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OFFICE OF NAVAL RESEARCH

Contract Nonr-562(10)

NR-064-406

Technical Report No. 7

ON THE EQUATIONS OF THE THEORY OF PLASTICITY

by

V. V. Sokolovsky

DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

January, 1956

C11-7

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V. V. Sokolovsky

translated by J. R. M. Radok

from

Priklanaya Matematika i Mechanika

Vol. XIX, No. 1, 1955, pp. 41-54

On the equations of the theory of plasticity

V. V. Sokolovsky
(Moscow)

Abstract. The present paper deals with the formulation of the basic relationship between the components of the stress and the velocity strain for yield conditions of a general form, and does not involve the usual assumption of incompressibility of the material. A detailed investigation of the equations of plane plastic equilibrium is given, and different methods of transforming these equations are indicated. A comparatively simple method of solving these equations in the form of trigonometric series is presented. Certain forms of the yield condition are considered, for which the equations of plane plastic equilibrium have particularly simple coefficients.

1. The basic equations.

We consider the three dimensional problem and use the rectangular coordinates x,y,z. We denote the stress components by σ_x, σ_y, σ_z, τ_{yz}, τ_{zx}, τ_{xy} and the components of the velocity strain by ε_x, ε_y, ε_z, γ_{yz}, γ_{zx}, γ_{xy}, the latter being related to the velocity components u,v,w by

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad 2\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$

$$\varepsilon_y = \frac{\partial v}{\partial y}, \quad 2\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$

$$\varepsilon_z = \frac{\partial w}{\partial z}, \quad 2\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

Indicate the principal axes of stress and velocity strain by the subscripts 1, 2, 3 and denote the principal components of stresses and velocity strain by σ_1, σ_2, σ_3 and ε_1, ε_2, ε_3, respectively.

The mean normal stress σ and mean rate of extension ε are defined by

$$3\sigma = \sigma_x + \sigma_y + \sigma_z, \quad 3\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

and the intensities τ and γ of stress and velocity strain, respectively, by

$$\tau^2 = \frac{1}{6}[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2] + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2$$

$$\gamma^2 = \frac{1}{6}[(\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + (\varepsilon_x - \varepsilon_y)^2] + \gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2.$$

Following R. v. Mises [1],* a general yield condition will be introduced in the form

$$\Phi(\sigma,\tau) = 0 \quad \text{or} \quad \tau = \tau(\sigma) \tag{1.1}$$

and Φ will be taken as the plastic potential; thus one obtains the relations

$$\frac{\varepsilon_x}{\frac{\partial\Phi}{\partial\sigma_x}} = \frac{\varepsilon_y}{\frac{\partial\Phi}{\partial\sigma_y}} = \frac{\varepsilon_z}{\frac{\partial\Phi}{\partial\sigma_z}} = \frac{2\gamma_{yz}}{\frac{\partial\Phi}{\partial\tau_{yz}}} = \frac{2\gamma_{zx}}{\frac{\partial\Phi}{\partial\tau_{zx}}} = \frac{2\gamma_{xy}}{\frac{\partial\Phi}{\partial\tau_{xy}}} \tag{1.2}$$

which have recently been considered by D. Drucker and W. Prager [2].

The derivatives $\partial\Phi/\partial\sigma_x, \ldots, \partial\Phi/\partial\tau_{xy}$ may be expressed in the form

$$\frac{\partial\Phi}{\partial\sigma_x} = \frac{1}{3}\frac{\partial\Phi}{\partial\sigma} + \frac{\sigma_x - \sigma}{2\tau}\frac{\partial\Phi}{\partial\tau}, \ldots, \frac{\partial\Phi}{\partial\tau_{xy}} = \frac{\tau_{xy}}{\tau}\frac{\partial\Phi}{\partial\tau}$$

* Numbers in square brackets refer to the bibliography at the end of the paper.

Hence (1.2) gives the equations

$$\begin{aligned} \varepsilon_x - \varepsilon &= \frac{\Upsilon}{\tau}(\sigma_x - \sigma), & \gamma_{yz} &= \frac{\Upsilon}{\tau}\tau_{yz} \\ \varepsilon_y - \varepsilon &= \frac{\Upsilon}{\tau}(\sigma_y - \sigma), & \gamma_{zx} &= \frac{\Upsilon}{\tau}\tau_{zx} \\ \varepsilon_z - \varepsilon &= \frac{\Upsilon}{\tau}(\sigma_z - \sigma), & \gamma_{xy} &= \frac{\Upsilon}{\tau}\tau_{xy} \end{aligned} \tag{1.3}$$

and for the mean rate of extension one has the equation

$$\varepsilon = \frac{2}{3}\Upsilon \frac{\partial\Phi/\partial\sigma}{\partial\Phi/\partial\tau} = \Upsilon\chi \tag{1.4}$$

where χ is a known function of σ and τ. Since clearly

$$\frac{\partial\Phi/\partial\sigma}{\partial\Phi/\partial\tau} = -\frac{d\tau}{d\sigma} \geq 0,$$

the mean rate of extension $\varepsilon \geq 0$. It will be noted that in the particular case where (1.1) has the form τ = const., equation (1.4) yields $\varepsilon = 0$.

The problem of plane strain will now be considered in detail. The relevant stress components are σ_x, σ_y, τ_{xy} and σ_z, the components of the velocity strains are ε_x, ε_y, γ_{xy} and ε_z. The principal axes 1 and 2 lie in the plane xy while the principal axis 3 is parallel to the z axis.

