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RESEARCH MEMORANDUM

DIFFERENTIAL GAMES I: INTRODUCTION

Rufus Isaacs

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SUMMARY

This report is the opening chapter of the theory and application of differential games. It embodies the role of differential games in general game theory as well as its contrast thereto. The basic concepts are informally presented in terms of the more cogent applications. The latter are largely military such as pursuit, battle, and aiming games. Later pages assess the relevancy of this theory to practical ends, show that the best schemes for automatic guidance devices are really optimal strategies, and compare mechanical guidance with the human pilot.

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DIFFERENTIAL GAMES I: INTRODUCTION

Rufus Isaacs

After orienting them in and contrasting them to current game theory, we shall give an informal description of the concept of differential games. Sharper definitions will be reserved until the next chapter. Here we approach the subject through its applications.

1. The Theory of Games. It is only a few years since J. von Neumann and O. Morgenstern innovated the theory of games.* Their work gave us such essential items as the concept of a strategy, the value of a game and, probably most important of all, a workable and sound delineation of an optimal strategy which might be either pure or mixed.

At once the new field assumed the status of an accepted branch of science. The subject was innundated by papers nurturing the theory and expanding the applications. Prominent in the latter category were problems of military strategy. During the past several years, many clever people have attempted to utilize this new instrument so as to transfer both the classical and new problems of warfare from the domain of military (and hence human) judgment to the realm of a mathematical science. The ideal is a mathematical mill which, upon the receipt of data descriptive of the conflict, yields exact formulas for the making of decisions.

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* The Theory of Games and Economic Behavior. It is desirable, but not absolutely necessary, that the reader peruse this volume or one of the other basic texts that have since appeared. At any rate, he should be conversant with the basic terms, such as those we have used in our opening paragraph. To this end we have set forth in an appendix those few rudiments of general game theory requisite for the present work.

The same attitude has prevailed in other fields. Witness, for example, the recent growth spurts in linear programming and operations research. There seems to be a new vista, a new facet of current technical progress.

The von Neumann-Morgenstern work has become a kind of classic in what must be record time. It is devoted exclusively to discrete games. We remark the parallelism to the historic first tracts on probability by Pascal and Fermat. These works, too dealt only with the discrete; in fact, games of chance. But it was not until these narrow bounds were burst that the theory of probability realized the full scope it has today.

In the theory of games, it was not long after its origin that its proponents turned to continuous models. But the compass was still shackled. The founders had presented their ideas largely in terms of the game matrix. Excellent as is this device for certain theoretical purposes such as the basic proofs of theory, it is cumbersome for games where the sequence of moves is lengthy.

Perhaps we put the blame wrongly here; games of many moves are essentially difficult (in practice, if not in principle) when there are no systematic connections between the moves. In chess, for example, there is no recourse for the analyst save an enumeration of the consequences (and the consequences of consequences, etc.) of moves; the number is, of course, astronomical. If there is no rational and systematic way to link one move with the next, the mathematician can hardly be expected to get a grip on the problem. In recreational games -- an instructive arena as the specimens are familiar to all -- we know of only two exceptions. One is the game of nim,* where the information is complete and the optimal strategies pure; the other is "Guess it," a game of incomplete information with mixed optimal strategies.

The situation in the case of continuous games is somewhat the same. There is the same difficulty in handling cases where each player must make a multitude of decisions. Worse, there has been a tendency of the subject to float away on the rarified air of pure mathematics. Great effort has been expended on rigorous and recondite theorems, emphasizing such details as the sharp delineation of the function classes involved, all at the expense of a widening horizon of practical applications.

In a sense, our present subject, differential games, falls between the discrete and continuous. But actually it is different from both the two extremes, and, we believe, vastly richer in application than either. Some applications will be taken up in the succeeding sections; now we will endeavor to depict the subject in broad terms.

In a differential game the concept of a move dissolves. Instead the players maneuver continuously. For what in a discrete game is a sequence of moves, the counterpart here is a duration of continuous action. This duration, rather than being the cause of difficulties, is now at the heart of the matter. Airplanes dogfighting, football, fencing, a marksman

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* A Card Game with Bluffing, Isaacs. To appear in the Amer. Math. Monthly.

with a voluntarily moving target, the logistics of a battle, a torpedo pursuing a ship or a missile intercepting a bomber are all differential games. The players each control certain variables with conflicting objectives. The latter, as usual in game theory, is to be framed as a numerically valued payoff, with the players respectively seeking to minimize and maximize it.

The permissible types of control which the players may exert on the variable become what -- to be consistent to our elected genre of "game" -- we must call rules. The motion which ensues from this control is to be at least piecewise differentiable. Thus a large portion of our methods will be interfused with differential equations, long a most fertile tool of the engineer and physicist. Their connection with our work is through what we call the tenet of transition. This is simply a statement that for a small interval between two states of a game the players endeavor to maximize and minimize the change in payoff during the transition. The differential equation arises when we let the duration of the interval approach zero.

It would be terse, but somewhat misleading, to describe differential games as a two-handed calculus of variations. Now our technique, simply by rendering one of the players inert, becomes applicable to minimizing problems. Sometimes they can actually be identified with the classical variational problems. It is even possible to perform the curious feat of incorporating a second, opposing player into some of the traditional problems, thus making games of them.

However, our methods are not the old ones. What we will later call the path equations may sometimes be considered as the analogue of the Euler equations, but as the reader will later see, there is a wide divergence of viewpoint. Our technique has utility for solving minimizing problems (one-player games) and the verification theorem (Chapter III) furnishes a new method of proving the validity of solutions for some of the old problems.

