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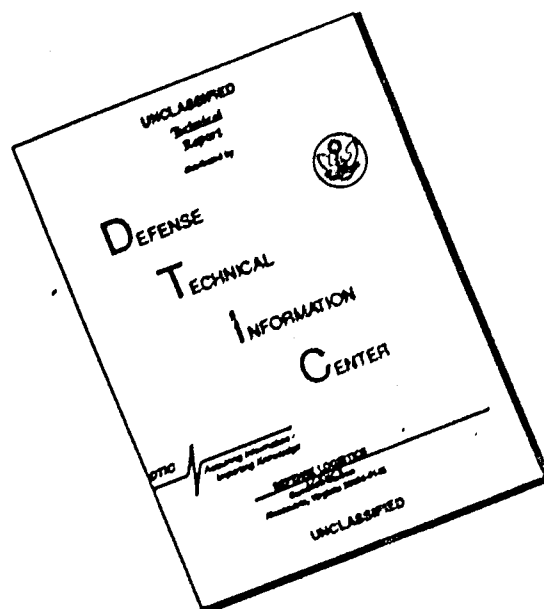
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FUNDAMENTAL RESEARCH IN APPLIED MATHEMATICS

ON THE NUMERICAL SOLUTION OF LINEAR AND NON-LINEAR
PARABOLIC EQUATIONS ON THE CRDVAC

by

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On the Numerical Solution of Linear and Non-Linear
Parabolic Equations on the Ordvac

by

David Young and Louis Ehrlich

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1. Introduction

Let us consider the problem of finding a function $u(x,y)$ which satisfies the parabolic partial differential equation

$$(1.1) \quad u_t = u_{xx} + f(x,t,u)$$

in R : $0 \leq x \leq A$, $0 \leq t \leq T$ and which satisfies the initial conditions

$$(1.2) \quad u(x,0) = g(x) \quad 0 \leq x \leq A$$

and the boundary conditions

$$(1.3) \quad \begin{cases} \alpha_1(t)u_x(0,t) + \beta_1(t)u(0,t) = \phi(t) & 0 \leq t \\ \alpha_2(t)u_x(A,t) + \beta_2(t)u(A,t) = \psi(t) & 0 \leq t. \end{cases}$$

The function $u(x,y)$ is required to be continuous and bounded in $R + S$, where S is the boundary of R , except that continuity of $u(x,y)$ is not required at a point of S where a jump discontinuity occurs in any of the other functions involved in (1.2) and (1.3). We assume that all functions are continuous and bounded in R .

An analytic solution for the above problem is available only under very special conditions. We shall not seek such solutions here. Moreover, we shall not even consider the very important questions of existence and uniqueness. Rather, we shall be concerned with the study of numerical procedures for solving certain difference equations which one obtains when one uses finite difference methods. Of course it is recognized that any solutions so obtained would be of no value in the event that no solution of the original problem existed or in the event that more than one solution existed. In certain cases existence

and uniqueness are available (See for instance [1] and [6]).

The numerical methods which we shall consider include the well-known Forward Difference method, the Crank-Nicolson method [4] and the DuFort-Frankel method [5]. With the Crank-Nicolson method one is usually required to solve at each step a system of simultaneous equations. This can be done effectively in many cases using the Successive Overrelaxation method of iteration [22]. Alternatively if the system is linear, or is replaced by a related linear system, one can use a non-iterative method, apparently first used for this type of problem by Bruce, Peaceman, Rachford, and Rice [2].

In section 2 we first derive finite difference representations of the boundary conditions including two well-known methods and one which we have not seen in the literature. We then derive several difference equations and obtain local error estimates using methods of numerical quadrature. Next, we consider the process known as "extrapolation to zero grid size", first used by L.F. Richardson [20]. Using this process one can, under favorable conditions, appreciably improve the accuracy of the finite difference methods. We indicate how results obtained using several different grid sizes can often be used to obtain a more accurate extrapolation formula than one would obtain using Richardson's formula wherein it is assumed that the error tends to zero like the square of the grid size.

In order to compare the various finite difference methods we prepared a comprehensive routine for Ordvac which is capable of solving a large class of problems by each method. Details of this program are given in section 4 and Appendix. The specific problems which we considered are discussed in section 3.

The program may be considered as a research tool to enable one to study the effectiveness of the various methods under different conditions. When used together with a more complete theoretical analysis than we have been able to carry out here, it is to be expected that new and significant results on finite difference methods should be obtained.

In section 5 the results obtained on the Ordvac are analyzed and tentative conclusions are reached as to the order of convergence and the overall effectiveness of each of the methods. For the linear case our results indicate strongly that the use of the Crank-Nicolson method with the non-iterative procedure, is superior to any of the other methods. In the non-linear case however, one loses accuracy with this method if one modifies the difference equation in an attempt to avoid all iterations. Further work appears to be indicated in order to preserve the accuracy of the difference equation without requiring too many iterations. We suggest one possible scheme for doing this, but we have not yet tried it out on the machine.

For the Forward Difference method and for the Crank-Nicolson method, used with the iterative process, we found that the accuracy could usually be improved considerably by the use of the proper extrapolation to zero grid size. Such extrapolation was more difficult to carry out for the Crank-Nicolson method, used without iterations.

As expected the use of the proposed method for representing the boundary conditions appeared to be more accurate than either of the other two considered.

Although this study is far from complete, it was felt that the work which has been done to date should be reported now, especially since no further work can be done during the next academic year. The work may be regarded

h

as a natural extension of research which was begun by the Interior Ballistics Laboratory, Aberdeen Proving Ground. One of the objects was to determine the order of convergence of the Forward Difference method. Extensive calculations were carried out on the Bell Relay Computer, as described in [6], and on the Eniac. The latter calculations have not previously been reported. We shall study them in section 5 along with the Ordvac results.

In addition to acknowledging the sponsorship of the Office of Ordnance Research, U.S. Army, we should also like to express our appreciation for the encouragement and assistance furnished by the Interior Ballistic Laboratory at the Aberdeen Proving Ground. Discussions with Dr. W.C. Taylor of the Interior Ballistic Laboratory and with Dr. B.L. Hicks, formerly of that laboratory, were of great value. We are also pleased to acknowledge the assistance of Dr. R.M. Conlan, Miss J.M. Wood, Mr. B.T.C. Koo, and Mr. S. Soscia who worked at various times on the programming of the problem for Ordvac.

2. Difference Equations

Let h and k be positive numbers such that A/h is an integer, which we denote by M , and let $I_{h,k}$ denote the set of points (x,t) such that x/h and t/k are integers. Let $R_{h,k}$ and $S_{h,k}$ denote respectively the set of points of $I_{h,k}$ belonging to R and to S , (See Figure 1). We replace the problem of the previous section by that of finding a function $u(x,t)$ defined on $R_{h,k} + S_{h,k}$, satisfying a certain difference equation on $R_{h,k}$ and satisfying on $S_{h,k}$ certain other conditions derived from (1.2) and (1.3).

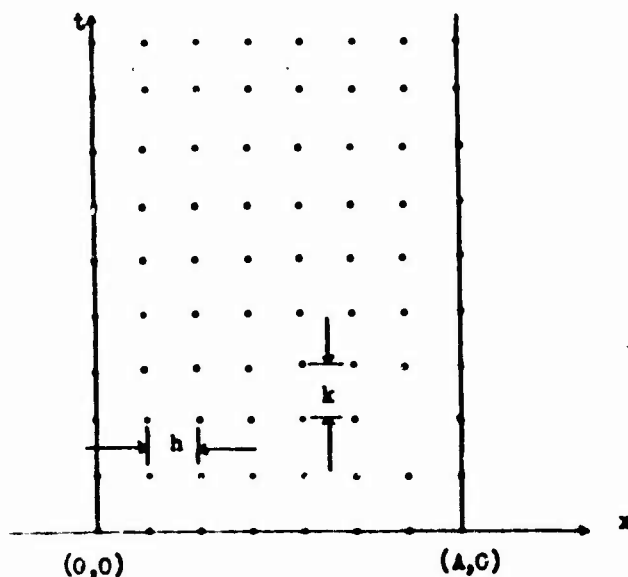


Figure 1

In this section we derive finite difference ^{equations} Δ which are obtained from the differential equation and from the boundary conditions, and seek to estimate the accuracy. Our methods are by no means rigorous. Indeed they are merely formal methods utilising Taylor series expansions whenever convenient. Nevertheless, it is felt that these methods will give some basis for making comparisons of the accuracies of the various finite difference methods, at least

for some cases. We will later test the comparisons thus obtained against numerical results obtained for typical problems on the Ordvac.

For discussion of finite difference analogues of problems involving the infinite region ~~---(---)---~~ the reader is referred to papers by Courant, Friedrichs and Lewy [3] and by Pitts John [10].

In addition to deriving the difference equations we will also describe here the associated numerical procedures. In order that we may be able to give a complete description of these procedures it is convenient to first consider the treatment of the boundary conditions.

2.1 Boundary conditions. Although our discussion could be extended to cover more general situations, we shall consider only the following two types of boundary conditions

$$(2.1.1) \quad \begin{cases} (a) & u(0,t) = \phi(t) \quad , \quad (t \geq 0) \\ (b) & u(A,t) = \psi(t) \quad , \quad (t \geq 0), \end{cases}$$

and

$$(2.1.2) \quad \begin{cases} (a) & u_x(0,t) = -H(u_g - u(0,t)) \quad , \quad (t > 0) \\ (b) & u_x(A,t) = 0 \quad , \quad (t > 0) . \end{cases}$$

Here H and u_g are constants.

We shall at times refer to the boundary conditions represented by (2.1.1) and (2.1.2) as B V and N D conditions, respectively.

We first note that the initial conditions (1.2) are easily represented by

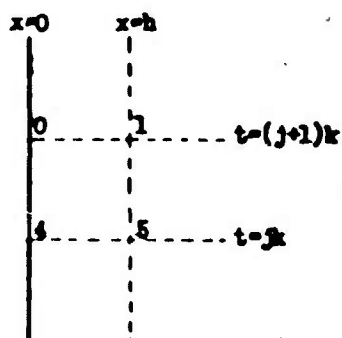
$$(2.1.3) \quad u_{i,0} = g_i \quad 1 \leq i \leq M-1 .$$

Here and subsequently we let $u_{i,j} = u(ih, jk)$, $g_i = g(ix)$, etc., where i and j are integers. Moreover, the B V conditions (2.1.1) can be replaced by

$$(2.1.4) \quad \begin{cases} u_{0,j} = \phi_j \quad , \quad (0 \leq j) \\ u_{M,j} = \psi_j \quad , \quad (0 \leq j) . \end{cases}$$

However, with the ND condition (2.1.2a) we shall consider several alternative procedures. A number of these are considered by Price and Slack [9]. We shall discuss two of these together with one which may very well be known also, although we have not seen it in the literature.

In order to simplify our notation we shall consider a configuration of four points near the line $x = 0$, as shown in Figure 2. We denote by u_i , t_i , x_i , f_i the values of the respective variables at the point numbered i , ($i = 0, 1, 4, 5$). We seek relations between u_0 and one or more of the other u_i . It



Typical Boundary Configuration - Left Hand Boundary

Figure 2

is assumed that u_4 and u_5 have already been obtained by previous calculations.

Perhaps the most obvious method is to replace $u_x(0,t)$ by $(u_1 - u_0)h^{-1}$ in (2.1.2a) obtaining

$$(2.1.5) \quad u_0 \approx \frac{u_1 + h u_x}{1 + h h}.$$

Formula (2.1.5) defines a procedure for representing (2.1.2a) which we shall refer to as Option I.

To study the accuracy of (2.1.5) we expand $u(x,t)$ in a Taylor series about the point (x_0, t_0) obtaining

$$u_1 = u_0 + h u_x + \frac{1}{2} h^2 u_{xx} + \dots$$

Using (2.1.2a) we obtain

$$(2.1.6) \quad u_0 = \frac{u_1 - h u_x}{1 + h h} = -\frac{h^2}{2(1+h h)} u_{xx} + \dots$$

We note that the leading term of the local error, given by the right member of (2.1.6) is $O(h^2)$. we shall not try to estimate here the effect of local errors on the accuracy of the solution of the difference equation as a solution of the

differential equation. Instead we shall attempt to infer this from the results of Ordvac computations.

If one makes use of the differential equation at the point (x_4, t_4) , one can obtain more accurate formulas. Thus, using a Taylor series expansion about (x_4, t_4) we have

$$u_0 = u_4 + ku_t + \frac{1}{2}k^2 u_{tt} + \dots$$

$$u_5 = u_4 + hu_x + \frac{1}{2}h^2 u_{xx} + \frac{1}{6}h^3 u_{xxx} + \dots$$

Here the derivatives are evaluated at (x_4, t_4) . Using (1.1) and the relation

$u_x = -H(u_g - u_4)$ we obtain

$$(2.1.7) \quad u_0 \approx u_4(1-2r) + 2ru_5 + 2rhH(u_g - u_4) + kf_4$$

where the leading terms of the difference between the right and the left members of (2.1.7), that is, the right member minus the left member, are

$$(2.1.8) \quad -\frac{kh}{3} u_{xxx} + \frac{k^2}{2} u_{tt}.$$

Here and subsequently we let

$$(2.1.9) \quad r = \frac{k}{h^2}.$$

Price and Slack [19] show that this formula introduces instability for $r > \frac{1}{2}$, in contrast to (2.1.5) which is stable for all r . Thus assuming $k = rh^2$, where r is a constant not greater than $\frac{1}{2}$, we obtain from (2.1.8)

$$(2.1.10) \quad -\frac{rh^3}{3} u_{xxx} + \frac{r^2 h^4}{2} u_{tt}.$$

Thus the leading term of the local error is $O(h^3)$ as compared with $O(h^2)$ with Option 1. Unfortunately, however, as we shall see later, the limitation which must be imposed on k is very severe.