It is convenient to use, in addition to σ and τ, the quantities s and t defined by

$$s = \tfrac{1}{2}(\sigma_1 + \sigma_2) = \tfrac{1}{2}(\sigma_x + \sigma_y),$$

$$t = \tfrac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2};$$

similarly, in addition to ε and γ, the quantities e and g will be used which are defined by $(\varepsilon_1 \geq \varepsilon_2)$

$$e = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) = \frac{1}{2}(\varepsilon_x + \varepsilon_y),$$

$$g = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) = \sqrt{\frac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^{\ 2}}$$

It is easily seen on the basis of (1.3) that

$$\frac{g}{t} = \frac{\gamma}{\tau} .$$

In the case of plane strain, putting $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$, one finds the mean rate of extension and the intensity of the velocity strain in the form

$$\varepsilon = \frac{1}{3}(\varepsilon_x + \varepsilon_y) = \frac{2}{3} e,$$

$$\gamma^2 = \frac{1}{3}(\varepsilon_x^{\ 2} - \varepsilon_x\varepsilon_y + \varepsilon_y^{\ 2}) + \gamma_{xy}^{\ 2} = g^2 + \frac{1}{3} e^2.$$

It follows from this and equation (1.3) and (1.4) that

$$\tau^2 - t^2 = 3(\sigma - s)^2, \quad s = \sigma + \frac{1}{2} \chi\tau, \quad t^2 = \tau^2(1 - \frac{3}{4} \chi^2)$$

which, on the basis of (1.1), enable us to write

$$\Phi(\sigma,\tau) = F(s,t) = 0.$$

Note also, the differential relations

$$\frac{d\tau^2}{d\sigma} = \frac{dt^2}{ds}, \quad \frac{d\gamma^2}{\partial\varepsilon} = e + \frac{3}{2} \frac{dg^2}{de},$$

which follow from the above expressions.

In the case of plane stress, assuming $\sigma_z = \tau_{yz} = \tau_{zx} = 0$, one finds for the mean normal stress and shear stress intensity

$$\sigma = \frac{1}{3}(\sigma_x + \sigma_y) = \frac{2}{3} s, \; \tau^2 = \frac{1}{3}(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2) + \tau_{xy}^2 = t^2 + \frac{1}{3}s^2.$$

Hence, by (1.3) and (1.4), follow the relations

$$\gamma^2 - g^2 = 3(\varepsilon - e)^2, \quad \sigma = \frac{2}{3} s, \quad \tau^2 = t^2 + \frac{1}{3} s^2.$$

which, as a consequence of (1.1), give

$$\Phi(\sigma,\tau) = F(s,t) = 0.$$

The following differential relations should be noted, which follow from the preceding expressions:

$$\frac{d\gamma^2}{d\varepsilon} = \frac{dg^2}{de}, \qquad \frac{d\tau^2}{d\sigma} = s + \frac{3}{2}\frac{dt^2}{ds}.$$

Thus, for plane strain as well as for plane stress the general condition of plasticity takes the form

$$F(s,t) = 0 \quad \text{or} \quad t = t(s) \tag{1.5}$$

and (1.2) becomes

$$\frac{\varepsilon_x}{\frac{\partial F}{\partial \sigma_x}} = \frac{\varepsilon_y}{\frac{\partial F}{\partial \sigma_y}} = \frac{\varepsilon_z}{\frac{\partial F}{\partial \sigma_z}} = \frac{2\gamma_{xy}}{\frac{\partial F}{\partial \tau_{xy}}}. \tag{1.6}$$

The derivatives $\partial F/\partial\sigma_x$, ... , $\partial F/\partial\tau_{xy}$ may be expressed in the form

$$2\frac{\partial F}{\partial \sigma_x} = \frac{\partial F}{\partial s} + \frac{\sigma_x - s}{t}\frac{\partial F}{\partial t}, \; \ldots \; , \frac{\partial F}{\partial \tau_{xy}} = \frac{\tau_{xy}}{t}\frac{\partial F}{\partial t}.$$

Hence (1.6) gives

$$\varepsilon_x - e = \frac{g}{t}(\sigma_x - s),$$

$$\varepsilon_y - e = \frac{g}{t}(\sigma_y - s), \qquad \gamma_{xy} = \frac{g}{t}\tau_{xy} \tag{1.7}$$

$$\varepsilon_z - e = \frac{g}{t}(\sigma_z - s),$$

and

$$e = g\,\frac{\partial F/\partial s}{\partial F/\partial t} = gh, \tag{1.8}$$

where h is a known function of s and t. Clearly

$$h = \frac{e}{g} = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 - \varepsilon_2}$$

and hence

$$|h| \leq 1 \quad \text{for} \quad \varepsilon_1\varepsilon_2 \leq 0, \qquad |h| \geq 1 \quad \text{for} \quad \varepsilon_1\varepsilon_2 \geq 0.$$

It will be noted that the equations (1.7) may be easily obtained directly from (1.3).