To crystallize this general discussion into something definite, we will in the next section use pursuit games both for themselves and to exemplify our ideas. We will extract from the general problem of pursuit and evasion the genesis of differential games. The line of thought is typical enough to be generally instructive. Throughout we will assume perfect and instantaneous information as to the situation on the parts of both players. We deal only with zero-sum, two-person games.

2. Pursuit Games; Objectives. This application was the origin of the present subject. The various guises pursuit games may assume in warfare are legion.

To obtain a very general picture we will denote the pursuer by P and the evader by E. In simple pursuit games, P and E can each be considered as some kind of craft such as a torpedo and ship or an interceptor missile and invading bomber. These craft can be steered by either a human pilot or an automatic mechanism. In more complex versions we may have more craft such as several small fighter planes opposing an enemy fleet of bombers or --

to leap to a new arena -- several tacklers against a ball carrier with interference runners in a football game. Here P and E must be unified; in game theoretic jargon they are the "players" and are to be thought of as a single controlling brain for each side.

Actually pursuit games will be but one of the applications of the methods to be developed in later chapters. But if we wish a distinguishing characteristic, we can say that pursuit games are those which end when capture occurs. Capture, in turn, is said to occur when distance between the pursuing and evading craft becomes less than a certain prescribed positive quantity δ.

To clarify our ideas let us settle on some definite typical instance. For E we take an invading bomber, either a plane or guided missile, and for P a defending interceptor, likewise either a plane or missile. First we ask the question: how best should P pursue E? That is, if at each instant of time, P knows his own and E's position, how should P at this instant regulate the various navigation or steering variables under his control? By position we mean not only the geometric location of P or E but also all other relevant magnitudes such as flight direction, orientation, velocity, etc. Position in this sense is a basic concept; it will be discussed in detail shortly.

Secondly, we must decide what is meant by "best." In terms of game theory, we must select a payoff. The most obvious criterion is whether capture can be achieved at all. In such a case, where we are interested only in two (or any finite number) of outcomes, we shall speak of the problem as a game of kind.

(in contrast to <u>games of degree</u> which have a continuum of outcomes). But P may be an interceptor with a limited fuel supply. Then a more realistic criterion would be based on whether capture can be achieved prior to the elapse of certain stipulated time. If E is a bomber whose objective is to reach a certain target, the point of interest may be whether capture can be attained before E reaches his destination. If P is to use a gun, rocket or some such weapon, capture will consist of bringing E within range. If there is but a probability of hit less than certainty, P may wish to keep E within range for a stipulated period.

All of the above criteria are discrete, or rather of a "yes-or-no" variety, and we catalogue them as games of kind. But there are also cases where there is some variable quantity which the opponents conflictingly seek to maximize or minimize. This quantity is the <u>payoff</u> and the game becomes one of degree. It is often possible to find such a continuous payoff in such a way that the aforementioned discrete criteria are automatically subsumed within it. For example, suppose we are interested only in whether capture can be achieved or not. We can pick as payoff the time of capture, with P's objective being to make this quantity as small as possible and E's, as great. No capture at all corresponds to infinite time. Then if P acts in accordance with this dictate, he will certainly achieve his primary objective of just capture whenever possible. And more. He will do it as rapidly as possible. Again suppose the objective was initially capture before a stipulated time T. By minimizing the time of capture, P will surely succeed if it is possible for him to do so. We need but look at the minimal value of the capture time that P can attain and see whether or not it exceeds T.

This idea is fairly general. If the original desideration was whether or not E could attain a certain proximity to a certain target, we can make the payoff the distance from the target when capture takes place. By having P strive to maximize this quantity, we are assured that he will not only attain his objective of protecting the target when possible but also the biggest margin of safety, or smallest deficiency if he can't frustrate E.

We answer our question as to what is meant by "best" in all cases by deciding upon a numerically valued payoff. For games of kind we do this a bit artificially by assigning two (or more) numerical values to be the payoff for the two (or more) outcomes. "Best" for P means making this payoff as small* as possible.

Supposing that a payoff is elected, how shall P minimize it? If he is pursuing a missile E, how should he act? Should he, for example, using data culled from measurements of E's position, endeavor to extrapolate E's future course and meaneuver so as to head him off?

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*It is convenient to canonize matters by always letting P be the minimizing player. When things appear otherwise, as in the distance from target example above, we can maintain this uniformity by working with the negative of the natural payoff.

A brief reflection shows that such questions are meaningless. Their answer depends on how E is going to behave. If he adopts the naive policy of traveling in a straight line at constant speed, then of course P should head him off and it is a simple matter to compute the best way of his doing so. But should P always act so and if E is astute he can easily frustrate P by feinting false directions and decoying P into false predictions. There is no plan of pursuit for P which will be optimal under all types of opposition.

It is here that we see that the theory of games enters and permeates the subject. We cannot talk of optimal pursuit without also speaking of optimal evasion. The behavior possibilities for both opponents must be considered together before we can evolve a technique for analyzing the situation. Such is precisely what is done by the present theory of differential games.

Optimal evasion then enjoys a status ranking with optimal pursuit. All the remarks made a few paragraphs back about P's objectives for pursuit have their counterparts about E's objectives for evasion. For example, we might have (and indeed should have) spoken of E's methods for avoiding capture or at least forestalling it until the elapse of T. We might have spoken, in case distance from the target at capture was the payoff, of how E is to maximize it. In actual warfare, of course, both sides will have to consider both classes of questions. It was only to ease our exposition that we wrote the earlier discussion of objectives and payoffs from P's point of view.