We shall refer to (2.1.7) as Option II.

We now seek a formula which is as accurate as Option II and which is, at the same time, stable for all r . With this aim we expand $u(x, t)$ in a Taylor series about the point (x_0, t_0) obtaining

$$u_1 = u_0 + hu_x + \frac{1}{2}h^2u_{xx} + \frac{1}{6}h^3u_{xxx} + \dots$$

$$u_4 = u_0 - ku_t + \frac{1}{2}k^2u_{tt} - \frac{1}{6}k^3u_{ttt} + \dots$$

Using (2.1.2a) and (1.1) we have

$$(2.1.11) \quad u_0 \approx \frac{u_1 + hHu_x + \frac{1}{2r}u_4 + \frac{1}{2}h^2f_0}{1 + hH + \frac{1}{2r}}$$

where the leading term of the difference between the right and left members of (2.1.11) is

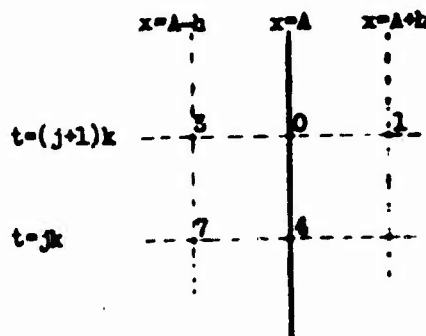
$$(2.1.12) \quad \frac{-\frac{1}{4}h^2ku_{tt} - \frac{1}{6}h^3u_{xxx}}{1 + hH + \frac{1}{2r}}.$$

Evidently the leading term of the local error for (2.1.11), which we shall call Option III, is $O(h^2k + h^3)$. Moreover, since all coefficients in (2.1.11) are positive, the formula is stable for all r .

In some cases it is convenient to replace f_0 by f_4 in (2.1.11), obtaining

$$(2.1.13) \quad u_0 \approx \frac{u_1 + hHu_x + \frac{1}{2r}u_4 + \frac{1}{2}h^2f_4}{1 + hH + \frac{1}{2r}},$$

where the numerator in (2.1.12) is changed by a term whose absolute value is $\frac{1}{2}h^2kf_t^*$, where $f_t^* = f_t + f_{uu}u_t$. Since this term is $O(h^2k)$, the order of the local accuracy of (2.1.13) is the same as that of (2.1.11). We will refer to (2.1.13) as Option III'.



Typical Boundary Configuration - Right hand Boundary

Figure 3

We now consider the treatment of (2.1.2b). For the simplest representation we replace u_x by $(u_0 - u_3)h^{-1}$ obtaining

$$(2.1.14) \quad u_0 \approx u_3.$$

It is easily seen that the leading term of the difference between the right and left members of (2.1.14) is

$$-\frac{h^2}{2} u_{xx}$$

so that (2.1.14) has the same accuracy as Option I. Actually, of course, (2.1.14) is a special case of (2.1.5) where $H = 0$.

Formula (2.1.14) represents the only method which we used for treating (2.1.2b). Perhaps a better procedure would have been to consider (x_0, y_0) as an interior point and to use

$$(2.1.15) \quad u_1 \approx u_3.$$

Here, the leading term of the error is

$$\frac{h^3}{3} u_{xxx},$$

which is $O(h^3)$.

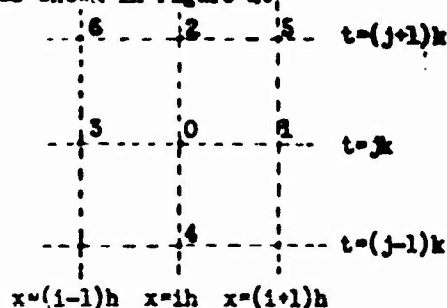
If one does not wish to use the point (x_1, t_1) , which is outside the region, one may obtain a formula of the same order of accuracy by using the point (x_4, t_4) instead. The formula may be obtained immediately from (2.1.11) by letting $\mathbb{H} = 0$. We get

$$(2.1.16) \quad u_0 \approx \frac{u_1 + \frac{1}{2r} u_4 + \frac{h^2}{2} f_0}{1 + \frac{1}{2r}}.$$

The leading term of the error is $O(h^3)$. As before the accuracy would be preserved if we replaced f_0 by f_4 in (2.1.16), obtaining

$$(2.1.17) \quad u_0 \approx \frac{u_1 + \frac{1}{2r} u_4 + \frac{h^2}{2} f_4}{1 + \frac{1}{2r}}.$$

2.2 Forward difference method. For the derivation of this and other difference equations it will be convenient to consider a typical configuration of points such as shown in Figure h.



Typical Configuration - Interior Points

Figure h

We assume here that $1 \leq i \leq M-1$, but that j may equal zero in which case the point (x_i, t_j) would lie outside of the region.

Probably the simplest difference equation is obtained by replacing u_t by $(u_2 - u_0) k^{-1}$ and by replacing u_{xx} by $(u_1 + u_3 - 2u_0) h^{-2}$. These are familiar finite difference representations of the respective derivatives. However, for this method and for the others given below we shall present a derivation based on methods of numerical quadrature. Although the actual finite difference formulas come out the same, nevertheless it seems simpler to obtain the error estimates by the method which we use.

By (1.1) and the rectangle rule for integration we have

$$(2.2.1) \quad u_2 - u_0 = \int_{t_0}^{t_0+k} u_t dt = \int_{t_0}^{t_0+k} (u_{xx} + f) dt = k(u_{xx} + f) + \frac{k^2}{2}(u_{xxt} + f_t),$$

where $f_t^0 = f_t + f_u u_t$. Using Taylor's series we also obtain

$$(2.2.2) \quad u_{xx} = (u_1 + u_3 - 2u_0)h^{-2} - \frac{1}{12}h^2 u_{xxxx} + \dots$$

Substituting in the previous equation we get

$$(2.2.3) \quad u_2 \approx (1-2r)u_0 + r(u_1 + u_3) + kf_0$$

where the leading term of the difference between the right and left members of (2.2.3) is

$$(2.2.4) \quad k \left\{ \left(\frac{k}{2} - \frac{h^2}{12} \right) u_{xxxx} \right\}.$$

Since the number of time steps needed to reach a given value of t is proportional to k^{-1} it seems reasonable to take

$$(2.2.5) \quad \left(\frac{k}{2} - \frac{h^2}{12} \right) u_{xxxx}$$

as a measure of the local error. It is well known, (See for instance [18]) that

k must satisfy the condition

$$(2.2.6) \quad k \leq \frac{1}{2} h^2$$

in order for the numerical procedure to be stable. Thus the local error for the Forward Difference method is $O(h^2)$.

Of course, such an analysis does not, by any means, indicate that the accuracy in the large is of second order. The question of the order of convergence in the large has been studied for a special class of problems by Juncosa and Young [12]. For an analysis of the situation in the large for the case where $u(x,t)$ is assumed to possess certain partial derivatives in R^+S , see Milne [17].

For B V problems with f, ϕ, ψ vanishing, the convergence of the sequence of solutions of the Forward Difference method to the exact solution of the differential equation has been proved by Leutert [16] for $r \leq 1/4$, under the assumption that $g(x)$ is piecewise continuous, that is, continuous except for a finite number of finite jumps, and that $g(x)$ has a one sided first derivative everywhere. Hildebrand [9] proved convergence for $0 < r \leq \frac{1}{2}$ under the assumption that $g(x)$

is piecewise continuous and has bounded variation. More severe restrictions were imposed on $g(x)$ for the case $r = \frac{1}{2}$. Juncosa and Young [11] proved convergence for $r \leq \frac{1}{2}$ under the assumption that $g(x)$ is piecewise continuous.

The numerical computational procedure is extremely simple. Since one knows $u_{i,0}$, $0 \leq i \leq M$, one can compute $u_{i,1}$, $1 \leq i \leq M-1$ using (2.2.5) which for a general point becomes

$$(2.2.7) \quad u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j}) + kf_i.$$

For B V problems $u_{0,1}$ and $u_{M,1}$ are given and we can proceed at once to the next row. For M D problems we can compute $u_{0,1}$ directly from either (2.1.5)/(2.1.7) or (2.1.13) and we can use (2.1.14), (2.1.15) or (2.1.17) to find $u_{M,1}$. Having thus determined all values of $u_{i,1}$ we proceed to compute values of $u_{i,2}$ then $u_{i,3}$ etc.

Were it not for the condition (2.2.6) on the size of k , the Forward Difference method would be ideal for use on computing machines. Unfortunately, when one attempts to improve the accuracy by reducing h , say by a factor of 2, then the number of time steps is increased by a factor of 4, making the total work increase by a factor of 8. More generally, one can assert that the work increases with the third power of h^{-1} , or equivalently, as the third power of M .

2.3 Crank-Nicolson method. If, instead of using the rectangle rule of integration as in the case of the forward difference method, we use the trapezoidal rule, we obtain

$$(2.3.1) \quad u_2 - u_0 = \int_{t_0}^{t_0+k} u_t dt = \int_{t_0}^{t_0+k} (u_{xx} + f) dt \\ = \frac{k}{2} [(u_{xx})_0 + (u_{xx})_2 + f_0 + f_2] - \frac{k^3}{12} (u_{xxtt} + f_{tt}) + \dots$$

Using Taylor's series we have

$$(u_{xx})_0 = h^{-2}(u_1 + u_3 - 2u_0) - \frac{h^2}{12} u_{xxxx} + \dots$$

$$(u_{xx})_2 = h^{-2}(u_5 + u_6 - 2u_2) - \frac{h^2}{12} u_{xxxx} + \dots$$

Substituting in (2.3.1) we get

$$u_2 - u_0 = \frac{k}{2h^2} [u_1 + u_3 + u_5 + u_6 - 2u_0 - 2u_2] + \frac{k}{2} [f_0 + f_2] - \frac{kh^2}{12} u_{xxxx} - \frac{k^3}{12} (u_{xxtt} + f_{tt}).$$

Solving for u_2 we have

$$(2.3.2) \quad u_2 = \frac{1-r}{1+r} u_0 + \frac{r}{2(1+r)} [u_1 + u_3 + u_5 + u_6] + \frac{k}{2(1+r)} [f_0 + f_2]$$

where we have neglected the remainder whose leading terms are

$$(2.3.3) \quad \frac{k}{1+r} \left\{ -\frac{h^2}{12} u_{xxxx} - \frac{k^2}{12} (u_{xxtt} + f_{tt}) \right\}.$$

We note that the expression in the braces is $O(h^2)$ as long as $k = O(h)$.

Juncosa and Young [13] showed that under certain conditions the error in the large is $O(h^2)$ provided $k = O(h/|\log h|)$. It is not known, however, whether this result can be extended to include cases where $k = O(h)$.

It is sometimes convenient to replace f_2 by f_0 in (2.3.2). The additional error which is introduced is proportional to k^2 and hence the expression in braces occurring in (2.3.3) is increased by a term proportional to k . Thus if k tends to zero like h , the expression in the braces would be $O(h)$ instead of $O(h^2)$.

Another modification of (2.3.2) is obtained by replacing the term $\frac{1}{2}[f_0 + f_2]$ by $f(x_0, \frac{1}{2}(t_0 + t_2), \frac{1}{2}(u_0 + u_2))$. This change was made in the program prepared for the Ordvac. For simplicity, we assume here that $f(x, t, u)$ is a function of

u alone. Expanding in a Taylor series about the point $(x, \bar{t})$, where $\bar{t} = \frac{1}{2}(t_0 + t_2)$

we have
$$u_2 = \bar{u} + \frac{1}{2} k u_t + \frac{1}{8} k^2 u_{tt} + \dots$$

$$u_0 = \bar{u} - \frac{1}{2} k u_t + \frac{1}{8} k^2 u_{tt} + \dots$$

$$f_2 = \bar{f} + \frac{1}{2} k f_t + \frac{1}{8} k^2 f_{tt} + \dots$$

$$f_0 = \bar{f} - \frac{1}{2} k f_t + \frac{1}{8} k^2 f_{tt} + \dots$$

where $\bar{f} = f(\bar{u})$, $\bar{u} = u(x, \bar{t})$. It is evident that $\frac{1}{2}(f_0 + f_2) = \bar{f} + \frac{1}{8} k^2 f_{tt} + \dots$

and
$$\frac{1}{2}(u_0 + u_2) = \bar{u} + \frac{1}{8} k^2 u_{tt} + \dots$$

Therefore
$$f\left(\frac{1}{2}(u_0 + u_2)\right) = \bar{f} + \frac{1}{8} k^2 u_{tt} f_t + \dots$$

and
$$\frac{1}{2}(f_0 + f_2) - f\left(\frac{1}{2}(u_0 + u_2)\right) = \frac{1}{8} k^2 f_{tt} - \frac{1}{8} k^2 u_{tt} f_t + \dots = O(k^2).$$

By (2.3.2) and (2.3.3) it is evident that the order of accuracy has not been changed.