The known transformation formulae

$$\left.\begin{matrix}\sigma_x\\ \sigma_y\end{matrix}\right\} = \tfrac{1}{2}(\sigma_1 + \sigma_2) \pm \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos 2\varphi, \quad \tau_{xy} = \tfrac{1}{2}(\sigma_1 - \sigma_2)\sin 2\varphi$$

where φ is the angle between the x axis and the direction of the principal stress σ, reduce for the stress components to

$$\left.\begin{matrix}\sigma_x\\ \sigma_y\end{matrix}\right\} = s \pm t\cos 2\varphi, \qquad \tau_{xy} = t\sin 2\varphi \tag{1.9}$$

and for the components of the velocity strain to

$$\left.\begin{matrix}\varepsilon_x\\ \varepsilon_y\end{matrix}\right\} = e \pm g\cos 2\varphi, \qquad \gamma_{xy} = g\sin 2\varphi. \tag{1.10}$$

2. Investigation of the hyperbolic equations of the plane theory.

The basic equations of the plane theory of plasticity will now be investigated by the usual methods.

Introducing into the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

the expressions (1.9), one finds

$$\frac{\partial s}{\partial x} + \cos 2\varphi \frac{\partial t}{\partial x} + \sin 2\varphi \frac{\partial t}{\partial y} - 2t(\sin 2\varphi \frac{\partial \varphi}{\partial x} - \cos 2\varphi \frac{\partial \varphi}{\partial y}) = 0$$

$$\frac{\partial s}{\partial y} + \sin 2\varphi \frac{\partial t}{\partial x} - \cos 2\varphi \frac{\partial t}{\partial y} + 2t(\cos 2\varphi \frac{\partial \varphi}{\partial x} + \sin 2\varphi \frac{\partial \varphi}{\partial y}) = 0. \qquad (2.1)$$

Substituting in (1.10) the expressions

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad 2\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

one has

$$2 \sin 2\varphi \frac{\partial u}{\partial x} - (h + \cos 2\varphi)(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = 0$$

$$2 \sin 2\varphi \frac{\partial v}{\partial y} - (h - \cos 2\varphi)(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = 0. \qquad (2.2)$$

The differential equations of the characteristics of the deduced system of equations (2.1) and (2.2) have the form

$$d\varphi \pm \frac{\sqrt{1 - h^2}}{2h} d \ln t = 0,$$

$$\frac{dy}{dx} = \frac{\sin 2\varphi \mp \sqrt{1 - h^2}}{\cos 2\varphi - h},$$

$$\frac{dv}{du} = \frac{\sin 2\varphi \pm \sqrt{1 - h^2}}{\cos 2\varphi + h}. \qquad (2.3)$$

Hence it is clear that the basic system of equations belongs to the <u>hyperbolic</u> or the <u>elliptic</u> type depending on whether

$|h| < 1$ or $|h| > 1$.

The equations of the characteristics will now be studied for the case $|h| < 1$, i.e., when the basic system of equations is hyperbolic. Introduce the new variable ψ, defined by

$$\frac{dt}{ds} = - h = \cos 2\psi,$$

and the new function λ

$$2d\lambda = \sqrt{1 - h^2}\,\frac{ds}{t} = \sin 2\psi\,\frac{ds}{t} = \tan 2\psi\; d\ln t \tag{2.4}$$

For plane strain in the particular case $\tau = k$, one has obviously

$$t = k, \quad h = 0, \quad \psi = \frac{1}{4}\pi,$$

whence λ, apart from an arbitrary constant, has the form

$$\lambda = \frac{s}{2k}.$$

For plane stress, in the same case $\tau = k$, one has

$$\frac{s^2}{3} + t^2 = k^2, \quad h = \frac{s}{3t} = - \cos 2\psi$$

whence follows

$$s = \frac{3kh}{\sqrt{1 + 3h^2}} = \frac{- 3k \cos 2\psi}{\sqrt{1 + 3\cos^2 2\psi}},$$

$$t = \frac{k}{\sqrt{1 + 3h^2}} = \frac{k}{\sqrt{1 + 3\cos^2 2\psi}}$$

and hence λ, apart from an arbitrary constant, may be written

$$\lambda = - \operatorname{artan}\left[\left(\frac{1 - h}{1 + h}\right)^{3/2}\right] = - \operatorname{artan}(\cot^3\psi).$$

Here and later on artan refers to its principal value, lying between $-\pi/2$ and $+\pi/2$.

The characteristics of the system of equations (2.1) and (2.2) form two families and they are determined by the equations

$$\eta = \lambda(\psi) - \varphi = \text{const}, \quad \frac{dy}{dx} = \tan(\varphi - \psi), \quad \frac{du}{dv} = -\tan(\varphi - \psi), \quad (2.5)$$

$$\xi = \lambda(\psi) + \varphi = \text{const}, \quad \frac{dy}{dx} = \tan(\varphi + \psi), \quad \frac{du}{dv} = -\tan(\varphi + \psi). \quad (2.6)$$

It is clear that the characteristics are inclined to the x axis by an angle $\varphi \mp \psi$; the angle 2ψ between the characteristics in the plane x,y will, in general, differ from point to point. Since for each family

$$\frac{dy}{dx}\frac{dv}{du} = -1,$$

the characteristics in the plane x,y are orthogonal to the corresponding characteristics in the plane u,v.

Using different variables, the equations for the characteristics may be obtained in a more convenient form. Instead of x and y, introduce the coordinates

$$\bar{x} = \frac{1}{\sin 2\psi}[x \sin(\varphi + \psi) - y \cos(\varphi + \psi)],$$

$$\bar{y} = \frac{1}{\sin 2\psi}[y \cos(\varphi - \psi) - x \sin(\varphi - \psi)],$$

and instead of the velocity components u and v the components

$$\bar{u} = \frac{1}{\sin 2\psi}[u \cos(\varphi - \psi) + v \sin(\varphi - \psi)],$$

$$\bar{v} = \frac{1}{\sin 2\psi}[u \cos(\varphi + \psi) + v \sin(\varphi + \psi)].$$

Further, the new variables X, Y, U, V will be used which are defined by

$$X/\bar{x} = Y/\bar{y} = \sqrt{(t/k)\sin 2\psi}, \quad U/\bar{u} = V/\bar{v} = \sqrt{(t/k)\sin 2\psi},$$

where k is an arbitrary constant which has the dimension of a stress and which is chosen beforehand.