There is a kind of inevitability about the introduction of game principles into certain pursuit situations. Suppose that Nation A, at war with Nation B, designs an interceptor missile for protection from B's bombing missiles. In the design of the guidance mechanism, A's engineers presuppose a fixed behavior of their targets. This supposition is well founded and the interceptors prove highly successful. Disconcerted, B examines some specimen interceptors that have fallen into his hands and notes that their navigation scheme is predicated on the bomber missile's flying a very direct, nearly straight course to their destinations. The situation is remedied; B incorporates in his aircraft an evading mechanism, designed to frustrate A. If the improvement is successful, A will be forced to retaliate with a less naive pursuit mechanism.

The result will be a kind of technological see-sawing with each side striving to outdo the other. Indeed, this is a common phenomenon in modern warfare. As it progresses, the pursuit and evasion schemes will more and more resemble optimal strategies. The introduction of game considerations at an early stage by either side is thus an anticipation of the ultimate status, a rational recognition of reality.

3. Pursuit Games; Navigation. Our next step is to clarify the kinematics of the craft involved in a pursuit game. As in all applications of mathematics to physical and engineering

problems, we must settle upon a model that simulates reality simplified to some degree. This simplification is here, as always, a compromise between our desire for realistic answers and our ability to solve the mathematics the model betokens.

To display our ideas with the utmost lucidity, we shall pick, as an instance of the kinematic type of craft participating in pursuit games, an automobile. Our motive is simply that here is a vehicle whose behavior is known to all. The ideas hold, with at most small changes, for almost any craft or vehicle; we can apply them to tanks or ships -- outboards to cruisers. Aircraft, of course, move in three-dimensional space, but the principles are the same.

Three variables will be needed to describe the auto's geometrical position, namely x_1 and x_2, the cartesian coordinates of some fixed point on the car, and x_3, the inclination or direction in which the car is pointing. That is, x_3 is the angle between the main axis of the car and the x_1 axis as drawn on the unbounded, empty parking lot which we will take for our theatre of operations. But if the auto is to figure in a differential game, we need more. Let us assume that the car is navigated by an accelerometer and steering wheel. The former we will suppose controls the tangential acceleration. This quantity, being under the control of a player, will be termed a navigation variable, and be denoted by ϕ_1. The speed x_4 is not <u>directly</u> under the control of the driver, but it is a quantity both players of a pursuit game would have to take into account as much as x_1, x_2, x_3. Hence it, too, will be considered a descriptive variable.

The position of the steering wheel determines the curvature of the car's path. But to say that the driver can vary it from instant to instant is not realistic. Rather we will take the curvature of the car's path as another descriptive variable x_5 (physically, it is visible from the inclination of the front wheels) and its rate of change as another navigation variable ϕ_2.

Under these assumptions, the motion of the automobile will be governed by the following differential equations:

$$\dot{x}_1 = x_4 \cos x_3 \tag{1}$$

$$\dot{x}_2 = x_4 \sin x_3 \tag{2}$$

$$\dot{x}_3 = x_4 x_5 \tag{3}$$

$$\dot{x}_4 = A\,\phi_1 \qquad (0 \le \phi_1 \le 1) \tag{4}$$

$$\dot{x}_5 = W\,\phi_2\,. \qquad (-1 \le \phi_2 \le 1) \tag{5}$$

Here (1), (2) is simply the decomposition of the car's velocity into its axiswise components; (3) states that the rate of change of direction is the speed times the curvature. In (4) and (5) we have made a slight alteration of our definitions. In (4), A is greatest acceleration of which the car is capable and ϕ_1 is fraction of it under the driver's control. The convenience of having uniform bounds for the navigation variables will appear later. Likewise in (6) W is assumed to be the greatest rate at which it is possible to change the curvature.

To recapitulate: $x_1, \cdots, x_5$ describe the aspects of our vehicle which would be relevant were it a participant in a pursuit game and are called descriptive variables. The driver controls ϕ_1 (the accelerometer) and ϕ_2 (the fractional speed at which he turns the steering wheel). They are the navigational variables. In a pursuit game it is they that are at each instant under the control of a player. They are not measurable by the opponent; the descriptive variables are.

The reader will at once perceive the deficiencies in our model. The most glaring is that there is no check on the speed. This could be remedied by putting a bound on x_4, but a more realistic diagnosis would lay the blame on (4). First, it is probably oversimplifying automobile dynamics to state that the force exerted by the motor is proportional to the amount we depress the accelerator pedal; secondly, and more important, this force is proportional to the acceleration of the car only if we neglect friction. If we assume, for the sake of simplicity, that friction is negatively proportional to the speed, a better version of (4) would be

$$\dot{x}_4 = F(A\,\phi_1) - \kappa\, x_4 .$$

Here $A\,\phi_1$ $(0 \leq \phi_1 \leq 1)$ is the amount the accelerator pedal is depressed,* F the resulting force (per unit mass of the car) exerted by the engine, and κ a friction coefficient. The speed now will be bounded by $F(A)/\kappa$.**

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*Thus ϕ_1 has changed in meaning again.

**If the car travels in a straight line, with the accelerator fully depressed, a simple application of differential equation theory shows that this quantity is always the limiting value of the speed.

Another essential correction consists of bounding the curvature x_5. (Anyone who has tried to turn his car in a narrow street needs be told no more.)

Thus the equations of motions may be complicated for a closer simulation of reality or simplified for easier mathematics. In most of the sample problems to be given in later chapters we will modify in the latter direction, as we do not wish to encumber these pages with tortuous mathematical derivations.