For a general point (2.3.2) may be written

$$(2.3.4) \quad u_{1,j+1} = \frac{1-r}{1+r} u_{1,j} + \frac{r}{2(1+r)} [u_{1+1,j} + u_{1-1,j} + u_{1+1,j+1} + u_{1-1,j+1}]$$

Two alternative numerical procedures are available for using the Crank-Nicolson method. The first is an iterative method wherein, given all values of $u_{1,j}$, $0 \leq j \leq M$, one chooses a first approximation $u_{1,j+1}^{(0)}$, $0 \leq j \leq M$ and proceeds to iterate using the formula

$$(2.3.5) \quad u_{1,j+1}^{(n)} = \omega \left\{ \frac{1-r}{1+r} u_{1,j} + \frac{r}{2(1+r)} [u_{1+1,j} + u_{1-1,j} + u_{1+1,j+1}^{(n-1)} + u_{1-1,j+1}^{(n-1)}] + \frac{k}{2(1+r)} [f_{1,j} + f_{1,j}^{(n)}] \right\} - (\omega-1) u_{1,j+1}^{(n)}, \quad i=1,2,\dots,M.$$

Here the parameter ω known as the relaxation factor, is chosen in the range

$1 \leq \omega < 2$ to accelerate the convergence. At the end of each iteration one obtains $u_{0,j+1}^{(n)}$ by solving (2.1.5), (2.1.7), (2.1.11), or (2.1.13) for $u_{0,j+1}$ using the latest values of $u_{1,j+1}$ when appropriate. Similarly $u_{M,j+1}^{(n)}$ is obtained.

The iterative process is repeated until

$$(2.3.6) \quad \max_{0 \leq j \leq M} |u_{1,j+1}^{(n)} - u_{1,j+1}^{(n-1)}| < \epsilon$$

where ϵ is a preassigned tolerance. When the iterative process has converged, one proceeds to determine the values in the next row in the same way.

The choice of the relaxation factors α_b , which will yield the fastest convergence can easily be computed for linear BV problems. It is shown in [22] that

$$(2.3.7) \quad \alpha_b = 1 + \left[\frac{\mu}{1 + \sqrt{1 - \mu^2}} \right]^2$$

where μ is the maximum eigenvalue of the $(M-1) \times (M-1)$ matrix

$$\begin{pmatrix} 0 & \beta & 0 & 0 & \dots & 0 \\ \beta & 0 & \beta & 0 & \dots & 0 \\ 0 & \beta & 0 & \beta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \beta & 0 \end{pmatrix}$$

where $\beta = \frac{r}{2(1+r)}$. It is easy to show that

$$(2.3.8) \quad \mu = \frac{r}{1+r} \cos \frac{\pi}{M}.$$

Let us now assume that $k = O(h)$ and that, as a result, r is proportional to M . Since for large M we have $1 - \mu \approx \frac{1}{M}$, we can show, as in [22], section 4, that the rate of convergence of (2.3.5) varies as $M^{-1/2}$; hence the number of iterations increases as $M^{1/2}$. Since the number of points on each row increases with M , it follows that the work per row is proportional to $M^{3/2}$. Moreover, since the number of time steps increases as M , (instead of M^2 as with the Forward Differences method) we conclude that the total time increases as $M^{5/2}$ with the Crank-Nicolson method compared with M^3 for the Forward Difference method. Thus the Crank-Nicolson method would appear to be superior to the Forward Difference method, at least for large M .

The advantage of the Crank-Nicolson method can be increased still further by using a non-iterative procedure. The effectiveness of this procedure depends on the fact that for a linear system involving a Jacobi matrix, i.e., a matrix whose only non-zero elements occupy the main diagonal and the two diagonals adjacent to the main diagonal, the solution can be obtained very easily. This fact was observed by L.H. Thomas in unpublished notes [21] and was applied to parabolic equations by Bruce, Peaceman, Rachford and Rice [2]. Thus following their discussion, let us suppose we have the system

$$(2.3.9) \quad \begin{cases} B_1 x_1 + C_1 x_2 = d_1 \\ A_1 x_{i-1} + B_1 x_i + C_1 x_{i+1} = d_i \quad 2 \leq i \leq M-1 \\ A_{M-1} x_{M-1} + B_M x_M = d_M \end{cases}$$

If one carries out the familiar Gauss elimination procedure, eliminating the elements below the main diagonal of the matrix, and then uses the method of back substitution, one obtains

$$(2.3.10a) \quad \begin{cases} x_M = q_M \\ x_i = q_i - b_i x_{i+1} \end{cases} \quad 1 \leq i \leq M-1$$

where

$$(2.3.10b) \quad \begin{cases} q_1 = d_1/B_1 \\ q_i = \frac{d_i - A_i q_{i-1}}{B_i - A_i b_{i-1}} \end{cases}$$

and

$$(2.3.10c) \quad \begin{cases} b_0 = 0 \\ b_i = \frac{C_i}{B_i - A_i b_{i-1}} \end{cases} \quad 1 \leq i \leq M-1.$$

The stability of this computation procedure has been investigated by Thomas in [2].

No more than about 3 or 4 times as much work per row would appear to be required with this method as with the Forward Difference method. Thus, if we assume $k = O(h)$, then the amount of work required increases as M^2 instead of $M^{5/2}$ with the iterative procedure and M^3 with the Forward Difference method.

Unfortunately, when (1.1) is non-linear it is not always easy to obtain linear equations and still retain the desired accuracy. For, if we use (2.3.2), we find that the unknown value of $u_{i,j+1}$ is involved, in general, in a non-linear manner in the evaluation of the function $f(x,t,u)$. On the

other hand, if one replaces $u_{1,j+1}$ by $u_{1,j}$, one obtains a less accurate formula, as discussed above. To get around this difficulty in an analogous situation, Bruce, Peaceman, Rachford, and Rice [2] used an iterative procedure together with the non-iterative method for solving the linear equations. Although this procedure might appear to defeat the original purpose of avoiding all iterations, it is reported that convergence is obtained after a very few iterations.

It might not be too unreasonable to preassign a fixed number of iterations, independent of h and k , to be carried out at each time step. Thus, the Heun method for solving ordinary differential equations is the same as the modified Euler method (see for instance, Milne [17]) except that one performs only one iteration. The local error for the Heun method is somewhat larger than, but of the same order of magnitude as the modified Euler method.

Another possible solution to this difficulty would be to use a difference equation involving more points. However, care must be taken to avoid formulas which might lead to unstable procedures, and also to avoid obtaining a matrix which was not a Jacobi matrix.

2.4 DuFort-Frankel method. In [5] DuFort and Frankel obtained a difference equation which is stable for all values of r and which, at the same time, yields explicit formulas for values of u in the new row in terms of values on previous rows. We may derive the formula by use of numerical quadrature as follows.

$$\begin{aligned}
 (2.4.1) \quad u_2 - u_0 &= \int_{t_0}^{t_2} u_t dt = \int_{t_0}^{t_2} (u_{xx} + f) dt \\
 &= 2k [u_{xx} + f_0] + \frac{1}{3}k^3 [u_{xxx} + f_t'] + \dots
 \end{aligned}$$

Using Taylor's series, as before, we get

$$(2.4.2) \quad u_2 = u_4 + 2k[(u_1 + u_3 - 2u_0)h^{-2} + f_0] + \left\{ -2k \frac{h^2}{12} u_{xxxx} + \frac{k^3}{3} (u_{xxt} + f_t') \right\} + \dots$$

If we were to neglect the expression in the braces and all higher order terms in (2.4.2) we would obtain Richardson's method [20], which is unstable for all values of r and is thus unsuited for numerical calculations unless very special precautions are taken [15]. Nevertheless, at this point we remark that the local error for Richardson's method is

$$(2.4.3) \quad k \left\{ -\frac{h^2}{6} u_{xxxx} + \frac{k^2}{3} (u_{xxt} + f_t') \right\}.$$

In order to obtain a stable method, DuFort and Frankel replace u_0 by $\frac{1}{2}(u_4 + u_2)$ in (2.4.2) and obtain, after solving for u_2 ,

$$(2.4.4) \quad u_2 = \frac{1-2r}{1+2r} u_4 + \frac{2r}{1+2r} (u_1 + u_3) + \frac{2k}{1+2r} f_0 + \frac{k}{1+2r} \left\{ -\frac{h^2}{6} u_{xxxx} + \frac{k^2}{3} (u_{xxt} + f_t') - \frac{2k^2}{h^2} u_{tt} \right\} + \dots$$

Now if $k = O(h^2)$, it is evident that the expression in the braces is $O(h^2)$. On the other hand, if $k = O(h)$, then the expression inside the braces need not even tend to zero with h . Consequently, although stability is assured for all h and k , for convergence the ratio k/h must tend to zero with h , (see [5]).

Neglecting remainder terms we have for a general point

$$(2.4.5) \quad u_{1,j+1} = \frac{1-2r}{1+2r} u_{1,j-1} + \frac{2r}{1+2r} (u_{1+1,j} + u_{1-1,j}) + \frac{2k}{1+2r} f_{1,j}.$$

Evidently, to compute $u_{1,j+1}$, one needs the values of u on the j -th and $(j-1)$ st rows. At the beginning one has only the values for $j = 0$. To obtain the values

of u for $j = 1$ another procedure must be used. This point is discussed further in Section 4.

The convergence of the DuFort-Frankel method for a certain class of problems has been given by Juncosa [14].

2.5. Extrapolation to zero grid size. For the process known as extrapolation to zero grid size, proposed by L. F. Richardson [20], one assumes that the error $e_M(x, t) = u_M(x, t) - u(x, t)$ is proportional to h^a . Here $u(x, t)$ is the exact solution of the differential equation and $u_M(x, t)$ is the solution of the difference equation with $h = A/M$. Usually certain restrictions are assumed for k . Even though $e_M(x, t)$ might vanish like M^{-a} , as is indeed true for certain cases as shown in [12], [13], nevertheless it does not necessarily follow that $e_M(x, t)$ is exactly proportional to M^{-a} . This can be easily seen by considering the case where $e_M(x, t) = M^{-a} \sin M$.

If $e_M(x, t)$ were directly proportional to M^{-a} , one could determine $u(x, t)$ in terms of $u_M(x, t)$ and $u_{2M}(x, t)$. Indeed we would have

$$(2.5.1) \quad u(x, t) = u_{2M}(x, t) + \frac{1}{2^a - 1} [u_{2M}(x, t) - u_M(x, t)].$$

If one solves the difference equation for three different mesh sizes, obtaining $u_M(x, y)$, $u_{2M}(x, y)$, and $u_{4M}(x, y)$ we can perform a simple test for the assumption that $e_M(x, y)$ is proportional to M^{-a} . Indeed, if the assumption were valid, we would have

$$(2.5.2) \quad \frac{e_M(x, y)}{e_{2M}(x, y)} = 2^a.$$

By computing the ratio, ρ , appearing in (2.5.2) one can determine the proper extrapolation to zero grid size, provided such exists, and also estimate the

degree of convergence of the sequence of solutions of the difference equation. It is not necessary to compute α itself since 2^α appears in (2.5.1). We wish to emphasize once more, that the considerations of this section are not rigorous.

3. Specific Problems Considered

In order to obtain some indication as to the validity of the non-rigorous mathematical considerations of the previous section, we have solved some sample problems on the Ordvac computer at Aberdeen. We chose problems involving either boundary conditions (2.1.1) or (2.1.2) and either linear and non-linear differential equations. Actually, as will be seen in the next section, the program which was prepared for Ordvac is capable of handling entire classes of problems rather than merely the specific problems described below.

3.1. Boundary value problem - linear differential equation (BVL). Here we selected a problem for which an analytic solution is readily available. The equations are

$$(3.1.1) \quad \begin{cases} u_t = u_{xx} & , \quad 0 < x < 1, \quad t > 0, \\ u(x,0) = g(x) & , \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & , \quad t \geq 0. \end{cases}$$

The analytic solution for the case $g(x) = 1$ is given by

$$(3.1.2) \quad u(x,t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin n\pi x e^{-n^2 \pi^2 t}$$

We shall, by solving the difference equations with various mesh sizes, attempt to determine the rate of convergence for both methods. For the Crank-Nicolson method we shall let $k = O(h)$ as $h \rightarrow 0$, even though the theoretical results have been proved only under stronger restrictions on k , [13], in an attempt to determine whether a generalization is to be expected.

3.2. Boundary value problem - non-linear differential equation (BVN).

The following problem described by Hartree [7] appears to be typical of this class of problems.

$$(3.2.1) \quad \begin{cases} u_t = u_{xx} + \gamma e^u & 0 < x < 1, \quad t > 0 \\ u(x, 0) = 0 & 0 < x < 1 \\ u(0, t) = u(1, t) = 0 & t \geq 0 \end{cases}$$

Here $u(x, t)$ represents the temperature in a substance where there is generation of heat increasing exponentially with the temperature. When γ exceeds a certain critical value there is no steady state solution and the solution increases beyond all bounds. We shall not study this aspect of the problem but, rather, will use a fixed value of 0.8 for γ , which is below the critical value. We shall solve the problem by the Forward Difference and Crank-Nicolson method for several values of h . Solutions for various values of γ have been obtained by Hartree using a differential analyzer and the results agreed quite well with experiments.

3.3. Normal derivative problems (NDL and NDN). Here we consider boundary conditions (2.1.2) with both the linear equation

$$(3.3.1) \quad u_t = u_{xx} \quad 0 < x < A, \quad t > 0$$

and the non-linear equation

$$(3.3.2) \quad u_t = u_{xx} + e^{-1/u}, \quad 0 < x < A, \quad t > 0.$$

The initial conditions are

$$(3.3.3) \quad f(x) = c \quad 0 \leq x \leq A$$

where in the cases which we shall consider $c = .031$. Actually, in our computations, we shall take for our initial function not $g(x) = .031$ but a set of

values obtained for a later value of t using the analytic solution of the linear problem (see for instance [8], page 22). The values are given in Table I. Even in the non-linear case, these values are sufficiently accurate since the non linear term is negligible for small values of t . This adjustment in the initial conditions was made to eliminate any possible effect of the singularity caused by the discontinuity of u_x when $t = 0$. It is by no means certain that such a precaution would be necessary, and for the linear case at least, it should be possible to determine analytically the effect of the singularity.

