part of the eigenfunction). The i-spin provides the means for distinguishing between these various symmetries.

(Ref 4: 400)

Bibliography

1. Adair, Robert K. "Conservation of Isotopic Spin in Nuclear Reactions." *The Physical Review*, *87*:1041-3 (1952).

2. Ajzenberg, F., and Lauritsen, T. "Energy Levels of Light Nuclei. IV." *Reviews of Modern Physics*, *24*:321-402 (1952).

3. Hummel, H. H., and Inglis, D. R. "Low States of Li^7 in Intermediate Coupling." *The Physical Review*, *81*:910-14 (1951).

4. Inglis, D. R. "The Energy Levels and the Structure of Light Nuclei." *Reviews of Modern Phycics*, *25*:390-450 (1953)..

5. Jordan, Pascual. *Anschauliche Quantentheorie*. Berlin: Julius Springer, 1937, pp. 98-9.

6. Kemble, Edwin C. *The Fundamental Principles of Quantum Mechanics.* New York: McGraw-Hill Book Company, Inc., 1937, Chapter X.

7. Pauling, Linus, and Wilson, Edgar Bright. *Introduction to Quantum Mechanics*. New York: McGraw-Hill Book Company, Inc., 1935, Chapters III-IX.

8. Sachs, Robert G. *Nuclear Theory*. Cambridge: Addison-Wesley Publishing Company, Inc., 1953.

9. Schiff, Leonard, I. *Quantum Mechanics*. New York: McGraw-Hill Book Company, Inc., 1949. Chapter VI.

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