Performing the above transformation and introducing the function

$$P(\lambda) = \frac{1 - d\psi/d\lambda}{\sin 2\psi},$$

we write the equations of the characteristics (2.5) and (2.6) in the form

$$\eta = \lambda(\psi) - \varphi = \text{const}, \quad dY + XPd\lambda = 0, \quad dU = VPd\lambda, \tag{2.7}$$

$$\xi = \lambda(\psi) + \varphi = \text{const}, \quad DX + YPd\lambda = 0, \quad dV = UPd\lambda. \tag{2.8}$$

From these equations one deduces readily the so-called canonical systems of equations for the coordinates

$$\frac{\partial Y}{\partial \xi} + \frac{P}{2} X = 0, \quad \frac{\partial X}{\partial \eta} + \frac{P}{2} Y = 0 \tag{2.9}$$

and the velocities

$$\frac{\partial U}{\partial \xi} = \frac{P}{2} V, \quad \frac{\partial V}{\partial \eta} = \frac{P}{2} U. \tag{2.10}$$

There also exist particular solutions, corresponding to the variables ξ and η, which may be called integrals of the equations of plasticity.

If $\xi \neq$ const, $\eta = \eta_0 =$ const, then $\eta = \lambda - \varphi = \eta_0$ and hence λ and φ are constant along the family of characteristics (2.8). Thus (2.8) gives

$$dX = 0, \quad dV = 0.$$

It is clear that X and V are arbitrary functions of ξ :

$$\lambda - \varphi = \eta_0, \quad X = X(\xi), \quad V = V(\xi), \tag{2.11}$$

and the family of characteristics (2.8) consists of straight lines ξ = const.

The family of characteristics (2.7) in the plane xy and the quantity U are found by integrating the equations

$$\frac{\partial Y}{\partial \xi} + \frac{P}{2} X = 0, \quad \frac{\partial U}{\partial \xi} = \frac{P}{2} V.$$

If $\xi = \xi_0$ = const, $\eta \neq$ const, then $\xi = \lambda + \varphi = \xi_0$ and hence along the family of characteristics (2.7) λ and φ are constant. Therefore (2.7) gives

$$dY = 0, \quad dU = 0.$$

Thus Y and V are arbitrary functions of

$$\lambda + \varphi = \xi_0, \quad Y = Y(\eta), \quad U = U(\eta) \tag{2.12}$$

and the family of characteristics (2.7) in the plane x,y consists of straight lines η = const. The family of characteristics (2.8) and the quantity V are found by integration of the equations

$$\frac{\partial X}{\partial \eta} + \frac{P}{2} Y = 0, \quad \frac{\partial V}{\partial \eta} = \frac{P}{2} U.$$

Finally, if $\xi = \xi_0$ = const, $\eta = \eta_0$ = const, then $\psi = \psi_0$ and $\varphi = \varphi_0$ are identical constants. In the plane x,y, the families of characteristics (2.7) and (2.8) represent two systems of parallel straight lines

$$Y = \text{const}, \quad X = \text{const},$$

and U, V are arbitrary functions of Y and X

$$U = U(Y), \quad V = V(X).$$

For plane strain, in the particular case $\tau = k = \text{const}$, one has

$$t = k, \quad \psi = \tfrac{1}{4}\pi, \quad P = 1$$

and the preceding equations are greatly simplified.

It will be noted that by replacing the velocity components u and v by the coordinates y and x and the components U and V by the coordinates Y and X, all the equations in terms of velocity components are transformed into the corresponding equations in terms of the coordinates.

3. Transformation of the equations of the hyperbolic and elliptic types.

Another method of transforming the equations of the characteristics will now be considered which applies when $|h| < 1$ and $|h| > 1$, i.e., when the basic system is hyperbolic or elliptic. In the particular case $\tau = k$, this method was used by the author in his book [3].

Instead of the coordinates x and y, introduce the coordinates

$$\bar{x} = x \cos \varphi + y \sin \varphi, \quad \bar{y} = y \cos \varphi - x \sin \varphi$$

and instead of the components u and v the components

$$\bar{u} = u \cos \varphi + v \sin \varphi, \quad \bar{v} = v \cos \varphi - u \sin \varphi.$$

Further, use will be made of the new variables X, Y and U, V, defined by

$$X/\bar{x} = V/\bar{v} = \sqrt{\frac{t}{k}} \exp\left(-\int \frac{d \ln t}{2h}\right),$$

$$Y/\bar{y} = U/\bar{u} = \sqrt{\frac{t}{k}} \exp\left(\int \frac{d \ln t}{2h}\right),$$

where k is an arbitrary constant, having the dimension of a stress.