Problems of this type involving (3.3.1) are designated MDL and those involving (3.3.2) are designated MDN. For MDN problems the solution $u(x,t)$ represents the temperature in a propellant where the reaction rate is determined by the Arrhenius term $e^{-1/u}$. The propellant is assumed to be a semi-infinite slab, $0 \leq x < \infty$. The propellant is heated by a gas at temperature u_g which occupies the region $x < 0$ and whose influence on the propellant depends on the heat transfer coefficient h . In order to be able to use finite difference methods we consider a slab of finite thickness L and replace the condition $\lim_{x \rightarrow \infty} u_x = 0$ by the condition (2.1.2b). Further details on the physical motivation for these problems may be found in [8].

In order to represent more accurately the physical situation, it was sometimes necessary to introduce "shutoff", that is after a certain value of t , u_g is greatly reduced. However, we shall not do this in our study.

We shall also not consider the determination of the "ignition time" which is the time required for the value of $u(x,t)$ to increase above a preassigned value.

We shall assume the following values for the parameters

$$A = 73,600$$

$$c = .031$$

$$u = .18$$

$$H = 2 \times 10^{-6}$$

Prior to 1952 a number of calculations were done on this problem at the Aberdeen Proving Ground using the Bell Relay Computer and the Eniac. The Forward Difference method was used with $r = \frac{1}{2}$ and several values of h . We shall make use of some of these results in Section 5.

4. Description of the Program for Drives

To run a particular problem with the program as presently coded, it is necessary to specify three major conditions; i.e. (1) the difference equation to be used, (2) the boundary conditions, and (3) the necessary parameters such as h , r , etc.

All of the difference equations and boundary conditions described in Section 2 have been coded and any problem involving these can be computed using the program. Except for the CN method, the interior points of the $(j+1)$ st row are computed followed by the computation or insertion of the boundary values of that row. This permits the boundary conditions to be easily modified. The appendix contains the detailed equations used in the present coding.

Unless otherwise instructed, the machine will assume that the differential equation is linear. In the non-linear case it is necessary to instruct the machine to compute the non-linear term.

For the DR method, since the j -th and $(j-1)$ st rows are necessary to compute the $(j+1)$ st row (see appendix), it was necessary to compute the first row by a different method. We used the FD method. The number of rows necessary to be computed by the FD method depends upon r , which involves k , the increment in time. This is noted in the Appendix.

For the CN method, NON case, when the values of the function became large, the iterative scheme would not converge. A feature was added in the program whereby if the number of iterations exceeded a given K_0 for the $(j+1)$ st row, the coding would revert back to the FD method and commence computation from the j -th row using r' , where $r = r' \cdot 2^6$ and $0 < r' \leq \frac{1}{2}$.

Shutoff, another feature, can be described as reducing the gas temperature

(u_g) or shutting off the gas flow after time t_g (see Section 3). After computing j_g rows (ND only), the value of u_g can be changed for the (j_g+1)st row and the remaining rows.

The parameters that must be chosen are:

$$h = \Delta x / A$$

$$r = k/h^2$$

S switch - which is the stopping procedure where $S = S_1$ stops the machine after m rows and $S = S_2$ stops machine when $\max |u_{i,j}|$ exceeds or falls below a given tolerance.

Besides these parameters, certain specific cases will require a specification of other information, such as:

X_0 (CN) - as explained above

ω (CN) - the relaxation factor. If ω is not specified, $\omega = 1$ is assumed

γ (BVN): (.8 was used)

H(ND): ($2 \cdot 10^{-6}$ was used)

u_g (ND): (.18 was used)

A(ND): (73,600 was used)

$j_g, u_g^{(1)}$ (ND) - (shutoff); replaces u_g by $u_g^{(1)}$ after computing j_g rows.

The non-linear term was computed by a floating point subroutine with the exponential factor being computed by taking the first 15 terms of the series expansion of e^x . Initially, the exponent was normalized; i.e. placed in the form $a \cdot 2^m$ where $\frac{1}{2} \leq |a| < 1$, or $a = 0$ where appropriate. (For the BV cases, $a = 4$ since all values were scaled by 2^{-4} . For ND cases, the presence of the exponent $(-\frac{1}{m})$ necessitated

the normalizing procedure.) Then, $2^{-2} e^a$ was computed followed by a normalization of e^a . Since $e^{f(u)}$ was desired, the scaling problem was solved by the following:

$$e^{f(u)} = e^a \cdot 2^a = (e^a)^{2^a}$$

Therefore, it was necessary to square e^a a times in computing the non-linear term.

5. Analysis of Results

In Tables II - IX we give the values of $u(x,t)$ for the respective problems as obtained by the various finite difference methods. Scaling factors are indicated on the tables. In each case the columns headed $FD(\frac{1}{2})$ and $FD(\frac{1}{4})$ contain results obtained using the Forward Difference method with $r = \frac{1}{2}$ and $r = 1/4$, respectively. Columns headed $CN(I)$ and $CNN(I)$ contain the results obtained by using the Crank-Nicolson method using the iterative and non-iterative numerical procedures, respectively. For different values of $M = A/h$, the following values of r are used

M	r
8	$1/2$
16	1
32	2
64	4

The column headed $CNN(II)$ contains results obtained using the Crank-Nicolson method, non-iterative and with the following values of r

A/h	r
8	1
16	2
32	4
64	8

The value of t can be obtained from the row number by multiplying the row number by $\frac{1}{128} A^2$. Thus the row number equals t/k_0 where k_0 is the value of k corresponding to $M = 8$ and $r = 1/2$. The column headed "Eniac" gives results obtained on the Eniac using the Forward Difference method and $r = 1/2$.

Corresponding to each of the row numbers 4, 8, 12, 16, and 20 and to the indicated values of x/A we give the following results, when available,

<u>M</u>	
8	u_8
	e_{16}
16	u_{16}
	A/B
32	u_{32}
	C/D
64	u_{64}
	[X]

Here u_M represents the solution of the difference equation being considered with $h = A/M$. Also, $e_{16} = u_{16} - u_8$, $A = e_{16}/e_{32}$, $B = e_{32} = u_{32} - u_{16}$, $C = e_{32}/e_{64}$, and $D = e_{64} = u_{64} - u_{32}$. In the brackets we give the extrapolated value. The estimated value of p used in the extrapolation is given at the bottom of the column, (see section 2.5). For the extrapolation e_{64} is used when available, otherwise e_{32} is used. An asterisk in place of $e_{M/2}/e_M$ indicates that the computed value is not regarded as significant, usually because e_M is too small. An asterisk near an extrapolated value means that the value obtained from the use of the smallest mesh size is given in place of the extrapolated value.

5.1. Boundary value linear (BVL). In this case the results obtained from the Crank-Nicolson method with the iterative and the non-iterative numerical procedures were identical. Since the ratios e_M/e_{2M} are close to 4 it would appear that $e_M = O(M^{-2})$ and that extrapolation to zero grid size based on $a = 2$ might give good results. Indeed, if one uses this procedure one obtains almost identical results for all methods and very close agreement with the exact value as calculated from (3.3.2). We note that the values obtained from the Forward Difference method with $r = \frac{1}{2}$ and $X = 32$ are considerably different

from the exact values and that the extrapolation process has greatly improved the accuracy. We also remark that for this particular problem the Crank-Nicolson method is much more accurate than the Forward Difference method although the extrapolated values are practically the same.

5.2. Boundary value non-linear (BVN). As in the linear case the values of e_M/e_{2M} are, with the FD and CN methods, close to 4. Extrapolation to zero grid size using $p = 4$ and $a = 2$ yields results which are nearly the same for both the FD method (with $r = \frac{1}{2}$ as well as with $r = 1/4$) and for CN(I). On the other hand with the non-iterative procedures CNN(I) and CNN(II), (where the less accurate difference equation obtained by replacing f_2 by f_0 in (2.3.2) was used), the values of e_M/e_{2M} were usually between 3 and 3.5. Thus as expected, the order of convergence was less than for the other methods. Nevertheless with an extrapolation using $p = 3$ we obtained fairly accurate results, especially for CNN(I).

We believe that combining CNN with a few iterations, not more than five, would increase the order of convergence.

The error due to using f_0 instead of f_2 in (2.3.2) varies approximately as M^{-1} as can be seen from comparing results of CN(I) and CNN(I). Thus for the 10th row and for $x = \frac{1}{2}$ we have

M	CN(I)	CNN(I)	CN(I)-CNN(I)
8	72534	72400	134
16	72666	72621	67
32	72977	72943	34

(All values have been multiplied by 10^5 .)

Examining the differences 134, 67, 34 we observe that they are approximately in the ratios 4:2:1. Hence the differences appear to tend to zero like M^{-1} . This tends to confirm our prediction that the error introduced by using the less accurate difference equation is of order M^{-1} .

5.3. Normal derivative linear (NNL). In both the linear and the non-linear cases we are concerned with three options for representing the condition (2.1.2a). We first consider the results obtained for the linear case for Options I, II, III as given in Tables IV, V, VI respectively. As in the boundary value linear case it did not matter whether one used the iterative or the non-iterative procedure for the Crank-Nicolson method.

For Option I the values obtained by the Forward Difference method for $r = 1/4$ and $r = 1/2$ agree closely with each other and with those obtained with CNN(I) and CNN(II). Based on the observed value of p , extrapolated values were obtained as indicated. Of course, in this case we can conclude little from the fact that the extrapolated values obtained from the different methods agree with each other. However, the extrapolated values do agree rather closely with those obtained from results using Option II and III, and the latter two agree with each other. Moreover, the difference $u_{64} - u_{32}$ obtained for CNN(I) with Option III was very small, and this indicates that u_{64} is probably very close to the true value of u . Thus it would appear that in this case (2.1.5) (Option I) is a reasonably satisfactory representation of (2.1.2a) provided one uses the proper extrapolation procedure.

Finally we compare our results with the "exact" values obtained under the assumption that A is infinite, using the procedure given in [8]. These values are given on Table IV in braces. Near the line $x = A$ our values will not agree

with these "exact" values, because in our case A is finite. Nevertheless the agreement is generally rather close.

We give in Table V the values obtained on the Eniac using Option II and the FD method with $r = \frac{1}{2}$. Although the Eniac values and the Ordvac values listed under FD, again with $r = \frac{1}{2}$, agree exactly for $M = 8$, exact agreement cannot be expected for other values of M . The reason for this is that $A=64,400$ was used for the Eniac results instead of $A=73,600$ as used on the Ordvac. On the Eniac, instead of (2.1.14), the condition

$$u_M = u_{M+1}$$

was used to represent (2.1.2b). Thus, since $64,400/73,600 = 7/8$ we have exact agreement in the case $M = 8$.

Near the left hand boundary, ρ appears to be rather close to 4.0 so that we use extrapolation based on $a = 2$. We note that $u_{32} - u_{16}$ is small, and this indicates that u_{32} is already quite accurate. For $x/A = \frac{1}{2}$ the use of $\rho = 3$ seems indicated, and for $x/A = 1$ we used $\rho = 2$. It appears that the largest errors occurred at the right hand boundary.

Similar results were obtained for Option III as indicated in Table V. Of course for Option III it was possible to use CNM(I) and thus use many fewer time steps than were necessary for the Forward Difference method.

It seems reasonable to suppose that, by using either (2.1.15) or (2.1.17) to represent more accurately the right hand boundary condition, and by using Option III or III', the ratio ρ would be nearly 4 everywhere and that extrapolation based on $a = 2$ could have been used for all points.

5.4. Total derivative non-linear (TDN). The Eniac results obtained

using the FD method for Option II indicate that extrapolation to zero grid size using $\alpha = 2$ for $x/A = 0, 1/4$ and $\alpha = 1$ for $x/A = 1/2, 1$ should be quite accurate. The same also appears to be true for our Ordvac results. Of course the extrapolated values obtained in the respective cases do not agree because of the difference in the boundary conditions at $x/A = 1$, as mentioned above. The extrapolated values obtained on the Ordvac using Option II and the FD method also agree very closely with those extrapolated values obtained using CN(1) with Option III. It is difficult to make definite statements as to the behavior of the CNN method except that the errors are small and irregular. Here, of course, Option III' was used.

For Option I the ratios ρ are all approximately 2 with all difference equations as in the linear case. The errors are somewhat larger and the extrapolated values somewhat less accurate than with Option III.

5.5. Machine times. Figures 5 and 6 show the approximate times required to compute $t = 10k_0$ rows by the various methods in both the linear and in the non-linear cases respectively. The times for BV problems and for ND problems were nearly the same. The slopes for the curves corresponding to the FD, CN and CNN methods, after proper corrections have been made to account for the use of different logarithmic bases, are approximately 3, $5/2$, and 2 respectively. Hence as expected, the times increase as k^3 , $k^{5/2}$ and k^2 respectively. For the non-linear cases about 10 times as much machine time was required as for the linear cases.