Let us take a second example. Here a point will move in the plane with no other restriction* save that its speed v is constant. The controlling player picks the direction of travel, and may change it abruptly at any instant. There is a single navigation variable ϕ, the direction. The kinematic equations are

$$\dot{x}_1 = v \cos \phi \qquad (6)$$

$$\dot{x}_2 = v \sin \phi \qquad (7)$$

Here we have but two descriptive variables, x_1 and x_2. This situation will be spoken of as <u>simple</u> motion.

Descriptive variables enjoy the two properties:

1. Their values must be known at the outset in order to determine the outcome of a game.

2. They are exactly the values that are relevant to a player making decisions as to how to play.

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*That is, within reason. We outlaw highly pathological paths such as will occur to the mathematician. If he demands a sharper definition, let the direction be piecewise continuous.

Remark, that in the case of simple motion, the velocity is not a descriptive variable; in fact, both properties fail.*

In a pursuit game, there will be a number (at least two) of craft. Let the descriptive variables for each be listed and let the aggregate of all the lists be $x_1, x_2, \cdots, x_n$. Likewise let $\phi_1, \cdots, \phi_\lambda$ be the aggregate of all navigation variables under the control of P and $\psi_1, \cdots, \psi_\mu$ those controlled by E.

For example, if P controlled one "automobile" and E controlled two craft, each with simple motion, we would have $n = 5 + 2 + 2 = 9$, $\lambda = 2$, $\mu = 2$.

The motion is governed by the <u>kinematic</u> <u>equations</u>:

$$\dot{x}_j = f_j(x_1, \cdots, x_n, \phi_1, \cdots, \phi_\lambda, \psi_1, \cdots, \psi_\mu) \quad (j = 1, \cdots, n) \tag{8}$$

of which (1), $\cdots$, (7) are instances.

4. <u>Pursuit Games; Strategies</u>. In conventional game theory** a strategy for a player consists of a set of decisions which tell what move he will make for every possible position that may arise in the course of a play. If each player selects a strategy, the outcome of the game is completely determined thereby.

The natural analogue in differential games is the selection of the navigation variables as functions of the descriptive variables. Thus things are as they should be; for each position that may arise --

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*See Chapter 2.

**We are speaking of games with perfect information.

that is, set of values of the descriptive variables -- each player has decided upon his course of action -- that is, he has chosen values for his navigation variables. It is easy to see that here, too, the outcome is determined. Let P select the ϕ_j and E select the ψ_j as functions of the x_j. If these functions are reasonably simple and they are substituted in (8), the right side becomes a function of the x_j. Then (8) becomes a set of simultaneous ordinary differential equations. They are to be integrated, employing as initial conditions the particular values of x_j at which the play starts. The solutions will give the x_j as functions of the time t and will describe the action that ensues from the chosen strategies.

The payoff can now be computed. We find ourselves in a game theoretic terrain. The objectives of the players are to pick the $\phi_j(x)$ and $\psi_j(x)$ so as to minimize and maximize the payoff respectively.

Although this heuristic description of the mechanism of our theory consorts well with practice, in the next chapter we shall give a sharper definition of a strategy. The principle will remain the same, but certain difficulties, which the mathematician will at once perceive, will be alleviated.

In an actual pursuit and evasion contest, the pilots may be human or automatic guidance mechanisms. If the situation is pre-analyzed by the methods of the present text, the human pilot will have to be instructed accordingly. The nature of this instruction will be that the pilot is to simulate more or less an automatic mechanism.

A mechanical guidance system must comprise, first, a detection device to obtain the relevant information about the opponent's current status and, secondly, a decision device which regulates the steering controls in accordance with this information. Our concern here is how best to design the decision mechanism. The answer is immediate and striking. <u>A design scheme for the decision device is exactly a choice of a strategy</u> in the sense just discussed. For what the detection device does is to ascertain the values of the descriptive variables; what the decision device does is to select values of the navigation variables dependent upon them. Thus a design scheme simply prescribes that the navigation variables will be certain functions of the descriptive variables. But this is precisely what we termed a strategy.

Thus our theory is directly applicable to guidance mechanisms for such missiles and other craft used in combat. If the combat situation has been realistically formulated as a differential game, the game analyzed and the optimal strategies found, these strategies will provide the best possible type of decision device. A human pilot can do no better. Of course there are always the possibilities of the unforeseen circumstance, the deviation of the actual system from our mathematical model, unknown values of certain parameters of the enemy craft, etc., possibilities which render the advantages of a human pilot obvious. In a later section we shall dissect some very simple examples, throwing a bit of light on such matters.

5. Battle Games. We will continue to use the letters P and E, but now they will denote the antagonists of a war. Or rather a mathematical model of some phase enjoying as much realism as is mathematically tractable.

We compile a list of variables (for both sides) that are to be indicative of the current state. These will include such items as numbers of men, aircraft, tanks, ships and other munitions, possibly subdivided by allocation to different sectors or by classification into different types, the number of miles advanced by a front, etc. They will be the descriptive variables.

The kind of decisions P and E are to make should next be scrutinized and framed in terms of navigation variables (no longer an apt term). We require here a deviation from reality. To fit our theory, the decisions must be continuous processes rather than discrete. For example, one factor might be the percentage of available aircraft to be used daily to attack a particular enemy target. While in actuality this figure will be selected once a day, for our purposes we take it to be a continuous function of time. In fact, it will be a typical instance of what we have called a navigation variable. The amount of aircraft involved will also be taken as not necessarily a whole number; it will be a descriptive variable. Thus our model yields a smoothed over, large scale, approximate picture. The compensating advantage is the more efficient techniques available for solving, when we exploit the extant tools of differential analysis.*

*We do not advocate complete abandonment of the discrete model. In later chapters, we will show how our "differential" approach can be replaced by a "difference" approach, suitable for discrete cases. Of course, the computations are generally more laborious.