The differential equations of the characteristics (2.3) take now the form

$$d\varphi \pm \frac{\sqrt{1 - h^2}}{2h} d \ln t = 0, \quad dX \pm \sqrt{\frac{1 - h}{1 + h}} \exp\left(-\int \frac{d \ln t}{h}\right) dY = 0,$$

$$dV \mp \sqrt{\frac{1 - h}{1 + h}} \exp\left(-\int \frac{d \ln t}{h}\right) dU = 0.$$

First, the transformation of the equations of the characteristics will be investigated for the case when $|h| < 1$ and dY/dX, dV/dU, $d\varphi/dt$ are real. For this purpose functions L and λ will be introduced which are defined by

$$L = \sqrt{\frac{1 - h}{1 + h}} \exp\left(-\int \frac{d \ln t}{h}\right), \quad d\lambda = -\frac{\sqrt{1 - h^2}}{2h} d \ln t.$$

The equations of the characteristics may then be rewritten

$$\eta = \lambda - \varphi = \text{const}, \quad dX + LdY = 0, \quad dV - LdU = 0, \quad (3.1)$$

$$\xi = \lambda + \varphi = \text{const}, \quad dX - LdY = 0, \quad dV + LdU = 0. \quad (3.2)$$

The canonical systems of equations for the coordinates and for the velocities are now easily deduced:

$$\frac{\partial X}{\partial \xi} = - L \frac{\partial Y}{\partial \xi}, \quad \frac{\partial X}{\partial \eta} = L \frac{\partial Y}{\partial \eta} \tag{3.3}$$

$$\frac{\partial V}{\partial \xi} = L \frac{\partial U}{\partial \xi}, \quad \frac{\partial V}{\partial \eta} = - L \frac{\partial U}{\partial \eta}. \tag{3.4}$$

Reverting in these equations from the variables ξ and η to the earlier variables λ and φ, one finds the systems of equations

$$\frac{\partial X}{\partial \varphi} = - L \frac{\partial Y}{\partial \lambda}, \quad \frac{\partial X}{\partial \lambda} = - L \frac{\partial Y}{\partial \varphi}, \tag{3.5}$$

$$\frac{\partial V}{\partial \varphi} = L \frac{\partial U}{\partial \lambda}, \quad \frac{\partial V}{\partial \lambda} = L \frac{\partial U}{\partial \varphi}. \tag{3.6}$$

It is also not difficult to obtain the particular solutions, corresponding to constant ξ and η which have already been found above in a different form.

For plane strain, when $\tau = k$, it is obvious that $t = k$, $h = 0$, and hence L and λ are given by

$$L = \exp(2\lambda), \quad \lambda = \frac{s}{2k}$$

where the latter is only determined apart from a constant.

For plane stress, when $\tau = k$, one has

$$\frac{s^2}{3} + t^2 = k^2, \quad h = \frac{s}{3t},$$

whence

$$s = \frac{3kh}{\sqrt{1 + 3h^2}}, \quad t = \frac{k}{\sqrt{1 + 3h^2}}.$$

The functions L and λ have now the form

$$L = \sqrt{\frac{1 - h}{1 + h}} \exp[\sqrt{3}\, \text{artan}(\sqrt{3}h)], \quad \lambda = - \text{artan}\left[\left(\frac{1 - h}{1 + h}\right)^{3/2}\right],$$

where the expression for λ is exact apart from a constant. Eliminating h, one finds

$$L = (\tan|\lambda|)^{1/3}\exp\left[\sqrt{3}\ \mathrm{artan}\left(\sqrt{3}\,\frac{1-(\tan\lambda)^{2/3}}{1+(\tan\lambda)^{2/3}}\right)\right].$$

As before, artan refers here to its principal value lying between $-\pi/2$ and $+\pi/2$.

Next, a new method will be demonstrated of transforming the equations (3.5) for the coordinates and the equations (3.6) for the velocities. For the sake of definiteness, attention will be concentrated on the equations for the coordinates, since the equations for the velocities are quite analogous.

Introduce the new variables

$$A = \frac{X}{\sqrt{L}} = \bar{x}\ \sqrt{\frac{t}{k}}\left(\frac{1+h}{1-h}\right)^{1/4}, \qquad B = Y\sqrt{L} = \bar{y}\ \sqrt{\frac{t}{k}}\left(\frac{1-h}{1+h}\right)^{1/4}$$

by use of which (3.5) takes the form

$$\frac{\partial A}{\partial\lambda} + \frac{\partial B}{\partial\varphi} = -\,PA, \qquad \frac{\partial B}{\partial\lambda} + \frac{\partial A}{\partial\varphi} = PB \tag{3.7}$$

where

$$P = \frac{1}{2}\,\frac{d\,\ln L}{d\lambda} = \frac{1}{\sqrt{1-h^2}}\left(1 - \frac{1}{2\sqrt{1+h^2}}\,\frac{dh}{d\lambda}\right).$$

The first order system of equations (3.7) may be reduced to a single second order equation. Eliminating B or A, one finds

$$\frac{\partial^2 A}{\partial\lambda^2} - \frac{\partial^2 A}{\partial\varphi^2} = Aa, \qquad \frac{\partial^2 B}{\partial\lambda^2} - \frac{\partial^2 B}{\partial\varphi^2} = Bb \tag{3.8}$$

where the coefficients a and b are given by

$$a = P^2 - \frac{dP}{d\lambda}, \qquad b = P^2 + \frac{dP}{d\lambda}.$$

Now, the transformation of the equations of the characteristics will be considered for the case where $|h| > 1$ and dY/dX, dV/dU, $d\varphi/dt$ are imaginary. For this purpose introduce the functions M and μ

$$M = \sqrt{\frac{h - 1}{h + 1}} \exp\left(-\int \frac{d \ln t}{h}\right), \qquad d\mu = -\frac{\sqrt{h^2 - 1}}{2h} d \ln t.$$