The number of iterations required using the Crank-Nicolson method seemed to be nearly independent of whether the differential equation was linear or non-linear or whether BV or ND boundary conditions were used. The following

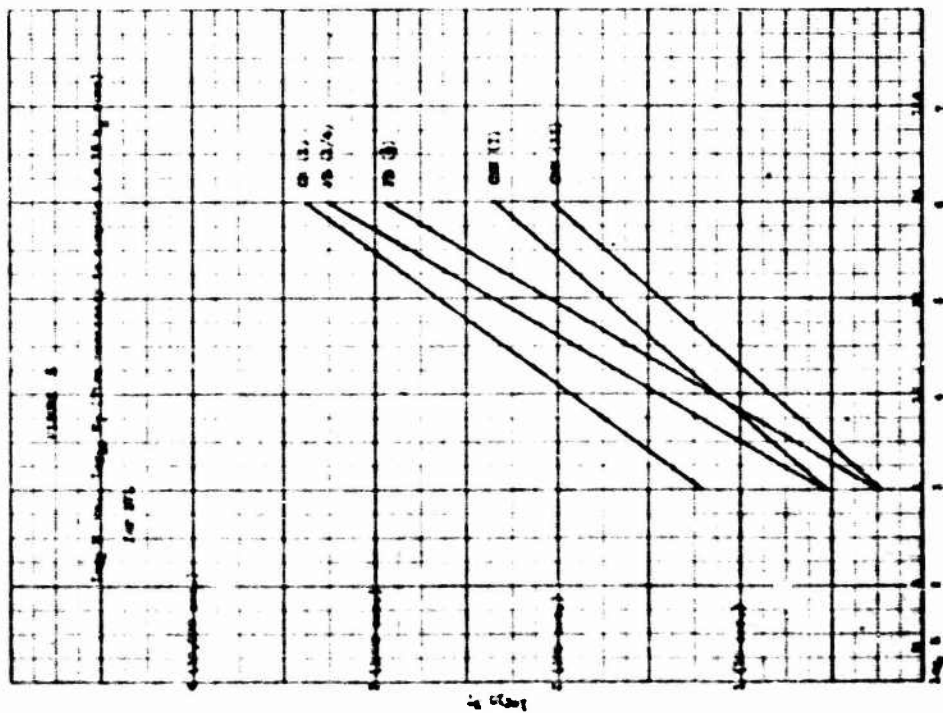
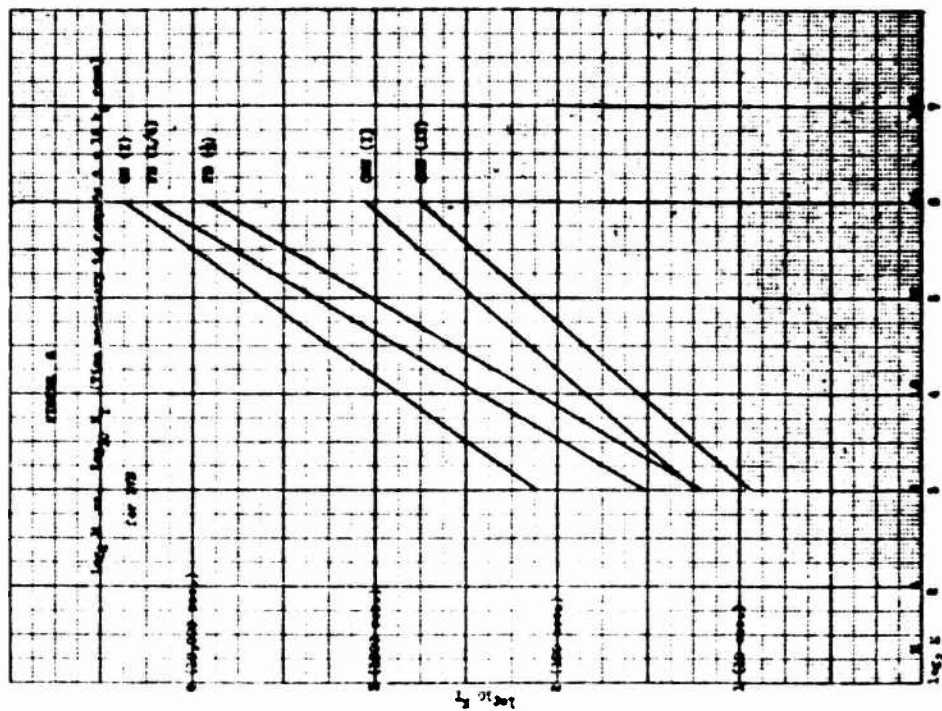


table gives the value of ω used for each M and the approximate number of iterations required per row.

M	ω	No. of Iterations Per Row
8	1.025	8
16	1.069	11
32	1.144	15
64	1.375	20

5.6. Tentative conclusions. Considering the analysis of the results as a whole we are led to the following tentative conclusions:

1. In the linear case, the error caused by the use of any of the difference equations rather than the differential equations varies as h^2 . In the non-linear case this is true for the ED and CN methods but for the $CNN(I)$ and $CNN(II)$ methods the error varies as h .

2. The use of Option I and the condition (2.1.14) used at $x/A = 1$ introduce errors of order h . On the other hand, the use of Option II, when it can be used, and Options III and III' introduces errors of order h^2 .

Further work should include the following:

1. In the non-linear cases, BVN and MDN , try CNN with a fixed number of iterations independent of h .

2. For MDN use CNN with Option III' for the left hand boundary condition and (2.1.15) or (2.1.17) to represent the right hand boundary conditions.

5.7. Runge-Kutta method. We have not given the results which were obtained using the RY method. The results which were obtained indicated that the method would not be as effective as the CN or CNN method. However, this should be studied further both analytically and by running some cases on the

machine. Results obtained in [5] indicate that one must require that $k = o(h)$ in order to obtain convergence, and that, moreover, the order of convergence is dependent on the order of h/k as $h \rightarrow 0$. Also, the fact that a special procedure appears to be needed to start the process is a disadvantage of the method.

Bibliography

1. Dorothy L. Bernstein, Existence theorems in partial differential equations, Princeton University Press, Princeton, New Jersey, (1950).
2. G. H. Bruce, D. A. Peaceman, H. H. Rachford, Jr., and J. D. Rice, Calculations of unsteady-state gas flow through porous media, Petroleum Transactions, AIME, Vol. 198 (1953), pp. 79-92.
3. R. Courant, K. Friedrichs and H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik, Mathematische Annalen, Vol. 100 (1928), pp. 32-74.
4. J. Crank and P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, Proc. Camb. Phil. Soc., Vol. 45 (1947), pp. 50-67.
5. E. C. Dufort and S. P. Frankel, Stability conditions in the numerical treatment of parabolic differential equations, Math. Tables and Other Aids to Comp., Vol. 7 (1953), No. 43, pp. 135-152.
6. P. Hartman and A. Wintner, On the solutions of the equation of heat conduction, Amer. Jour. of Math., Vol. 72 (1950), pp. 367-395.
7. Douglas R. Hartree, Calculating instruments and machines, The University of Illinois Press, Urbana, Illinois, (1949).
8. Bruce L. Hicks, J. W. Kelso and Julian Davis, Mathematical theory of the ignition process considered as a thermal reaction, Ballistic Research Laboratories Report No. 56, Aberdeen Proving Ground, Maryland, (1951).
9. F. Hildebrand, On the convergence of numerical solutions of the heat-flow equation, Jour. of Math. and Physics, Vol. 31 (1952), pp. 35-41.
10. Fritz John, On integration of parabolic equations by difference methods, Comm. Pure Appl. Math., Vol. 5 (1952), pp. 155-211.
11. M. L. Juncosa and David Young, On the convergence of a solution of a difference equation to a solution of the equation of diffusion, Proc. of the Am. Math. Soc., Vol. 5 (1954), No. 1, pp. 168-174.
12. M. L. Juncosa and D. L. Young, On the order of convergence of solutions of a difference equation to a solution of the diffusion equation, Jour. of the Soc. for Ind. and Applied Math., Vol. 1 (1953), No. 2, pp. 111-135.
13. M. L. Juncosa and D. L. Young, On the Crank-Nicolson procedure for parabolic partial differential equations, submitted for publication.

Bibliography

1. Dorothy L. Bernstein, Existence theorems in partial differential equations, Princeton University Press, Princeton, New Jersey, (1950).
2. G. H. Bruce, D. L. Peaceman, H. H. Rachford, Jr., and J. D. Rice, Calculations of unsteady-state gas flow through porous media, Petroleum Transactions, AIME, Vol. 198 (1953), pp. 79-92.
3. R. Courant, K. Friedrichs and H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik, Mathematische Annalen, Vol. 100 (1928), pp. 32-74.
4. J. Crank and P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, Proc. Camb. Phil. Soc., Vol. 45 (1947), pp. 50-67.
5. E. C. Dufort and S. P. Frankel, Stability conditions in the numerical treatment of parabolic differential equations, Math. Tables and Other Aids to Comp., Vol. 7 (1953), No. 43, pp. 135-152.
6. P. Hartman and A. Wintner, On the solutions of the equation of heat conduction, Amer. Jour. of Math., Vol. 72 (1950), pp. 367-395.
7. Douglas R. Hartree, Calculating instruments and machines, The University of Illinois Press, Urbana, Illinois, (1949).
8. Bruce L. Hicks, J. W. Kelso and Julian Davis, Mathematical theory of the ignition process considered as a thermal reaction, Ballistic Research Laboratories Report No. 56, Aberdeen Proving Ground, Maryland, (1951).
9. F. Hildebrand, On the convergence of numerical solutions of the heat-flow equation, Jour. of Math. and Physics, Vol. 31 (1952), pp. 35-41.
10. Fritz John, On integration of parabolic equations by difference methods, Comm. Pure Appl. Math., Vol. 5 (1952), pp. 155-211.
11. M. L. Juncosa and David Young, On the convergence of a solution of a difference equation to a solution of the equation of diffusion, Proc. of the Am. Math. Soc., Vol. 5 (1954), No. 1, pp. 166-174.
12. M. L. Juncosa and D. L. Young, On the order of convergence of solutions of a difference equation to a solution of the diffusion equation, Jour. of the Soc. for Ind. and Applied Math., Vol. 1 (1953), No. 2, pp. 111-136.
13. M. L. Juncosa and D. L. Young, On the Crank-Nicolson procedure for parabolic partial differential equations, submitted for publication.

List of Abbreviations

1. Differential Equations and Boundary Conditions

Symbol	The function $f(x,t,u)$ in (1.1)	Boundary conditions
FVL	0	See (3.1.1) with $g(x)=1$.
FVN	$\bullet u$	See (3.2.1).
NDL	0	See (2.1.2).
NDN	$\bullet -1/u$	See (2.1.2).

For both NDL and NDN the initial values used are given in Table 1.

2. Difference Equations

FD Forward Difference

FD($\frac{1}{2}$) uses $r = \frac{1}{2}$

FD($\frac{1}{4}$) uses $r = \frac{1}{4}$

CN Crank-Nicolson (iterative)

CN(I) uses $r = \frac{1}{2}, 1, 2, 4$ with $M = 8, 16, 32, 64$ respectively.

CNN Crank-Nicolson (non-iterative)

CNN(I) uses $r = \frac{1}{2}, 1, 2, 4$ with $M = 8, 16, 32, 64$ respectively.

CNN(II) uses $r = 1, 2, 4, 8$ with $M = 8, 16, 32, 64$ respectively.

DeV DuFort-Frankel

3. Representations of (2.1.2a) - (for ND problems only)

Option	Equation
I	(2.1.5)
II	(2.1.7)
III	(2.1.11)
III'	(2.1.15)

TABLE I

Initial values for MD cases
(M = 64)

x/A	$u_{1,0} \cdot 10^7$	x/A	$u_{1,0} \cdot 10^7$	x/A	$u_{1,0} \cdot 10^7$
0	*** 34304	• 51123		** 51000	
	33980	51100		51000	
• 33677		3/8 *** 51081		• 51000	
33395		51065		51000	
** 33134		• 51052		3/4 *** 51000	
32894		51041		51000	
• 32674		** 51033		• 51000	
32473		51026		51000	
1/8 *** 32290		• 51020		** 51000	
32125		51016		51000	
• 31977		1/2 *** 51012		• 51000	
31844		51009		51000	
** 31726		• 51007		7/8 *** 51000	
31621		51005		51000	
• 31529		** 51004		• 51000	
31449		51003		51000	
1/4 *** 31378		• 51002		** 51000	
31318		51002		51000	
• 31265		5/8 *** 51001		• 51000	
31220		51001		51000	
** 31182		• 51001		1 *** 51000	
31150		51001			

• - M = 32

•• - M = 32, 16

••• - M = 32, 16, 8

TABLE II

SOLUTIONS OF DIFFERENCE EQUATION FOR EVC
(All values multiplied by 10^5 .)

$x=1/4, 3/4$						$x=1/2$			
Row	N	FD(1)	FD(2)	CNN(I)*	CNN(II)*	FD(1)	FD(2)	CNN(I)	CNN(II)
20	0	17520	18823	19376	19333	21777	26619	27402	27341
		1303	328		-54	1842	164		-77
	16	16823	19151		19279	26619	27083		27264
		3.97/328	4.0/82		3.86/-14	3.97/164	4.0/116		3.95/-20
	32	19151	19233	19268	19265	27083	27199	27248	27244
	EXACT	[19260]	[19260]		[19260]	[27238]	[27238]		[27237]
16	0	21048	25674	26278	26230	31009	36084	37159	37044
		1626	409	-44	-8	2299	801	-80	-11
	16	25674	26083	26234	26222	36308	36885	37079	37083
		3.98/409	4.0/102	4.0/-11	4.0/-2	3.98/577	5.5/145	3.4/-16	2.75/-4
	32	26083	26185	26223	26220	36885	37030	37083	37079
	EXACT	[26219]	[26219]	[26219]	[26219]	[37077]	[37079]	[37079]	[37079]
12	0	33008	35023	35647	35588	46680	49520	50377	50316
		2015	506		81	2840	708		113
	16	35023	35529		35669	49520	50228		50429
		3.98/506	3.98/127		3.68/22	4.01/708	4.0/177		4.35/26
	32	35529	35656	35695	35691	50228	50405	50459	50465
	EXACT	[35698]	[35698]		[35698]	[50464]	[50464]		[50464]
8	0	45312	47839	48485	48322	61063	67451	68114	68125
		2527	646		278	3388	822		327
	16	47839	48485		48600	67451	68273		68452
		4.0/630	3.99/162		3.48/80	4.12/822	4.0/204		4.61/70
	32	48469	48647	48688	48680	68273	68477	68519	68522
	EXACT	[48679]	[48691]		[48707]	[68517]	[68545]		[68545]
4	0	62500	66534	67581	66330	87500	90192	90783	90477
		4034	1095		1517	2692	537		370
	16	66534	67629		67847	90192	90729		90847
		3.68/1095	3.95/277		21.87/108	5.01/537	4.16/129		11.56/32
	32	67629	67906	67982	67943	90729	90858	90820	90879
	EXACT	[67994]	[67999]		[67977]	[90908]	[90901]		[90898]
	C	h	h	h	h	h	h	h	h

* - see Section 5

TABLE IV

SOLUTIONS OF DIFFERENCE EQUATIONS FOR BOX, OPTION I.
(All values multiplied by 10^4 .)