As before, we denote by ϕ_j the navigation variables under the control of P and by ψ_j those of E.

Kinematic equations of the type (6) must be constructed as best we can. The fidelity of our model to actuality will largely depend upon how well this is done. Each equation expresses how the rate of growth of a descriptive variable is affected by other factors. For example, suppose x_1 represents the number of men at a certain sector for side P. Enemy air raids may deplete this number. Suppose x_2 is the number of E's aircraft available for this purpose and he decides to send over at a certain time a fraction ψ_1 of them. We must now decide from experience or otherwise how the expected number of casualties depends on the number $\psi_1 x_2$ of present enemy aircraft. Let us suppose it is directly proportional with a constant c picked empirically. Let us also suppose P furnishes replacements to the post in question at a fixed rate r. Then one kinematic equation will begin to emerge:

$$\dot{x}_1 = r - c\,\psi_1 x_2 + \cdots$$

The dots signify various other terms such as more negative ones arising from other means of attack by E or the deployment of the men for other purposes by P and positive terms such as may arise from defensive measures of P against E's attacks.

The final step in the formulation of the problem as a differential game is a critical and important one. We seek "best" strategies and again we must decide what is meant by "best." In other words, we must choose a payoff. Whatever its nature, we can always arrange matters so that again it is a quantity which P is seeking to minimize and E to maximize.

The objective of the battle may be of some concrete, yes-or-no, type. We have a game of kind. For the payoff we select two numbers corresponding to whether or not the objective is attained, the smaller number pertaining to P's desideratum. As before, it is often sensible to imbed the game of kind in a game of degree. For example, a side may endeavor to attain the objective with a minimum expenditure of some item. The payoff will then generally be of the form of an integral of some quantity extending over time from the beginning to the end of the play.

In a war of extermination, the game is over when one side is annihilated. For a payoff we might take the number of survivors of the victorious side. In other cases, the payoff might be taken as the excess of strength of E over P at the end of a stipulated time T. Here strength may relate to men, aircraft, or, in general, any one or combination of the descriptive variables.

6. Dogfight and Aiming Games. The prototype here is aircraft or other maneuvering vehicles equipped with some type of medium range fire power. The kinematics are as for pursuit games, but the objective is to bring the opponent within range of fire. The games differ essentially from pursuit games only when they concern such craft as single seater airplanes with a gun that can be aimed only by turning the entire ship.

We obtain a problem quite similar to a pursuit game when E cannot retaliate. The objectives are for P to get E within range and for E to stay out of it. If P's hit probability is less than certainty, we can formulate a payoff as follows: We fix on a function of the descriptive variables which assumes low values if and only if the range situation is favorable to P. Using T as a stipulated value of the duration, the payoff will be the time integral of this function from 0 to T.

If both players are armed, a similar function can be written for E, and a promising payoff is the same time integral over the difference of the two functions. On the other hand, if a hit is regarded as fairly certain for the player who first gets his opponent into a vulnerable position, a payoff can readily be constructed expressive of this situation and will lead to a game of kind.

There are games of this type in which P has no motion at all. Such occurs for a gunner versus a mobile target; the variables for the former player relate to the aiming of the gun. However, the trenchant aspects of the problem relate to time lag and other such uncertainties. The game is one of incomplete information.

7. Programming, Economic, and Athletic Games. We have already remarked that our methods are applicable to certain minimizing problems. All we need to do is consider E inactive, that is, deprived of control over his navigation variables. We are thus provided with a tool for a certain class of problems

where the goal is an optimal program of operations. Such occur in the field today known as operations research. Again we must remark that the benefits of our methods are most advantageous when the processes are continuous rather than discrete.

For all we know there may be an opulence of applications of proper (that is, two-sided) differential games to economics, but at the present time we have not been able to coin any that have an authentic ring. Business firms -- at least in theoretical models -- have the single incentive of maximizing profits. If we try to fabricate a gamelike situation between competitors with a payoff of this kind, we find the firms engaged in an unconvincing joust where the losses of one's rival are as desirable as profits to oneself.* Employment of direct and drastic means to eliminate competition leads to fantastic games with a Machiavellian air.

On the other hand, we can fit our theory nicely to certain athletic games. Football, for example, is a natural. A single ball carrier opposed by a single tackler is nothing other than a simple pursuit game with distance from the goal to the point of capture as payoff. The situation is still in our scope if there are several tacklers and interference runners.

8. Information. All the games that we have discussed are games of complete information. Technically this simply means that at all times both players know the current values of the

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*The source of the difficulty is that such games are not zero-sum.

descriptive variables.* Practically -- say, in a pursuit game -- it means that each player has perfect knowledge of his opponent's whereabouts and can act instantaneously on this knowledge. (For if he could not, his action is tantamount to that based on information with a time lag which in turn is equivalent to a situation with ignorance of the current state.)

Admittedly this situation is unrealistic. We have endeavored to cope with the difficulty and have succeeded only in some simple, discrete examples which will be expounded in later chapters. Judging from their solutions, differential games with incomplete information will be an astonishingly difficult subject. But it is not hopeless. Nor are our methods rendered nugatory by this hiatus. Let us orient ourselves to obtain a more rounded understanding.