Then the equations of the characteristics assume the form

$$\eta = i\mu - \varphi = \text{const}, \quad dX + iMdY = 0, \quad dV - iMdU = 0, \tag{3.9}$$

$$\xi = i\mu + \varphi = \text{const}, \quad dX - iMdY = 0, \quad dV + iMdU = 0. \tag{3.10}$$

The canonical system of the equations for the coordinates is now easily found:

$$\frac{\partial X}{\partial \xi} + iM \frac{\partial Y}{\partial \xi} = 0, \qquad \frac{\partial X}{\partial \eta} - iM \frac{\partial Y}{\partial \eta} = 0. \tag{3.11}$$

Similarly one has for the velocities

$$\frac{\partial V}{\partial \xi} - iM \frac{\partial U}{\partial \xi} = 0, \qquad \frac{\partial V}{\partial \eta} + iM \frac{\partial U}{\partial \eta} = 0. \tag{3.12}$$

Reverting in these equations from the variables ξ and η to the variables μ and φ, using the formulae

$$+ 2 \frac{\partial X}{\partial \xi} = \frac{\partial X}{\partial \varphi} - i \frac{\partial X}{\partial \mu}, \qquad - 2 \frac{\partial X}{\partial \eta} = \frac{\partial X}{\partial \varphi} + i \frac{\partial X}{\partial \mu}, \ \ldots$$

and separating real and imaginary parts, one obtains the following canonical systems of equations for the coordinates and velocities:

$$\frac{\partial X}{\partial \varphi} = - M \frac{\partial Y}{\partial \mu}, \quad \frac{\partial X}{\partial \mu} = M \frac{\partial Y}{\partial \varphi}, \tag{3.13}$$

$$\frac{\partial V}{\partial \varphi} = M \frac{\partial U}{\partial \mu}, \quad \frac{\partial V}{\partial \mu} = - M \frac{\partial U}{\partial \varphi}. \tag{3.14}$$

For plane stress, in the particular case $\tau = k$, one has again the earlier relations

$$s = \frac{3kh}{\sqrt{1 + 3h^2}}, \quad t = \frac{k}{\sqrt{1 + 3h^2}}.$$

The functions M and μ have the form

$$M = \sqrt{\frac{h - 1}{h + 1}} \exp[\sqrt{3} \operatorname{artan}(\sqrt{3}h)], \quad \mu = \varkappa \operatorname{artanh} \left[\left(\frac{|h| - 1}{|h| + 1}\right)^{3/2}\right]$$

where $\varkappa$ = sign h and the last expression is exact apart from a constant term. Eliminating h one finds

$$M^{\varkappa} = (\tanh|\mu|)^{1/3} \exp\left[\sqrt{3} \operatorname{artan}\left(\sqrt{3} \frac{1 + (\tanh \mu)^{2/3}}{1 - (\tanh \mu)^{2/3}}\right)\right].$$

As before, artan refers to its principal value.

Returning now to the equations (3.13) for the coordinates or (3.14) for the velocities, another method of transformation will be shown. For example, consider the equations for the coordinates. Introduce the new variables

$$A = \frac{X}{\sqrt{M}} = \bar{x} \sqrt{\frac{t}{k}}\left(\frac{h + 1}{h - 1}\right)^{1/4}, \quad B = Y\sqrt{M} = \bar{y} \sqrt{\frac{t}{k}}\left(\frac{h - 1}{h + 1}\right)^{1/4}$$

as a result of which (3.13) becomes

$$\frac{\partial A}{\partial \mu} + \frac{\partial B}{\partial \varphi} = - QA, \quad \frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \varphi} = QB \tag{3.15}$$

where

$$Q = \frac{1}{2}\frac{d\ln M}{d\mu} = \frac{1}{2\sqrt{h^2 - 1}}\left(1 + \frac{1}{2\sqrt{h^2 - 1}}\frac{dh}{d\mu}\right).$$

The first order system of equations (3.15) may be reduced to a single second order equation. Eliminating A or B, one finds

$$\frac{\partial^2 A}{\partial\mu^2} + \frac{\partial^2 A}{\partial\varphi^2} = Aa, \quad \frac{\partial^2 B}{\partial\mu^2} + \frac{\partial^2 B}{\partial\varphi^2} = Bb, \tag{3.16}$$

where

$$a = Q^2 - \frac{dQ}{d\mu}, \quad b = Q^2 + \frac{dQ}{d\mu}.$$

In the preceding study other substitutions could have been used. For example, it is convenient to introduce a function ω defined by [3]

$$h = \frac{\cot\,\omega}{\sqrt{3}},$$

having in mind that the interval $\frac{1}{6}\pi < \omega < \frac{5}{6}\pi$ corresponds to $|h| < 1$, while $0 \leq \omega < \frac{1}{6}\pi$, $\frac{5}{6}\pi < \omega \leq \pi$ refer to $|h| > 1$. Alternatively one may use the function ψ defined by $h = -\cos 2\psi$ or $h = \mp \cosh 2\psi$ depending on whether $|h| < 1$ or $|h| > 1$. Use of these functions make it possible to transform the coefficients of the corresponding equations into a simpler form.

4. Solution of the equations by trigonometric series.

A method of integration of the above equations will now be considered which is based on the application of trigonometric series, analogous to the series used by S. A. Chaplygin in the theory of plane gas flow. Only the reasoning applying to the equations for the coordinates will be given here, since the equations for the velocities are quite analogous.