$x/\lambda = \frac{1}{2}$				$x/\lambda = \frac{1}{3}$				$x/\lambda = 1$			
$FD(r-1)$	$CNH(I)$	$CNH(II)$		$FD(r-1)$	$FC(r-1)$	$CNH(I)$	$CNH(II)$	$FD(r-1)$	$FD(r-1)$	$CNH(I)$	$CNH(II)$
37184	37156	37160		36183	36205	36203	36181	36360	36377	36317	36379
-365	-335	-316		-196	-213	-214	-197	-229	-259	-275	-261
36839	36831	36826		33989	33972	33989	33986	36131	36136	36112	36138
2.12/-158	2.08/-154			2.06/-95		2.23/-96	2.11/-92	2.24/-96		2.5/-110	2.14/-107
36673	36672			33996		33993	33992	36098		36032	36031
2.00/-76						2.13/-85				2.24/-89	
36597						33988				36011	
[36521]	[36520]			[33799]		[33603]	[33600]	[36027]		[35936]	[35976]
				[33750]				[35927]			
36302	36280	36259		35468	35473	35470	35468	35822	35863	35864	35869
-317	-306	-283		-116	-166	-168	-151	-166	-178	-196	-186
35995	35976	35969		33302	33375	33302	33277	35676	35683	35688	35682
2.14/-114	2.1/-114	2.1/-113		2.97/-74	2.16/-77	2.26/-75	2.13/-71	2.21/-66	2.51/-71	2.61/-75	2.52/-73
35837	35836	35832		33278	33279	33277	33276	35610	35612	35611	35612
	2.06/-59					2.14/-35				2.27/-33	
	35765					33298				35580	
[35689]	[35689]	[35689]		[33154]	[33151]	[33157]	[33155]	[35514]	[35511]	[35517]	[35519]
				[33110]				[35476]			
35328	35324	35272		32709	32735	32736	32713	35388	35404	35450	35460
-281	-266	-242		-94	-123	-121	-106	-99	-98	-119	-111
35047	35038	35030		32614	32617	32615	32609	35319	35326	35331	35329
2.16/-114	2.2/-114	2.1/-114		1.81/-59	2.19/-57	2.26/-53	2.17/-48	2.08/-33	2.51/-39	2.77/-43	2.71/-41
34917	34914	34912		32562	32563	32562	32561	35286	35287	35288	35288
	2.07/-60					2.12/-25				2.53/-17	
	34854					32537				35271	
[34767]	[34772]	[34794]		[32510]	[32502]	[32512]	[32511]	[35257]	[35264]	[35254]	[35276]
				[32509]				[35227]			
34216	34190	34151		32173	32006	32012	31989	35106	35111	35153	35150
-228	-212	-182		-36	-54	-78	-53	-14	-34	-52	-69
33968	33978	33959		31937	31972	31961	31936	35022	35027	35010	35018
2.2/-103	2.2/-75	2.07/-59		1.69/-26	2.12/-39	2.14/-29	2.04/-26	1.56/-9	2.41/-13	3.25/-16	2.84/-17
33805	33802	33800		31911	31912	31912	31910	35001	35006	35005	35006
	2.09/-46					2.07/-14				2.79/-7	
	33736					31898				35079	
[33702]	[33720]	[33717]		[31885]	[31877]	[31884]	[31881]	[35071]	[35071]	[35071]	[35067]
				[31829]				[35078]			
32883	32862	32806		31306	31254	31263	31250	34806	34812	34809	34811
-129	-117	-73		76	-8	-73	-23	1	-8	-10	-11
32754	32745	32733		31132	31136	31136	31117	34802	34808	34810	34808
2.22/-58	2.29/-51	1.76/-42		0/-1	1.12/-5	2.88/-8	2.4/-5	0/-1	1.13/-3	1.67/-3	
32636	32634	32631		31111	31132	31132	31132	34806	34807	34807	34807
	2.13/-24					2.67/-3				1.0/-1	
	32670					31127				34804	
[32636]	[32646]	[32643]		[31130]	[31124]	[31129]	[31127]	[34800]	[34806]	[34805]	[34804]
				[31126]				[34803]			
2	2	2		2	2	2	2	2	2	2	2

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NDL, OPTION II ($x = \frac{1}{2}$)
(All values multiplied by 10^6 .)

[illegible]

OF DIFFERENCE EQUATIONS FOR MDL, OPTION II ($\lambda = \frac{1}{2}$)

[illegible]

TABLE VI
SOLUTIONS OF DIFFERENCE EQUATIONS FOR EDA, OPTION III
(all values multiplied by 10^6)

Row	N	I=0		I=1		I=2		I=3	
		FD(I)	CNW(I)	FD(I)	CNW(I)	FD(I)	CNW(I)	FD(I)	CNW(I)
20	8	10022	10781	36521	36508	13820	13814	32198	32181
	13	13	23	2	13	-2	-16	-125	-170
	16	10815	10804	36529	36521	13818	13818	32063	32076
	22	0/-2	4.6/5	0/-3	6.5/7	0/-7	2.3/-7	2.0/-64	2.3/-73
	24	10813	10829	36526	36523	13811	13811	31999	32081
16	8		5.0/1		0/0		1.7/-4		2.3/-32
	15		10810		36523		13807		31969
	22		[10810]		[36523]		[13807]		[31937]
	24								
	26								
12	8	39917	39893	35690	35674	13155	13170	31726	31776
	13	17	26	11	18	8	-5	-90	-129
	16	39934	39921	35701	35697	13161	13164	31626	31647
	22	0/-2	4.0/7	0/-2	4.5/4	0/-2	2.0/-3	2.3/-43	2.5/-52
	24	39932	39928	35699	35696	13161	13161	31597	31575
8	8		3.5/2		4.0/1		3.0/-1		2.2/-23
	16		39930		35697		13160		31572
	22		[39930]		[35697]		[13160]		[31549]
	24								
	26								
4	8	38924	38896	34766	34768	12538	12515	31348	31397
	13	20	34	13	11	15	-1	-16	-44
	16	38944	38930	34777	34789	12523	12511	31302	31313
	22	0/-1	1.25/3	0/-3	1.8/6	0/1	0/0	1.3/-24	2.4/-32
	24	38943	38938	34756	34775	12514	12511	31278	31281
0	8		4.0/2		6.0/1		0/0		2.3/-11
	16		38940		34796		12514		31267
	22		[38940]		[34796]		[12514]		[31253]
	24								
	26								
2	8	37766	37737	33782	33765	11846	11896	31101	31139
	13	26	38	11	19	16	1	-12	-12
	16	37792	37775	33773	33784	11882	11885	31089	31097
	22	0/-2	3.8/10	1.2/9	3.8/5	5.3/3	0/0	1.5/-8	3.0/-13
	24	37790	37785	33792	33789	11885	11885	31081	31083
2	8		3.3/3		2.5/2		0/0		2.8/-5
	16		37788		33791		11885		31078
	22		[37788]		[33791]		[11885]		[31074]
	24								
	26								
2	8	36337	36305	32642	32611	11306	11314	31006	31019
	13	31	3.8/42	5	11	16	-1	1	-9
	16	36363	36347	32647	32642	11302	11308	31007	31010
	22	0/0	2.6/13	0/0	2.8/4	5.3/3	6.0/-1	0/-1	3.0/-3
	24	36363	36358	32647	32646	11303	11307	31006	31007
2	8		5.5/2		0/0		0/0		3.0/-1
	16		36360		32646		11307		31006
	22		[36360]		[32646]		[11307]		[31006]
	24								
	26								

TABLE VII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR HMA , ORDER 1.
(All values multiplied by 10^6 .)

		$\alpha/\lambda = 0$						$\alpha/\lambda = \frac{1}{2}$						$\alpha/\lambda = \frac{1}{4}$					
Row	N	FD($r=1$)	FD($r=2$)	CH(I)	CH(II)	CH(I)	FD($r=1$)	FD($r=2$)	CH(I)	CH(II)	CH(I)	FD($r=1$)	FD($r=2$)	CH(I)	CH(II)	CH(I)	FD($r=1$)		
20	8	13615	13653	13621	13625	13653	13615	13616	13611	13609	13616	13666	13601	13680	13612	13653	13612		
	16	2970	15823	1211	822	B.L.	13616	13616	13616	13616	B.L.	13616	13616	13616	13616	B.L.			
	32			0.37/1783	0.18/1730				0.27/181	0.36/177				0.14/-64	0.43/-18				
	64			13715	15577				13611	13616				13616	13616				
16	8	11553	11595	11606	11601	11586	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553		
	16	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553	11553		
	32	2.3/-71		1.2/-44	1.27/-71	1.1199	1.8/-119		2.17/-105	2.14/-84		1.74/-68		2.38/-59	2.17/-59		2.20/-66		
	64	11320		11259	11199		13617		13617	13617		11358		11358	11358		11631		
12	8	10160	10176	10117	10068	10169	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160		
	16	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160	10160		
	32	2.06/-193		2.72/-175	2.75/-157	1.96/-171	1.96/-171		2.15/-111	2.05/-109		1.75/-89		2.30/-50	2.18/-44		2.06/-31		
	64	10573		10558	10532		10589		10589	10589		10589		10589	10589		10589		
8	8	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573		
	16	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573		
	32	10573		10573	10573	10573	10573		10573	10573		10573		10573	10573		10573		
	64	10573		10573	10573	10573	10573		10573	10573		10573		10573	10573		10573		
4	8	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573		
	16	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573	10573		
	32	10573		10573	10573	10573	10573		10573	10573		10573		10573	10573		10573		
	64	10573		10573	10573	10573	10573		10573	10573		10573		10573	10573		10573		
		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2		

- - no fraction
.. overflow

TABLE VII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR HEM. OPTION 1.
(All values multiplied by 10^3 .)

$x/\lambda = \frac{1}{2}$				$x/\lambda = \frac{1}{2}$				$x/\lambda = 1$				$x/\lambda = 1$			
FD(r_{-1})	CH(I)	CH(II)	CH(I)	FD(r_{-1})	FD(r_{-1})	CH(I)	CH(II)	CH(I)	FD(r_{-1})	FD(r_{-1})	CH(I)	CH(II)	CH(I)		
36116 116 36168 2.77/181 36161 B.L.	36171 69 36180 2.77/181 36161 B.L.	37908 60 37969 2.77/177 36166 B.L.	36116 B.L.	36166 36501 -116 36354 2.61/-64 36260 B.L.	36501 -116 36354 2.61/-64 36260 B.L.	36180 -156 36324 2.61/-64 36260 B.L.	36112 -126 36286 2.61/-64 36260 B.L.	36534 B.L.	36112 36158 -263 36195 2.51/-112 36205 B.L.	36158 -263 36195 2.51/-112 36205 B.L.	36178 -281 36197 2.51/-112 36205 B.L.	36152 -262 36199 2.51/-112 36205 B.L.	36187 B.L.		
36696 -714 36152 2.12/-105 36305 2.06/-51 [36708]	36611 -223 36110 2.12/-105 36305 2.06/-51 [36708]	36516 -180 36366 2.12/-105 36305 2.06/-51 [36708]	36690 -742 36168 2.12/-105 36305 2.06/-51 [36708]	36516 -170 36125 1.76/-68 36305 [36708]	36581 -150 36131 1.76/-68 36305 [36708]	36574 -152 36172 2.38/-68 36305 [36708]	36535 -126 36107 2.17/-59 36305 [36708]	36589 -158 36131 2.17/-59 36305 [36708]	36112 -115 36197 2.20/-66 36305 [36708]	36158 -129 36205 2.20/-66 36305 [36708]	36178 -199 36205 2.20/-66 36305 [36708]	36152 -185 36205 2.20/-66 36305 [36708]	36187 -200 36205 2.20/-66 36305 [36708]		
35749 -257 35712 2.15/-111 35080 2.00/-55 [35955]	35730 -239 35191 2.15/-111 35080 2.00/-55 [35955]	35376 -205 35171 2.05/-100 35071 [35955]	35449 -216 35203 2.05/-100 35071 [35955]	35762 -86 35654 2.75/-49 35607 [35955]	35772 -111 35661 2.75/-49 35607 [35955]	35771 -115 35654 2.30/-50 35606 2.08/-26 35582 [35955]	35743 -96 35647 2.18/-44 35603 [35955]	35777 -118 35659 2.06/-33 35603 [35955]	35395 -68 35327 2.06/-33 35603 [35955]	35111 -99 35134 2.06/-33 35603 [35955]	35159 -120 35139 2.06/-33 35603 [35955]	35168 -111 35137 2.06/-33 35603 [35955]	35160 -20 35160 2.06/-33 35603 [35955]		
36257 -219 36238 2.17/-93 35932 2.11/-44 [35919]	36227 -208 36025 2.17/-93 35932 2.11/-44 [35919]	36181 -170 36011 2.05/-83 35928 [35919]	36234 -205 36029 2.05/-83 35928 [35919]	36182 -33 35953 2.32/-25 35924 [35919]	36182 -33 35953 2.32/-25 35924 [35919]	36022 -69 35953 2.37/-29 35924 2.23/-13 35911 [35919]	36024 -51 35954 2.06/-25 35922 [35919]	36024 -70 35954 2.06/-25 35922 [35919]	36108 -13 36098 1.9/-10 36084 [35919]	36115 -35 36100 1.9/-10 36084 [35919]	36157 -52 36105 1.06/-17 36084 2.81/-4 36082 [35919]	36154 -58 36104 1.06/-17 36084 2.81/-4 36082 [35919]	36157 -62 36105 1.06/-17 36084 2.81/-4 36082 [35919]		
32790 -126 32764 2.75/-51 31703 2.72/-23 [32636]	32669 -115 32754 2.75/-51 31703 2.72/-23 [32636]	32811 -70 32761 1.71/-43 32700 [32636]	32871 -116 32755 1.71/-43 32700 [32636]	32708 26 32761 0/0 32761 [32636]	32764 -4 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]	32764 -23 32761 2.04/-4 32761 2.04/-4 32761 [32636]		

TABLE VIII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NEW, OPTION II ($\rho = 1$).(All values multiplied by 10^4 .)