Extant game theory has much to say on the subject of information. Proven results for discrete games are that when both players have full information, an optimal (pure) strategy exists for each. In general, if the information is incomplete this is not true, but there do exist optimal mixed strategies. Such means the selection of a pure strategy in accordance with some probability distribution. What is meant is that there is no definite prescription as to how a player should best act, but at each position, there will be specified a best probability distribution for the several moves at his disposal. His mixed optimal strategy employs some chance device, so that his decisions

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*Past values are known too, but are not relevant.

are made with these specified probabilities.

We can confidently expect a similar situation to hold for differential games without complete information. To fix our ideas, let us take the case of simple pursuit.

There are several possibilities for information shortages which may occur in practice. We will speak from P's viewpoint; reciprocal possibilities exist for E. The pursuer may know E's position* only approximately such as would occur with an imprecise detection device. Or P may know only E's relative orientation and not his distance away. Or P may know E's position only at certain instants such as happens with certain detection devices which function intermittently. Or P may know E's position with a time lag. This lag is always to be taken as the elapsed time between E's assumption of a position and P's responsive action; it matters not if it is all or partly due to sluggishness in the response of P's controls. Observe that a time lag makes feasible E's deceiving P by "feinting." Finally, there is the case when P has no information at all. Here it seems appropriate to term P's action a <u>search</u>, rather than a pursuit.

We delve only into the first of these possibilities. Suppose P at each instant knows only a probability distribution of E's position. We can simplify matters further by thinking of a large sphere at each instant; all P knows is that E is somewhere inside. If at the outset the players are far apart relative to the diameter

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*As usual, "position" entails values of all descriptive variables germane to E.

of the sphere, P will chase its center. Thus in the early phases, play will proceed as in a game with complete information. Generally something analogous holds for any other type of information dearth, so that initially our results for the full information case are at least approximately applicable.

The later phase of this sample game will be of a steady state or "stationary" character. For when P catches up with the sphere,* his policy must be a mixture of searching it and pursuing it. On the other hand, E's policy must be a mixture of hiding and fleeing. Both players persist until capture occurs, and hence the "stationary" quality. We see now how a mixed strategy enters the picture. Suppose P were to conduct his search according to some fixed (pure) strategy. If it dictated, say, his always searching the left side of the sphere first, E could score by then hiding in the right side. It is clear that P must introduce randomness in his search measures; similarly for E's hiding technique.

A promising approach, which would yield a good, if not the best, answer, would be to resort to a direct mixture of the solution of the pursuit problem, where the center of the sphere acts as evader, and the search problem, where P searches a fixed sphere with no information at all. Unfortunately, to the best of our knowledge, the solution of the latter problem is still unknown.

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*We have tacitly supposed P to have kinematic superiority, that is, he could capture were the information complete.

9. Optimal Strategies in Operation. We will now endeavor to elucidate the nature -- and also the limitations -- of the kind of solutions to which our later methods will lead. We importune the reader's tolerance for the artificiality of the two sample pursuit games we shall here display. As we have at this point developed no techniques of solving, the examples will necessarily be of a very simple kind -- ones for which the solution can be spotted intuitively. In both, each player will use simple motion. Thus they take place in an ideal world where there is no inertia and instantaneous changes of velocity are possible.

Ordinarily we define capture as occurring when the two points involved are a distance δ apart. To simplify things here, we take δ very small -- virtually 0.

Example 1. Let P and E have speeds W and w with $W > w$. They move in the plane. The payoff is to be the time of capture.

In this simple framework, with no other extraneous contingencies to worry the players, the solution is apparent. Both players travel along the straight line which joins their initial positions, E fleeing and P pursuing. They travel at their given speeds until P catches E.*

An example is mapped in Figure 1(a), the speeds being shown at (b). Capture occurs at X and the payoff is roughly 15.5 time units. This number is the value of the game, by which we mean:

*A proof hardly appears necessary for this very elementary example, but one can be supplied, for this as well as the next example, by the method of Chapter III.