First, the equations of the hyperbolic type (3.7) and (3.8), when $|h| < 1$, will be considered and particular solutions of these equations for different integral values of n will be sought in the form

$$A = A_n(\lambda)\cos(n\varphi + \gamma_n), \quad B = B_n(\lambda)\sin(n\varphi + \gamma_n), \tag{4.1}$$

where γ_n are constants and A_n, B_n depend only on λ. Substituting from (4.1) in (3.7), one obtains

$$\frac{dA_n}{d\lambda} + PA_n = -nB_n, \quad \frac{dB_n}{d\lambda} - PB_n = nA_n, \tag{4.2}$$

while substitution in (3.8) gives

$$\frac{d^2A_n}{d\lambda^2} = A_n(a - n^2), \quad \frac{d^2B_n}{d\lambda^2} = B_n(b - n^2). \tag{4.3}$$

It is now easy to construct the solution with period 2π in the form of trigonometric series

$$A = \frac{A_0}{\sqrt{L}} + \sum_{n=1}^{\infty} A_n(\lambda)\cos(n\varphi + \gamma_n),$$

$$B = B_0\sqrt{L} + \sum_{n=1}^{\infty} B_n(\lambda)\sin(n\varphi + \gamma_n). \tag{4.4}$$

It will be noted that the terms of both series satisfy equations (3.7) and (3.8) for any n. Hence it is not difficult to deduce different solutions of the equations of plasticity.

The preceding equations, for plane strain, when $\tau = k$, were obtained by the author in his book [3].

Next consider the equations of the elliptic type (3.15) and (3.16) corresponding to $|h| > 1$, and, as before, look for particular solutions of these equations in the form

$$A = A_n(\mu)\cos(n\varphi + \gamma_n), \quad B = B_n(\mu)\sin(n\varphi + \gamma_n) \tag{4.5}$$

where γ_n are constants and A_n, B_n are functions of μ only. Introducing (4.5) in (3.15), one finds

$$\frac{dA_n}{d\mu} + QA_n = -nB_n, \quad \frac{dB_n}{d\mu} - QB_n = -nA_n \tag{4.6}$$

while (3.16) gives

$$\frac{d^2A_n}{d\mu^2} = A_n(a + n^2), \quad \frac{d^2B_n}{d\mu^2} = B_n(b + n^2). \tag{4.7}$$

Solutions with period 2π may now be easily constructed in the form

$$A = \frac{A_o}{\sqrt{M}} + \sum_{n=1}^{\infty} A_n(\mu)\cos(n\varphi + \gamma_n),$$

$$B = B_o\sqrt{M} + \sum_{n=1}^{\infty} B_n(\mu)\sin(n\varphi + \gamma_n). \tag{4.8}$$

Since the terms in both series satisfy (3.15) and (3.16), different particular solutions of the equations of plasticity are easily found.

Finally it will be noted that the difficulty of solving the equations (3.8), (4.3) or (3.16), (4.7), studied above, depends on the form of their coefficients. Hence, the question as to which of the conditions (1.5) corresponds to the simpler form of the functions: $a(\lambda)$ and $b(\lambda)$ or $a(\mu)$ and $b(\mu)$, has a definite meaning. This question, in the case of $a(\lambda)$ and $b(\lambda)$ entering into the equations (3.8) and (4.3), is resolved by integrating the equations

$$\frac{dh}{d\lambda} = 2\sqrt{1 - h^2}(1 - P\sqrt{1 - h^2}), \quad \frac{d\ln t}{d\lambda} = -\frac{2h}{\sqrt{1 - h^2}},$$

$$\frac{dt}{ds} = -h, \tag{4.9}$$

while for $a(\mu)$ and $b(\mu)$ in (3.16) and (3.17) one has to integrate the equations

$$\frac{dh}{d\mu} = 2\sqrt{h^2 - 1}(Q\sqrt{h^2 - 1} - 1), \quad \frac{d\ln t}{d\mu} = -\frac{2h}{\sqrt{h^2 - 1}},$$

$$\frac{dt}{ds} = -h. \tag{4.10}$$

Equations (4.9) and (4.10) may be simplified. This will only be done here for equation (4.9), since (4.10) is quite analogous.

If $a(\lambda)$ is given, one may introduce the new variables

$$\alpha_1 = -P, \quad \alpha_2 = \alpha_1 + \sqrt{\frac{1 - h}{1 + h}}$$

which satisfy the Riccati equations

$$\frac{d\alpha_1}{d\lambda} + \alpha_1{}^2 = a, \quad \frac{d\alpha_2}{d\lambda} + \alpha_2{}^2 = a - 1. \tag{4.11}$$

After determining α_1 and α_2, the functions $s = s(\lambda)$ and $t = t(\lambda)$ which give the condition (1.5) in parametric form, are easily found from the equations

$$d\ln t = (\alpha - \frac{1}{\alpha})d\lambda, \quad ds = t(\alpha + \frac{1}{\alpha})d\lambda, \quad \alpha = \alpha_2 - \alpha_1. \tag{4.12}$$

If $b(\lambda)$ is known, then one must take as new variables

$$\beta_1 = P, \quad \beta_2 = \beta_1 - \sqrt{\frac{1+h}{1-h}}$$

which likewise satisfy the Riccati equations

$$\frac{d\beta_1}{d\lambda} + \beta_1{}^2 = b, \quad \frac{d\beta_2}{d\lambda} + \beta_2{}^2 = b - 1. \qquad (4.13)$$

After obtaining β_1 and β_2, the functions $s = s(\lambda)$ and $t = t(\lambda)$, giving (1.5) in parametric form, are easily found from

$$d \ln t = (\beta - \frac{1}{\beta})d\lambda, \quad -ds = t(\beta + \frac{1}{\beta})d\lambda, \quad \beta = \beta_2 - \beta_1. \qquad (4.14)$$

It will be noted that replacement of b, β and $-s$ by a, α and $+s$ in equations (4.13) and (4.14) leads to (4.11) and (4.12).