$\rho = 0$		$\rho = 1/2$		$\rho = 1$		$\rho = 3/2$		$\rho = 2$		$\rho = 5/2$		$\rho = 3$		$\rho = 7/2$		$\rho = 4$		$\rho = 9/2$		$\rho = 5$		$\rho = 11/2$		$\rho = 6$		$\rho = 13/2$		$\rho = 7$		$\rho = 15/2$		$\rho = 8$		$\rho = 17/2$		$\rho = 9$		$\rho = 19/2$		$\rho = 10$		$\rho = 21/2$		$\rho = 11$		$\rho = 23/2$		$\rho = 12$		$\rho = 25/2$		$\rho = 13$		$\rho = 27/2$		$\rho = 14$		$\rho = 29/2$		$\rho = 15$		$\rho = 31/2$		$\rho = 16$		$\rho = 33/2$		$\rho = 17$		$\rho = 35/2$		$\rho = 18$		$\rho = 37/2$		$\rho = 19$		$\rho = 39/2$		$\rho = 20$		$\rho = 41/2$		$\rho = 21$		$\rho = 43/2$		$\rho = 22$		$\rho = 45/2$		$\rho = 23$		$\rho = 47/2$		$\rho = 24$		$\rho = 49/2$		$\rho = 25$		$\rho = 51/2$		$\rho = 26$		$\rho = 53/2$		$\rho = 27$		$\rho = 55/2$		$\rho = 28$		$\rho = 57/2$		$\rho = 29$		$\rho = 59/2$		$\rho = 30$		$\rho = 61/2$		$\rho = 31$		$\rho = 63/2$		$\rho = 32$		$\rho = 65/2$		$\rho = 33$		$\rho = 67/2$		$\rho = 34$		$\rho = 69/2$		$\rho = 35$		$\rho = 71/2$		$\rho = 36$		$\rho = 73/2$		$\rho = 37$		$\rho = 75/2$		$\rho = 38$		$\rho = 77/2$		$\rho = 39$		$\rho = 79/2$		$\rho = 40$		$\rho = 81/2$		$\rho = 41$		$\rho = 83/2$		$\rho = 42$		$\rho = 85/2$		$\rho = 43$		$\rho = 87/2$		$\rho = 44$		$\rho = 89/2$		$\rho = 45$		$\rho = 91/2$		$\rho = 46$		$\rho = 93/2$		$\rho = 47$		$\rho = 95/2$		$\rho = 48$		$\rho = 97/2$		$\rho = 49$		$\rho = 99/2$		$\rho = 50$		$\rho = 101/2$		$\rho = 51$		$\rho = 103/2$		$\rho = 52$		$\rho = 105/2$		$\rho = 53$		$\rho = 107/2$		$\rho = 54$		$\rho = 109/2$		$\rho = 55$		$\rho = 111/2$		$\rho = 56$		$\rho = 113/2$		$\rho = 57$		$\rho = 115/2$		$\rho = 58$		$\rho = 117/2$		$\rho = 59$		$\rho = 119/2$		$\rho = 60$		$\rho = 121/2$		$\rho = 61$		$\rho = 123/2$		$\rho = 62$		$\rho = 125/2$		$\rho = 63$		$\rho = 127/2$		$\rho = 64$		$\rho = 129/2$		$\rho = 65$		$\rho = 131/2$		$\rho = 66$		$\rho = 133/2$		$\rho = 67$		$\rho = 135/2$		$\rho = 68$		$\rho = 137/2$		$\rho = 69$		$\rho = 139/2$		$\rho = 70$		$\rho = 141/2$		$\rho = 71$		$\rho = 143/2$		$\rho = 72$		$\rho = 145/2$		$\rho = 73$		$\rho = 147/2$		$\rho = 74$		$\rho = 149/2$		$\rho = 75$		$\rho = 151/2$		$\rho = 76$		$\rho = 153/2$		$\rho = 77$		$\rho = 155/2$		$\rho = 78$		$\rho = 157/2$		$\rho = 79$		$\rho = 159/2$		$\rho = 80$		$\rho = 161/2$		$\rho = 81$		$\rho = 163/2$		$\rho = 82$		$\rho = 165/2$		$\rho = 83$		$\rho = 167/2$		$\rho = 84$		$\rho = 169/2$		$\rho = 85$		$\rho = 171/2$		$\rho = 86$		$\rho = 173/2$		$\rho = 87$		$\rho = 175/2$		$\rho = 88$		$\rho = 177/2$		$\rho = 89$		$\rho = 179/2$		$\rho = 90$		$\rho = 181/2$		$\rho = 91$		$\rho = 183/2$		$\rho = 92$		$\rho = 185/2$		$\rho = 93$		$\rho = 187/2$		$\rho = 94$		$\rho = 189/2$		$\rho = 95$		$\rho = 191/2$		$\rho = 96$		$\rho = 193/2$		$\rho = 97$		$\rho = 195/2$		$\rho = 98$		$\rho = 197/2$		$\rho = 99$		$\rho = 199/2$		$\rho = 100$		$\rho = 201/2$		$\rho = 101$		$\rho = 203/2$		$\rho = 102$		$\rho = 205/2$		$\rho = 103$		$\rho = 207/2$		$\rho = 104$		$\rho = 209/2$		$\rho = 105$		$\rho = 211/2$		$\rho = 106$		$\rho = 213/2$		$\rho = 107$		$\rho = 215/2$		$\rho = 108$		$\rho = 217/2$		$\rho = 109$		$\rho = 219/2$		$\rho = 110$		$\rho = 221/2$		$\rho = 111$		$\rho = 223/2$		$\rho = 112$		$\rho = 225/2$		$\rho = 113$		$\rho = 227/2$		$\rho = 114$		$\rho = 229/2$		$\rho = 115$		$\rho = 231/2$		$\rho = 116$		$\rho = 233/2$		$\rho = 117$		$\rho = 235/2$		$\rho = 118$		$\rho = 237/2$		$\rho = 119$		$\rho = 239/2$		$\rho = 120$		$\rho = 241/2$		$\rho = 121$		$\rho = 243/2$		$\rho = 122$		$\rho = 245/2$		$\rho = 123$		$\rho = 247/2$		$\rho = 124$		$\rho = 249/2$		$\rho = 125$		$\rho = 251/2$		$\rho = 126$		$\rho = 253/2$		$\rho = 127$		$\rho = 255/2$		$\rho = 128$		$\rho = 257/2$		$\rho = 129$		$\rho = 259/2$		$\rho = 130$		$\rho = 261/2$		$\rho = 131$		$\rho = 263/2$		$\rho = 132$		$\rho = 265/2$		$\rho = 133$		$\rho = 267/2$		$\rho = 134$		$\rho = 269/2$		$\rho = 135$		$\rho = 271/2$		$\rho = 136$		$\rho = 273/2$		$\rho = 137$		$\rho = 275/2$		$\rho = 138$		$\rho = 277/2$		$\rho = 139$		$\rho = 279/2$		$\rho = 140$		$\rho = 281/2$		$\rho = 141$		$\rho = 283/2$		$\rho = 142$		$\rho = 285/2$		$\rho = 143$		$\rho = 287/2$		$\rho = 144$		$\rho = 289/2$		$\rho = 145$		$\rho = 291/2$		$\rho = 146$		$\rho = 293/2$		$\rho = 147$		$\rho = 295/2$		$\rho = 148$		$\rho = 297/2$		$\rho = 149$		$\rho = 299/2$		$\rho = 150$		$\rho = 301/2$		$\rho = 151$		$\rho = 303/2$		$\rho = 152$		$\rho = 305/2$		$\rho = 153$		$\rho = 307/2$		$\rho = 154$		$\rho = 309/2$		$\rho = 155$		$\rho = 311/2$		$\rho = 156$		$\rho = 313/2$		$\rho = 157$		$\rho = 315/2$		$\rho = 158$		$\rho = 317/2$		$\rho = 159$		$\rho = 319/2$		$\rho = 160$		$\rho = 321/2$		$\rho = 161$		$\rho = 323/2$		$\rho = 162$		$\rho = 325/2$		$\rho = 163$		$\rho = 327/2$		$\rho = 164$		$\rho = 329/2$		$\rho = 165$		$\rho = 331/2$		$\rho = 166$		$\rho = 333/2$		$\rho = 167$		$\rho = 335/2$		$\rho = 168$		$\rho = 337/2$		$\rho = 169$		$\rho = 339/2$		$\rho = 170$		$\rho = 341/2$		$\rho = 171$		$\rho = 343/2$		$\rho = 172$		$\rho = 345/2$		$\rho = 173$		$\rho = 347/2$		$\rho = 174$		$\rho = 349/2$		$\rho = 175$		$\rho = 351/2$		$\rho = 176$		$\rho = 353/2$		$\rho = 177$		$\rho = 355/2$		$\rho = 178$		$\rho = 357/2$		$\rho = 179$		$\rho = 359/2$		$\rho = 180$		$\rho = 361/2$		$\rho = 181$		$\rho = 363/2$		$\rho = 182$		$\rho = 365/2$		$\rho = 183$		$\rho = 367/2$		$\rho = 184$		$\rho = 369/2$		$\rho = 185$		$\rho = 371/2$		$\rho = 186$		$\rho = 373/2$		$\rho = 187$		$\rho = 375/2$		$\rho = 188$		$\rho = 377/2$		$\rho = 189$		$\rho = 379/2$		$\rho = 190$		$\rho = 381/2$		$\rho = 191$		$\rho = 383/2$		$\rho = 192$		$\rho = 385/2$		$\rho = 193$		$\rho = 387/2$		$\rho = 194$		$\rho = 389/2$		$\rho = 195$		$\rho = 391/2$		$\rho = 196$		$\rho = 393/2$		$\rho = 197$		$\rho = 395/2$		$\rho = 198$		$\rho = 397/2$		$\rho = 199$		$\rho = 399/2$		$\rho = 200$		$\rho = 401/2$		$\rho = 201$		$\rho = 403/2$		$\rho = 202$		$\rho = 405/2$		$\rho = 203$		$\rho = 407/2$		$\rho = 204$		$\rho = 409/2$		$\rho = 205$		$\rho = 411/2$		$\rho = 206$		$\rho = 413/2$		$\rho = 207$		$\rho = 415/2$		$\rho = 208$		$\rho = 417/2$		$\rho = 209$		$\rho = 419/2$		$\rho = 210$		$\rho = 421/2$		$\rho = 211$		$\rho = 423/2$		$\rho = 212$		$\rho = 425/2$		$\rho = 213$		$\rho = 427/2$		$\rho = 214$		$\rho = 429/2$		$\rho = 215$		$\rho = 431/2$		$\rho = 216$		$\rho = 433/2$		$\rho = 217$		$\rho = 435/2$		$\rho = 218$		$\rho = 437/2$		$\rho = 219$		$\rho = 439/2$		$\rho = 220$		$\rho = 441/2$		$\rho = 221$		$\rho = 443/2$		$\rho = 222$		$\rho = 445/2$		$\rho = 223$		$\rho = 447/2$		$\rho = 224$		$\rho = 449/2$		$\rho = 225$		$\rho = 451/2$		$\rho = 226$		$\rho = 453/2$		$\rho = 227$		$\rho = 455/2$		$\rho = 228$		$\rho = 457/2$		$\rho = 229$		$\rho = 459/2$		$\rho = 230$		$\rho = 461/2$		$\rho = 231$		$\rho = 463/2$		$\rho = 232$		$\rho = 465/2$		$\rho = 233$		$\rho = 467/2$		$\rho = 234$		$\rho = 469/2$		$\rho = 235$		$\rho = 471/2$		$\rho = 236$		$\rho = 473/2$		$\rho = 237$		$\rho = 475/2$		$\rho = 238$		$\rho = 477/2$		$\rho = 239$		$\rho = 479/2$		$\rho = 240$		$\rho = 481/2$		$\rho = 241$		$\rho = 483/2$		$\rho = 242$		$\rho = 485/2$		$\rho = 243$		$\rho = 487/2$		$\rho = 244$		$\rho = 489/2$		$\rho = 245$		$\rho = 491/2$		$\rho = 246$		$\rho = 493/2$		$\rho = 247$		$\rho = 495/2$		$\rho = 248$		$\rho = 497/2$		$\rho = 249$		$\rho = 499/2$		$\rho = 250$		$\rho = 501/2$		$\rho = 251$		$\rho = 503/2$		$\rho = 252$		$\rho = 505/2$		$\rho = 253$		$\rho = 507/2$		$\rho = 254$		$\rho = 509/2$		$\rho = 255$		$\rho = 511/2$		$\rho = 256$		$\rho = 513/2$		$\rho = 257$		$\rho = 515/2$		$\rho = 258$		$\rho = 517/2$		$\rho = 259$		$\rho = 519/2$		$\rho = 260$		$\rho = 521/2$		$\rho = 261$		$\rho = 523/2$		$\rho = 262$		$\rho = 525/2$		$\rho = 263$		$\rho = 527/2$		$\rho = 264$		$\rho = 529/2$		$\rho = 265$		$\rho = 531/2$		$\rho = 266$		$\rho = 533/2$		$\rho = 267$		$\rho = 535/2$		$\rho = 268$		$\rho = 537/2$		$\rho = 269$		$\rho = 539/2$		$\rho = 270$		$\rho = 541/2$		$\rho = 271$		$\rho = 543/2$		$\rho = 272$		$\rho = 545/2$		$\rho = 273$		$\rho = 547/2$		$\rho = 274$		$\rho = 549/2$		$\rho = 275$		$\rho = 551/2$		$\rho = 276$		$\rho = 553/2$		$\rho = 277$		$\rho = 555/2$		$\rho = 278$		$\rho = 557/2$		$\rho = 279$		$\rho = 559/2$		$\rho = 280$		$\rho = 561/2$		$\rho = 281$		$\rho = 563/2$		$\rho = 282$		$\rho = 565/2$		$\rho = 283$		$\rho = 567/2$		$\rho = 284$		$\rho = 569/2$		$\rho = 285$		$\rho = 571/2$		$\rho = 286$		$\rho = 573/2$		$\rho = 287$		$\rho = 575/2$		$\rho = 288$		$\rho = 577/2$		$\rho = 289$		$\rho = 579/2$		$\rho = 290$		$\rho = 581/2$		$\rho = 291$		$\rho = 583/2$		$\rho = 292$		$\rho = 585/2$		$\rho = 293$		$\rho = 587/2$		$\rho = 294$		$\rho = 589/2$		$\rho = 295$		$\rho = 591/2$		$\rho = 296$		$\rho = 593/2$		$\rho = 297$		$\rho = 595/2$		$\rho = 298$		$\rho = 597/2$		$\rho = 299$		$\rho = 599/2$		$\rho = 300$		$\rho = 601/2$		$\rho = 301$		$\rho = 603/2$		$\rho = 302$		$\rho = 605/2$		$\rho = 303$		$\rho = 607/2$		$\rho = 304$		$\rho = 609/2$		$\rho = 305$		$\rho = 611/2$		$\rho = 306$		$\rho = 613/2$		$\rho = 307$		$\rho = 615/2$		$\rho = 308$		$\rho = 617/2$		$\rho = 309$		$\rho = 619/2$		$\rho = 310$		$\rho = 621/2$		$\rho = 311$		$\rho = 623/2$		$\rho = 312$		$\rho = 625/2$		$\rho = 313$		$\rho = 627/2$		$\rho = 314$		$\rho = 629/2$		$\rho = 315$		$\rho = 631/2$		$\rho = 316$		$\rho = 633/2$		$\rho = 317$		$\rho = 635/2$		$\rho = 318$		$\rho = 637/2$		$\rho = 319$		$\rho = 639/2$		$\rho = 320$		$\rho = 641/2$		$\rho = 321$		$\rho = 643/2$		$\rho = 322$		$\rho = 645/2$		$\rho = 323$		$\rho = 647/2$		$\rho = 324$		$\rho = 649/2$		$\rho = 325$		$\rho = 651/2$		$\rho = 326$		$\rho = 653/2$		$\rho = 327$		$\rho = 655/2$		$\rho = 328$		$\rho = 657/2$		$\rho = 329$		$\rho = 659/2$		<	
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TABLE VIII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NON, OPTION II ($r = 1$).