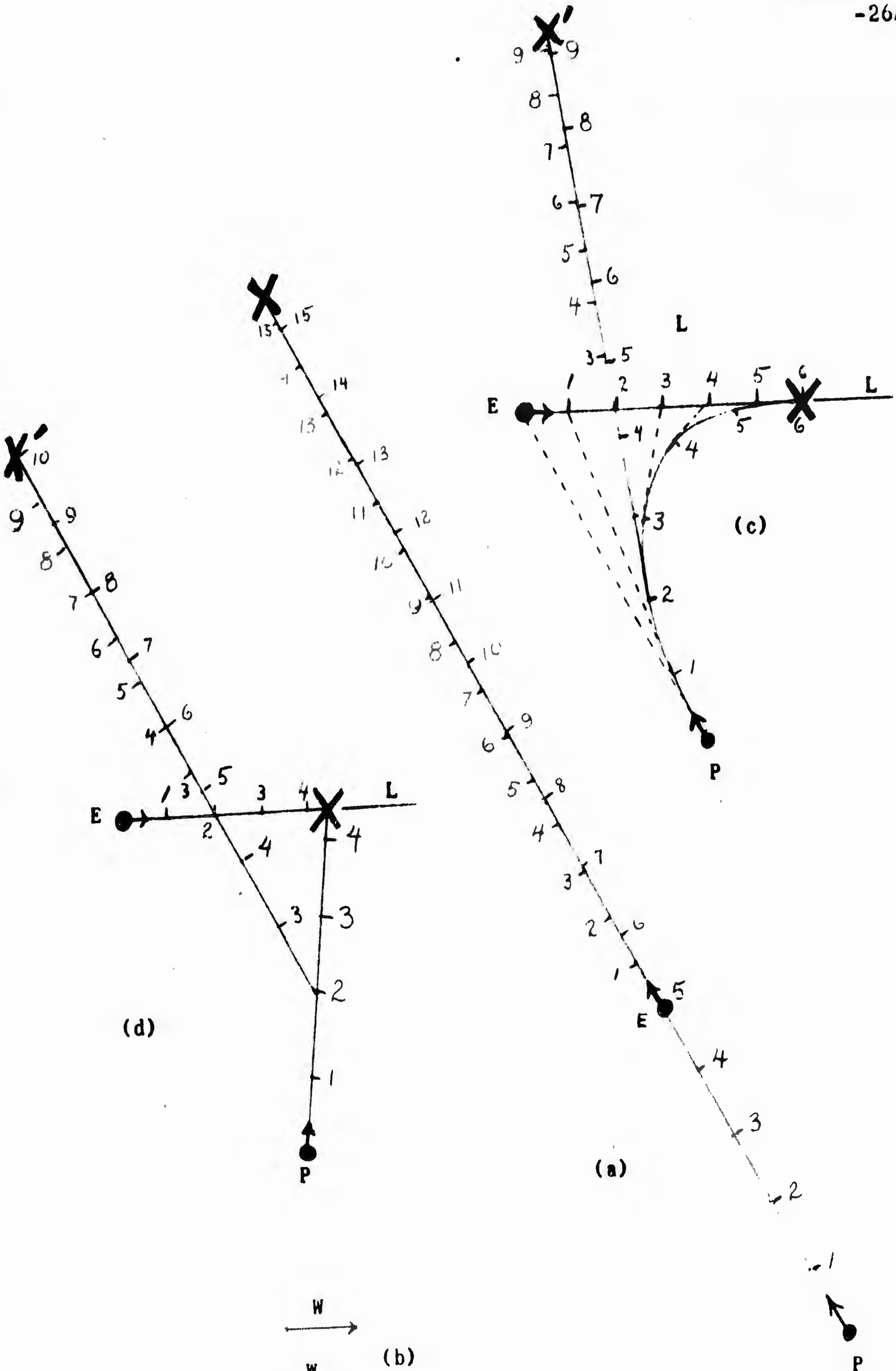

FIGURE 1

No matter what E does, he cannot hold out for longer than 15.5. If he does anything other than direct fleeing, P can capture in less than 15.5 time units. Reciprocally, P cannot capture in less than 15.5, and if he does not directly pursue, E can remain at large for longer than 15.5.

Recalling our concept of a strategy -- navigation variables equal functions of descriptive variables -- we see what the optimal strategies are here. For P, it is to always travel toward E whatever his position. Figure 1(c) represents the same game with E behaving nonoptimally; he travels along the line L. But P behaves decorously and nails E in 6 time units at X.

But, one may interject, P can do better. If E is to travel along this line L, P can extrapolate his future position, calculate a collision point, and go straight to it. The outcome is shown at (d) where P now captures in about 4.3. Why is this policy outside the optimal strategy?

The answer is that P has no grounds for his prediction. We have formulated the game so that E can change direction abruptly at will. Our mathematical model at no time gives P the right to assume that E will persevere on L.

Suppose E remains oblivious of the predatory P until 2 time units have elapsed. Alerted then, he belatedly assumes his optimal strategy, leaving L and fleeing directly away from P. These alternatives are shown on (c) and (d), capture occurring at X' with payoffs 9.3 and 10, respectively. The former figure is the best that P can attain if he exploits all the knowledge at his disposal and no more.

The same situation prevails throughout the play depicted at (c). At each instant P acts as if he will face optimal opposition in the future. This trait of an optimal strategy applies to both players and in all pursuit games as we shall formulate them.

If E is known to be unproficient at detecting pursuers and we wish to take this into account, the thing to do is formulate a new game. We might, say, estimate a probability distribution for the time when E is alerted to P's presence. In the new game the payoff would be the expected value of the capture time and P, presumably, will have a new optimal strategy.

On the other hand, when P notes E's oblivious course along L, is he to construe it to mean that E has no detection or evasive capacity at all? If so, of course, the collision course strategy of (d) is the best. Again a revised game might be constructed incorporating an estimated probability that E is impervious to P's presence. But such seems a jejune approach; this is simply a case where a human pilot's judgment may excel a formal strategy.

Example 2. Both P and E travel with the same speed. The motive of P is to guard a target C, which we take as an area in the plane, from an attack by E. The payoff is the distance from C to the point where capture occurs. We take matters to be as in Figure 2(a) where P and E denote starting positions.