Now, examples of the functions $a(\lambda)$ and $b(\lambda)$ will be given which correspond to the simplest forms of the equations (3.8). As first example one may take a = const, or b = const, when one of the equations (3.8) reduces to the telegraph form.

If a = const, the unknown functions α_1 and α_2 are determined in closed form. The first and second Riccati equations (4.11) have such solutions:

for $a = + c^2$

$$\alpha_1 = c \coth [c(\lambda - \lambda_1)],$$

$$\alpha_2 = \begin{cases} \sqrt{1 - c^2} \cot [\sqrt{1 - c^2}(\lambda - \lambda_2)] & (c < 1) \\ \sqrt{c^2 - 1} \coth [\sqrt{c^2 - 1}(\lambda - \lambda_2)] & (c > 1) \end{cases}$$

for $a = - c^2$

$$\alpha_1 = c \cot [c(\lambda - \lambda_1)],$$

$$\alpha_2 = \sqrt{c^2 + 1} \cot [\sqrt{c^2 + 1}(\lambda - \lambda_2)]$$

where λ_1, λ_2 are arbitrary constants and $c > 0$.

For $c \to 0$ and $c \to 1$

$$\alpha_1 = \frac{1}{\lambda - \lambda_1} \quad \text{and} \quad \alpha_2 = \frac{1}{\lambda - \lambda_2},$$

while for $|\lambda_1| \to \infty$ and $|\lambda_2| \to \infty$

$$\alpha_1 = \pm c \quad \text{and} \quad \alpha_2 = \pm \sqrt{c^2 - 1}.$$

If b = const, then β_1 and β_2 are determined by formulae following from the preceding expressions by replacing a and α by b and β.

Substituting α_1 and α_2 in (4.12) or β_1 and β_2 in (4.14) and finding the integrals, one obtains the classes of functions $s = s(\lambda)$ and $t = t(\lambda)$ which determine the condition (1.5) in parametric form.

In the simplest particular case where

$$-\alpha_1 = \frac{1}{\sin 2\psi_o}, \quad \alpha_2 = \cot 2\psi_o, \quad c = \frac{1}{\sin 2\psi_o},$$

equations (4.12) take the form

$$d\,\ln t = 2 \cot 2\psi_o d\lambda, \quad ds = \frac{2t}{\sin 2\psi_o} d\lambda.$$

Solving these equations, one obtains $t = k \exp(2\lambda \cot 2\psi_o)$, and at the same time the condition of plasticity

$$t = k + s \cos 2\psi_o$$

involving the constants ψ_o and k which was proposed by C. Coulomb.

$b = n(n - 1)\lambda^{-2}$, where n is an integer, when one of the equations (3.8) reduces to an integrable form.

If $a = n(n - 1)\lambda^{-2}$, the functions α_1 and α_2 likewise are determined in closed form. The first Riccati equation (4.11) has the solution

$$\alpha_1 = \frac{n(\lambda/\lambda_1)^{2n-1} + n - 1}{\lambda[(\lambda/\lambda_1)^{2n-1} - 1]}$$

which involves an arbitrary constant λ_1, and the second Riccati equation (4.11) by help of the substitution

$$\alpha_2 = \frac{w}{\lambda^{2n}} + \frac{n}{\lambda}, \quad z = \lambda^{1-2n}$$

takes the form

$$(1 - 2n)\frac{dw}{dz} + w^2 = - z^N, \quad N = \frac{4n}{1 - 2n}$$

and it will have a solution in quadratures, likewise containing an arbitrary constant λ_2. For n = 0 and n = 1 one finds

$$\alpha_1 = \frac{1}{\lambda - \lambda_1}, \quad \alpha_2 = \cot(\lambda - \lambda_2)$$

and for $|\lambda_1| \to \infty$

$$\alpha_1 = \frac{1 - n}{\lambda}.$$

If $b = n(n - 1)\lambda^{-2}$, the functions β_1 and β_2 are given by the formulae, following from the preceding ones by replacing a and α by b and β.

Substituting α_1 and α_2 in (4.12) or β_1 and β_2 in (4.14) and finding the integrals, one determines the classes of functions $s = s(\lambda)$ and $t = t(\lambda)$, giving (1.5) in parametric form.

In the simplest particular case, when $\alpha_1 = 0$, $\alpha_2 = \cot\lambda$, equations (4.12) become

$$d \ln t = 2 \cot 2\lambda d\lambda, \qquad ds = \frac{2t}{\sin 2\lambda} d\lambda.$$

Solving these equations, one finds $t = k \sin 2\lambda$ and at the same time the condition of plasticity

$$t = k \sin\left(\frac{s - s_o}{k}\right)$$

involving the constants s_o and k which has already been considered earlier by the author [3].

The above examples determine sufficiently wide the classes of conditions of plasticity (1.5) which may be used for the solution of different problems.

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