(All values multiplied by 10^6 .)

[illegible]

Appendix

Formulas Used for Ordvac ProgramNotations:

$u_{1,j}$ - as discussed in Section 3

$u_{1,j}^{(n)}$ - value of function for n-th iteration

Bracketed term included only for the non-linear case.

FD - BV

$$u_{i,j+1} = r(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + u_{i,j} + [ke^{u_{i,j}}], \quad 1 \leq i \leq M-1; j \geq 0$$

$$\left. \begin{aligned} u_{0,j+1} &= \phi_{j+1} \\ u_{M,j+1} &= \psi_{j+1} \end{aligned} \right\} j \geq 0$$

FD - ND

$$u_{i,j+1} = r(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + u_{i,j} + [ke^{-1/u_{i,j}}], \quad 1 \leq i \leq M-1; j \geq 0$$

$$\text{Op I} \quad u_{0,j+1} = \frac{Hhu_g + u_{1,j+1}}{1 + Hh} \quad j \geq 0$$

$$\text{Op II} \quad u_{0,j+1} = u_{1,j} + Hh(u_g - u_{0,j}) + [ke^{-1/u_{0,j}}] \quad j \geq 0$$

 $r = \frac{1}{2}$ only

$$\text{Op III} \quad u_{0,j+1} = \frac{u_{1,j+1} + Hhu_g + \frac{1}{2r} \{ u_{0,j} + [ke^{-1/u_{0,j}}] \}}{1 + Hh + \frac{1}{2r}} \quad j \geq 0$$

All Options:

$$u_{M,j+1} = u_{M-1,j+1}$$

CN - BV

$$u_{i,j+1}^{(0)} = u_{i,j+1} \text{ by FD method if } r \leq \frac{1}{2}$$

$$u_{i,j+1}^{(0)} = u_{i,j} \text{ if } r > \frac{1}{2}$$

$$1 \leq i \leq M-1; j \geq 0$$

$$u_{i,j+1}^{(n)} = (\omega-1)(u_{i,j+1}^{(n)} - u_{i,j+1}^{(n-1)}) + u_{i,j+1}^{(n-1)}$$

where:

$$\begin{aligned} u_{i,j+1}^{(n)} = & \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r}u_{i,j} \\ & + \frac{r}{2+2r}(u_{i-1,j+1} + u_{i+1,j+1}^{(n-1)}) \\ & + \left[\frac{1}{1+r} k \gamma \cdot \frac{(u_{i,j} + u_{i,j+1}^{(n-1)})}{2} \right] \end{aligned}$$

where:

$$u_{i-1,j+1}^{*} = u_{0,j+1}^{(n-1)} \quad i=1; j \geq 0$$

$$u_{i-1,j+1}^{*} = u_{i-1,j+1}^{(n)} \quad 2 \leq i \leq M-1; j \geq 0$$

$$u_{0,j+1}^{(n)} = \phi_{j+1}$$

$$M \geq 0; j \geq 0$$

$$u_{M,j+1}^{(n)} = \psi_{j+1}$$

CN - ND

$$u_{1,j+1}^{(0)} = u_{1,j+1} \text{ by FD method if } r \leq \frac{1}{2}$$

$$u_{1,j+1}^{(0)} = u_{1,j} \text{ if } r > \frac{1}{2}$$

$$0 \leq i \leq M; j \geq 0$$

$$u_{i,j+1}^{(n)} = (n-1)(u_{1,j+1}^{(n)} - u_{1,j+1}^{(n-1)}) + u_{1,j+1}^{(n)}$$

where:

$$u_{1,j+1}^{(n)} = \frac{r}{2+2r}(u_{1-1,j} + u_{1+1,j} - 2u_{1,j}) + \frac{1}{1+r}u_{1,j} \\ + \frac{r}{2+2r}(u_{1-1,j+1} + u_{1+1,j+1}^{(n-1)}) \\ + \left[\frac{1}{1+r} k \exp \left\{ - \frac{1}{\left(\frac{u_{1,j} + u_{1,j+1}^{(n-1)}}{2} \right)} \right\} \right]$$

where:

$$u_{1-1,j+1}^{(n)} = u_{0,j+1}^{(n-1)}$$

$$i = 1; j \geq 0$$

$$u_{1-1,j+1}^{(n)} = u_{1-1,j+1}^{(n)}$$

$$2 \leq i \leq M-1; j \geq 0$$

$$\text{Op I} \quad u_{0,j+1}^{(n)} = \frac{Hhu + u_{1,j+1}^{(n)}}{1 + Hh} \quad n > 0; j \geq 0$$

$$\text{Op II} \quad u_{0,j+1}^{(n)} = u_{1,j} + Hh(u_g - u_{0,j}) + \left[k e^{-1/u_{0,j}} \right] \quad n > 0; j \geq 0$$

 $r = \frac{1}{2}$ only

$$\text{Op III} \quad u_{0,j+1}^{(n)} = \frac{u_{1,j+1}^{(n)} + Hhu + \frac{1}{2r} \left\{ u_{0,j} + \left[k e^{-1/u_{0,j+1}^{(n-1)}} \right] \right\}}{1 + Hh + 1/2r} \quad n > 0; j \geq 0$$

All Options:

$$u_{M,j+1}^{(n)} = u_{M-1,j+1}^{(n)}$$

CMM - BV

$$d_0 = 0$$

$$d_1 = \frac{r}{2+2r}(u_{0,j} + u_{2,j} - 2u_{1,j}) + \frac{1}{1+r} u_{1,j} + \frac{r}{2+2r} u_{0,j+1} + \left[\frac{1}{1+r} k r \circ u_{1,j} \right] \quad j \geq 0$$

$$d_i = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r} u_{i,j} + \left[\frac{1}{1+r} k r \circ u_{i,j} \right] \quad 2 \leq i \leq M-2; j \geq 0$$

$$d_{M-1} = \frac{r}{2+2r}(u_{M-2,j} + u_{M,j} - 2u_{M-1,j}) + \frac{1}{1+r} u_{M-1,j} + \frac{r}{2+2r} u_{M,j+1} + \left[\frac{1}{1+r} k r \circ u_{M-1,j} \right] \quad j \geq 0$$

$$d_M = 0$$

$$b_0 = 0$$

$$b_1 = -\frac{r}{2+2r}$$

$$b_i = \frac{-r/2+2r}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \quad 2 \leq i \leq M-2$$

$$b_{M-1} = 0$$

$$q_0 = 0$$

$$q_1 = d_1$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r}\right)q_{i-1}}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \quad 2 \leq i \leq M-1$$

$$u_{M-1,j+1} = q_{M-1}$$

$$u_{i,j+1} = q_i - b_i u_{i+1,j+1} \quad 1 \leq i \leq M-2$$

CNS - ND Op I

$$d_0 = Hh_g$$

$$d_i = \frac{r}{2+2r} (u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r} u_{i,j} + \left[\frac{1}{1+r} k_0^{-1/u_{i,j}} \right]$$

$$1 \leq i \leq M-1$$

$$d_M = 0$$

$$b_0 = \frac{-1}{1+Hh}$$

$$b_i = \frac{-r/2+2r}{1 + \left(\frac{r}{2+2r} \right) b_{i-1}}$$

$$1 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = \frac{d_0}{1+Hh}$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r} \right) q_{i-1}}{1 + \left(\frac{r}{2+2r} \right) b_{i-1}}$$

$$1 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1+b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{i,j+1} = q_i - b_i u_{i+1,j+1}$$

$$0 \leq i \leq M-1$$

CNN - ND Op II $r = \frac{1}{2}$ only

$$d_0 = 0$$

$$d_1 = \frac{r}{2+2r}(u_{0,j} + u_{2,j} - 2u_{1,j}) + \frac{1}{1+r} u_{1,j} + \frac{r}{2+2r} u_{0,j+1} + \left[\frac{1}{1+r} k_0 - 1/u_{1,j} \right]$$

$$d_i = \frac{r}{2+2r}(u_{i-2,j} + u_{i+2,j} - 2u_{i,j}) + \frac{1}{1+r} u_{i,j} + \left[\frac{1}{1+r} k_0 - 1/u_{i,j} \right] \quad 2 \leq i \leq M-1$$

$$d_M = 0$$

$$b_0 = 0$$

$$b_1 = -\frac{r}{2+2r}$$

$$b_i = \frac{-r/2+2r}{1 + \left(\frac{r}{2+2r} \right) b_{i-1}}$$

$$2 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = 0$$

$$q_1 = d_1$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r} \right) q_{i-1}}{1 + \left(\frac{r}{2+2r} \right) b_{i-1}}$$

$$2 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1 + b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{1,j+1} = q_1 - b_1 u_{1+1,j+1}$$

$$1 \leq i \leq M-1$$

CNM - ND Op III'

$$d_0 = Hh u_g + \frac{1}{2r} \left\{ u_{0,j} + \left[k e^{-1/u_{0,j}} \right] \right\}$$

$$d_1 = \frac{r}{2+2r} (u_{1-1,j} + u_{1+1,j} - 2u_{1,j}) + \frac{1}{1+r} u_{1,j} + \left[\frac{1}{1+r} k e^{-1/u_{1,j}} \right]$$

$$1 \leq i \leq M-1$$

$$d_M = 0$$

$$b_0 = \frac{-1}{1+Hh+1/2r}$$

$$b_1 = \frac{-r/2+2r}{1 + \left(\frac{r}{2+2r} \right) b_{1-1}}$$

$$1 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = \frac{d_0}{1+Hh+1/2r}$$

$$q_1 = \frac{d_1 + \left(\frac{r}{2+2r} \right) q_{1-1}}{1 + \left(\frac{r}{2+2r} \right) b_{1-1}}$$

$$1 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1+b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{1,j+1} = q_1 - b_1 u_{1+1,j+1}$$

$$0 \leq i \leq M-1$$

DEF

Compute 2^k rows by FD technique using r^1 , where $r = r^1 \cdot 2^k$; $0 \leq r^1 \leq \frac{1}{2}$.

Let 2^k th row by FD be $j = 1$ (first row) for DEF. Then:

$$u_{1,j+1} = \frac{2r}{1+2r}(u_{1-1,j} + u_{1+1,j} - 2u_{1,j-1}) + u_{1,j-1} + \left[\frac{2}{1+r} k \epsilon \right]$$

$$1 \leq i \leq M-1; j \geq 1$$

$$\text{where } \epsilon = \begin{cases} u_{1,j} & \text{BV} \\ -1/u_{1,j} & \text{ND} \end{cases}$$

The boundary conditions are the same as for the FD method, both BV and ND.

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