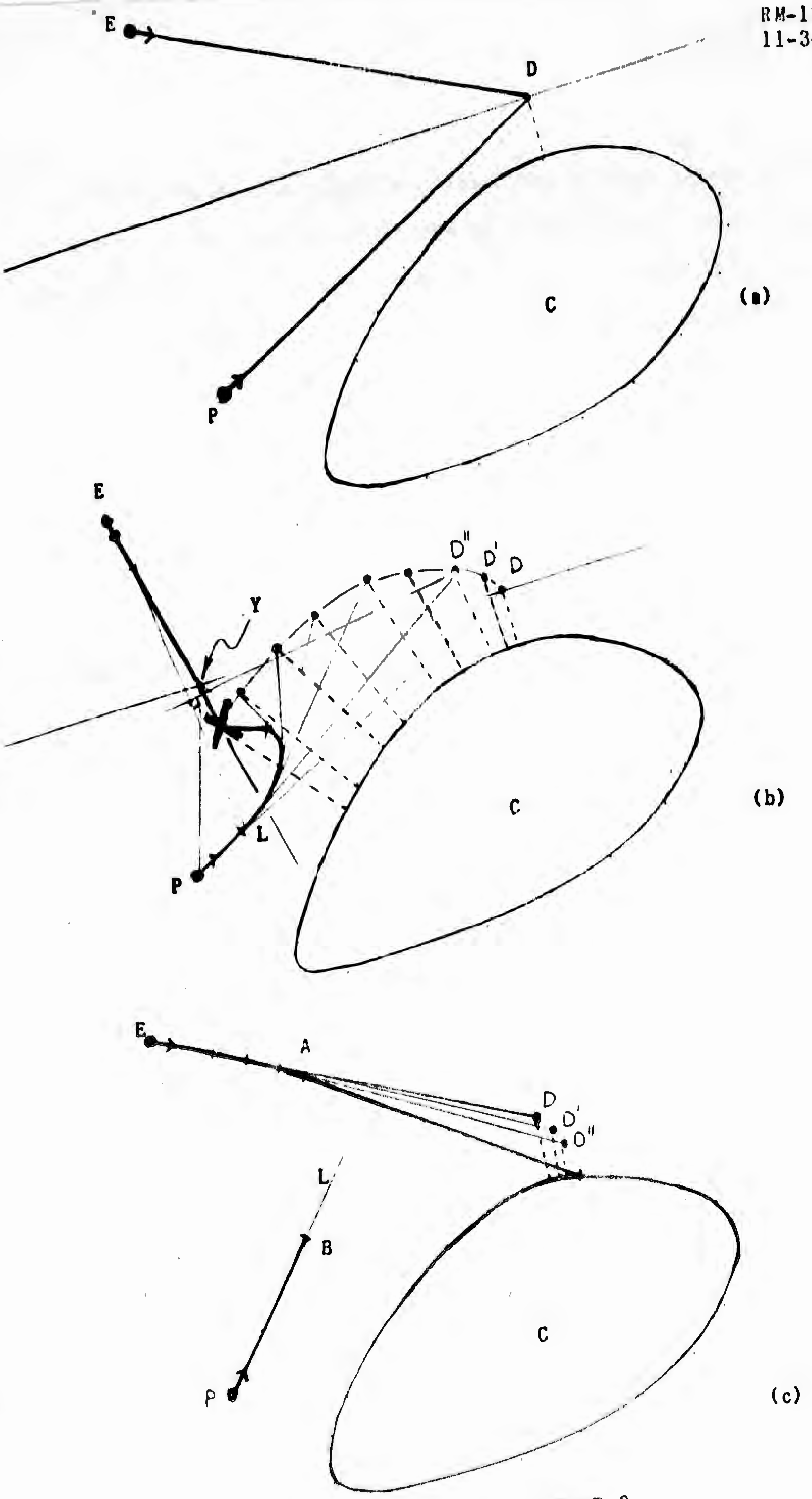

FIGURE 2

The optimal strategies are found thus: Draw the perpendicular bisector of PE. Any point in the half-plane above this line can be reached by E prior to P, and this property fails in the lower half-plane. Clearly E should head for the best of his accessible points. Let D be the point of the bisector nearest C. The optimal strategies for both players decree that they travel toward D. Capture occurs there and the length of the dotted line is the value of the game.

Let us see what happens if P plays optimally, but E does not; say he decides to traverse the line L of Figure 2(b). Always P heads for the nearest point to C on the perpendicular bisector, now drawn relative to the current positions of P and E. Some typical positions of this point are marked D, D', D", ···. Note the increase in the lengths of the dotted segments. They represent the progressive penalty E pays for his poor strategy. Each length is the payoff were E at the corresponding instant to revert to optimal play and hence is the best that E can hope to recoup.

As P aims at the moving point D, (D', D", ···) he describes a curved path until D reaches L. From then on D remains fixed, both players move straight, and, in fact, the play is optimal on both sides. Capture occurs at X.

The same discussion of unwarranted prognosis, given for Example 1, applies here. If P had advance assurance that E would never leave L, his best policy is to travel straight to Y (where the bisector meets L). As it is, he swerves to the right before capture. He knows that when E is near his starting position, the upper right part of C is most vulnerable and he moves to protect it; as E advances the danger relents and P moves in for the capture.

At (c), we see P playing nonoptimally, traversing a line L. Now E should always head toward the moving point D. The dotted segments shrink; when E is at A, D actually reaches C. From then on E moves straight toward this point. He actually reaches C; nothing P can do now will stop him. (P is at B when E is at A.)

10. The Scope of this Work. We have listed in the preceding sections some impressive problems. They will not all be solved in the subsequent pages. But methods for solving them will be given.

Complex differential games can present imposing technical mathematics, routine in principle but oppressive in practice. The straits consist of the integration of systems of first order ordinary differential equations and elimination of variables. No applied mathematician needs be told how formidable such matters may become when the number of variables is large.

However, in this era of analogue computers and such devices, these difficulties should not frustrate a determined effort. The more elaborate problems are fitting subjects for group projects. Reflection should be bestowed until the instances relevant to practice are delineated. With a judicial amount of reason and trial and error, an answer should be attainable.

As this work is by a single author, the examples to follow are but illustrative. None will be included which cannot be solved by elementary means.